n-inner products from n-Homs

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Abstract

Sections of n-vector bundles are morphisms from the tensor unit into the corresonding transport n-functor. Cosections go the other way around. The natural pairing on sections and cosections yields the pointwise inner product. But, by abstract nonsense, this pairing has itself the structure of an $(n-1)$-transport. Hence we get an entire hierarchy of higher order sections.

For $n = 2$, first order sections correspond to boundary insertions, while second order sections correspond to D-branes.

Contents

1 Concept Formation

I assume that the reader is familiar with the basic setup of functorial parallel transport theory, as described elsewhere.

1.1 Sections and their pairing

Let tra_* be any n-transport on some configuration space, and let 1_* be the trivial such n-transport.

We write

$$\mathrm{obs} := \mathrm{End}(1_*, 1_*)\,.$$

A **section** of tra_* is an object in

$$\mathrm{sect} := \mathrm{Hom}(1_*, \mathrm{tra}_*)\,.$$

A **cosection** of tra_* is an object in

$$\mathrm{cosect} := \mathrm{Hom}(\mathrm{tra}_*, 1_*)\,.$$

The **pointwise inner product** between a section

$$1_* \xrightarrow{e_1} \mathrm{tra}_*$$

and a cosection

$$\mathrm{tra}_* \xrightarrow{\bar{e}_2} 1_*$$

is the $(n-1)$-functor

$$\mathrm{Hom}\left(\begin{tikzcd} \mathrm{tra}_* \\ 1_* \arrow[u, "e_1"] \end{tikzcd}\ ,\ \begin{tikzcd} \mathrm{tra}_* \arrow[d, "\bar{e}_2"] \\ 1_* \end{tikzcd}\right)\ :\ \mathrm{End}(\mathrm{tra}_*) \to \mathrm{obs}$$

or rather its value on the identity

$$(\cdot,\cdot) :\ \mathrm{sect} \times \mathrm{cosect} \xrightarrow{\mathrm{Hom}(\cdot,\cdot)} [\mathrm{End}(\mathrm{tra}_*), \mathrm{obs}] \xrightarrow{\mathrm{ev}(\cdot,\mathrm{Id})} \mathrm{obs}\ .$$

Of course this is just a fancy way of saying that

$$(\bar{e}_2, e_1) =\ 1_* \xrightarrow{e_1} \mathrm{tra}_* \xrightarrow{\bar{e}_2} 1_*\ .$$

Notice that a modification of sections translates into a transformation of their inner product $(n-1)$-functor, and so on. The inner product is completely natural.

1.2 Higher order sections

Since obs is itself monoidal, with tensor unit Id_{1_*}, we may identify **second order sections** of a given observable o

$$D_1 \in \mathrm{Hom}_{\mathrm{obs}}(\mathrm{Id}_{1_*}, o)$$

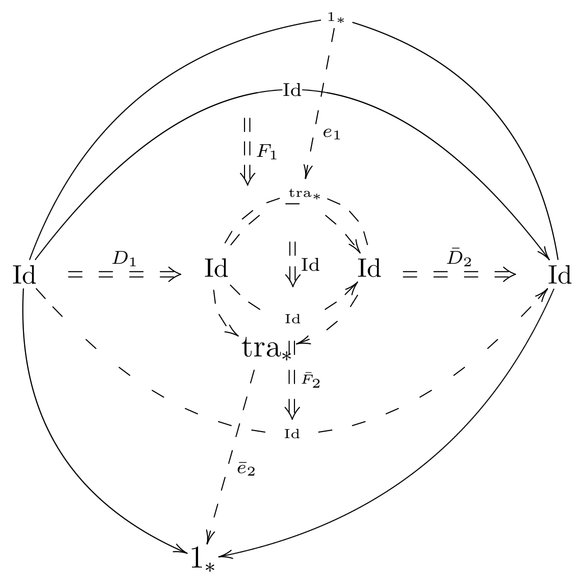
as well as second order cosections in

$$\bar{D}_2 \in \mathrm{Hom}_{\mathrm{obs}}(o, \mathrm{Id}_{1_*})\,.$$

All the above reasoning goes through as before, with everything one dimension lower. In particular, for $o = (\bar{e}_2, e_1)$, we have the second order inner product pairing

$$(\bar{D}_2, D_1) \;=\; \begin{tikzcd}[row sep=large]
 & 1_* \arrow[d, "e_1"] \arrow[ddd, bend right=80, "\mathrm{Id}"'] \arrow[ddd, bend left=80, "\mathrm{Id}"] & \\
 {} \arrow[r, phantom, "\overset{D_1}{\Rightarrow}"] & \mathrm{tra}_* \arrow[d, bend right=50, "\mathrm{Id}"'] \arrow[d, bend left=50, "\mathrm{Id}"] \arrow[d, phantom, "\overset{\mathrm{Id}}{\Rightarrow}"] & {} \arrow[l, phantom, "\overset{\bar{D}_2}{\Rightarrow}"'] \\
 & \mathrm{tra}_* \arrow[d, "\bar{e}_2"] & \\
 & 1_* &
\end{tikzcd} .$$

For n-transport, there are higher order sections up to and including order n. For 3-transport we find higher order pairings of the form



where the 3-arrows F_1 and $\bar{F}_2$ denote a third order section and cosection, respectively.

2 Examples

I spell out what all this sophisticated formalism reduces to for the case of ordinary vector bundles. Then I show how the construction of gerbe disk holonomy, including boundary insertions and gerbe modules (D-branes) on the boundary is captured by the concept of pairing of second order sectins of line 2-bundles.

This reproduces, from a more conceptually well-founded point, the expression for gerbe disk holonomy which I had already discussed elsewhere.

2.1 Ordinary sections of vector bundles

Example 1 *Let* $\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$ *be a vector bundle with connection, and let configuration space be*

$$\mathrm{conf} := \mathrm{Disc}(X) \subset [\{\bullet\}, \mathcal{P}_1(X)]\,.$$

Then $\mathrm{tra}_* : \mathrm{conf} \to \mathrm{Vect}$ *is the same vector bundle, but without connection.* 1_* *is the trivial line bundle over* X*.* obs *is the monoidal 0-category of scalar functions on* X*.*

A section of tra_* *is nothing but a section of the corresponding vector bundle in the familiar sense. Same for cosections. The pairing*

$$(\bar{e}_2, e_1) = \; 1_* \xrightarrow{e_1} \mathrm{tra}_* \xrightarrow{\bar{e}_2} 1_*$$

is the familiar pointwise pairing of sections and cosections of vector bundles.

Notice that we would need extra structure, like a Riemannian structure on X*, to get from this pointwise inner product to an integrated inner product.*

2.2 Gerbe modules and second order sections

Example 2 *Let* $\mathrm{tra} : \mathcal{P}_2(X) \to \mathrm{Bim}$ *be a line 2-bundle. Over objects this is a* $\mathrm{PU}(H)$*-associated bundle of compact operators. Over each path this is a bimodule of the respective algebras of compact operators, coming from the algebra automorphism associated to parallel transport over that path, coming from a* $\mathrm{PU}(H)$ *connection* ∇

$$\begin{tikzcd} \mathrm{tra}: x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow\Sigma"] & y & \mapsto & A_x \arrow[r, bend left, "{(A_x, \mathrm{tra}_\nabla(\gamma))}"] \arrow[r, bend right, "{(A_x, \mathrm{tra}_\nabla(\gamma'))}"'] \arrow[r, phantom, "\Downarrow \mathrm{tra}(\Sigma)"] & A_y \end{tikzcd}$$

Let

$$\mathrm{conf} = \mathrm{Disc}(\mathrm{Obj}([\{a \to b\}, \mathcal{P}_2(X)])) \subset [\{a \to b\}, \mathcal{P}_2(X)]$$

be the space of open string configurations in X*. A section*

$$e : \; 1_* \longrightarrow \mathrm{tra}_*$$

is, over each path $x \xrightarrow{\gamma} y$ *, a morphism*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(\gamma)"'] \arrow[dr, phantom, "\swarrow e_{a\to b}(\gamma)"] & \mathbb{C} \arrow[d, "e_b(\gamma)"] \\
A_x \arrow[r, "(A_x, \mathrm{tra}_\nabla(\gamma))"'] & A_y
\end{tikzcd}$$

in $[a \to b, \mathrm{Bim}]$*. Given a flow on configuration space which carries* γ_+ *to* γ_- *across a surface* Σ*, we may transport that section along that flow. It will then be given, over* $x \xrightarrow{\gamma_-} y$ *, by the morphism*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(\gamma)"'] \arrow[dr, phantom, "\swarrow e_{a\to b}(\gamma)"] & \mathbb{C} \arrow[d, "e_b(\gamma)"] \\
A_{x_+} \arrow[r, "(A_{x_+}, \mathrm{tra}_\nabla(\gamma_+))"'] \arrow[d, "(A_{x_+}, \mathrm{tra}_\nabla(\gamma_a))"'] \arrow[dr, phantom, "\swarrow \mathrm{tra}(\Sigma)"] & A_{y_+} \arrow[d, "(A_{y_+}, \mathrm{tra}_\nabla(\gamma_b))"] \\
A_{x_-} \arrow[r, "(A_x, \mathrm{tra}_\nabla(\gamma_-))"'] & A_{y_-}
\end{tikzcd} .$$

Pairing this with any cosection $\bar{e}$, gives, over γ_-, the morphism

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(\gamma_+)"'] \arrow[dr, phantom, "\swarrow e_{a\to b}(\gamma_+)"] & \mathbb{C} \arrow[d, "e_b(\gamma_+)"] \\
A_{x_+} \arrow[r, "(A_{x_+}, \mathrm{tra}_\nabla(\gamma_+)"'] \arrow[d, "(A_{x_+}, \mathrm{tra}_\nabla(\gamma_a))"'] \arrow[dr, phantom, "\swarrow \mathrm{tra}(\Sigma)"] & A_{y_+} \arrow[d, "(A_{y_+}, \mathrm{tra}_\nabla(\gamma_b))"] \\
A_{x_-} \arrow[r, "(A_x, \mathrm{tra}_\nabla(\gamma_-)"'] \arrow[d, "\bar{e}_a(\gamma_-)"'] \arrow[dr, phantom, "\swarrow \bar{e}_{a\to b}(\gamma_-)"] & A_{y_-} \arrow[d, "\bar{e}_b(\gamma_-)"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C}
\end{tikzcd}
.
$$

This is the component of the pairing

$$
1_* \xrightarrow{e} \mathrm{tra}_* \xrightarrow{\bar{e}} 1_*
$$

over each point in configuration space. Certainly the component of the identity

$$
1_* \xrightarrow{\mathrm{Id}} 1_*
$$

is just

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & \mathbb{C} \arrow[d, "\mathrm{Id}"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"] & \mathbb{C}
\end{tikzcd}
.
$$

A second order section

$$
\begin{tikzcd}
1_* \arrow[d, "e_1"] \arrow[ddd, bend right=60, "\mathrm{Id}"'] \arrow[dd, phantom, "\overset{D}{\Longrightarrow}" near start, shift right=6ex] \\
\mathrm{tra}_* \arrow[d, bend right, "\mathrm{Id}"'] \arrow[d, bend left, "\mathrm{Id}"] \arrow[d, phantom, "\overset{\mathrm{Id}}{\Longrightarrow}"] \\
\mathrm{tra}_* \arrow[d, "\bar{e}_2"] \\
1_*
\end{tikzcd}
$$

is, in components, a tin can

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[ddd, "\mathrm{Id}"'] \arrow[dddr, phantom, "\Swarrow \mathrm{Id}"] & \mathbb{C} \arrow[ddd, "\mathrm{Id}"] \arrow[dr, "e_b(\gamma_+)"] & \\
& & A_{y_+} \arrow[d, "{(A_{y_+}, \mathrm{tra}_\nabla(\gamma_b))}"] \arrow[l, phantom, "\Lleftarrow D_b(\gamma_-)"] \\
& & A_{y_-} \arrow[dl, "\bar{e}_b(\gamma_-)"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"] & \mathbb{C} &
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id"] \arrow[d, "e_a(\gamma_+)"'] \arrow[dddd, bend right=50, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow e_{a\to b}(\gamma_+)"] & \mathbb{C} \arrow[d, "e_b(\gamma_+)"] \\
A_{x_+} \arrow[r, "{(A_{x_+}, \mathrm{tra}_\nabla(\gamma_+))}"'] \arrow[dd] \arrow[ddr, phantom, "\Swarrow \mathrm{tra}(\Sigma)"] & A_{y_+} \arrow[dd, "{(A_{y_+}, \mathrm{tra}_\nabla(\gamma_b))}"] \\
\arrow[r, phantom, "\Lleftarrow D_a(\gamma_-)", near start] & \\
A_{x_-} \arrow[r, "{(A_x, \mathrm{tra}_\nabla(\gamma_-))}"'] \arrow[d, "\bar{e}_a(\gamma_-)"'] \arrow[dr, phantom, "\Swarrow \bar{e}_{a\to b}(\gamma_-)"] & A_{y_-} \arrow[d, "\bar{e}_b(\gamma_-)"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C}
\end{tikzcd}
,
$$

since these are the 2-morphisms in $[\{a \to b\}, \mathrm{Bim}]$. *Clearly,* D_b *is completely determined by* D_a*, so this is really just a choice of* D_a.

The situation for a second order cosection is completely analogous

$$
\begin{tikzcd}
1_* \arrow[d, "e_1"] \arrow[ddd, bend left=60, "\mathrm{Id}"] \\
\mathrm{tra}_* \arrow[d, bend right=50, "\mathrm{Id}"'] \arrow[d, bend left=50, "\mathrm{Id}"] \arrow[d, phantom, "\overset{\mathrm{Id}}{\Longrightarrow}"] \arrow[d, phantom, shift left=9ex, "\overset{\bar{D}}{\Longrightarrow}"] \\
\mathrm{tra}_* \arrow[d, "\bar{e}_2"] \\
1_*
\end{tikzcd}
.
$$

From that we find that the pairing of the second order sections

$$
(\bar{D}_2, D_1) \quad = \quad
\begin{tikzcd}
1_* \arrow[d, "e_1"] \arrow[ddd, bend right=60, "\mathrm{Id}"'] \arrow[ddd, bend left=60, "\mathrm{Id}"] \\
\mathrm{tra}_* \arrow[d, bend right=50, "\mathrm{Id}"'] \arrow[d, bend left=50, "\mathrm{Id}"] \arrow[d, phantom, "\overset{\mathrm{Id}}{\Longrightarrow}"] \arrow[d, phantom, shift right=9ex, "\overset{D_1}{\Longrightarrow}"] \arrow[d, phantom, shift left=9ex, "\overset{\bar{D}_2}{\Longrightarrow}"] \\
\mathrm{tra}_* \arrow[d, "\bar{e}_2"] \\
1_*
\end{tikzcd}
$$

is, in components, for each path γ_- given by the diagram

$$
\begin{tikzcd}[row sep=large, column sep=huge]
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(\gamma_+)"] \arrow[ddd, bend right=60, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow e_{a\to b}(\gamma_+)"] & \mathbb{C} \arrow[d, "e_b(\gamma_+)"] \arrow[ddd, bend left=60, "\mathrm{Id}"] \\
A_{x_+} \arrow[r, "{(A_{x_+},\mathrm{tra}_\nabla(\gamma_+))}"'] \arrow[d] \arrow[dr, phantom, "\Swarrow \mathrm{tra}(\Sigma)"] \arrow[d, phantom, shift right=8ex, "\Swarrow D_a(\gamma_-)"] & A_{y_+} \arrow[d] \arrow[d, phantom, shift left=8ex, "\Swarrow D_b(\gamma_-)"] \\
A_{x_-} \arrow[r, "{(A_x,\mathrm{tra}_\nabla(\gamma_-))}"'] \arrow[d, "\bar{e}_a(\gamma_-)"] \arrow[dr, phantom, "\Swarrow \bar{e}_{a\to b}(\gamma_-)"] & A_{y_-} \arrow[d, "\bar{e}_b(\gamma_-)"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C}
\end{tikzcd}
.
$$

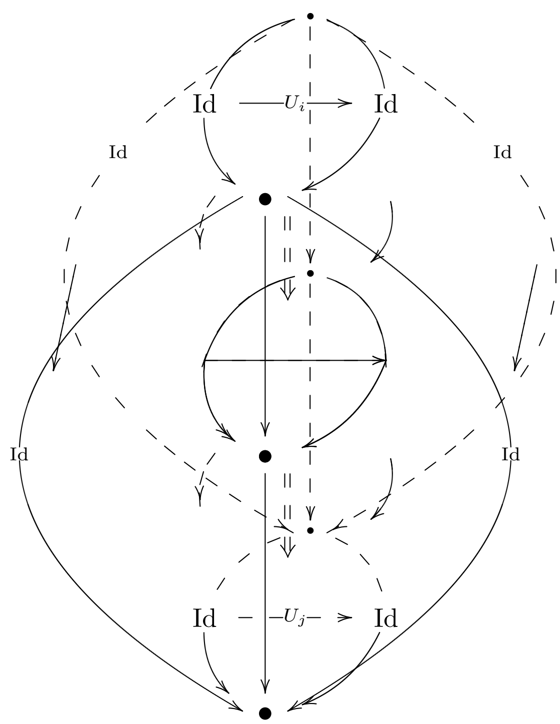
We had discussed elsewhere how diagrams like that reproduce the expected expression for the holonomy of a line bundle gerbe with connection over a disk with gerbe modules on the boundary given by D and $\bar{D}$ and with boundary insertions coming from the sections e and $\bar{e}$.

Remark. I am not sure if I should call these second order sections D and $\bar{D}$. The gerbe module, hence the D-brane, is really encoded in e_a, e_b as well as $\bar{e}_a$ and $\bar{e}_b$. So this is the part of the data that is shared by first and second order sections. A first order section here describes how the string stretches from brane a to brane b, while the second order section describes how its endpoint is parallel transported over the given D-brane.

2.3 Components of third order sections

A pairing of third order sections looks, roughly, like a suspension of the diagram for the pairing of second order sections. It is a cylinder over that diagram, with all points at either one of the ends of the cylinder identified.

I have so far only coded the following rough sketch:



There are also lens-shaped volumes at the front and the back of this diagram, not drawn, coming from modifications of modifications of pseudonatural transformations of 3-functors.

As you can see, most labels have been omitted. In the central application, that of Chern-Simons theory, we are dealing with a 3-vector transport with values in 1-dimensional 3-vector spaces, whose 3-category is modeled by

$$\Sigma(\mathrm{Bim}(\mathcal{C})) \xrightarrow{\subset} 3\mathrm{Vect} = (\mathrm{Vect}\text{-}\mathrm{Mod})\text{-}\mathrm{Mod}\ .$$

In the case that the tensor category $\mathcal{C}$ has several simple objects $\{U_i\}$ (or rather isomorphism classes of these), we will have to consider sections with respect to each of these. To indicate that, the top and bottom of the diagram are labelled by two such objects, regarded as bimodules over the tensor unit in $\mathcal{C}$.