Framed bicategories and locally strict 2-functors

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Abstract

Some bicategories, like that of spans, or that of bimodules, have composition of 1-morphisms defined by universal properties, such as pullbacks or coequalizers, but secretly remember a more algebraic composition rule on some of their 1-morphisms, such as composition of ring homomorphisms. It is useful to make this extra property explicit by promoting it to an extra structure. In the literature this is known as *bicategories with equipment* [5, 6, 4, 1] or as *framed bicategories* [3]. I try to quickly recall the basic ideas and then relate it to the notion of locally strict 2-functors [2].

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1 Framed Bicategories

The standard motivating example to keep in mind is this:

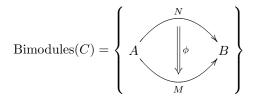
For C be some abelian monoidal category, like that of vector spaces, we obtain

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• the strict 2-category

whose objects are all algebra objects in C, whose morphisms are all algebra homomorphisms in C and whose 2-morphisms are all algebra homomorphism intertwiners. Horizontal composition is ordinary composition of algebra homomorphisms. • the non-strict 2-category (bicategory)

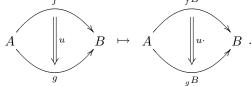


whose objects are also all algebra objects in C, whose morphisms are all bimodules in C, and whose 2-morphisms all bimodule homomorphisms. Horizontal composition is tensor product of bimodules.

It is standard to name the first 2-category by its objects, and the second one by its morphisms, which, as argued in [3], is part of the phenomenon that the notion of framed bicategories is supposed to clarify.

That bimodules generalize algebra homomorphisms can be formalized by noticing that there is a weak 2-functor (pseudo 2-functor)





Here ${}_{f}B$ denotes the A-B bimodule which, as an object of C is B, equipped with the canonical right action of B and with the left action of A induced by f. Once checks that the map $u : {}_{f}B \to {}_{g}B$ induced by left multiplication by an element u in B is a bimodule homomorphism precisely if u intertwines f and g, and that all bimodule homomorphisms ${}_{f}B \to {}_{g}B$ are of this form.

Therefore the 2-functor i

- is the identity on objects;
- sends each 1-morphism f to a 1-morphism $_fB$ which has a right adjoint, B_f ; (compare proposition 6.3 in [3]);
- is full and faithful on each Hom-category.

These kind of properties have been summarized in the concept of a *proarrow* equipment (taken literally from appendix C of [3]):

Definition 1 (proarrow equipment) A proarrow equipment is a pseudo 2-functor $\overline{(-)}: \mathcal{K} \to \mathcal{M}$ between bicategories such that

1. \mathcal{K} and \mathcal{M} have the same objects and $\overline{(-)}$ is the identity on objects;

- 2. For every 1-morphism f in \mathcal{K} , \overline{f} has a right adjoint;
- 3. $\overline{(-)}$ is locally (meaning: on each Hom-category) full and faithful.

It is argued in [3] that it is useful to restrict attention here to the case where \mathcal{K} is indeed strict, as in our example above. In that case, lemmas C.2 and C.3 of [3] show that proarrow equipments are tantamount to *framed bicategories*:

Definition 2 (framed bicategory) A framed bicategory is a weak double category (a category internal to categories) in which the source-target functor is a bifibration of categories and the composition functor is a strong morphism of bifibrations.

Unwrapping this definition one finds that it merges the bicategories \mathcal{K} and \mathcal{M} from above into a single double category whose *horizontal* 1-morphisms are those of \mathcal{M} , while the *vertical* 1-morphisms are those of \mathcal{K} . So vertical composition is strict, while horizontal composition may be non-strict.

In our example, a 2-cell of the framed bicategory of bimodules looks like

$$\begin{array}{c} A \xrightarrow{N} B \\ f \downarrow & \downarrow \phi & \downarrow g \\ C \xrightarrow{N'} D \end{array}$$

where $f : A \to C$ and $g : B \to D$ are algebra homomorphisms, N is an A-B bimodule, N' a C-D bimodule and $\phi : N \to {}_{f}N'_{g}$ is an A-B bimodule homomorphism from N to the A-B-bimodule induced on N' by f and g.

2 Locally strict 2-functors

Elsewhere we made a big deal out of the canonical 2-representation of any strict 2-group $G = (H \xrightarrow{t} G \xrightarrow{\alpha} \operatorname{Aut}(H))$ on 2-vector spaces regarded as Vect-module categories. The fact underlying this construction is, in the present language, nothing but the fact that bimodules form a framed bicategory, since that representation factors as

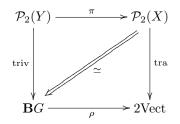
$$\rho: \mathbf{B}G \xrightarrow{\rho'} \mathbf{Groups} \longrightarrow \mathbf{Algebras}(\mathbf{Vect}) \longrightarrow \mathbf{Bimodules}(\mathbf{Vect}) \longrightarrow 2\mathbf{Vect}$$

where Groups denotes the 2-category of groups, group homomorphisms and intertwiners. The first map is the obvious one obtained from thinking of the example G = AUT(H). The second map is forming group algebras. The next one is the "proarrow equipment" discussed above. Then the last one sends algebras to their categories of modules.

We say a 2-functor

tra :
$$\mathcal{P}_2(X) \to 2$$
Vect

is locally ρ -trivializable if locally it factors through this ρ :



So in particular, such a 2-functor is **locally strict**. And this is just another aspect of framed bicategories.

References

- Aurelio Carboni, Stefano Kasangian and Robert Walters, An axiomatics for bicategories of modules, J. Pure Appl. Algebra, 45(2):127-141, 1987
- [2] Urs Schreiber and Konrad Waldorf, Some title that makes the reader think of gerbes with connection
- [3] Michael Shulman, Framed bicategories and monoidal fibrations, [arXiv:0706.1286]
- [4] Ross Street, Cauchy characterization of enriched categories, Rend. Sem. Mat. Fis. Milano, 51:217-233 (1983), 1981. Reprinted as Repr. Theory Appl. Categ. 4:1-16
- [5] R. J. Wood, Abstract proarrows I, Cahier Topologie Géom. Diff'erentielle, 23(3):279-290, 1982
- [6] R. J. Wood, Abstract proarrows II, Cahier Topologie Géom. Diff'erentielle, 26(2):135-168, 1985