# disk holonomy of a line 2-bundle

## October 18, 2006

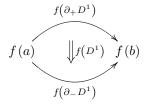
#### Abstract

A complex line 2-bundle (U(1)-gerbe) with connection on a smooth space X is, in particular, a smooth 2-functor

 $\operatorname{tra}: \mathcal{P}_2(X) \to \operatorname{RIBim}(\operatorname{\mathbf{Vect}}_{\mathbb{C}})$ 

from thin-homotopy classes of 2-paths in X to right-induced bimodules in  $\mathbf{Vect}_{\mathbb{C}}$ .

Given a disk  $f: D^1_{a,b} \to X$  in X, with two marked points  $a, b \in \partial D^1$  on the boundary, regarded as a 2-morphism

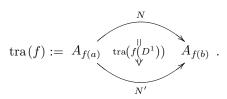


in  $\mathcal{P}_2(X)$ , there are two different notions of surface transport over that disk:

- **parallel surface transport** over the disk, taking values in bimodule homomorphisms;
- surface holonomy of the disk, taking values in the complex numbers.

These differ by whether or not the boundary  $f|_{\partial D^1}$  of the disk is regarded as an entity in its own right or just as an artefact of our description.

The transport in the first case would be



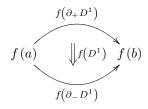
In terms of this, the disk  $D^1$  is potentially just part of a larger surface. Accordingly, the surface transport along it is such that it may eventually be continued across the boundary.

But if there is extra data assigned to the boundary of the disk, we may *close* this transport using a *trace* which pulls it back along

$$\Sigma(\operatorname{\mathbf{Vect}}_{\mathbb{C}}) \to \operatorname{Bim}(\operatorname{\mathbf{Vect}}_{\mathbb{C}})$$
,

analogous to the situation for transport over closed surfaces. This involves conjugation by one-sided modules  $\mathbb{C} \xrightarrow{A} A$  that exist along the boundary:

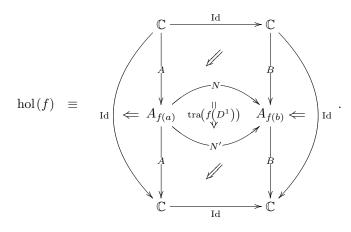
#### Definition 1 The surface holonomy over a disk



of the 2-transport

$$\operatorname{tra}: \mathcal{P}_2(X) \to \operatorname{Bim}(\operatorname{Vect}_{\mathbb{C}})$$

is the complex number



The unlabeled 2-morphisms here are the canonical ones.

### Example 1

Assume there exists a complex vector bundle  $V \to \partial D^1$  with connection  $(V, \nabla)$  over the boundary, such that

$$A_{f(p)} \equiv \operatorname{tra}(f(p)) = \operatorname{End}_{V_p}$$

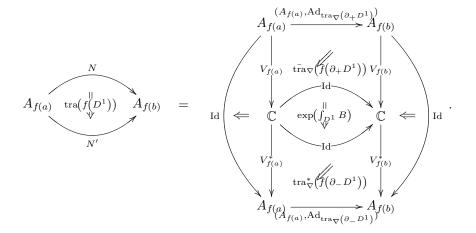
for all  $p \in \partial D^1$ . In the language of bundle gerbes, this says that the gerbe module descends over the boundary to an untwisted vector bundle.

Furthermore, let

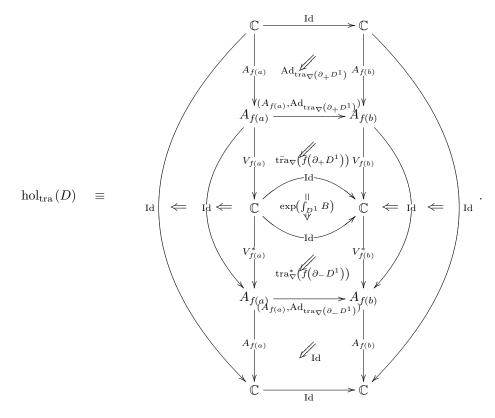
 $B \in \Omega^2 \left( D^1 \right)$ 

be the globally defined curving 2-form of the 2-transport trivialized over the disk.

Then we have the equality



Inserting this into def. 1 yields



The right hand side is a complex number, whose value is

$$\operatorname{hol}_{\operatorname{tra}}(D) = \exp\left(\int_{D} B\right) \operatorname{Tr}\left(\operatorname{tra}_{\nabla}(\partial D)\right)$$

.