disk holonomy of a line 2-bundle

October 18, 2006

Abstract

A complex line 2-bundle ($U(1)$-gerbe) with connection on a smooth space X is, in particular, a smooth 2-functor

$$\mathrm{tra} : \mathcal{P}_2(X) \to \mathrm{RIBim}(\mathbf{Vect}_{\mathbb{C}})$$

from thin-homotopy classes of 2-paths in X to right-induced bimodules in $\mathbf{Vect}_{\mathbb{C}}$.

Given a disk $f : D^1_{a,b} \to X$ in X, with two marked points $a, b \in \partial D^1$ on the boundary, regarded as a 2-morphism

$$\begin{tikzcd} f(a) \arrow[r, bend left=50, "f(\partial_+ D^1)"] \arrow[r, bend right=50, "f(\partial_- D^1)"'] \arrow[r, phantom, "\Downarrow f(D^1)"] & f(b) \end{tikzcd}$$

in $\mathcal{P}_2(X)$, there are two different notions of surface transport over that disk:

- **parallel surface transport** over the disk, taking values in bimodule homomorphisms;
- **surface holonomy** of the disk, taking values in the complex numbers.

These differ by whether or not the boundary $f|_{\partial D^1}$ of the disk is regarded as an entity in its own right or just as an artefact of our description.

The transport in the first case would be

$$\mathrm{tra}(f) := \begin{tikzcd} A_{f(a)} \arrow[r, bend left=50, "N"] \arrow[r, bend right=50, "N'"'] \arrow[r, phantom, "\Downarrow \mathrm{tra}(f(D^1))"] & A_{f(b)} \end{tikzcd} .$$

In terms of this, the disk D^1 is potentially just part of a larger surface. Accordingly, the surface transport along it is such that it may eventually be continued across the boundary.

But if there is extra data assigned to the boundary of the disk, we may *close* this transport using a *trace* which pulls it back along

$$\Sigma(\mathbf{Vect}_{\mathbb{C}}) \to \mathrm{Bim}(\mathbf{Vect}_{\mathbb{C}}) \;,$$

analogous to the situation for transport over closed surfaces. This involves conjugation by one-sided modules $\mathbb{C} \xrightarrow{A} A$ that exist along the boundary:

Definition 1 *The* **surface holonomy over a disk**

$$\begin{tikzcd}
f(a) \arrow[rr, bend left=50, "f(\partial_+ D^1)"] \arrow[rr, bend right=50, "f(\partial_- D^1)"'] \arrow[rr, phantom, "\Downarrow f(D^1)"] & & f(b)
\end{tikzcd}$$

of the 2-transport

$$\mathrm{tra} : \mathcal{P}_2(X) \to \mathrm{Bim}(\mathbf{Vect}_{\mathbb{C}})$$

is the complex number

$$\mathrm{hol}(f) \quad \equiv \quad \begin{tikzcd}
\mathbb{C} \arrow[rr, "\mathrm{Id}"] \arrow[d, "A"] \arrow[dd, bend right=70, "\mathrm{Id}"'] & & \mathbb{C} \arrow[d, "B"] \arrow[dd, bend left=70, "\mathrm{Id}"] \\
A_{f(a)} \arrow[rr, bend left, "N"] \arrow[rr, bend right, "N'"'] \arrow[rr, phantom, "\Downarrow \mathrm{tra}(f(D^1))"] \arrow[d, "A"] & & A_{f(b)} \arrow[d, "B"] \\
\mathbb{C} \arrow[rr, "\mathrm{Id}"'] & & \mathbb{C}
\end{tikzcd} \,.$$

The unlabeled 2-morphisms here are the canonical ones.

Example 1

Assume there exists a complex vector bundle $V \to \partial D^1$ with connection (V, ∇) over the boundary, such that

$$A_{f(p)} \equiv \mathrm{tra}(f(p)) = \mathrm{End}_{V_p}$$

for all $p \in \partial D^1$. In the language of bundle gerbes, this says that the gerbe module descends over the boundary to an untwisted vector bundle.

Furthermore, let

$$B \in \Omega^2\left(D^1\right)$$

be the globally defined curving 2-form of the 2-transport trivialized over the disk.

Then we have the equality

$$
\begin{tikzcd}
A_{f(a)} \arrow[rr, bend left, "N"] \arrow[rr, bend right, "N'"'] \arrow[rr, phantom, "\Downarrow \mathrm{tra}\left(f\left(D^1\right)\right)"] & & A_{f(b)}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A_{f(a)} \arrow[rr, "(A_{f(a)}, \mathrm{Ad}_{\mathrm{tra}_\nabla(\partial_+ D^1)})"] \arrow[d, "V_{f(a)}"] \arrow[dd, bend right=60, "\mathrm{Id}"'] \arrow[drr, phantom, "\Swarrow \bar{\mathrm{tra}}_\nabla\left(f\left(\partial_+ D^1\right)\right)"] & & A_{f(b)} \arrow[d, "V_{f(b)}"] \arrow[dd, bend left=60, "\mathrm{Id}"] \\
\mathbb{C} \arrow[rr, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"'] \arrow[rr, phantom, "\Downarrow \exp\left(\int_{D^1} B\right)"] \arrow[d, "V^*_{f(a)}"] \arrow[l, phantom, "\Leftarrow"] & & \mathbb{C} \arrow[d, "V^*_{f(b)}"] \arrow[r, phantom, "\Leftarrow"] & {} \\
A_{f(a)} \arrow[rr, "(A_{f(a)}, \mathrm{Ad}_{\mathrm{tra}_\nabla(\partial_- D^1)})"'] \arrow[urr, phantom, "\Swarrow \mathrm{tra}^*_\nabla\left(f\left(\partial_- D^1\right)\right)"] & & A_{f(b)}
\end{tikzcd}
.
$$

Inserting this into def. 1 yields

$$
\mathrm{hol}_{\mathrm{tra}}(D) \quad \equiv \quad
\begin{tikzcd}[row sep=large, column sep=huge]
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "A_{f(a)}"'] \arrow[dddd, bend right=80, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow\, \mathrm{Ad}_{\mathrm{tra}_\nabla(\partial_+ D^1)}"] & \mathbb{C} \arrow[d, "A_{f(b)}"] \arrow[dddd, bend left=80, "\mathrm{Id}"] \\
A_{f(a)} \arrow[r, "{(A_{f(a)}, \mathrm{Ad}_{\mathrm{tra}_\nabla(\partial_+ D^1)})}"] \arrow[d, "V_{f(a)}"'] \arrow[dd, bend right=60, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow\, \overline{\mathrm{tra}}_\nabla\big(f(\partial_+ D^1)\big)"] & A_{f(b)} \arrow[d, "V_{f(b)}"] \arrow[dd, bend left=60, "\mathrm{Id}"] \\
\mathbb{C} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\exp\left(\int_{D^1} B\right)"] \arrow[d, "V^*_{f(a)}"'] \arrow[dr, phantom, "\Swarrow\, \mathrm{tra}^*_\nabla\big(f(\partial_- D^1)\big)"] & \mathbb{C} \arrow[d, "V^*_{f(b)}"] \\
A_{f(a)} \arrow[r, "{(A_{f(a)}, \mathrm{Ad}_{\mathrm{tra}_\nabla(\partial_- D^1)})}"'] \arrow[d, "A_{f(a)}"'] \arrow[dr, phantom, "\Swarrow\, \mathrm{Id}"] & A_{f(b)} \arrow[d, "A_{f(b)}"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C}
\end{tikzcd}
.
$$

The right hand side is a complex number, whose value is

$$\mathrm{hol}_{\mathrm{tra}}(D) = \exp\left(\int_D B\right) \mathrm{Tr}\left(\mathrm{tra}_\nabla(\partial D)\right) .$$