Integration without Integration
Integration by Transgression
Integration by Internal Homs

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Abstract

On how transgression and integration of forms comes from internal homs applied on transport n-functors, on what that looks like after passing to a Lie ∞-algebraic description and how it realizes the notion of *integration without integration*.

Contents

1 Introduction

For X a smooth space, there is a canonical isomorphism of n-categories [1, 5, 6]

$$\{\text{smooth } n\text{-functors from } n\text{-paths in } X \text{ to } \mathbf{B}^n\mathbb{R}\} \simeq \{n\text{-forms on } X\}\,. \tag{1}$$

Here and in the following, for G any abelian group we write $\mathbf{B}^n G$ for the n-groupoid with just one element in degrees below n and with G in degree n.

Integration of an n-form ω on X over an n-dimensional subspace Σ

$$\phi : \Sigma \to X$$

is the same thing as

- first transgressing ω to a 0-form $\mathrm{tg}_\Sigma \omega$ on the space $\mathrm{maps}(\Sigma, X)$,
- then evaluating that at $\phi \in \mathrm{maps}(\Sigma, X)$.

$$\int_\Sigma \phi^* \omega = (\mathrm{tg}_\Sigma \omega)(\phi)\,.$$

By itself, this statement is just a weird reformulation of the comparativly more elementary notion of integration, based on the fact that transgression consists of pullback followed by fiber integration.

But the important point is: it is transgression, not integration itself, which is naturally represented on the left hand of our equivalence 1.

Theorem 1 ([6]) *Let $\Sigma = S^n$ be the n-sphere. Let*

$$\mathrm{tra} : \mathcal{P}_n(X) \to \mathbf{B}^n U(1)$$

be a smooth n-functor on n-paths in X, coming from an n-form ω under the above equivalence. Then the smooth 0-functor

$$S^n\mathrm{tra} := \mathrm{Hom}(\mathbf{B}^n\mathbb{Z}, \mathrm{tra}) : \mathcal{P}_0(S^n X) \to U(1)$$

comes from the 0-form $\mathrm{tg}_{S^n}\omega$.

Here $S^n X = \mathrm{maps}(S^n, X)$ denotes the n-sphere space of X.

So

$$\begin{array}{ccc} n\mathrm{Funct}^\infty(\mathcal{P}_n(X), \mathbf{B}^n U(1)) & \xrightarrow{\simeq} & \Omega^n(X) \\ \downarrow{\scriptstyle \mathrm{Hom}(\mathbf{B}^n\mathbb{Z},-)} & & \downarrow{\scriptstyle \mathrm{tg}_{S^n}} \\ 0\mathrm{Func}^\infty(\mathcal{P}_0(S^n X), U(1)) & \xrightarrow{\simeq} & \Omega^0(S^n X) \end{array} \qquad \begin{array}{ccc} \mathrm{tra} & \overset{\simeq}{\longmapsto} & \omega \\ \downarrow{\scriptstyle \mathrm{Hom}(\mathbf{B}^n\mathbb{Z},-)} & & \downarrow{\scriptstyle \mathrm{tg}_{S^n}} \\ S^n\mathrm{tra} & \overset{\simeq}{\longmapsto} & \int_\Sigma (-)^*\omega \end{array}$$

This says that transgression, and hence integration of forms, is the same, under our equivalence 1, as forming an inner hom on n-functors.

That this works so easily has to do with the fact that $\mathbf{B}^n\mathbb{Z}$ is an "integral" model for the n-sphere, which we should think of as the n-groupoid freely generated from the the fundamental n-path on S^n:

$$\mathbf{B}^n\mathbb{Z} \hookrightarrow \mathcal{P}_n(S^n)\ .$$

If we want to describe everything Lie ∞-algebraically, we have to use the path n-groupoid on the right, instead. We will find that the above statement still remains true *after we divide out equivalences* of n-functors. We show that this quotienting is the forming of equivalence classes which underlies the idea of *integration without integration* [2, 3].

2 Integration over S^1

Not to get distracted by inessential distractions, we'll focus on S^1 for the time being.

2.1 Functorial integration on S^1

Consider $\mathcal{P}_1(S^1)$, the path 1-groupoid of the circle, where we had $\mathbf{B}\mathbb{Z}$ before.

Let X be a smooth space and

$$\mathrm{tra} : P_1(X) \to \mathbf{B}U(1)$$

a smooth 1-functor coming from a 1-form $A \in \Omega^1(X)$. Hitting this with

$$\mathrm{Hom}(\mathcal{P}_1(S^1), -)$$

produces

$$\mathrm{Hom}(\mathcal{P}_1(S^1), \mathrm{tra}) : \mathrm{Hom}(\mathcal{P}_1(S^1), \mathcal{P}_1(X)) \to \mathrm{Hom}(\mathcal{P}_1(S^1), \mathbf{B}U(1)) \tag{2}$$

By again appliyng the equivalence 1, we know that the groupoid $\mathrm{Hom}(\mathcal{P}_1(S^1), \mathbf{B}U(1))$ has as objects all 1-forms ω on the circle, and morphisms

$$\lambda : \omega \to \omega'$$

all 0-forms (functions) such that

$$\omega' = \omega + d\lambda\,.$$

So unlike for $\mathrm{Hom}(\mathbf{B}\mathbb{Z}, \mathbf{B}U(1))$, an object is not quite an element in $U(1)$, but instead a 1-form which induces an element in $U(1)$ by integrating it over the circle. In fact, it's the *isomorphism classes* of objects in $\mathrm{Hom}(\mathcal{P}_1(S^1), \mathbf{B}U(1))$ which correspond to elements of $U(1)$.

So we do get our expected holonomy functor by quotienting out isomorphism

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{P}_1(S^1), \mathcal{P}_1(X)) & \xrightarrow{\mathrm{Hom}(\mathcal{P}_1(S^1),\mathrm{tra})} & \mathrm{Hom}(\mathcal{P}_1(S^1), \mathbf{B}U(1)) \\ \downarrow{\scriptstyle /\sim} & & \downarrow{\scriptstyle /\sim} \\ LX & \xrightarrow{\exp(\int i(-))} & U(1) \end{array} \quad . \tag{3}$$

We have integrated the 1-form without integrating it. We have just waved an abstract-nonsense wand above it. If you like abstract nonsense, that's pleasing. If not, I should describe how this is actually useful for quantization and for understanding BV-formalism. But that's a longer story, which I won't complete here. But let's start with some aspects.

2.2 Lie ∞-algebraic integration on S^1

I now consider the situation from the point of view of [4].

Then a smooth 1-functor from paths in X to $\mathbf{B}U(1)$, hence a 1-form on X, is the same as a morphisms of differential graded-commutative algebras (DGCAs)

$$\Omega^\bullet(X) \xleftarrow{(A,dA)} \mathrm{W}(\mathfrak{u}(1)) \ .$$

Here $\mathrm{W}(\mathfrak{u}(1))$ denotes the Weil algebra of $\mathfrak{u}(1)$, which is the free differential graded-commutative algebra on a single degree 1 generator.

As described in [4], the notion of inner hom we want to use here is, for any two DGCAs A and B

$$\Omega^\bullet(\mathrm{maps}(B, A))\,,$$

the DGCA of differential forms on the smooth space of DGCA morphisms from B to A, which is the presheaf of sets on open subsets U of Euclidean spaces given by

$$U \mapsto \mathrm{Hom}_{\mathrm{DGCAs}}(B, A \otimes \Omega^\bullet(U))\,.$$

The functor

$$\Omega^\bullet(\mathrm{maps}(--, \Omega^\bullet(S^1))) : \mathrm{DGCAs} \to \mathrm{DGCAs}$$

is the analog of our $\mathrm{Hom}(\mathcal{P}_1(S^1), -)$ above.

Applying it to (A, dA) we get a DGCA morphism

$$\Omega^\bullet(\mathrm{maps}(\Omega^\bullet(X), \Omega^\bullet(S^1))) \xleftarrow{\mathrm{tg}_{S^1}(A,dA)} \Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1)))$$

as the Lie analog of 2.

We need to understand what it means, in this picture, to form the quotient which amounts to forming the integral, as in 3.

I shall be making the following

Claim. *Forming the integral of A over circles $\phi : S^1 \to X$ amounts to restricting the transgressed $\mathfrak{u}(1)$-connection morphism to the* **characteristic forms** *in* $\Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1)))$.

Recall from [4] what that means:

Hitting the sequence of forms on the universal $\mathfrak{u}(1)$-bundle

$$\begin{array}{ccc} & & \mathrm{CE}(\mathfrak{u}(1)) \\ & & \uparrow \\ \Omega^\bullet(X) & \xleftarrow{(A,dA)} & \mathrm{W}(\mathfrak{u}(1)) \\ & & \uparrow \\ & & \mathrm{inv}(\mathfrak{u}(1)) = \mathrm{CE}(b\mathfrak{u}(1)) \end{array}$$

with our transgression functor yields

$$\begin{array}{ccc} & & \Omega^\bullet(\mathrm{maps}(\mathrm{CE}(\mathfrak{u}(1)), \Omega^\bullet(S^1))) \\ & & \uparrow \\ \Omega^\bullet(\mathrm{maps}(\Omega^\bullet(X), \Omega^\bullet(S^1))) & \xleftarrow{\mathrm{tg}_{S^1}(A,dA)} & \Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1))) \\ & & \uparrow \\ & & \Omega^\bullet(\mathrm{maps}(\mathrm{CE}(b\mathfrak{u}(1)), \Omega^\bullet(S^1))) \end{array}\,.$$

With respect to the surjection

$$\begin{array}{c} \Omega^\bullet(\mathrm{maps}(\mathrm{CE}(\mathfrak{u}(1)), \Omega^\bullet(S^1))) \\ \uparrow{\scriptstyle i^*} \\ \Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1))) \end{array}$$

we can form the basic elements in $\Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1)))$ (those invariant under vertical derivations) and the claim is that the image of these knows about the integral of A over the circles $\phi : S^1 \to X$:

$$
\begin{array}{ccc}
 & & \Omega^\bullet(\mathrm{maps}(\mathrm{CE}(\mathfrak{u}(1)), \Omega^\bullet(S^1))) \quad . \\
 & & \uparrow \\
\Omega^\bullet(\mathrm{maps}(\Omega^\bullet(X), \Omega^\bullet(S^1))) & \xleftarrow{\mathrm{tg}_{S^1}(A, dA)} & \Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1))) \\
 & \nwarrow^{\text{integration}} & \uparrow \\
 & & \Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1)))_{\mathrm{basic}}
\end{array}
$$

I now describe some details of this. We need to understand what $\Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1)))$ looks like. This was described in [4] already for $\mathfrak{u}(1)$ replaced by an arbitrary Lie algebra, but in the simple case of interest here this deserves to be said again:

Denote by a and σa the canonical degree 1 and 2 generators of $\mathrm{W}(\mathfrak{g})$, with $d_{\mathrm{W}(\mathfrak{g})} : a \mapsto \sigma a$. Then for U any open subset of a Euclidean space, on which we consider the canonical coordinate functions $\{x^\mu\}$, we find that a general morphism $f^*\ \mathrm{W}(\mathfrak{u}(1)) \to \Omega^\bullet(S^1) \otimes \Omega^\bullet(U)$ is like

$$
\begin{array}{ccc}
a & \xmapsto{f^*} & \omega + \lambda_\mu dx^\mu \\
\downarrow{\scriptstyle d_{\mathrm{W}(\mathfrak{u}(1))}} & & \downarrow{\scriptstyle d_{S^1} + d_U} \\
\sigma a & \xmapsto{f^*} & \begin{array}{c} d_U\omega + (d_{S^1}\lambda) \wedge dx^\mu \\ = \alpha_\mu \wedge dx^\mu + \beta_{\mu\nu} dx^\mu \wedge dx^\nu \end{array}
\end{array}
$$

where

- $\omega \in \Omega^1(S^1)$
- $\lambda \in \Omega^0(S^1)$
- $\alpha_\mu \in \Omega^1(S^1)$
- $\beta_{\mu\nu} \in \Omega^0(S^1)$.

By postcomposing this with a 1-current $c_1 : \Omega^1(S^1) \to \mathbb{R}$ on the circle we obtain the 0-form

$$U \mapsto c(\omega) \in \Omega^0(U)$$

and the 1-forms

$$U \mapsto c(\alpha_\mu) \in \Omega^1(U)$$

on $\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1))$.

Similarly for postcompositon with currents of other degrees.

Notice, in particular, that a function $c(\omega)$ on this space of forms in the circle has, by the equality on the bottom right corner of the above diagram, the differential

$$d(c(\omega)) = c(d_{S^1}\lambda) + c(\alpha_\mu)\,.$$

The first contribution on the right is the change due to a gauge transformation of 1-forms $A \mapsto A + d\rho$, the second one that due to a shift which is not pure gauge.

So what are the *basic* (the invariant) 0-forms here? One sees that the projection

$$\begin{array}{c} \Omega^\bullet(\mathrm{maps}(\mathrm{CE}(\mathfrak{u}(1)), \Omega^\bullet(S^1))) \\ \Big\uparrow{\scriptstyle i^*} \\ \Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{u}(1)), \Omega^\bullet(S^1))) \end{array}$$

restricts functions on the space of 1-forms on S^1 to the subset of *closed* 1-forms, and sends $c(\alpha_\mu)$ and $c(\beta_{\mu\nu})$ to 0.

A basic form on the space of 1-forms on the circle is one which all whose indecomposable components are annihilated by this projection and such that the same is true for its differential.

But this means that a basic 0-form on the space of 1-forms on the circle is a 0-form which comes from a 1-current c on S^1

- such that $U \mapsto c(\omega)$ vanishes on flat 1-forms
- such that in $U \mapsto d(c(\omega)) = c(d_{S^1}\lambda) + c(\alpha_\mu)$ the first term after the equality sign vanishes.

A little reflection shows that this is true precisely for one single current, namely the integration operation

$$c : \Omega^1(S^1) \to \mathbb{R}$$

$$\alpha \mapsto \int_{S^1} \alpha \,.$$

Conclusion Passing to equivalence classes of objects has a nice description in the DGCA-formulation of the Lie ∞-algebraic perspective: it amounts to looking at "basic 0-forms" aka characteristic 0-forms.

These are precisely the elements that compute integrals, in the above sens.

Again, we have done *integration without integration*: we integrated a 1-form simply by passing to the characteristic 0-forms on the space of 1-forms on the circle.

3 Transgression of differential forms

We have identified integrals of forms with characteristic 0-forms on the space of all forms. To exhibit the relation to transgression of forms and to fiber integration more explicitly, it may be helpful to consider generally forms on mapping spaces obtained from transgression of forms.

Definition 1 (forms on mapping spaces from transgression) *For A any DGCA, Y any smooth manifold of dimension d, and $\omega \in A$ any element of degree $n \geq d$, we denote by*

$$\int_Y \mathrm{ev}^*\omega \;\; \in \;\; \Omega^{n-d}(\mathrm{maps}(A, \Omega^\bullet(Y)))$$

the $(n-d)$-form on the space of morphisms from A to $\Omega^\bullet(Y)$ given by the assignment

$$\int_Y \mathrm{ev}^*\omega\Big|_U \; : \; (\mathrm{ev}_U^* \in \mathrm{Hom}_{\mathrm{DGCAs}}(A, \Omega^\bullet(U)\otimes\Omega^\bullet(Y))) \;\mapsto\; (\int_Y \mathrm{ev}_U^*(\omega) \;\in\; \Omega^\bullet(U))$$

for all test domains $U \in S$.

Here ev_U^* is just our suggestive name for any element in $\mathrm{Hom}_{\mathrm{DGCAs}}(A, \Omega^\bullet(U) \otimes \Omega^\bullet(Y)$.

Example. (ordinary transgression forms) For $A = \Omega^\bullet(X)$ with X some smooth manifold, a DGCA morphism

$$\mathrm{ev}_U^* : \Omega^\bullet(X) \to \Omega^\bullet(Y) \otimes \Omega^\bullet(U)$$

comes from a smooth map $\mathrm{ev}_U : U \times Y \to X$.

4 Chern-Simons integrals

We now compute the parallel transport of a membrane (the 3-particle) propagating on the classifying space BG and coupled to the canonical Chern-Simons 3-bundle with connection on BG, along the lines described in [4].

Let $\mathfrak{g}$ be any semisiple Lie algebra, μ its canonical 3-cocycle and P the corresponding invariant polynomial and cs the corresponding transgression element. The canonical Chern-Simons 3-bundle on BG is then given by the $b^2\mathfrak{u}(1)$-connection descent object

$$\begin{array}{ccc}
\mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^2\mathfrak{u}(1)) \\
\uparrow & & \uparrow \\
\mathrm{W}(\mathfrak{g}) & \xleftarrow{(\mathrm{cs},P)} & \mathrm{W}(b^2\mathfrak{u}(1)) \\
\uparrow & & \uparrow \\
\mathrm{inv}(\mathfrak{g}) & \xleftarrow{P} & \mathrm{CE}(b^3\mathfrak{u}(1))
\end{array} .$$

Now let par be a 3-dimensional manifold, the parameter space (worldvolume) of the 3-particle (membrane). Coupling the membrane to the above 3-connection amounts to applying the functor

$$\Omega^\bullet(\mathrm{maps}(-, \Omega^\bullet(\mathrm{par}))) : \mathrm{DGCAs} \to \mathrm{DGCAs}$$

to the above diagram, thus transgressing the 3-bundle to configuration space

$$\mathrm{conf} = \mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(\mathrm{par}))\,.$$

Entirely analogously to the discussion in 2.2 this yields

$$\begin{array}{ccc}
\Omega^\bullet(\mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(\mathrm{par})) & \xleftarrow{\mathrm{tg}_{\mathrm{par}}\mu} & \Omega^\bullet(\mathrm{maps}(\mathrm{CE}(b^2\mathfrak{u}(1)), \Omega^\bullet(\mathrm{par})) \\
\uparrow & & \uparrow \\
\Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(\mathrm{par})) & \xleftarrow{\mathrm{tg}_{\mathrm{par}}(\mathrm{cs},P)} & \Omega^\bullet(\mathrm{maps}(\mathrm{W}(b^2\mathfrak{u}(1)), \Omega^\bullet(\mathrm{par})) \\
 & \nwarrow^{\text{integration}} & \uparrow \\
 & & \Omega^\bullet(\mathrm{maps}(\mathrm{W}(b^2\mathfrak{u}(1)), \Omega^\bullet(\mathrm{par}))_{\mathrm{basic}}
\end{array} .$$

The morphism labeled "integration" here indeed computes the Chern-Simons intgeral over the worldvolume of the membrane, as described now.

For $k \in \mathbb{R}$ any constant let again $\omega \in \Omega^0(\mathrm{maps}(\mathrm{W}(b^2\mathfrak{u}(1))), \Omega^\bullet(\mathrm{par}))$ given on each test domain U by

$$(f^* : \mathrm{W}(b^2\mathfrak{u}(1)) \to \Omega^\bullet(\mathrm{par}) \otimes \Omega^\bullet(\mathrm{U})) \xmapsto{\omega_U} (k \int_{\mathrm{par}} f^*(a) \in \Omega^0(U))$$

be the characteristic 0-form on the space of 3-forms on par, as in 2.2, where now

$$a \in \mathrm{W}(b^2\mathfrak{u}(1))$$

denotes the canonical degree 3-generator.

Mapping this along

$$\Omega^\bullet(\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(\mathrm{par})) \xleftarrow{\mathrm{tg}_{\mathrm{par}}(\mathrm{cs}, P)} \Omega^\bullet(\mathrm{maps}(\mathrm{W}(b^2\mathfrak{u}(1)), \Omega^\bullet(\mathrm{par}))$$

produces the 0-form

$$(\mathrm{tg}_{\mathrm{par}}(\mathrm{cs}, P))(\omega) \in \Omega^0(\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(\mathrm{par}))) = \Omega^0(\Omega^1(\mathrm{par}, \mathfrak{g}))$$

on the space of $\mathfrak{g}$-valued 1-forms on par which is given on each test domain U by

$$\begin{aligned}(A \in \Omega^\bullet(\mathrm{par}, \mathfrak{g})) \longmapsto\; & k \int_{\mathrm{par}} (A, F_A)((\mathrm{cs}, P)(a)) \quad . \\ & = k \int_{\mathrm{par}} \mathrm{CS}(A, F_A) \in \Omega^0(U)\end{aligned}$$

As claimed, this is indeed the Chern-Simons action functional. Recall that we obtain this merely by transgressing the canonical Chern-Simons 3-bundle to the configuration space of the membrane and then looking at the characteristic 0-forms.

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