BV-formalism and the charged n-particle

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Abstract

We describe the general framework of the *charged n-particle* in the differential realm (Lie ∞-algebroids instead of Lie ∞-groupoids) and show how it reproduces aspects of the standard BV formalism. We spell out the example of ordinary gauge theory in detail.

Contents

$$\begin{array}{ccccc} \text{tangent category} & \begin{array}{c}\text{inner automorphism}\\ (n+1)\text{-group}\end{array} & \begin{array}{c}\text{inner derivation}\\ \text{Lie } (n+1)\text{-algebra}\end{array} & \text{Weil algebra} & \begin{array}{c}\text{shifted}\\ \text{tangent bundle}\end{array} \end{array}$$

$$\mathrm{CE}(\mathrm{Lie}(T\mathbf{B}G)) = \mathrm{CE}(\mathrm{Lie}(\mathrm{INN}(G))) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) = \mathrm{W}(\mathfrak{g}) = C^\infty(T[1]\mathfrak{g})$$

Figure 1: **A remarkable coincidence of concepts** relates the notion of tangency to the notion of universal bundles. See [1] and [2].

1 Introduction

2 The charged n-particle

We address as the concept of the "charged n-particle" a general diagrammatic formulation for quantum field theories of general σ-model type. The idea is to fix the general structure of such QFTs in as abstract and clean terms as possible, such that any internalization of this concept into any suitable context provides us with all the information about what σ-model like quantum field theory means in that context.

The only property of the ambient context category T which we require here is that it is monoidal and that it comes equipped with a functor

$$\mathrm{maps}(-,-) : T^{\mathrm{op}} \times T \to T$$

which behaves sufficiently like an internal hom. For the time being we will not actually require this functor to be necessarily the internal hom with respect to the tensor product, for reasons discussed in more detail in 4.1.

Then a **charged n-particle internal to T** is a diagram

$$\begin{tikzcd} & \mathrm{maps}(\mathrm{par},\mathrm{tar}) \otimes \mathrm{par} \arrow[dl] \arrow[dr, "\mathrm{ev}"] & & \\ \mathrm{par} & & \mathrm{tar} \arrow[r, "\mathrm{tra}"] & \mathrm{phas} \end{tikzcd}$$

in T.

We write

$$\mathrm{conf} := \mathrm{maps}(\mathrm{par}, \mathrm{tar})\,.$$

The morphism tra induces the **action**

$$\exp(S) : 1 \to \mathrm{maps}(\mathrm{conf}, \mathrm{maps}(\mathrm{par}, \mathrm{tar}))\,.$$

3 BV-formalism: horizontal inner derivations on configuration space

BV-formalism is usually described in the language of supermanifolds. In that language the basic object of interest is the shifted cotangent bundle to the supermanifold of physical fields and ghosts.

Using the relation between tangent categories, inner automorphisms $(n+1)$-groups, Weil algebras, inner derivation Lie $(n+1)$-algebras and shifted tangent bundles

$$\mathrm{CE}(\mathrm{Lie}(T\mathbf{B}G)) = \mathrm{CE}(\mathrm{Lie}(\mathrm{INN}(G))) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) = \mathrm{W}(\mathfrak{g}) = C^\infty(T[1]\mathfrak{g})$$

we see that there is a more Lie-algebraic interpretation of the setup of BV-formalism.

We will assume now, and demonstrate in concrete examples in 5, that in our setup of the charged n-particle internal to DGCAs we have that configuration space is the Weil algebra of something:

$$\mathrm{conf} = \mathrm{W}(\mathfrak{g}, V)\,.$$

(I write $\mathfrak{g}$ for Lie ∞-algebras and $(\mathfrak{g}, V)$ for Lie ∞-algebroids without, at the moment, explaining anything much about that. The main point of relevance here is that $\mathrm{CE}(\mathfrak{g})$ is of positive degree, while $\mathrm{CE}(\mathfrak{g}, V)$ is in general only of non-negative degree.)

Then notice the universal $(\mathfrak{g}, V)$-bundle

$$\begin{array}{c} \mathrm{CE}(\mathfrak{g}, V) \\ {\scriptstyle i*}\big\uparrow \\ \mathrm{W}(\mathfrak{g}, V) \end{array}$$

When translating between the two perspectives $\mathrm{W}(\mathfrak{g}, V)$ and $C^\infty(T^*[1](\mathfrak{g}, V))$ we find the dictionary displayed in table 1.

	BV-terminology	**DGCA interpretation**	**Lie ∞-groupoid interpretation**
fields and ghosts	fields	degree 0 generators of $\mathrm{CE}(\mathfrak{g}, V)$	objects of configuration space
	ghosts	degree 1 generators of $\mathrm{CE}(\mathfrak{g}, V)$	morphisms in configuration space
	n-fold ghosts	degree n generators of $\mathrm{CE}(\mathfrak{g}, V)$	n-morphisms in configuration space
antifields and antighosts	antifields	degree 1 horizontal derivations in $\mathrm{W}(\mathfrak{g}, V)$	paths of objects
	antighosts	degree 2 horizontal derivations in $\mathrm{W}(\mathfrak{g}, V)$	paths of 1-morphisms
	anti-ghosts-of-ghosts	degree 3 horizontal derivations in $\mathrm{W}(\mathfrak{g}, V)$	paths of 2-morphisms

Table 1: Dictionary between BV-terminology and Lie-∞-algebraic entities.

4 DGCAs of maps

We now explain and put into perspective the following definition, which is used extensively in 5.

Definition 1 *For A and B any two DGCAs, we write*

$$\mathrm{maps}(B, A) := \Omega^\bullet(\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(-)))$$

for the DGCA of differential forms on the presheaf over manifolds whose set of plots on any domain U is $\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(U))$.

4.1 An approximation to the internal hom in DGCAs

Throughout I write capital Hom for Hom-*sets* and lower case hom for internal Hom-objects, or their "approximations" to be discussed here.

We write DGCAs for the category whose objects are differential graded commutative algebras in non-negative degree, and

$$S^\infty := \mathrm{Set}^{S^{\mathrm{op}}}$$

for the category of generalized smooth spaces, namely of presheaves over the site S, which is any one of the sites of manifolds, the site of open subsets of $\mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \cdots$, or the like.

We have contravariant functors going back and forth between these two categories, forming an adjunction.

The functor

$$\Omega^\bullet(\cdot) : S^\infty \to \mathrm{DCGA}s$$

acts as

$$\Omega^\bullet : X \mapsto \mathrm{Hom}_{S^\infty}(X, \Omega^\bullet)\,,$$

where, in turn, here on the right $\Omega^\bullet$ denotes the smooth space of all differential forms, given by the object in S^∞ which acts as

$$\Omega^\bullet : U \mapsto \Omega^\bullet(U)\,,$$

where on the right we have the ordinary algebra of differential forms on U.

The contravariant functor

$$S^\infty \longleftarrow \mathrm{DGCA}s \;\; : \mathrm{Hom}(-, \Omega^\bullet(-))$$

acts as

$$(X_A : U \mapsto \mathrm{Hom}(A, \Omega^\bullet(U))) \longleftarrow\!\!\shortmid\; A \;\; : \mathrm{Hom}_{\mathrm{DGCA}s}(-, \Omega^\bullet(-))\,.$$

Notice that S^∞, being a topos, has lots of nice properties. In particular it is cartesian closed. The inner hom is

$$\mathrm{hom}_{S^\infty}(X, Y) : U \mapsto \mathrm{Hom}(X \times U, Y)\,.$$

On the other hand, the category DGCAs doesn't have these nice properties in general, except after one restricts to a suitably well behaved subcategory. (I suspect, though, that the above adjunction can be turned into an equivalence *on cohomology*, but I am not sure yet.)

But we can use the above adjunction to "pull back" the inernal hom in S^∞ to DGCAs, meaning that we consider

$$\mathrm{hom}_{\mathrm{DGCA}s}(-,-) : \; (\mathrm{DGCA}s)^{\mathrm{op}} \times \mathrm{DGCA}s \xrightarrow{\mathrm{Hom}(-,\Omega^\bullet(-))^{\mathrm{op}} \times \mathrm{Hom}(-,\Omega^\bullet(-))^{\mathrm{op}}} (S^\infty)^{\mathrm{op}} \times S^\infty \xrightarrow{\mathrm{hom}_{S^\infty}(-,-)} S^\infty \xrightarrow{\Omega^\bullet} \mathrm{DGCA}s\,.$$

So given DGCAs A and B, we get

$$\mathrm{hom}_{\mathrm{DGCA}s}(B, A) = \Omega^\bullet(\mathrm{hom}_{S^\infty}(X_A, X_B))\,.$$

Proposition 1 *We have a canonical surjection of DGCAs*

$$\Omega^\bullet(\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(-))) \twoheadleftarrow \mathrm{hom}_{\mathrm{DGCA}s}(B, A)\,.$$

Proof. This comes from the canonical inclusion of smooth spaces

$$\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(-)) \hookrightarrow \mathrm{Hom}_{S^\infty}(X_A \times -, X_B)$$

which comes, on each $U \in S$, from the inclusion of sets

$$\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(U)) \hookrightarrow \mathrm{Hom}_{S^\infty}(X_A \times U, X_B)$$

which is given by

$$(f^* : B \to A \otimes \Omega^\bullet(U)) \mapsto (V \mapsto (\mathrm{Hom}_{\mathrm{DGCA}s}(A, \Omega^\bullet(V)) \times \mathrm{Hom}_S(V,U) \stackrel{\circ f^*}{\to} \mathrm{Hom}_{\mathrm{DGCA}s}(B, \Omega^\bullet(V)))) \,.$$

$\square$

All this must mean something deeper than I can currrently appreciate. But the practical implication is that I am going to consider the construction $\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(-))$ in the following examples.

Definition 2 (currents) *For A any DGCA, we say that a current on A is a smooth linear map*

$$c : A \to \mathbb{R} \,.$$

For $A = \Omega^\bullet(X)$ this reduces to the ordinary notion of currents.

Proposition 2 *For each element $b \in B$ and current c on A, we get an element in $\Omega^\bullet(\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(-)))$ by mapping, for each $U \in S$*

$$\mathrm{Hom}_{\mathrm{DGCA}s}(B, A \otimes \Omega^\bullet(U)) \to \Omega^\bullet(U)$$

$$f^* \mapsto c(f^*(b)) \,.$$

If b is in degree n and c in degree $m \leq n$, then this differential form is in degree $n - m$.

5 Applications

5.1 Spaces of Lie ∞-algebra valued forms

Let $\mathfrak{g}$ be any (finite dimensional) Lie ∞-algebra, $\mathrm{CE}(\mathfrak{g})$ its Chevalley-Eilenberg DGC algebra and $\mathrm{W}(\mathfrak{g})$ its Weil DGCA.

For Y any smooth space, $\mathfrak{g}$-valued differential forms on Y are DGCA morphisms

$$\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \ \ .$$

So the *set* of $\mathfrak{g}$-valued differential forms is

$$\mathrm{Hom}_{\mathrm{DGCA}s}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y)) \,.$$

We want to consider the algebra of differential forms on the *smooth space* of $\mathfrak{g}$-valued forms on Y:

$$\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$$

according to definition 1.

5.2 The configuration space of ordinary gauge theory

Let $\mathfrak{g}$ be an ordinary Lie algebra and Y be some manifold. The configuration space of ordinary $\mathfrak{g}$-gauge theory (assuming trivial bundles for the moment) is

$$\Omega^\bullet(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}s}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y)) \,.$$

We now analyze the algebra

$$\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$$

and demonstrate that it is itself the Weil algebra of some Lie 2-algebroid.

To make contact with the physics literature and most of the BV-literature, we describe everything in components.

So let $Y = \mathbb{R}^n$ and let $\{x^\mu\}$ be the canonical set of coordinate functions on Y. Choose a basis $\{t_a\}$ of $\mathfrak{g}$ and let $\{t^a\}$ be the corresponding dual basis of $\mathfrak{g}^*$. Denote by

$$\delta_y \iota_{\frac{\partial}{\partial x^\mu}}$$

the delta-current on $\Omega^\bullet(Y)$ which sends a 1-form ω to

$$\omega_\mu(y) := \omega(\frac{\partial}{\partial x^\mu})(y)\,.$$

Summary of the main result. Recall that the Weil algebra $\mathrm{W}(\mathfrak{g})$ is generated from the $\{t^a\}$ in degree 1 and the σt^a in degree 2, with the differential defined by

$$dt^a = -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c + \sigma t^a$$

$$d(\sigma t^a) = -C^a{}_{bc}t^b \wedge (\sigma t^c)\,.$$

We will find that maps$(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$ does look pretty much entirely like this, only that all generators are now forms on Y.

fields	$\left\{A^a_\mu(y), (F_A)_{\mu\nu}(y) \in \Omega^0(\Omega(Y,\mathfrak{g}))\ \mid\ y \in Y, \mu,\nu \in \{1,\cdots,\dim(Y), a \in \{1,\ldots,\dim(\mathfrak{g})\}\}\right\}$
ghosts	$\left\{c^a(y)\Omega^1(\Omega(Y,\mathfrak{g}))\ \mid\ y \in Y, a \in \{1,\ldots,\dim(\mathfrak{g})\}\}\right\}$
antifields	$\left\{\frac{\partial}{\partial \delta A^a_\mu(y)} \in \mathrm{Hom}(\Omega^1(\Omega(Y,\mathfrak{g})),\mathbb{R})\ \mid\ y \in Y, \mu \in \{1,\cdots,\dim(Y), a \in \{1,\ldots,\dim(\mathfrak{g})\}\}\right\}$
anti-ghosts	$\left\{\frac{\partial}{\partial \beta^a(y)} \in \mathrm{Hom}(\Omega^2(\Omega(Y,\mathfrak{g})),\mathbb{R})\ \mid\ y \in Y, \dim(Y), a \in \{1,\ldots,\dim(\mathfrak{g})\}\}\right\}$

Table 2: **The BV field content of gauge theory** obtained from our almost internal hom of dg-algebras, definition 1. The dgc-algebra maps$(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$ is the algebra of differential forms on a smooth space of maps from Y to the smooth space underlying $\mathrm{W}(\mathfrak{g})$.

Remark. Before looking at the details of the computation, recall that an n-form ω in maps$(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$ is an assignment

$$\begin{tikzcd}
U \arrow[d, "\phi"] & \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y \times U)) \arrow[r, "\omega_U"] & \Omega^\bullet(U) \\
V & \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y \times V)) \arrow[u, "\phi^*"] \arrow[r, "\omega_V"] & \Omega^\bullet(V) \arrow[u, "\phi^*"]
\end{tikzcd}$$

of forms on U to $\mathfrak{g}$-valued forms on $Y \times U$ for all manifolds U, conatural in U.

We concentrate on those n-forms ω which arise in the way of proposition 2.

0-Forms. The 0-forms on the space of $\mathfrak{g}$-value forms are constructed as in proposition 2 from an element $t^a \in \mathfrak{g}^*$ and a current $\delta_y \iota_{\frac{\partial}{\partial x^\mu}}$ using

$$t^a \delta_y \iota_{\frac{\partial}{\partial x^\mu}}$$

and from an element $\sigma t^a \in \mathfrak{g}^*[1]$ and a current

$$\delta_y \iota_{\frac{\partial}{\partial x^\mu}} \iota_{\frac{\partial}{\partial x^\nu}} .$$

This way we obtain the families of functions (0-forms) on the space of $\mathfrak{g}$-valued forms:

$$A^a_\mu(y) : (\Omega^\bullet(Y \times U) \leftarrow \mathrm{W}(\mathfrak{g}) : A) \mapsto (u \mapsto \iota_{\frac{\partial}{\partial x^\mu}} A(t^a)(y,u))$$

and

$$F^a_{\mu\nu}(y) : (\Omega^\bullet(Y \times U) \leftarrow \mathrm{W}(\mathfrak{g}) : F_A) \mapsto (u \mapsto \iota_{\frac{\partial}{\partial x^\mu}} \iota_{\frac{\partial}{\partial x^\nu}} F_A(\sigma t^a)(y,u))$$

which pick out the corresponding components of the $\mathfrak{g}$-valued 1-form and of its curvature 2-form, respectively.

This are the *fields*.

1-Forms. A 1-form on the space of $\mathfrak{g}$-valued forms is obtained from either starting with a degree 1 element and contracting with a degree 0 delta-current

$$t^a \delta_y$$

or starting with a degree 2 element and contracting with a degree 1 delta current:

$$(\sigma t^a)\delta_y \frac{\partial}{\partial x^\mu} .$$

To get started, consider firstt the case where $U = I$ is the interval. Then a DGCA morphism

$$(A, F_A) : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(Y) \otimes \Omega^\bullet(I)$$

can be split into its components proportional to $dt \in \Omega^\bullet(I)$ and those not containing dt.

We hence can write the general $\mathfrak{g}$-valued 1-form on $Y \times I$ as

$$(A, F_A) : t^a \mapsto A^a(y,t) + g^a(y,t) \wedge dt$$

and the corresponding curvature 2-form as

$$\begin{aligned}(A, F_A) : \sigma t^a \mapsto (d_Y + d_t)(A^a(y,t) + g^a(y,t) \wedge dt) + \frac{1}{2} C^a{}_{bc}(A^a(y,t) + g^a(y,t) \wedge dt) \wedge (A^b(y,t) + g^b(y,t) \wedge dt) \\ = F^a_A(y,t) + (\partial_t A^a(y,t) + d_Y g^a(y,t) + [g, A]^a) \wedge dt \,.\end{aligned}$$

By contracting this again with the current $\delta_y \frac{\partial}{\partial x^\mu}$ we obtain the 1-forms

$$t \mapsto g^a(y,t) dt$$

and

$$t \mapsto (\partial_t A^a_\mu(y,t) + \partial_\mu g^a(y,t) + [g, A_\mu]^a) dt$$

on the interval.

We will identify the first one with the component of the 1-forms on the space of $\mathfrak{g}$-valued forms on Y called the *ghosts* and the second one with the 1-forms which are killed by the derivations called the *anti-fields*.

To see more of this structure, consider now $U = I^2$, the unit square.

Then a DGCA morphism

$$(A, F_A) : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(Y) \otimes \Omega^\bullet(I^2)$$

can be split into its components proportional to $dt^1, dt^2 \in \Omega^\bullet(I^2)$.

We hence can write the general $\mathfrak{g}$-valued 1-form on $Y \times I$ as

$$(A, F_A) : t^a \mapsto A^a(y,t) + g_i^a(y,t) \wedge dt^i \,,$$

and the corresponding curvature 2-form as

$$\begin{aligned}(A, F_A) : \sigma t^a &\mapsto (d_Y + d_{I^2})(A^a(y,t) + g_i^a(y,t) \wedge dt^i + h^a(y,t)dt^1 \wedge dt^2) \\ &+\frac{1}{2}C^a{}_{bc}(A^a(y,t) + g_i^a(y,t) \wedge dt^i + h^a(y,t)dt^1 \wedge dt^2) \wedge (A^b(y,t) + g_i^b(y,t) \wedge dt^i) \\ &= F_A^a(y,t) + (\partial_{t^i} A^a(y,t) + d_Y g_i^a(y,t) + [g_i, A]^a) \wedge dt^i \\ &\quad +(\partial_i g_j^a + [g_i, g_j]^a)dt^i \wedge dt^j \,.\end{aligned}$$

By contracting this again with the current $\delta_y \frac{\partial}{\partial x^\mu}$ we obtain the 1-forms

$$t \mapsto g_i^a(y,t)dt^i$$

and

$$t \mapsto (\partial_t A_\mu^a(y,t) + \partial_\mu g_i^a(y,t) + [g_i, A_\mu]^a)dt^i$$

on the unit square.

This are again the local values of our

$$c^a(y) \in \Omega^1(\Omega^\bullet(Y, \mathfrak{g}))$$

and

$$\delta A_\mu^a(Y) \in \Omega^1(\Omega^\bullet(Y, \mathfrak{g}))\,.$$

The second 1-form vanishes in directions in which the variation of the $\mathfrak{g}$-valued 1-form A is a pure gauge transformation induced by the function g^a which is measured by the first 1-form.

Notice that it is the sum of the exterior derivative of the 0-form $A_\mu^a(y)$ with another term.

$$\delta A_\mu^a(y) = d(A_\mu^a(y)) + \delta_g A_\mu^a(y)\,.$$

The first term on the right measure the change of the connection, the second subtracts the contribution to this change due to gauge transformations. So the 1-form $\delta A_\mu^a(y)$ on the space of $\mathfrak{g}$-valued forms vanishes along all directions along which the form A is modfied purely by a gauge transformation.

The $\delta A_\mu^a(y)$ are the 1-forms the derivations dual to which will be the *antifields*.

2-Forms. We have already seen the 2-form appear on the standard square. We call this 2-form

$$\beta^a \in \Omega^2(\Omega^\bullet(Y, \mathfrak{g}))\,,$$

corresponding on the unit square to the assignment

$$\beta^a : (\Omega^\bullet(Y \times I^2) \leftarrow \mathrm{W}(\mathfrak{g}) : A) \mapsto (\partial_i g_j^a + [g_i, g_j]^a)dt^i \wedge dt^j \,.$$

There is also a 2-form in the game, coming from $(\sigma t^a)\delta_y$.

Then one immediately sees that our forms on the space of $\mathfrak{g}$-valued forms satisfy the relations

$$dc^a(y) = -\frac{1}{2}C^a{}_{bc}c^b(y) \wedge c^c(y) + \beta^a(y)$$

and

$$d\beta^a(y) = -C^a{}_{bc}c^a(y) \wedge c^b(y)\,.$$

The 2-form β on the space of $\mathfrak{g}$-valued forms is what is being contracted by the horizontal derivations called the *antighosts*.

We see, in total, that $\Omega^\bullet(\Omega^\bullet(Y, \mathfrak{g}))$ is the Weil algebra of a DGCA, which is obtained from the above formulas by setting $\beta = 0$ and $\delta A = 0$. This DGCA is the algebra of the gauge groupoid, that where the only morphisms present are gauge transformations.

I just did this computation here over $U = I^2$. But I think it is clear how the computation generalizes and that this result is indeed true.

References

[1] D. Roberts and U.S., *The inner automorphism 3-group of a strict 2-group*

[2] H. Sati, U.S. and J. Stasheff, *Lie ∞-connections and applications to String- and Chern-Simons transport*