Abstract

We would like to achieve a good explicit understanding of higher morphisms of Lie n-algebras. We notice that various formerly puzzling aspects of this seem to become clearer as one passes from Lie n-algebras $\mathfrak{g}_{(n)}$ to their Lie $(n+1)$-algebras of inner derivations $\mathrm{inn}(\mathfrak{g}_{(n)})$ in a certain way. Using this, we define higher morphisms of Lie n-algebras explicitly and in general. These should constitute an $(\infty, 1)$-category. While we fall short of verifying this in full generality, we do obtain the Baez-Crans 2-category of Lie 2-algebras in the special case where we restrict everything to Lie 2-algebras.

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1 Introduction

The crucial point of the discussion is possibly best exhibited by the following simple example:

Let G be any abelian group. Write ΣG for the corresponding one-object groupoid, and $\Sigma^2 G$ for the corresponding one-object one-morhisms 2-groupoid. Write $\mathrm{INN}(G)$ for the codiscrete groupoid over G.

We have canonically a short exact sequence of groupoids

$$G \hookrightarrow \mathrm{INN}(G) \twoheadrightarrow \Sigma G$$

of groupoids.

Using this, we may consider what happens when instead of looking at transformations of ΣG-valued functors

$$\begin{tikzcd} C \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "F_2"'] \arrow[r, phantom, "\Downarrow"] & \Sigma G \end{tikzcd}$$

directly, we first "open up" the corresponding bigon

$$\begin{tikzcd} & \Sigma G \arrow[dr, hook] & \\ C \arrow[ur, bend left, "F_1"] \arrow[dr, bend right, "F_2"'] \arrow[rr, phantom, "\swarrow"] & & \Sigma\mathrm{INN}(G) \\ & \Sigma G \arrow[ur, hook] & \end{tikzcd} ,$$

thus allowing more general transformations, but then in turn imposing the restriction that everything must collaps as we push forward to $\Sigma^2 G$:

$$\begin{tikzcd} & \Sigma G \arrow[dr, hook] & & & & & \\ C \arrow[ur, bend left, "F_1"] \arrow[dr, bend right, "F_2"'] \arrow[rr, phantom, "\swarrow"] & & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma^2 G & = & C \arrow[r] & \{\mathrm{pt}\} \arrow[r] & \Sigma^2 G \\ & \Sigma G \arrow[ur, hook] & & & & & \end{tikzcd} .$$

For this simple example, it is easy to see that the transformations obtained by first "opening up" and then restricting as above are in bijection with the ordinary transformations of the original functors.

As we pass to higher n-groups, though, the situation becomes more flexible.

We find that at the level of Lie n-algebras, it is only the "opened up" transformations which have a good direct description in the first place. We here *define* ordinary higher morphisms of Lie n-algebras by the analogue of "opened up" and then restricted transformations.

2 Arrow-theoretic differential theory

In the world of Lie, one commonly finds the following dichotomy

- In the *integral picture* of Lie n-groupoids and their morphisms, the concepts tend to be rather clear, but all operations tend to be rather technical and unwieldy.

- In the *differential picture* of Lie n-algebroids and their morphisms, the concepts tend to become a little more mysterious, but the computations tend to be comparatively easy.

We shall want the best of both worlds. Our discussion here is focused on constructions and considerations concerning the world of Lie n-algebroids, but we deem it important to keep the corresponding integral picture in mind. It shall be our GPS system whith which we navigate the world of differential graded algebra and coalgebra.

And abstract as it may seem, the following discussion of $(n+1)$-curvatures of n-functors proves to be exactly the right picture to keep in mind for interpreting the constructions in section 3.

2.1 Tangent n-categories

For the present discussion, we set $n = 2$ once and for all and place ourselves in the Gray category of strict 2-groupoids, strict 2-functors between them, pseudo-natural transformations between these and modifications between those. Everything we say ought to have a straightforward generalization to higher n and weaker notions of n-categories, once specified.

Definition 1 (the fat point) *Write*

$$\mathrm{pt} := \{\bullet\}$$

for the terminal n-category, called here the "point", and write

$$\mathbf{pt} := \{\, \bullet \xrightarrow{\sim} \circ \,\}$$

for the slightly puffed-up version of the point, called the "fat point".

Fix once and for all one of the two inclusive equivalences

$$i : \mathrm{pt} \overset{\sim}{\hookrightarrow} \mathbf{pt}\ .$$

Definition 2 (tangent n-category) *For C any n-category, let*

$$TC \subset \mathrm{Hom}_{n\mathrm{Cat}}(\mathbf{pt}, C)$$

be that maximal sub-n-category of all morphisms of the fat point into C, which has the property that it collapses to a 0-category when pulled back along i.

Proposition 1 (properties of the tangent n-category) *The tangent n-category has the following characteristic properties*

- *It is a fibered category over the space of objects of C*

  $$p : TC \to \mathrm{Obj}(C)$$

  which is a "deformation retract" in that

  $$TC \simeq \mathrm{Obj}(C)\,.$$

- *It sits inside the short exact sequence*

$$\mathrm{Mor}(C) \longrightarrow TC \longrightarrow C$$

of n-groupoids.

All these properties carry over to the world of Lie n-groupoids.

There are a priori two different, equivalent, choices for the projection p. We choose the one which is compatible with our choice of $i : \mathrm{pt} \hookrightarrow \mathbf{pt}$.

For the present purpose, we mostly need the dual description of the fibered category:

Definition 3 *We may regard TC as an n-functor*

$$TC : C^{\mathrm{op}} \to n\mathrm{Cat}$$

which sends

$$x \mapsto T_x C$$

for each object x of C.

2.2 Differentials of n-functors

Definition 4 (differential of an n-functor) *For*

$$F : C \to D$$

any n-functor, we write

$$\delta F : C^{\mathrm{op}} \to n\mathrm{Cat}$$

for the n-functor obtained by postcomposition with TD

$$\begin{tikzcd} C^{\mathrm{op}} \arrow[rr, "\delta F"] \arrow[dr, "F"'] & & n\mathrm{Cat} \\ & D^{\mathrm{op}} \arrow[ur, "TD"'] & \end{tikzcd} .$$

Proposition 2 *The differential δF extends essentially uniquely to the $(n+1)$-category* $\mathrm{Codisc}(C^{\mathrm{op}})$.

Definition 5 (curvature) *The $(n+1)$-functor*

$$\delta F : \mathrm{Codisc}(C^{\mathrm{op}}) \to n\mathrm{Cat}$$

provided by proposition 2 we call the curvature $(n+1)$-functor of F.

Remark. Due to the property from proposition 1, this curvature $(n+1)$-functor is, in the category of $(n+1)$-functors, equivalent to the terminal $(n+1)$-functor from $\mathrm{Codisc}(C^{\mathrm{op}})$ to $n\mathrm{Cat}$. What makes δF nontrivial is that we may regard it as sitting in the category of $(n+1)$-functors with values in n-groupoids over C.

Definition 6 *Let*

$$n\mathrm{Cat} \downarrow C$$

be the $(n+1)$-category of n-groupoids strictly over C. Objects in here are n-categories A equipped with a morphism

$$\begin{tikzcd} A \arrow[d] \\ C \end{tikzcd} ,$$

morphisms are n-functors strictly respecting this anchor

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd}$$

and higher morphisms are those strictly vanishing when pushed forward along the anchor

$$\begin{tikzcd} A \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[rr, phantom, "\Downarrow"] & & B \arrow[dl] & & A \arrow[dr] & \\ & C & & = & & C \end{tikzcd}$$

Proposition 3 *The differential $\delta F : C^{\mathrm{op}} \to n\mathrm{Cat}$ factors through $n\mathrm{Cat} \downarrow C$, i.e.*

$$\delta F : C^{\mathrm{op}} \to n\mathrm{Cat} \downarrow C \to n\mathrm{Cat}\,,$$

where the last morphism is the obvious forgetful n-functor.

2.3 Differentials of n-group valued n-functors

The special case of the above considerations of interest here is that where the codomain n-category D is a one-object n-groupoid

$$D = \Sigma G_{(n)}$$

obtained from suspending an n-group $G_{(n)}$.

Proposition 4 (Schreiber-D.M. Roberts) *The tangent n-category $T\Sigma G_{(n)}$ has the following properties.*

- *There is a canonical inclusion*

$$T\Sigma G_{(n)} \hookrightarrow \mathrm{INN}(G_{(n)})\,,$$

  where $\mathrm{INN}(G_{(n)})$ is the inner automorphism $(n+1)$-group of $G_{(n)}$, characterized by the fact that it sits in the exact sequence

$$1 \to Z(G_{(n)}) \to \mathrm{INN}(G_{(n)}) \to \mathrm{AUT}(G_{(n)}) \to \mathrm{OUT}(G_{(n)}) \to 1\,,$$

  where $Z(G_{(n)})$ is the suspension of the categorical center, which induces the structure of an $(n+1)$-group on $T\Sigma G_{(n)}$, called

$$\mathrm{INN}_0(G_{(n)})\,.$$

- *The short exact sequence form proposition 1 now reads*

$$G_{(n)} \to \mathrm{INN}_0(G_{(n)}) \to \Sigma G_{(n)}$$

  and plays the role of the fibration corresponding to the universal $G_{(n)}$-bundle.

- *When expressed in terms of complexes of ordinary groups, $\mathrm{INN}_0(G_{(n)})$ is the mapping cone of the identity on $G_{(n)}$.*

Remark. These three properties are the integral version of the following three aspects in the world of Lie n-algebras which are described in section 3.

- If $\mathfrak{g}_{(n)} := \mathrm{Lie}(G_{(n)})$ denotes the Lie n-algebra of the Lie n-group $G_{(n)}$, then the Lie $(n+1)$-algebra of $\mathrm{INN}_0(G_{(n)})$ is that which is denoted $\mathrm{inn}(\mathfrak{g}_{(n)})$ in 3.3.
- The differential Lie version of the above short exact sequence of Lie n-groupoids is the structure appearing in proposition 11. Notice that the fact that there only the left arrow is actually a morphism of Lie $(n+1)$-algebras, while the right arrow is just a morphisms of the underlying free graded-commutative algebras, corresponds to the fact that both $G_{(n)}$ and $\mathrm{INN}_0(G_{(n)})$ are Lie $(n+1)$-groups, while $\Sigma G_{(n)}$ is in general not (unless it is "sufficiently abelian".)
- The last property provides the easy construction of inner derivation Lie $(n+1)$-algebras: as stated in definition 14, these are just the mapping cones of the identity in the world of differential graded algebra.

Remark. It follows that for $F : C \to \Sigma G_{(n)}$ an n-group valued n-functor, its differential, regarded as an n-functor with values in n-groupoids over $\Sigma G_{(2)}$ as in proposition 3, looks like

$$\begin{tikzcd} \delta F : (x & y) \arrow[l, "\gamma"'] & \mapsto & \mathrm{INN}_0(G_{(n)}) \arrow[rr, "F(\gamma)^*"] \arrow[dr, two heads] & & \mathrm{INN}_0(G_{(n)}) \arrow[dl, two heads] \; . \\ & & & & \Sigma G_{(n)} & \end{tikzcd}$$

We are therefore interested in the n-category of functors which send each object to $\mathrm{INN}_0(G)$ and which do respect the canonical projection down to $\Sigma G_{(n)}$.

Definition 7 *Write*

$$\Sigma\mathrm{End}(\mathrm{INN}_0(G_{(n)})) \downarrow \Sigma G_{(n)} \subset n\mathrm{Cat} \downarrow \Sigma G_{(n)}$$

for the full one-object sub $(n+1)$-category of $n\mathrm{Cat} \downarrow \Sigma G_{(n)}$ sitting on the single object $\mathrm{INN}_0(G_{(n)})$.

Definition 8 *Write*

$$\mathrm{Funct}(C^{\mathrm{op}}, \Sigma\mathrm{INN}(G_{(n)}))| \subset \mathrm{Funct}(C^{\mathrm{op}}, \Sigma\mathrm{INN}(G_{(n)}))$$

for that sub n-category of all n-functors with the property that the component maps of all transformation – these take values in cylinders in $\Sigma\mathrm{INN}(G_{(n)})$), hence in $\mathrm{INN}(G_{(n)})$) itself – , send everything to the identity on the identity when pushed forward along the projection

$$\mathrm{INN}_0(G_{(n)}) \twoheadrightarrow \Sigma G_{(n)} \; .$$

Remark. This is precisely the integral analogue of the restriction on higher morphisms of Lie n-algebras given in definition 16.

Proposition 5 *There is a canonical equivalence of n-functor n-categories*

$$\mathrm{Funct}(C^{\mathrm{op}}, \Sigma\mathrm{End}(\mathrm{INN}_0(G_{(n)})) \downarrow \Sigma G_{(n)}) \simeq \mathrm{Funct}(C^{\mathrm{op}}, \Sigma\mathrm{INN}(G_{(n)}))| \, .$$

Proof. If you think about it, this is a corollary of the discussion in Schreiber-D.M.Roberts. Though not an entirly easy one. I need to describe the proof here in more detail. □

Remark. This means that we may replace the unwieldy n-category of n-functors respecting the projection down to $\Sigma G_{(n)}$ with the more direct n-category of n-functors with values in $\Sigma\mathrm{INN}_0(G_{(n)})$, subject to that straightforward restriction on the value of the component maps of their transformations. As remarked above, the latter is what we will be concerned with in the differential description in terms of morphisms of Lie n-algebras.

Example: the case $n = 1$. The above statement may look intricate, but that's mainly a matter of getting used to the notation. It all becomes quite obvious in the special case where we set $n = 1$.

The n-group $G = G_{(1)}$ now is an ordinary group, and $\mathrm{INN}_0(G) = \mathrm{INN}(G) = G//G = \mathrm{Codisc}(G)$ is nothing but the codiscrete groupoid over G.

The projection

$$\mathrm{INN}(G) \to \Sigma G$$

works simply as

$$(\, g \xrightarrow{h} h \cdot g \,) \;\mapsto\; (\, \bullet \xrightarrow{h} \bullet \,) \,.$$

A transformation of two functors F_1 and F_2 with values in $\Sigma\mathrm{INN}(G)$ is, in components, a functor which sends each morphism $(\, x \xrightarrow{\gamma} y \,)$ of the domain to a filled naturality square in $\Sigma\mathrm{INN}(G)$

$$\begin{tikzcd}
\bullet \arrow[r, "F_1(\gamma)"] \arrow[d, "\eta(x)"'] & \bullet \arrow[d, "\eta(y)"] \arrow[dl, Rightarrow, "h"] \\
\bullet \arrow[r, "F_2(\gamma)"'] & \bullet
\end{tikzcd} \,.$$

Requiring the corresponding morphism

$$\eta(y)F_1(\gamma) \xrightarrow{h} F_2(\gamma)\eta(x)$$

to vanish under the projection $\mathrm{INN}(G) \to \Sigma G$ means precisely that $h = \mathrm{Id}$. This, in turn, says nothing but that η behaves like an ordinary natural transformation of 1-functors with values in ΣG.

Similarly, it is easy to see that the same statement holds for functors with values in $\mathrm{End}(\mathrm{INN}(G))$ that respect the projection down to ΣG.

As a corollary, we find that for $n = 1$ the functor categories of restricted functors with values in $\mathrm{INN}(G)$ is actually isomorphic to just plain old functors with codomain ΣG.

For higher n the $\mathrm{INN}(\cdot)$ construction turns out to make room precisely for the tower of $(n-1)$ components of an n-curvature, while the constraint for respect of the projection $\mathrm{INN}(G_{(n)}) \to \Sigma G_{(n)}$ constrains the morphims of the curvature $(n+1)$-functors to look essentially like morphisms of the underlying n-functors.

The diagrams to keep in mind when following the construction of higher morphisms of Lie n-algebras in 3 are the following.

Any two functors with values in ΣG we may think of as functors with values in $\Sigma\mathrm{INN}(G)$ by using the inclusion $G \hookrightarrow \mathrm{INN}(G)$.

As such, morphism between them

$$
\begin{tikzcd}
 & \Sigma G \arrow[dr, hook] & \\
C \arrow[ur, bend left, "F_1"] \arrow[dr, bend right, "F_2"'] & \arrow[d, phantom, "\swarrow"] & \Sigma \mathrm{INN}(G) \\
 & \Sigma G \arrow[ur, hook] &
\end{tikzcd}
$$

are apriori more flexible than mere morphisms

$$
\begin{tikzcd}
C \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow"] & \Sigma G
\end{tikzcd} .
$$

But as we restrict the components of these transformations to vanish under $\mathrm{INN}(G) \to \Sigma G$, this flexibility is again reduced and the two kinds of transformations actually coincide.

Unfortunately, for nonabelian G this cannot be expressed nicely in one single diagram. But suppose G is abelian, such that $\Sigma^2 G$ exists. Then this constraint reads

$$
\begin{tikzcd}
 & \Sigma G \arrow[dr, hook] & & \\
C \arrow[ur, bend left, "F_1"] \arrow[dr, bend right, "F_2"'] & \arrow[d, phantom, "\swarrow"] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma^2 G \\
 & \Sigma G \arrow[ur, hook] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r] & \{\mathrm{pt}\} \arrow[r] & \Sigma^2 G
\end{tikzcd} .
$$

3 Towards an $(\infty, 1)$-category of Lie n-algebras

While Lie n-algebras are conveniently conceived in terms of n-term L_∞-algebras or, dually, in terms of qDGCAs generated in the first n degrees, this reformulation does not immediately make the right notion of higher morphisms of Lie n-algebras manifest.

Even though from abstract nonsense – and from the work of Boardman and Vogt – these concepts are in principle available, apparently no explicit description has appeared in the literature, nor does it seem to be known to the experts.

Here we do propose a general explicit definition of higher morphisms of L_∞-algebras, of qDGCAs and of Lie n-algebras by making use of the following observations.

- Higher morphisms of *free* qDGCAs are easy to describe.
- The inner derivation Lie $(n+1)$-algebra of any Lie n-algebra provides a bridge from general into free qDGCAs.
- By passing over that bridge, while remaining connected with the point one started from in a suitable way, the notion of higher morphisms of free qDGCAs may be "pulled back" to that of general qDGCAs.

This procedure is modelled after, and probably considerably illuminated by, a very general construction in what we call "arrow-theoretic differential theory", a survey of which is given in section 2.

While we do describe higher morphisms of Lie n-algebras as well as their composition laws, we examine the coherences of these higher compositions only rather partially. On general grounds one would expect there to be a weak $(\infty, 1)$-category of Lie n-algebras. To the extent that we do look into this issue our findings do seem to confirm this expectation. But a full discussion of such an $(\infty, 1)$-category structure in not attempted here.

On the other hand, we do look into the special case where everything is restricted to just Lie 2-algebras. For that case, we demonstrate that the 2-category of Lie 2-algebras which follows from our general prescription coincides exactly with that proposed by Baez and Crans.

The strategy. In more detail, this are the steps which we follow in order to define higher morphisms of Lie n-algebras.

- Ordinary (1-)morphisms of Lie n-algebras are maps of the corresponding qDGCAs which are at the same time chain maps and algebra homomorphisms. Respect for the free algebra structure implies that morphisms are fixed already by their restriction to generators. The problem is to retain this property for higher morphisms.
- As long as the source qDGCA is free as a differential algebra, there is an obvious notion of higher morphisms that are fixed by their restriction to generators. We give the explicit formula.
- For translating the notion of higher morphisms of free qDGCAs to that of higher morphisms for arbitray qDGCAs we make use of the Lie $(n+1)$-algebra $\mathrm{inn}(\mathfrak{g}_{(n)})$ of inner derivations associated with any Lie n-algebra $\mathfrak{g}_{(n)}$.

  This makes use of the following properties of $\mathrm{inn}(g_{(n)})$.

  - There is a canonical inclusion

    $$\mathfrak{g}_{(n)} \hookrightarrow \mathrm{inn}(\mathfrak{g}_{(n)})$$

    of Lie $(n+1)$-algebras. On the underlying graded-commutative algebras (the dual of) this inclusion is part of the short exact sequence

    $$\bigwedge\nolimits^{\bullet}(\mathfrak{g}^*_{(n)}) \twoheadleftarrow \bigwedge\nolimits^{\bullet}(\mathrm{inn}(\mathfrak{g}_{(n)})^*) \hookleftarrow \bigwedge\nolimits^{\bullet}(s\mathfrak{g}_{(n)})^*)$$

– There is a canonical nontrivial (1-)isomorphism

$$\mathrm{inn}(\mathfrak{g}_{(n)}) \xrightarrow{\sim} \mathrm{inn}_{\mathrm{sh}}(\mathfrak{g}_{(n)}) \ ,$$

where $\mathrm{inn}_{\mathrm{sh}}(\mathfrak{g}_{(n)})$ is the Lie $(n+1)$–algebra coming from the free qDGCA on the vector space underlying $\mathfrak{g}_{(n)}$.

- This yields first a notion of higher morphisms on 1-morphisms with target $\mathrm{inn}(\mathfrak{g}_{(n)})$ using the existing notion of higher morphisms on 1-morphisms with target $\mathrm{inn}_{\mathrm{sh}}(\mathfrak{g}_{(n)})$.

$$\begin{tikzcd} \mathfrak{h}_{(n)} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathrm{inn}(\mathfrak{g}_{(n)}) \end{tikzcd} \quad := \quad \begin{tikzcd} \mathfrak{h}_{(n)} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathrm{inn}_{\mathrm{sh}}(\mathfrak{g}_{(n)}) \arrow[r, "\sim"] & \mathrm{inn}(\mathfrak{g}_{(n)}) \end{tikzcd} \ .$$

- Finally, higher morphisms on arbitrary 1-morphisms

$$\begin{tikzcd} \mathfrak{h}_{(n)} \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow"] & \mathfrak{g}_{(n)} \end{tikzcd}$$

are taken to be those higher morphisms of the respective pushforwards to $\mathrm{inn}(\mathfrak{g}_{(n)})$

$$\begin{tikzcd} & \mathfrak{g}_{(n)} \arrow[dr, hook] & \\ \mathfrak{h}_{(n)} \arrow[ur, bend left, "f_1"] \arrow[dr, bend right, "f_2"'] \arrow[rr, phantom, "\Swarrow"] & & \mathrm{inn}(\mathfrak{g}_{(n)}) \\ & \mathfrak{g}_{(n)} \arrow[ur, hook] & \end{tikzcd}$$

whose dual component maps trivialize when pulled back along

$$\bigwedge\nolimits^\bullet(\mathrm{inn}(\mathfrak{g}_{(n)})^*) \hookleftarrow s\mathfrak{g}_{(n)})^* \ ,$$

i.e which are such that

$$\begin{tikzcd} & \bigwedge^\bullet(\mathfrak{g}^*_{(n)}) \arrow[dl, bend right, "f_1^*"'] & & \\ \bigwedge^\bullet(\mathfrak{h}^*_{(n)}) & & \bigwedge^\bullet(\mathrm{inn}(\mathfrak{g}_{(n)})^*) \arrow[ul, two heads] \arrow[dl, two heads] \arrow[ll, phantom, "\Swarrow"] & s\mathfrak{g}^*_{(n)} \arrow[l, hook'] \\ & \bigwedge^\bullet(\mathfrak{g}^*_{(n)}) \arrow[ul, bend left, "f_2^*"] & & \end{tikzcd}$$

vanishes.

3.1 1-Morphisms of Lie n-algebras

Definition 9 (morphisms of qDGCAs) *A morphisms of two qDGCAs*

$$f : (\bigwedge^\bullet(sW^*), d_W) \to (\bigwedge^\bullet(sV^*), d_V)$$

is a linear map which respects both the graded-commutative algebra structure and the differential. In other words, it is a chain map that at the same time is a homomorphism (of degree 0) of graded algebras.

Remark. It is crucial that both these properties are respected by the morphism. The respect for the free algebra structure implies that any morphism of qDGCAs is already determined by its value $f|_{sV^*}$ on generators. The respect for the differential then implies that the L_∞-structure on sV is respected.

qDGCAs with morphisms as above clearly form a 1-category. In order to get a handle on higher morphisms of qDGCAs it is helpful to first consider qDGCAs whose differential acts freely.

3.2 Free differential graded-commutative algebras

Definition 10 (free qDGCAs) *For V any graded vector space, we say the free qDGCA on V is the free graded-commutative algebra*

$$\bigwedge^\bullet(sV^* \oplus ssV^*)$$

together with the free differential defined by

$$d|_{sV^*} = \sigma$$

and

$$d|_{ssV^*} = 0\,,$$

where $\sigma : sV^* \to ssV^*$ is the canonical isomorphism of a graded vector space with its shifted copy.

Free qDGCAs by themselves are trivializable, in a sense to me made precise below, but they prove to be a useful tool for handling non-free qDGCAs.

Morphisms of free qDGCAs are given bydefinition 9. Following the remark below that definition, we want higher morphisms of free qDGCAs to be given in terms of higher chain homotopies which are fixed already by their value on generators.

Definition 11 (higher morphisms involving free qDGCAs) *Let $(\bigwedge^\bullet(sW^* \oplus ssW^*), d_W)$ be a free qDGCA and let $(\bigwedge^\bullet(sV^*), d_V)$ be any qDGCA.*

A 1-morphism

$$f^* : (\bigwedge^\bullet(sW^* \oplus ssW^*), d_W) \to (\bigwedge^\bullet(sV^*), d_V)$$

is, as in definition 9, an algebra homomorphism which is also a chain map.

For $j > 1$, a j-morphism h

$$\begin{tikzcd} {} \arrow[r, bend left=50, "\tau_1"{name=U}] \arrow[r, bend right=50, "\tau_2"'{name=D}] & {} \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd}$$

is a chain homotopy of order $(j-1)$ between between two $(j-1)$-morphisms τ_1 and τ_2, with the special property that it comes from a linear map of degree $-(j-1)$

$$h : sW^* \oplus ssW^* \to \bigwedge\nolimits^\bullet(sV^*)$$

from the generators of the domain to the GCA of the codomain, such that

$$\tau_2 - \tau_1 = [d, h]$$

on $(sW^* \oplus ssW^*)$, and which is extended to a chain homotopy

$$\bigwedge\nolimits^\bullet(sW^* \oplus ssW^*) \to \bigwedge\nolimits^{\bullet-(j-1)}(sV^*)$$

by setting

$$h : x_1 \wedge \cdots \wedge x_n \mapsto$$

$$\frac{1}{n!}\sum_{\sigma}\epsilon(\sigma)\sum_{k=1}^{n}(-1)^{\sum\limits_{i=1}^{k-1}(j-1)|x_{\sigma(i)}|} f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge h(x_{\sigma(k)})\wedge f_2^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})$$

for all $x_1, \cdots, x_n \in (sW^* \oplus ssW^*)$, where f_1^* and f_2^* are the underlying source and target 1-morphisms, respectively.

Remark. Notice where the assumption that the domain is a free qDGCA is crucial for this definition to make sense: we may indeed evaluate $[d, h]$ entirely without leaving the space $(sW^* \oplus ssW^*)$ of generators, since, by the very definition of free qDGCAs d, acts as an endomorphism of that space.

For instance, with $x \in sW^*$ a generator, with $\sigma x \in ssW^*$ its image shifted in degree, we have

$$\tau_2(x) - \tau_1(x) = dh(x) + (-1)^j h(\sigma(x))$$

and

$$\tau_2(\sigma x) - \tau_1(\sigma x) = dh(\sigma x)\,.$$

Proposition 6 *The extension used in the above definition is indeed well defined.*

Proof. We need to check that the chain homotopy property

$$\tau_2 - \tau_1 = [d, h]\,,$$

which was explicitly demanded to hold only on generators, in fact holds on arbitrary products of generators.

So let $x_1 \wedge \cdots \wedge x_n$ be the product of any collection of generators. The first thing to notice is that $[d, h](x_1 \wedge \cdots \wedge x_n)$ evaluates to

$$\frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1}(j-2)|x_{\sigma(i)}|} f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge[d,h](x_{\sigma(k)})\wedge f_2^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})\,.$$

All other terms cancel due to the way the signs work. This means already that for all $(j \geq 3)$-morphisms h we have indeed that $[d, h]$ is the difference of two $(j-1)$-morphisms.

For $j = 2$, in which case h has to be chain homotopy

$$f_1^* \xrightarrow{h} f_2^*$$

between the two algebra homomorphisms f_1^* and f_2^*, we need to check that indeed $f_2^* - f_1^* = [d, h]$ on all products of generators. This follows from rewriting the sum a little:

$$\begin{aligned}
& [d,\tau](x_1 \wedge \cdots \wedge x_n) \\
= \;& \frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge[d,\tau](x_{\sigma(k)})\wedge f_2^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)}) \\
= \;& \frac{1}{n!}\sum_\sigma \epsilon(\sigma)\sum_{k=1}^n f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge(f_2^* - f_1^*)(x_{\sigma(k)})\wedge f_2^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)}) \\
= \;& \frac{1}{n!}\sum_\sigma \epsilon(\sigma)\left(\sum_{k=1}^n f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k-1)})\wedge f_2^*(x_{\sigma(k)}\wedge\cdots\wedge x_{\sigma(n)})\right. \\
& \left. - \sum_{k=1}^n f_1^*(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k)})\wedge f_2^*(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n)})\right) \\
= \;& (f_2^* - f_1^*)(x_1 \wedge \ldots \wedge x_n)\,.
\end{aligned}$$

□

Remark 1 *Notice that this means in particular that not only do all j-morphisms of the above kind come from a degree $-(j-1)$-map h on generators, but also, for every choice of $(j-1)$-morphism τ_1 each such map h on generators provides a j-morphism* $\tau_1 \xrightarrow{h} (\tau_1 + [d,h])$.

Example. For examples of Lie n-algebras of low n, we mainly need the formula from definition 11 on products of two generators, where it becomes

$$\tau : a \wedge b \mapsto \frac{1}{2}\Big(\tau(a) \wedge (f_1^* + f_2^*)(b) + (-1)^{|a|}(f_1^* + f_2^*)(a) \wedge \tau(b)\Big) \tag{1}$$

for all $a, b \in (sW)^*$.

Proof. By applying the general formula to this special case, one gets

$$\begin{aligned}
\tau(a \wedge b) &= \frac{1}{2}\Big(\tau(a) \wedge f_2^*(b) + (-1)^{|a|} f_1^*(a) \wedge \tau(b)\Big) \\
&\quad + \frac{1}{2}(-1)^{|a||b|}\Big(\tau(b) \wedge f_2^*(a) + (-1)^{|b|} f_1^*(a) \wedge \tau(a)\Big) \\
&= \frac{1}{2}\Big(\tau(a) \wedge f_2^*(b) + (-1)^{|a|} f_1^*(a) \wedge \tau(b)\Big) \\
&\quad + \frac{1}{2}\Big((-1)^{|a|} f_2^*(a) \wedge \tau(b) + \tau(a) \wedge f_1^*(a)\Big)
\end{aligned}$$

□

Definition 12 (Composition of n-morphisms) *We let*

- *the composition of any n-morphism h with any 1-morphism f^* along an object be by the obvious pre- or postcomposition of the component maps.*

  For instance the component map of

  $$\begin{tikzcd}
  (\bigwedge^\bullet (sW)^*, d) \arrow[r, "f_1^*"] & (\bigwedge^\bullet (sV)^*, d) \arrow[rr, bend left, "f_2^*"] \arrow[rr, bend right, "{f_2'}^*"'] \arrow[rr, phantom, "\Downarrow \tau"] & & (\bigwedge^\bullet (sU)^*, d) \arrow[r, "f_3^*"] & (\bigwedge^\bullet (sT)^*, d)
  \end{tikzcd}$$

  on generators is $f_3^* \circ \tau \circ f_1^*$;

- *the composition of any two n-morphisms along any $(1 \leq k < n)$-morphism be given, on generators, by the sum of the respective component maps.*

  For instance, given 2-morphisms

  $$\begin{tikzcd}
  (\bigwedge^\bullet (sW)^*) \arrow[r, bend left=60, "f_1^*"] \arrow[r, "f_2^*" description] \arrow[r, bend right=60, "f_3^*"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \tau_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow \tau_2"] & (\bigwedge^\bullet (sV)^*)
  \end{tikzcd}$$

the component map of their composite along the 1-morphism f_2^* on generators is $\tau_1 + \tau_2$.

We need to check that this definition makes sense, in that the compound n-morphisms defined by the sum of their component maps of generators as above do have the right source and target.

For the time being, we check this only for 2-morphisms (and for 1-morphisms there is nothing to be checked).

Proposition 7 *The composition of 2-morphisms along a common 1-morphism, as above, does respect source and target 1-morphisms.*

Proof. Write $\tau_2 \circ \tau_1$ for the 2-morphism starting at f_1^* whose component map on generators is $\tau_1 + \tau_2$ and which is extended to a 2-morphism following def. 11. By remark 1 we are guaranteed that these component maps do correspond to some 2-morphism with target $f_1^* + [d, \tau_2 \circ \tau_1]$. We need to check that this target coincides with f_3^*.

On the other hand, we know that the ordinary sum $\tau_1 + \tau_2$ of these chain homotopies, directly interpreted as this sum not only on generators, but on all elements of the qDGCA, does have the right target, since

$$f_3^* = f_1^* + [d, \tau_1 + \tau_2]\,.$$

In order to show that also

$$f_3^* = f_1^* + [d, \tau_2 \circ \tau_1]$$

it is sufficient to exhibit a second order chain homotopy

$$\delta : \bigwedge\nolimits^{\bullet}(sW^*) \to \bigwedge\nolimits^{\bullet-2} sV^*$$

such that

$$\tau_2 + \tau_1 - \tau_2 \circ \tau_1 = [d, \delta]\,.$$

We claim that the assignment

$$\delta_{\tau_1,\tau} : (x_1 \wedge \cdots \wedge x_n) \mapsto$$

$$\frac{1}{n!}\sum_{\sigma} \epsilon(\sigma) \sum_{1 \le k_1 < k_2 \le n} (-1)^{\sum_{i=k_1}^{k_2-1} |x_{\sigma(i)}|} f_1^*(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k_1-1)}) \wedge \tau_1(x_{\sigma(k_1)}) \wedge f_2^*(x_{\sigma(k_1+1)} \wedge \cdots \wedge x_{\sigma(k_2-1)}) \wedge \tau_2(x_{\sigma(k_2)}) \wedge f_3^*(x_{\sigma(k_2+1)} \wedge \cdots \wedge x_{\sigma(n)})$$

does the job. This is a straightforward matter to check. $\square$

Remark. The way higher morphisms are composed simply by adding their component maps on generators implies that all $(n > 1)$-morphisms have strict inverses. One would therefore hope that free qDGCAs with n-morphisms and their composition as defined above form what is called an $(\infty, 1)$-category: a kind of weak ∞-category for which all $(n > 1)$-morphisms are equivalences. Here we don't attempt to exhibit that entire $(\infty, 1)$-category structure, but just remark that one cannot expect a *strict* ∞-category, for the following reason.

Proposition 8 *Composition of 2-morphisms of free qDGCAs satisfies the exchange law only up to a 3-isomorphism.*

Proof. The component map of the vertical composition of

$$
\begin{tikzcd}
(\bigwedge^\bullet (sW)^*, d) \arrow[rr, bend left, "f_1^*"] \arrow[rr, bend right, "{f'_1}^*"'] \arrow[rr, phantom, "\Downarrow\tau_1"] & & (\bigwedge^\bullet (sV)^*, d) \arrow[rr, bend left, "f_2^*"] & & (\bigwedge^\bullet (sU)^*, d) \\
(\bigwedge^\bullet (sW)^*, d) \arrow[rr, bend right, "{f'_1}^*"'] & & (\bigwedge^\bullet (sV)^*, d) \arrow[rr, bend left, "f_2^*"] \arrow[rr, bend right, "{f'_2}^*"'] \arrow[rr, phantom, "\Downarrow\tau_2"] & & (\bigwedge^\bullet (sU)^*, d)
\end{tikzcd}
$$

on generators is $f_2^* \circ \tau_1 + \tau_2 \circ {f'_1}^*$. On the other hand, the component map of the vertical composition of

$$
\begin{tikzcd}
(\bigwedge^\bullet (sW)^*, d) \arrow[rr, bend left, "f_1^*"] & & (\bigwedge^\bullet (sV)^*, d) \arrow[rr, bend left, "f_2^*"] \arrow[rr, bend right, "{f'_2}^*"'] \arrow[rr, phantom, "\Downarrow\tau_2"] & & (\bigwedge^\bullet (sU)^*, d) \\
(\bigwedge^\bullet (sW)^*, d) \arrow[rr, bend left, "f_1^*"] \arrow[rr, bend right, "{f'_1}^*"'] \arrow[rr, phantom, "\Downarrow\tau_1"] & & (\bigwedge^\bullet (sV)^*, d) \arrow[rr, bend right, "{f'_2}^*"'] & & (\bigwedge^\bullet (sU)^*, d)
\end{tikzcd}
$$

on generators is $\tau_2 \circ f_1^* + {f'_2}^* \circ \tau_1$. Notice that these two composites differ by an exact term

$$(\tau_2 \circ f_1^* + {f'_2}^* \circ \tau_1) - (f_2^* \circ \tau_1 + \tau_2 \circ {f'_1}^*) = [d, \tau_2] \circ \tau_1 - \tau_2 \circ [d, \tau_1] = [d, \tau_2 \circ \tau_1]\,,$$

which means that they are homotopic, which in turn means that the two ways to compose two 2-morphisms horizontally are connected by a 3-isomorphism. $\Box$

Even without the $(\infty, 1)$-category of free qDGCAs fully available, we have obtained enough structure to obtain the canonical notion of equivalence of free qDGCAs, using the fact that every $(n \geq 2)$-morphism is strictly invertible:

Definition 13 (equivalence of free qDGCAs) *Two free qDGCAs* $(\bigwedge^\bullet(sV^*), d_V)$ *and* $(\bigwedge^\bullet(sW^*), d_W)$ *are called equivalent precisely if there exist 1-morphisms*

$$(\bigwedge^\bullet(sV^*), d_V) \xrightarrow{f_1} (\bigwedge^\bullet(sW^*), d_W)$$

and

$$(\bigwedge^\bullet(sW^*), d_W) \xrightarrow{f_2} (\bigwedge^\bullet(sV^*), d_V)$$

both whose composites are isomorphic to the respective identity 1-morphism:

$$\begin{tikzcd}
(\bigwedge^\bullet(sV^*), d_V) \arrow[rr, "\mathrm{Id}"] \arrow[dr, "f_1"'] & {} \arrow[d, phantom, "\Downarrow"] & (\bigwedge^\bullet(sW^*), d_W) \\
& (\bigwedge^\bullet(sW^*), d_W) \arrow[ur, "f_2"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\bigwedge^\bullet(sW^*), d_W) \arrow[rr, "\mathrm{Id}"] \arrow[dr, "f_2"'] & {} \arrow[d, phantom, "\Downarrow"] & (\bigwedge^\bullet(sV^*), d_V) \\
& (\bigwedge^\bullet(sV^*), d_V) \arrow[ur, "f_1"'] &
\end{tikzcd}$$

Proposition 9 *Every free qfDGCA is – not isomorphic but – equivalent to the trivial qfDGCA.*

Proof. Let $(\mathfrak{g}_{(n)})^*$ denote any free qDGCA.

We need to show that there is a 2-morphism

$$\begin{tikzcd}
(\mathfrak{g}_{(n)})^* \arrow[rr, "\mathrm{id}"] \arrow[dr] & {} \arrow[d, phantom, "\Downarrow\sim"] & (\mathfrak{g}_{(n)})^* \\
& 0 \arrow[ur] &
\end{tikzcd} .$$

This means that we need to find a 2-morphism τ whose component map of degree -1 satisfies

$$[d_{\mathfrak{g}_{(n)}}, \tau] = \mathrm{Id}_{(\mathfrak{g}_{(n)})^*} .$$

By defining τ on generators by

$$\tau : a \mapsto 0$$

$$\tau : da \mapsto a$$

for all $a \in (sV)^*$ we get

$$[d_{\mathfrak{g}_{(n)}}, \tau] : a \mapsto a$$

$$[d_{\mathfrak{g}_{(n)}}, \tau] : da \mapsto da .$$

$\square$

Remark. This demonstrates that the $(\infty, 1)$-category of free qDGCAs is rather boring: all its objects are equivalent to the 0-object. Nevertheless, free qDGCAs will prove to be useful for defining and understanding general qDGCAs.

3.3 The inner derivation Lie $(n+1)$-algebra

To every Lie n-algebra $\mathfrak{g}_{(n)}$ we associate a Lie $(n+1)$-algebra which we denote $\mathrm{inn}(\mathfrak{g}_{(n)})$ and think of as the Lie $(n+1)$-algebra of *inner derivations* of $\mathfrak{g}_{(n)}$. This interpretation will be expounded on elsewhere. For the present purpose all that is required is the very structure of $\mathrm{inn}(\mathfrak{g}_{(n)})$.

Definition 14 *For $\mathfrak{g}_{(n)}$ any Lie n-algebra coming from the qDGCA $(\bigwedge^\bullet(sV^*), d)$ we write*

$$\mathrm{inn}(\mathfrak{g}_{(n)})$$

for the Lie $(n+1)$-algebra which comes from the qDGCA that is the mapping cone of the identity on $(\bigwedge^\bullet(sV^*), d)$.

This means in detail the following. The GCA underlying $\mathrm{inn}(\mathfrak{g}_{(n)})$ is

$$\bigwedge^\bullet(sV^* \oplus ssV^*)\,.$$

The differential on that may be thought of as acting on $sV^* \oplus ssV^*$ as

$$d' = \begin{pmatrix} d & 0 \\ \mathrm{Id} & -d \end{pmatrix}.$$

More in detail, if we denote by

$$\sigma : sV^* \to ssV^*$$

the canonical isomorphism of a vector space with its shifted copy, and by

$$\Sigma : \bigwedge^\bullet(sV^* \oplus ssV^*) \to \bigwedge^\bullet(sV^* \oplus ssV^*)$$

its extension as a graded derivation to the entire graded algebra, then

$$d'|_{sV^*} = d + \sigma$$

and

$$d'|_{ssV^*} = -\Sigma \circ d \circ \sigma^{-1} = -d' \circ d \circ \sigma^{-1}\,.$$

Hence for $a \in sV^*$ we find

$$d'd'a = d'(da + \sigma a) = \Sigma da - \Sigma da = 0\,.$$

and hence

$$d'd'\sigma a = -d'd'(da) = 0\,.$$

Proposition 10 *For any Lie n-algebra $\mathfrak{g}_{(n)}$, the qDGCA corresponding to the Lie $(n+1)$-algebra $\mathrm{inn}(\mathfrak{g}_{(n)})$ is connected by a 1-isomorphism to a free qDGCA.*

Proof. Let the qDGCA corresponding to $\mathfrak{g}_{(n)}$ be $(\bigwedge^\bullet(sV^*), d_{\mathfrak{g}_{(n)}})$. Write $F(V)$ for the free differential graded commutative algebra over $(sV)^*$. Define a morphism

$$f : F(V) \to (\mathrm{inn}(\mathfrak{g}_{(n)}))^*$$

by setting

$$f : a \mapsto a$$

$$f : d_{F(V)}a \mapsto d_{\mathrm{inn}(\mathfrak{g}_{(n)})}a$$

for all $a \in (sV)^*$. This clearly satisfies the morphism property. One checks that its inverse is given by

$$f^{-1} : a \mapsto a$$

$$f^{-1} : \sigma a \mapsto d_{F(V)}a - d_{\mathfrak{g}_{(n)}}a\,.$$

□

Remark. This implies that $\mathrm{inn}(\mathfrak{g}_{(n)})$ is always trivializable.

We can now translate the notion of higher morphisms of free qDGCAs to those of the form $\mathrm{inn}(\mathfrak{g}_{(n)})^*$.

Definition 15 *An n-morphism of qDGCAs where the source object is $(\mathrm{inn}(\mathfrak{g}_{(n)}))^*$ is an order $(n-1)$-chain homotopy which becomes an n-morphism of free qDGCAs when pulled back along the isomorphism of the proof of proposition 10.*

Proposition 11 *For $\mathfrak{g}_{(n)}$ any Lie n-algebra, we have a canonical epimorphism*

$$(\bigwedge^\bullet(\mathfrak{g}_{(n)})^*, d_{\mathfrak{g}_{(n)}}) \twoheadleftarrow (\bigwedge^\bullet(\mathrm{inn}(\mathfrak{g}_{(n)}))^*, d_{\mathrm{inn}(\mathfrak{g}_{(n)})})$$

of qDGCAs, whose underlying map of GCAs yields the exact sequence

$$\bigwedge^\bullet(\mathfrak{g}^*_{(n)}) \twoheadleftarrow \bigwedge^\bullet(\mathrm{inn}(\mathfrak{g}_{(n)})^*) \hookleftarrow \bigwedge^\bullet(s\mathfrak{g}_{(n)})^*) \;\; .$$

3.4 Higher morphisms of Lie n-algebras

We now define higher morphisms of arbitrary Lie n-algebras in terms of restricted higher morphisms involving their inner derivation Lie $(n+1)$-algebras.

Definition 16 *For source Lie n-algebra $\mathfrak{h}_{(n)}$ and target Lie n-algebra $\mathfrak{g}_{(n)}$, we take a j-morphism between them to be a j-morphism on the pushforward*

$$\mathfrak{g}_{(n)} \hookrightarrow \mathrm{inn}(\mathfrak{g}_{(n)})$$

as defined above, restricted to be such that its dual component map vanishes when pulled back along

$$\bigwedge^\bullet(\mathrm{inn}(\mathfrak{g}_{(n)})^*) \hookleftarrow \bigwedge^\bullet(s\mathfrak{g}_{(n)})^*) \;\; ,$$

Example. Here is what this means explicitly for 2-morphisms. A 2-morphism

$$\begin{tikzcd}
\mathfrak{h}_{(n)} \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[rr, phantom, "\Downarrow"] & & \mathfrak{g}_{(n)}
\end{tikzcd}$$

between 1-morphisms of Lie n-algebras is a 2-morphism

$$\begin{tikzcd}
 & \mathfrak{g}_{(n)} \arrow[dr, hook] & \\
\mathfrak{h}_{(n)} \arrow[ur, bend left, "f_1"] \arrow[dr, bend right, "f_2"'] \arrow[rr, phantom, "\Swarrow"] & & \mathrm{inn}(\mathfrak{g}_{(n)}) \\
 & \mathfrak{g}_{(n)} \arrow[ur, hook] &
\end{tikzcd}$$

as in definition 15, whose dual component map vanishes when pulled back along

$$\bigwedge\nolimits^\bullet(\mathrm{inn}(\mathfrak{g}_{(n)})^*) \hookleftarrow s\mathfrak{g}^*_{(n)} \;,$$

i.e which is such that the component map of the chain homotopy

$$\begin{tikzcd}
 & \bigwedge^\bullet(\mathfrak{g}^*_{(n)}) \arrow[dl, bend right, "f_1^*"'] & & \\
\bigwedge^\bullet(\mathfrak{h}^*_{(n)}) & & \bigwedge^\bullet(\mathrm{inn}(\mathfrak{g}_{(n)})^*) \arrow[ul] \arrow[dl] \arrow[ll, phantom, "\Swarrow"] & s\mathfrak{g}^*_{(n)} \arrow[l, hook'] \\
 & \bigwedge^\bullet(\mathfrak{g}^*_{(n)}) \arrow[ul, bend left, "f_2^*"] & &
\end{tikzcd}$$

vanishes.

Remark. Composition of j-morphisms of arbitrary Lie n-algebras is inherited from the composition law definition 12. We therefore expect these constructions to yield an $(\infty, 1)$-category of Lie n-algebras. Instead of trying to exhibit that structure in full generality, we shall here be content with checking that this does reproduce the 2-category of Lie 2-algebras as given by Baez and Crans.

3.5 The 2-category of Lie 2-algebras

We recall Baez and Crans's definition of the 2-category of Lie 2-algebras. Then we show that this is reproduced by restricting our definition of morphisms of Lie n-algebras to this case.

In order to set up the discussion of Baez-Crans 2-morphisms of 2-term L_∞-algebras, recall their notation for 1-morphisms of 2-term L_∞-algebras (which is of course just a special case of the general notion of 1-morphisms of L_∞-algebras).

Definition 17 *A morphism*

$$\varphi : V \to W$$

of 2-term L_∞-algebras V and W is a pair of maps

$$\phi_0 : V_0 \to W_0$$

$$\phi_1 : V_1 \to W_1$$

together with a skew-symmetric map

$$\phi_2 : V_0 \otimes V_0 \to W_1$$

satisfying

$$\phi_0(d(h)) = d(\phi_1(h))$$

as well as

$$\begin{aligned} d(\phi_2(x,y)) &= \phi_0(l_2(x,y)) - l_2(\phi_0(x), \phi_0(y)) \\ \phi_2(x, dh) &= \phi_1(l_2(x,h)) - l_2(\phi_0(x), \phi_1(h)) \end{aligned}$$

and finally

$$\begin{aligned} &l_3(\phi_0(x), \phi_0(y), \phi_0(z)) - \phi_1(l_3(x,y,z)) = \\ &\phi_2(x, l_2(y,z)) + \phi_2(y, l_2(z,x)) + \phi_2(z, l_2(x,y)) + \\ &l_2(\phi_0(x), \phi_2(y,z)) + l_2(\phi_0(y), \phi_2(z,x)) + l_2(\phi_0(z), \phi_2(x,y)) \,. \end{aligned}$$

for all $x, y, z \in V_0$ and $h \in V_1$.

Definition 18 (Baez-Crans) *A 2-morphism*

$$\tau : \phi \Rightarrow \psi$$

of 1-morphisms of 2-term L_∞-algebras is a linear map

$$\tau : V_0 \to W_1$$

such that

$$\psi_0 - \phi_0 = t_W \circ \tau$$

$$\psi_1 - \phi_1 = \tau \circ t_v$$

and

$$\phi_2(x,y) - \psi_2(x,y) = l_2(\phi_0(x), \tau(y)) + l_2(\tau(x), \psi_0(y)) - \tau(l_2(x,y))$$

Note that $[d,\tau] = d_W\tau + \tau d_V$ and that it restricts to $d_W\tau$ on V_0 and to τd_V on V_1.

The notion of 1-morphism here is obvious. The nontrivial part is

Proposition 12 *2-morphisms of 2-term L_∞-algebras as above are precisely the 2-morphism as in definition 16.*

Proof.

Let $(\bigwedge^\bullet(sV_0^* \oplus sV_1^*), d_V)$ and $(\bigwedge^\bullet(sW_0^* \oplus sW_1^*), d_W)$ be the corresponding qDGCAs and

$$\phi^*, \psi^* : (\bigwedge^\bullet(sW_0^* \oplus sW_1^*), d_W) \to (\bigwedge^\bullet(sV_0^* \oplus sV_1^*), d_V)$$

be the corresponding dual 1-morphisms.

A 2-morphism

$$\tau^* : \psi^* \to \phi^*$$

between these is a 2-morphism

$$\begin{tikzcd}
 & \bigwedge^\bullet(sW_0^* \oplus sW_1^*) \arrow[dl, bend right, "\psi^*"'] \arrow[dd, phantom, "\Swarrow"] & \\
\bigwedge^\bullet(sV_0^* \oplus sV_1^*) & & \bigwedge^\bullet(sW_0^* \oplus sW_1^* \oplus ssW_0^* \oplus ssW_1^*) \arrow[ul, two heads] \arrow[dl, two heads] \\
 & \bigwedge^\bullet(sV_0^* \oplus sW_1^*) \arrow[ul, bend left, "\phi^*"] &
\end{tikzcd} .$$

We now find it very helpful, if maybe somewhat bothersome on general grounds, to choose bases.

With $\{t^a\}$ a basis for sW_0^* and $\{b^i\}$ a basis for sW_1^*, this comes from a map

$$\tau^* : sW_0^* \oplus sW_1^* \oplus ssW_0^* \oplus ssW_1^* \to \bigwedge^\bullet(V_0^* \oplus V_1^*)$$

of degree -1 which acts on these basis elements as

$$\tau^* : b^i \mapsto \tau^i{}_a t'^a$$

and

$$\tau^* : a^a \mapsto 0$$

where we let $\{t'^a\}$ and $\{b'^i\}$ be a basis of sV_0^* and sW_1^*, respectively.

The requirement that

$$\begin{tikzcd}
 & \bigwedge^\bullet(sW_0^* \oplus sW_1^*) \arrow[dl, bend right, "\phi^*"'] \arrow[dd, phantom, "\Swarrow"] & & \\
\bigwedge^\bullet(sV_0^* \oplus sV_1^*) & & \bigwedge^\bullet(sW_0^* \oplus sW_1^* \oplus ssW_0^* \oplus ssW_1^*) \arrow[ul, two heads] \arrow[dl, two heads] & ssW_0^* \oplus ssW_1^* \arrow[l, hook'] \\
 & \bigwedge^\bullet(sV_0^* \oplus sW_1^*) \arrow[ul, bend left, "\psi^*"] & &
\end{tikzcd}$$

vanishes then restricts the value of τ^* on $dt^a = -\frac{1}{2}C^a{}_{bc}t^a \wedge t^b - t^a{}_i b^i + r^a$ to be

$$\tau^* : dt^a \mapsto -t^a{}_i \tau^i{}_b t'^b$$

and on $db^i = -\alpha^i{}_{aj} t^a \wedge b^j + c^i$ to be

$$\tau^*(db^i) = \tau^*(-\alpha^i{}_{aj} t^a \wedge b^j)\,.$$

This needs to be evaluated using the formula of definition 11, using the special case described in (1).

Using this we get

$$[d, \tau^*] : t^a \mapsto -t^a{}_i \tau^i{}_b t'^b$$

and

$$[d, \tau^*] : b^i \mapsto -\frac{1}{2}\tau^i{}_a C'^a{}_{bc} t'^b t'^c - \tau^i{}_a t'^a{}_j b'^j + \alpha^i{}_{aj} \frac{1}{2}(\phi + \psi)^a{}_b \tau^j{}_c t'^b t'^c\,.$$

Then

$$\phi^* - \psi^* = [d, \tau^*]$$

is equivalent to

$$(\psi^a{}_b - \phi^a{}_b) t'^b = t^a{}_i \tau^i{}_b t'^b$$

and

$$(\psi^i{}_j - \phi^i{}_j) b'^j = \tau^i{}_a t'^a{}_j b'^j$$

and

$$\frac{1}{2}(\phi^i{}_{ab} - \psi^i{}_{ab}) t'^a t'^b = -\frac{1}{2}\tau^i{}_a C'^a{}_{bc} t'^b t'^c + \alpha^i{}_{aj} \frac{1}{2}(\phi + \psi)^a{}_b \tau^j{}_c t'^b t'^c\,.$$

The first two equation express the fact that τ is a chain homotopy with respect to t and t'. The last equation is equivalent to

$$\begin{aligned}\phi_2(x, y) - \psi_2(x, y) &= -\tau([x, y]) + [q(x) + \frac{1}{2}t(\tau(x)), \tau(y)] - [q'(y) - \frac{1}{2}t(\tau(y)), \tau(x)] \\ &= -\tau([x, y]) + [q(x), \tau(y)] + [\tau(x), q'(y)]\end{aligned}$$

This is indeed the Baez-Crans condition on a 2-morphism. $\square$