Framed bicategories and locally strict 2-functors

March 26, 2008

Abstract

Some bicategories, like that of spans, or that of bimodules, have composition of 1-morphisms defined by universal properties, such as pullbacks or coequalizers, but secretly remember a more algebraic composition rule on some of their 1-morphisms, such as composition of ring homomorphisms. It is useful to make this extra property explicit by promoting it to an extra structure. In the literature this is known as *bicategories with equipment* [5, 6, 4, 1] or as *framed bicategories* [3]. I try to quickly recall the basic ideas and then relate it to the notion of locally strict 2-functors [2].

Contents

1 Framed Bicategories

The standard motivating example to keep in mind is this:

For C be some abelian monoidal category, like that of vector spaces, we obtain

- the strict 2-category

$$\mathrm{Algebras}(C) = \left\{ \begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow u"] & & B \end{tikzcd} \,|\, u \in B : \forall a \in A : uf(a) = g(a)u \right\}$$

  whose objects are all algebra objects in C, whose morphisms are all algebra homomorphisms in C and whose 2-morphisms are all algebra homomorphism intertwiners. Horizontal composition is ordinary composition of algebra homomorphisms.

- the non-strict 2-category (bicategory)

$$\mathrm{Bimodules}(C) = \left\{ \begin{tikzcd} A \arrow[rr, bend left, "N"] \arrow[rr, bend right, "M"'] \arrow[rr, phantom, "\Downarrow\phi"] & & B \end{tikzcd} \right\}$$

  whose objects are also all algebra objects in C, whose morphisms are all bimodules in C, and whose 2-morphisms all bimodule homomorphisms. Horizontal composition is tensor product of bimodules.

It is standard to name the first 2-category by its objects, and the second one by its morphisms, which, as argued in [3], is part of the phenomenon that the notion of framed bicategories is supposed to clarify.

That bimodules generalize algebra homomorphisms can be formalized by noticing that there is a weak 2-functor (pseudo 2-functor)

$$i : \ \mathrm{Algebras}(C) \hookrightarrow \mathrm{Bimodules}(C)$$

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow u"] & & B \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[rr, bend left, "{}_f B"] \arrow[rr, bend right, "{}_g B"'] \arrow[rr, phantom, "\Downarrow u\cdot"] & & B \end{tikzcd} .$$

Here ${}_fB$ denotes the A-B bimodule which, as an object of C is B, equipped with the canonical right action of B and with the left action of A induced by f. Once checks that the map $u\cdot : {}_fB \to {}_gB$ induced by left multiplication by an element u in B is a bimodule homomorphism precisely if u intertwines f and g, and that all bimodule homomorphisms ${}_fB \to {}_gB$ are of this form.

Therefore the 2-functor i

- is the identity on objects;
- sends each 1-morphism f to a 1-morphism ${}_fB$ which has a right adjoint, B_f; (compare proposition 6.3 in [3]);
- is full and faithful on each Hom-category.

These kind of properties have been summarized in the concept of a *proarrow equipment* (taken literally from appendix C of [3]):

Definition 1 (proarrow equipment) *A* proarrow equipment *is a pseudo 2-functor $\overline{(-)} : \mathcal{K} \to \mathcal{M}$ between bicategories such that*

1. *$\mathcal{K}$ and $\mathcal{M}$ have the same objects and $\overline{(-)}$ is the identity on objects;*

2. *For every 1-morphism f in $\mathcal{K}$, $\bar{f}$ has a right adjoint;*

3. $\overline{(-)}$ *is locally (meaning: on each Hom-category) full and faithful.*

It is argued in [3] that it is useful to restrict attention here to the case where $\mathcal{K}$ is indeed strict, as in our example above. In that case, lemmas C.2 and C.3 of [3] show that proarrow equipments are tantamount to *framed bicategories*:

Definition 2 (framed bicategory) *A* framed bicategory *is a weak* double *category (a category internal to categories) in which the source-target functor is a bifibration of categories and the composition functor is a strong morphism of bifibrations.*

Unwrapping this definition one finds that it merges the bicategories $\mathcal{K}$ and $\mathcal{M}$ from above into a single double category whose *horizontal* 1-morphisms are those of $\mathcal{M}$, while the *vertical* 1-morphisms are those of $\mathcal{K}$. So vertical composition is strict, while horizontal composition may be non-strict.

In our example, a 2-cell of the framed bicategory of bimodules looks like

$$\begin{tikzcd} A \arrow[r, "N"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\phi"] & B \arrow[d, "g"] \\ C \arrow[r, "N'"'] & D \end{tikzcd}$$

where $f : A \to C$ and $g : B \to D$ are algebra homomorphisms, N is an A-B bimodule, N' a C-D bimodule and $\phi : N \to {}_f N'_g$ is an A-B bimodule homomorphism from N to the A-B-bimodule induced on N' by f and g.

2 Locally strict 2-functors

Elsewhere we made a big deal out of the canonical 2-representation of any strict 2-group $G = (H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H))$ on 2-vector spaces regarded as Vect-module categories. The fact underlying this construction is, in the present language, nothing but the fact that bimodules form a framed bicategory, since that representation factors as

$$\rho : \mathbf{B}G \xrightarrow{\rho'} \mathrm{Groups} \longrightarrow \mathrm{Algebras(Vect)} \longrightarrow \mathrm{Bimodules(Vect)} \longrightarrow 2\mathrm{Vect}$$

where Groups denotes the 2-category of groups, group homomorphisms and intertwiners. The first map is the obvious one obtained from thinking of the example $G = \mathrm{AUT}(H)$. The second map is forming group algebras. The next one is the “proarrow equipment” discussed above. Then the last one sends algebras to their categories of modules.

We say a 2-functor

$$\mathrm{tra} : \mathcal{P}_2(X) \to 2\mathrm{Vect}$$

is locally ρ-trivializable if locally it factors through this ρ:

$$\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{triv}"'] & \mathcal{P}_2(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\simeq"] \\
\mathbf{B}G \arrow[r, "\rho"'] & 2\mathrm{Vect}
\end{tikzcd}$$

So in particular, such a 2-functor is **locally strict**. And this is just another aspect of framed bicategories.

References

[1] Aurelio Carboni, Stefano Kasangian and Robert Walters, *An axiomatics for bicategories of modules*, J. Pure Appl. Algebra, 45(2):127-141, 1987

[2] Urs Schreiber and Konrad Waldorf, *Some title that makes the reader think of gerbes with connection*

[3] Michael Shulman, *Framed bicategories and monoidal fibrations*, [arXiv:0706.1286]

[4] Ross Street, *Cauchy characterization of enriched categories*, Rend. Sem. Mat. Fis. Milano, 51:217-233 (1983), 1981. Reprinted as Repr. Theory Appl. Categ. 4:1-16

[5] R. J. Wood, *Abstract proarrows I*, Cahier Topologie Géom. Diff'erentielle, 23(3):279-290, 1982

[6] R. J. Wood, *Abstract proarrows II*, Cahier Topologie Géom. Diff'erentielle, 26(2):135-168, 1985