Nonablian cocycles and their σ-model QFTs

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Abstract

Nonabelian cohomology can be regarded as a generalization of group cohomology to the case where both the group itself as well as the coefficient object are allowed to be generalized to ∞-group*oid*s or even to general ∞-categories. Cocycles in nonabelian cohomology in particular represent higher principal bundles (gerbes) – possibly equivariant, possibly with connection – as well as the corresponding *associated* higher vector bundles.

We propose, expanding on considerations in [13, 34, 5], a systematic ∞-functorial formalization of the σ-model quantum field theory associated with a given nonabelian cocycle regarded as the background field for a brane coupled to it. We define propagation in these σ-model QFTs and recover central aspects of *groupoidification* [1, 2] of linear algebra.

In a series of examples we show how this formalization reproduces familiar structures in σ-models with finite target spaces such as Dijkgraaf-Witten theory and the Yetter model. The generalization to σ-models with smooth target spaces is developed elsewhere [24].

Contents

1 Introduction

A *σ-model* should, quite generally, be an n-dimensional quantum field theory which is canonically associated with the geometric structure given by a connection on a bundle whose fibers are n-categories – for instance a (higher) gerbe with connection.
For example for $n = 1$ a line bundle with connection over a Riemann manifold gives rise to the ordinary quantum mechanics of a charged particle. For $n = 2$ a line bundle gerbe with connection over a Lie group G gives rise to the 2-dimensional quantum field theory known as the WZW-model. For $n = 3$ a Chern-Simons 2-gerbe with connection over BG gives rise to Chern-Simons QFT.

The natural conceptual home of these higher connections appearing here is *differential nonabelian cohomology* [24], a joint generalization of sheaf cohomology, group cohomology and nonabelian group cohomology. One arrives at this general notion of cohomology for instance by first generalizing the coefficients of sheaf cohomology from complexes of abelian groups, via crossed complexes of groupoids and their equivalent ∞-groupoids [7], to general ∞-categories, and secondly by generalizing the domain spaces via orbifolds, hypercovers and their equivalent ∞-groupoids also to general ∞-categories. Therefore nonabelian cocycles are cocycles *on* ∞-categories with coefficients *in* ∞-categories. Moreover, when suitably interpreted such a cocycle is nothing but an ∞-functor from its domain to its coefficient object, hence a rather fundamental concept. In one way or other it is well known that such cocycles in particular classify fiber bundles – possibly equivariant – whose fibers are higher categories.

In a series of articles [3, 29, 30, 31] (see also [19]) it was shown for low n that by *internalizing* this notion of ∞-functorial nonabelian cocycles from the category of plain sets into a category of generalized *smooth spaces*, it yields a good notion of generalized *differential cohomology*: if the domain ∞-category is taken to be the smooth fundamental ∞-groupoid of a smooth space, then smooth ∞-functors out of it provide a higher dimensional notion of *parallel transport* and characterize higher connections on higher fiber bundles.

Indeed, regarding an ∞-functorial cocycle as a parallel transport functor generally provides a useful heuristic for the sense in which generalized cocycles are nothing but ∞-functors from their domain to their coefficient object, even if there is no smooth structure and no connection around: the ∞-functorial cocycles characterizing for instance a fiber bundle is the *fiber-assigning functor* which to each point in base space assigns the fiber sitting over that point, to each morphism in base space (be it a jump along an orbifold action, or a jump between points in the fiber of a Čech cover, or indeed a smooth path in base space) the corresponding morphisms between the fibers over its endpoints, and similarly for higher morphisms.

From this perspective much can already be learned from and achieved in finite approximations to full smooth differential cocycles. A central example is Dijkgraaf-Witten theory as a finite version of Chern-Simons theory: while Chern-Simons theory is a σ-model governed by a differential 3-cocycle on BG – in the smooth context– usually addressed as the *Chern-Simons 2-gerbe* –, Dijkgraaf-Witten theory is diagrammatically the same setup, but now internal to ∞-categories internal to Sets: the space BG is replaced by the finite groupoid $\mathbf{B}G$ with one object and Hom-set the finite group G. So among other things, ∞-functorial nonabelian cohomology, which treats group cocycles and higher bundles/higher gerbes intrinsically on the same footing, gives a precise formalization of the way in which finite group models such as Dijkgraaf-Witten theory are related to their smooth cousins such as Chern-Simons theory.

For that reason it is worthwhile to study the ∞-functorial nonabelian cohomology perspective on finite group σ-models before adding the further technical complication of working internal to smooth spaces. While discussion of differential nonabelian cohomology in the context of smooth spaces is in preparation in [24], here we develop some concepts and their applications in the simpler context of plain sets.

In sections 3 and 4 we set up the central concepts which we use to formalize the notion of a σ-model associated with a (differential) nonabelian cocycle. In particular we formalize in this context the notion of higher sections and higher spaces of states as indicated in [13, 34], and generalize to corresponding notions of branes and bibranes [15]. In section 4 we then go through a list of examples and applications illustrating these concepts.

2 ∞-Categories and Homotopy Theory

An ∞-category is a combinatorial model for higher directed homotopies, a combinatorial model for a *directed space*. The fact that it is *directed* means that not all cells in this space are necessarily *reversible*. If they are, the ∞-category is an ∞-groupoid, a combinatorial model for an ordinary space.

There are various definitions of ∞-categories and ∞-groupoids [17]. Most of them model ∞-categories as conglomerates of n-dimensional cells of certain shape, for all $n \in \mathbb{N}$, equipped with certain structure and certain properties.

Conglomerates of cells. A "conglomerate of n-dimensional cells of certain shape" technically means a presheaf on a category of basic cells.

Simplicial sets and $(\infty, 1)$-categories. The most familiar example is the simplicial category Δ whose objects are the standard cellular simplices and presheaves on which are simplicial sets. A popular model for ∞-groupoids are simplicial sets with the *Kan property*: *Kan complexes*. The Kan property can be interpreted as ensuring that for all adjacent simplices in the Kan simplicial set there exists a composite simplex and that for all simplices there exists a reverse simplex. Replacing the Kan property on simplicial sets by a slightly weaker property called the *weak Kan property* generalizes Kan complexes to a model of ∞-categories called *weak Kan complexes* or *quasicategories* or *$(\infty, 1)$-categories*: the weak Kan condition ensures just that for all n-simplices for $n \geq 2$ there exists a reverse simplex. A further weakening of the Kan condition such as to ensure only the existence of composites without any restriction on reversibility leads to a definition of weak ∞-categories based on simplicial sets with extra properties proposed by Ross Street []. This is very general but also somewhat unwieldy.

Two other basic shapes of relevance besides simplices are globes and cubes.

Globular sets and ω-categories.

Cubical sets and n-fold categories.

∞-Categories in terms of 1-categories. A general strategy to handle ∞-categories in practice is to regard them as categories (i.e. 1-categories) with extra bells and whistles. This notably involves the tools of *enriched category theory* and of *model category theory.*

Enriched categories. The definition of a category *enriched* over a monoidal category $\mathcal{V}$ [16] is like that of an ordinary category, but with the requirement that there is a *set* of morphisms between any two objects replaced by the requirement that there is an *object of C* for any two objects. If the enriching category $\mathcal{V}$ is a category of higher structures, such as simplicial sets, $\mathcal{V}$-enriched categories are models for ∞-categories. In practice the advantage of conceiving ∞-categories as suitably enriched categories is that enriched category theory is a well-developed subject with a supply of powerful general tools.

Model categories. From the modern perspective, a model category (Quillen model category), is the 1-categorical truncation of an $(\infty, 1)$-category, remembering which of the 1-morphisms retained used to be like isomorphisms, monomorphisms and epimorphisms up to higher coherent cells, in the original $(\infty, 1)$-category: in a model category these special 1-morphisms are, respectively, called *weak equivalences*, *cofibrations* and *fibrations* and satisfy a couple of properties.

One says that model categories are *presentations* of $(\infty, 1)$-categories in that they provide a convenient re-packaging of the information contained in an $(\infty, 1)$-category in purely 1-categorical terms. In practical computation the model category structure on a 1-category is in particular used to generalize morphisms between given objects to morphisms between suitable weakly equivalent *replacements* of these objects.

Our approach. The ∞-vector bundles which we want to describe are given by cocycles with values in ∞-categories (of models for ∞-vector spaces) which are not ∞-groupoids and are not $(\infty, 1)$-categories in that in general they have non-reversible cells in all degrees.

Among the simplicial models for ∞-categories this would force one to use models such as Street's weak ∞-categories. This model, however, we find unwieldy for our applications.

Among the remaining choices of models for ∞-categories for our developments in sections 3 and 4 we choose one which combines the "folk" model category structure [12] on $\omega\mathsf{Categories}$ with the enrichment of $\omega\mathsf{Categories}$ over itself [10]. For most considerations in section 4 and 5 this means effectively that we work in the 1-category $\omega\mathsf{Categories}$ while making use of the internal hom-functor and using the freedom to replace ω-categories by weakly equivalent replacements.

For handling $\omega\mathsf{Categories}$ the different shapes – globes, simplices, cubes – are useful for different purposes. Globular sets have the simplest boundary structure, simplicial sets provide powerful computational tools, cubical sets provide the important monoidal structure. In the following all three models of shapes are combined: following [7, 10] we conceive ω-categories as globular sets for general purposes and make use of their incarnation as cubical sets for describing their biclosed monoidal structure. Moreover, following [32, 7] we use cosimplicial ω-categories such as the orientals to pass between $\omega\mathsf{Categories}$ and $\mathsf{SimplicialSets}$, mostly for the purpose of constructing weakly equivalent replacements of ω-categories.

2.1 Shapes for ∞-cells

Three types of basic shapes are used frequently: globes, simplices and cubes. These are modeled, respectively, by the globular category G, the simplicial category Δ and the cubical category C. These three categories have as objects the integers, $n \in \mathbb{N}$, thought of as the standard cellular n-globe G^n, the standard cellular n-simplex Δ^n and the standard cellular n-cube C^n, respectively. Morphisms are all maps between these standard cellular shapes which respect the cellular structure.

Definition 2.1 (globular category) *The globular category G is the category whose objects are the integers $\mathbb{N}$ and whose morphisms are generated from morphisms*

$$\sigma_n, \tau_n : [n] \to [n+1]$$

subject to the relations

$$\begin{tikzcd}
{[n]} \arrow[r, "\sigma_n"] \arrow[d, "\sigma_n"'] & {[n+1]} \arrow[d, "\sigma_n"] \\
{[n+1]} \arrow[r, "\tau_n"'] & {[n+2]}
\end{tikzcd} \quad , \quad \begin{tikzcd}
{[n]} \arrow[r, "\tau_n"] \arrow[d, "\tau_n"'] & {[n+1]} \arrow[d, "\sigma_n"] \\
{[n+1]} \arrow[r, "\tau_n"'] & {[n+2]}
\end{tikzcd}$$

for all $n \in \mathbb{N}$.

Definition 2.2 (simplicial category) *The simplicial category Δ is the full subcategory of* Categories *on categories which are freely generated from connected linear graphs. Equivalently, Δ is the category with totally ordered finite sets as objects and order-preserving maps as morphisms.*

Definition 2.3 (cubical category) *The cubical category C is defined ... section 2 of [10]*

Definition 2.4 (monoidal structure on the cubical category) *section 2 of [10]*

2.2 ω-Categories

Recall the following standard facts:

- The category Sets is symmetric monoidal with respect to the standard cartesian product.

- For $\mathcal{V}$ a symmetric monoidal category, the category $\mathcal{V}-\mathrm{Cat}$ of $\mathcal{V}$-enriched categories is naturally itself symmetric monoidal.

Definition 2.5 (strict globular n-category, [11]) *The category of 0-categories is* $0\mathsf{Categories} := \mathsf{Sets}$*. For* $n \in \mathbb{N}$*,* $n \geq 1$ *the category of ("strict, globular") n-categories is defined inductively as the category*

$$n\mathsf{Categories} := (n-1)\mathsf{Categories}-\mathrm{Cat}$$

of categories enriched over $(n-1)\mathsf{Categories}$*.*

One notices that for all $n \in \mathbb{N}$ there is a canonical inclusion $n\mathsf{Categories} \hookrightarrow (n+1)\mathsf{Categories}$.

Definition 2.6 (ω-category, [33]) *The category of ω-categories is the direct limit over this chain of inclusions*

$$\omega\mathsf{Categories} := \lim_{\to n \in \mathbb{N}} n\mathsf{Categories}\,.$$

Unwrapping this definition shows that ω-categories are globular sets equipped with compatible structures of a strict 2-category on all sub-globular sets of length two:

Definition 2.7 (globular set) *A globular set S is a presheaf on the globular category G, i.e. a functor* $S : G^{\mathrm{op}} \to \mathsf{Sets}$*.*

We write $S(\ [n] \xrightarrow{\sigma_n,\tau_n} [n+1]\ ) := S_{n+1} \xrightarrow{s_n,t_n} S_n$ and call S_n the set of n-globes, s_n the n-source map and t_n the n-target map of S. The identities $s_n \circ s_{n+1} = s_n \circ t_{n+1}$ and $t_n \circ s_{n+1} = t_n \circ t_{n+1}$, called the globular identities, ensure that for all $n, k \in \mathbb{N}$ there are unique maps $S_{n+k} \xrightarrow{s,t} S_n$ themselves satisfying analogous globular identities.

Proposition 2.8 (ω-category, [32]) *An ω-category C is a globular set* $C : G^{\mathrm{op}} \to \mathsf{Sets}$ *equipped for all $n, k \in \mathbb{N}$ the structure of a category extending* $C_{n+k} \underset{t}{\overset{s}{\rightrightarrows}} C_n$ *such that this makes for all $n, k, l \in \mathbb{N}$* $C_{n+k+l} \underset{t}{\overset{s}{\rightrightarrows}} C_{n+k} \underset{t}{\overset{s}{\rightrightarrows}} C_n$ *into a strict 2-category.*

The elements in C_k are called k-morphisms. The composition in $C_{n+k} \underset{t}{\overset{s}{\rightrightarrows}} C_n$ is called composition of $n+k$-morphisms along n-morphisms. A morphism between ω-categories, called an ω-functor, is a morphism of the underlying globular sets respecting all the additional structure.

Definition 2.9 (standard globular globes) *The globular set G_n represented by $n \in \mathbb{N}$,* $G_n := \mathrm{Hom}_G(-,[n])$ *is the standard globular globe. There is a unique structure of an ω-category on G_n. This yields co-globular ω-category $G_\bullet$, i.e. a functor* $G^\bullet : G \to \omega\mathsf{Categories}$*.*

We also write

- $\emptyset =: G^{-1} =: \mathcal{I}^{-1}$ for the ω-category on the empty globular set (the initial object in $\omega\mathsf{Categories}$);
- $\mathrm{pt} =: I =: \mathcal{I}^0 := G^0 = \{\bullet\}$ for the ω-category with a single object and no nontrivial morphisms (the terminal object in $\omega\mathsf{Categories}$ and the tensor unit with respect to the Crans-Gray tensor product $\otimes$ described below);
- $\mathcal{I} =: \mathcal{I}^1 := G^1 = \{\ a \longrightarrow b\ \}$ for the ω-category with two objects and a single nontrivial morphism connecting them.

The first few n-globes can be depicted as follows:

$$\begin{tikzcd}
G^0 = \{d_0\} \arrow[r, shift left, "\sigma_0 : d_0 \mapsto d_0^-"] \arrow[r, shift right, "\tau_0 : d_0 \mapsto d_0^+"'] & G^1 = \{d_0^- \overset{d_1}{\to} d_0^+\} \arrow[r, shift left, "\sigma_1 : d_1 \mapsto d_1^-"] \arrow[r, shift right, "\tau_1 : d_1 \mapsto d_1^+"'] & G^2 = \{d_0^- \arrow[r, bend left, "d_1^-", ""{name=U, below}] \arrow[r, bend right, "d_1^+"', ""{name=D}] & d_0^+\} \arrow[r, shift left, "\sigma_2 : d_2 \mapsto d_2^-"] \arrow[r, shift right, "\tau_2 : d_2 \mapsto d_2^+"'] & G^3 = \{d_1^- \arrow[rr, bend left=80, dash, "d_0^-"] \arrow[rr, bend left=50, "d_2^-", ""{name=A, below}] \arrow[rr, dashed, bend right=30, "d_2^+"', ""{name=B}] \arrow[dr, bend right] & & d_1^+\} \arrow[dl, bend left] \, . \\
& & & & & d_0^+ &
\arrow[Rightarrow, from=U, to=D, "d_2"]
\arrow[Rightarrow, from=A, to=B, "\shortparallel \; d_3"]
\end{tikzcd}$$

2.3 ω-Groupoids

...ω-groupoids and crossed complexes...

2.4 Cosimplicial ω-categories

We can translate back and forth between simplicial sets and ω-categories by means of a fixed cosimplicial ω-categeory, i.e. a functor $O : \Delta \to \omega\mathsf{Categories}$ from the simplicial category Δ: from any such we obtain an ω-nerve functor $N : \omega\mathsf{Categories} \to \mathsf{SimplicialSets}$ by

$$N(C) : \; \Delta^{\mathrm{op}} \xrightarrow{O^{\mathrm{op}}} \omega\mathsf{Categories}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-,C)} \mathsf{Sets}$$

and its left adjoint $F : \mathsf{SimplicialSets} \to \omega\mathsf{Categories}$ given by the coend formula

$$F(S^\bullet) := \int^{[n]\in\Delta} S^n \cdot O([n])\,.$$

Ross Street defined such a cosimplicial ω-category called the orientals [32], for which $O([n])$ is the ω-category free on a single n-morphism of the shape of an n-simplex. To obtain more inverses, we can alternatively use the unorientals, for which $O([n])$ is the ω-category with n-objects, with 1-morphisms finite sequences of these objects, 2-morphisms finite sequences of such finite sequences, and so on.

2.5 Monoidal biclosed structure on $\omega\mathsf{Categories}$

The category $\omega\mathsf{Categories}$ is is equipped with the Crans-Gray tensor product [10], which is the extension to ω-categories of the tensor product on cubical sets which in turn is induced via Day convolution from the addition of natural numbers. This means that the Crans-Gray tensor product is dimension raising in a way analogous to the cartesian product on topological spaces:

for instance the tensor product of the interval ω-category $I = \{\; a \longrightarrow b \;\}$ with itself is the ω-category free on a single directed square

$$I \otimes I = \left\{ \begin{tikzcd} (a,a) \arrow[r] \arrow[d] & (a,b) \arrow[d] \arrow[dl, Rightarrow] \\ (b,a) \arrow[r] & (b,b) \end{tikzcd} \right\}.$$

Moreover, $\omega\mathsf{Categories}$ is biclosed with respect to this monoidal structure.

Definition 2.10 (internal hom) *For ω-categories C and D the ω-category $[C, D]$ is given by the globular set $\mathrm{Hom}(G_{[-]} \otimes C, D) : G^{\mathrm{op}} \to \mathsf{Sets}$ on which the composition of k-morphisms along an l-morphism is defined* *as the image of the diagram which glues two standard k-globes along a common l-globe*

$$\begin{tikzcd}
 & & G_k \sqcup_{G_l} G_k & & \\
 & G_k \arrow[ur, "\iota"] & & G_k \arrow[ul, "\tau"'] & \\
G_l \arrow[ur, "\sigma"] \arrow[drr, "\sigma"'] & & G_l \arrow[ul, "\tau"'] \arrow[ur, "\sigma"] & & G_l \arrow[ul, "\tau"'] \arrow[dll, "\tau"] \\
 & & G_k \arrow[uuu, bend right, "c_{k,l}"'] & &
\end{tikzcd}$$

under $\mathrm{Hom}(G_{[-]} \otimes C, D)$*:*

$$\begin{tikzcd}
 & & \mathrm{Hom}((G_k \sqcup_{G_l} G_k) \otimes C, D) \arrow[dl, "\iota^*"'] \arrow[dr, "\tau^*"] \arrow[ddd, bend left, "\mathrm{comp}:=c_{k,l}^*"] & & \\
 & \mathrm{Hom}(G_k \otimes C, D) \arrow[dl, "s"'] \arrow[dr, "t"] & & \mathrm{Hom}(G_k \otimes C, D) \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\mathrm{Hom}(G_l \otimes C, D) & & \mathrm{Hom}(G_l \otimes C, D) & & \mathrm{Hom}(G_l \otimes C, D) \\
 & & \mathrm{Hom}(G_k \otimes C, D) \arrow[ull, "s"] \arrow[urr, "t"'] & &
\end{tikzcd}$$

Remarks. Notice that everything in this definition works by abstract nonsense – for instance that the contravariant Hom takes colimits to limits – except the existence of the maps $c_{k,l}$, which encodes genuine information about pasting of standard globes [9]. For instance $G_2 \sqcup_{G_0} G_2 = \left\{ \begin{tikzcd} a \arrow[r, bend left, ""{name=U1, below}] \arrow[r, bend right, ""{name=D1, above}] & b \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2, above}] & c \arrow[Rightarrow, from=U1, to=D1] \arrow[Rightarrow, from=U2, to=D2] \end{tikzcd} \right\}$ while $G_2 \sqcup_{G_1} G_2 = \left\{ \begin{tikzcd} a \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & b \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=D] \end{tikzcd} \right\}$, where the right sides denote the free ω-categories on the indicated pasting diagram [9].

Proposition 2.11 *For every* $C \in \omega\mathsf{Categories}$ *this extends to a functor* $[C, -] : \omega\mathsf{Categories} \to \omega\mathsf{Categories}$ *which is right adjoint to* $- \otimes C : \omega\mathsf{Categories} \to \omega\mathsf{Categories}$.

Of particular interest to us are the internal hom-ω-categories of the form $A^{\mathcal{I}} := [\mathcal{I}, A]$ which satisfy $\mathrm{Hom}(I \otimes X, A) \simeq \mathrm{Hom}(X, A^{\mathcal{I}})$, where the set in question here is the set of <u>lax transformations</u> $\begin{tikzcd} X \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, bend right=50, "g"', ""{name=D, above}] & A \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$ $\Leftrightarrow$

$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[rr, bend left, "f \times g"] & A^{\mathcal{I}} \arrow[r, "d_0 \times d_1"] & A \times A \end{tikzcd}$ or *directed right homotopies* between ω-functors from X to A.

2.6 Model structure on $\omega\mathsf{Categories}$

That the 1-category $\omega\mathsf{Categories}$ is really an ∞-structure itself is remembered by a *model category structure* carried by it, due to [12], with respect to which the acyclic fibrations or hypercovers $f : C \overset{\simeq}{\twoheadrightarrow} D$ are those ω-functors which are k-surjective for all $k \in \mathbb{N}$, meaning that the universal dashed morphism in

$$\begin{tikzcd}
C_{k+1} \arrow[dr, dashed] \arrow[drr, bend left, "f_{k+1}"] \arrow[ddr, bend right, "s\times t"'] & & \\
& (f_k \times f_k)^* C_{k+1} \arrow[r] \arrow[d] & D_{k+1} \arrow[d, "s\times t"] \\
& C_k \times C_k \arrow[r, "f_k \times f_k"'] & D_k \times D_k
\end{tikzcd}$$

is epi, for all k. The weak equivalences $f : C \xrightarrow{\simeq} D$ are those ω-functors where these dashed morphisms become epi after projecting onto ω-equivalence classes of $(k+1)$-morphisms.

Using this we define an ω-anafunctor from an ω-category X to an ω-category A to be a span

$$(g : X \relbar\joinrel\mapstochar\joinrel\rightarrow A) := \begin{tikzcd}
\hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & A \\
X
\end{tikzcd}$$

whose left leg is a hypercover. (This terminology follows [18, 4].) One finds [6] that in the context of $\omega\mathsf{Groupoids}$ such ω-anafunctors represent morphisms in the homotopy category $[g] \in \mathrm{Ho}(X, A)$ which allows us to regard g as a cocycle in nonabelian cohomology on the ω-groupoid X with coefficients in the ω-groupoid A. Cocycles are regarded as distinct only up to refinements of their covers. This makes their composition by pullbacks

$$(X \overset{g}{\relbar\joinrel\mapstochar\joinrel\rightarrow} A \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} A') := \begin{tikzcd}
g^*\hat{A} \arrow[r] \arrow[d, two heads, "\simeq"] & \hat{A} \arrow[r, "r"] \arrow[d, two heads, "\simeq"] & A' \\
\hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & A & \\
X & &
\end{tikzcd}$$

well defined (noticing that acyclic fibrations are closed under pullback) and associative.

Definition 2.12 *We write* $\mathbf{Ho}$ *for the corresponding category of* ω*-anafunctors,*

$$\mathbf{Ho}(C, D) := \mathrm{colim}_{\hat{C} \in \mathsf{Hypercovers}(C)} \mathrm{Hom}(\hat{C}, D)\,.$$

(This is to be contrasted with the true homotopy category Ho, which is obtained by further dividing out homotopies.)

While cocycles in nonabelin cohomology are morphisms in $\mathbf{Ho}$, coboundaries should be morphisms between these morphisms. Hence $\mathbf{Ho}$ is to be thought of as enriched over $\omega\mathsf{Categories}$.

Definition 2.13 *Define a functor* $\mathrm{hom} : \mathbf{Ho}^{\mathrm{op}} \times \mathbf{Ho} \to \omega\mathsf{Categories}$ *by* $\mathrm{hom}(C, D) := F(\mathrm{Hom}(C \otimes O([\bullet]), D)))\,.$

3 Nonabelian cohomology and higher vector bundles

We consider fiber bundles whose fibers are ω-categories on which an ω-group G acts in a prescribed way. Here we conceive ω-groups as the hom-objects of one-object ω-groupoids.

Definition 3.1 (ω-monoid and ω-group) *Given a one-object ω-category $\mathbf{B}A$ the pullback*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & (\mathbf{B}A)^{\mathcal{I}} \arrow[d, "(d_0 \times d_1)"] \\
\mathrm{pt} \arrow[r] & \mathbf{B}A \times \mathbf{B}A
\end{tikzcd}$$

is the corresponding ω-monoid. If $\mathbf{B}A$ is an ω-groupoid then A is called an ω-group.

For X an ω-groupoid – addressed as target space – and for G an ω-group as above – the gauge ω-group or structure ω-group G – and given an ω-category F – the ω-category of typical fibers – together with a morphism $\rho : \mathbf{B}G \longrightarrow F$ into a pointed codomain, $\mathrm{pt}_F : \mathrm{pt} \to F$ – which we address as a representation – of G, we can speak of

- G-cocycles g on G;
- the G-principal ω-bundle $P := g^* \mathbf{E}G$ on X classified by these;
- the ρ-associated ω-bundles $V := g^* \rho^* \mathbf{E}_{\mathrm{pt}} F$
- the collection $\Gamma(V)$ of sections of V.

3.1 Principal ω-bundles

Definition 3.2 (universal G-principal ω-bundle) *The universal G-principal ω-bundle $\mathbf{E}G \twoheadrightarrow \mathbf{B}G$ is given by the pullback*

$$\begin{tikzcd}
\mathbf{E}G \arrow[r, two heads, "\simeq"] \arrow[d] \arrow[dd, bend right] & \mathrm{pt} \arrow[d] \\
(\mathbf{B}G)^{\mathcal{I}} \arrow[r, two heads, "\simeq", "d_0"'] \arrow[d, two heads, "d_1"', "\simeq"] & \mathbf{B}G \\
\mathbf{B}G &
\end{tikzcd} .$$

Proposition 3.3 *The morphism $\mathbf{E}G \twoheadrightarrow \mathbf{B}G$ defined this way is indeed a fibration and its kernel is G: we have a short exact sequence*

$$G \xhookrightarrow{i} \mathbf{E}G \xtwoheadrightarrow{p} \mathbf{B}G \; .$$

Proof. That p is a fibration follows from lemma 2 in [6]. To see that G is indeed the kernel of this fibration, consider the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & \mathbf{E}_{\mathrm{op}}G \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\mathbf{E}G \arrow[r] \arrow[d] & \mathbf{B}G^{\mathcal{I}} \arrow[r, "d_1"] \arrow[d, "d_0"] & \mathbf{B}G \\
\mathrm{pt} \arrow[r] & \mathbf{B}G &
\end{tikzcd} .$$

The right and bottom squares are pullback squares by definition. Moreover, G is by definition 3.1 the total pullback

$$\begin{tikzcd}
G \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{pt} \arrow[d] \\
& \mathbf{B}G^{\mathcal{I}} \arrow[r, "d_1"] \arrow[d, "d_0"] & \mathbf{B}G \\
\mathrm{pt} \arrow[r] & \mathbf{B}G &
\end{tikzcd} .$$

Therefore also the top left square exists and is a pullback itself and hence so is the pasting composite of the two top squares. This says that i is the kernel of p. ☐

Definition 3.4 (G-principal ω-bundles) *For X an ω-groupoid and G an ω-group, there is for every $\mathbf{B}G$-cocycle on X represented by an ω-anafunctor $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} \mathbf{B}G$ the corresponding G-principal ω-bundle $\pi_g : P \twoheadrightarrow X$ classified by g given by the pullback diagram*

$$\begin{tikzcd}
g^*\mathbf{E}G \arrow[r] \arrow[d, two heads] \arrow[dd, bend right, two heads] & \mathbf{E}G \arrow[d, two heads] \\
\hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & \mathbf{B}G \\
X &
\end{tikzcd} .$$

For $n \leq 2$ this way of describing (universal) principal n-bundles was described in [21].

Theorem 3.5 *For G a 1- or 2-group, this definition of G-principal bundles is equivalent to the existing definitions in the literature [Bartels, Baković, Wockel].*

Remarks.

- This statement involves higher categorical equivalences: for G a 2-group and $g : X \longrightarrow\!\!\!\!\!\!|\;\; \mathbf{B}G$ a cocycle, the pullback $g^*\mathbf{E}G$ is a priori a 2-groupoid, whereas in the literature on 2-bundles one expects this total space to be a 1-groupoid. But this desired 1-groupoid is obtained by dividing out 2-isomorphisms in $g^*\mathbf{E}G$ and the result is weakly equivalent to the original 2-groupoid $g^*\mathbf{E}G \overset{\simeq}{\twoheadrightarrow} (g^*\mathbf{E}G)_\sim$.
- For the purposes of this article we are glossing over the *internalization* of the entire setup from a context internal to Sets to a context internal to a category Spaces, for instance of topological spaces or of suitable generalized smooth spaces. The above statements generalize to such internal contexts by suitably lifting the model structure on ωGroupoids to the structure of a *category of fibrant objects* on ωGroupoids(Spaces). Further discussion of this point is relegated to [24].

3.2 Associated ω-bundles

Associated ω-bundles similarly arise from pullback along more general cocycles with prescribed factorization through $\mathbf{B}G$. For that purpose let F be some ω-category (*not* necessarily an ω-groupoid). An ω-anafunctor

$$\rho : \mathbf{B}G \longrightarrow\!\!\!\!\!\!|\;\; F$$

may be addressed as an ω-group cocycle with values in F. If F is equipped with a point, $\mathrm{pt} \xrightarrow{\mathrm{pt}_F} F$, we can address such a morphism ρ also as a representation of G. In analogy to the universal G-principal ω-bundle from definition 3.2 we obtain the universal F-bundle (with respect to the chosen point pt_F) as a pullback from the point:

Definition 3.6 (universal F-bundle) *For F an ω-category with chosen point $\mathrm{pt} \xrightarrow{\mathrm{pt}_F} F$ the universal F-bundle $\mathbf{E}_{\mathrm{pt}}F \twoheadrightarrow F$ is the pullback*

$$\begin{tikzcd}
\mathbf{E}_{\mathrm{pt}}F \arrow[r, two heads] \arrow[d] \arrow[dd, bend right, two heads] & \mathrm{pt} \arrow[d, "\mathrm{pt}_F"] \\
F^{\mathcal{I}} \arrow[r, "d_0", two heads] \arrow[d, "d_1", two heads] & F \\
F &
\end{tikzcd} .$$

So for $F = \mathbf{B}G$ with the unique $\mathrm{pt}_{\mathbf{B}G} : \mathrm{pt} \longrightarrow \mathbf{B}G$ comparison with definition 3.2 shows that with this notation we have $\mathbf{E}_{\mathrm{pt}}\mathbf{B}G = \mathbf{E}G$.

Definition 3.7 (ground ω-monoid) *The ground ω-monoid K, definition 3.1, corresponding to the pointed ω-category $\mathrm{pt} \xrightarrow{\mathrm{pt}_F} F$ is the further pullback of the universal F-bundle to the point:*

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & F^{\mathcal{I}} \arrow[d, "d_0 \times d_1"] & & K \arrow[r] \arrow[d, two heads] & \mathbf{E}_{\mathrm{pt}} \arrow[d, two heads] \\
\mathrm{pt} \arrow[r, "\mathrm{pt}_F \times \mathrm{pt}_F"] & F \times F \arrow[r, phantom, "\Leftrightarrow"] & & \mathrm{pt} \arrow[r, "\mathrm{pt}_F"] & F
\end{tikzcd} .$$

Definition 3.8 (associated F-bundle) *Given a representation morphism $\rho : \mathbf{B}G \to F$ we accordingly address the pullback*

$$\begin{tikzcd}
\rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[r] \arrow[d, two heads] & \mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\
\mathbf{B}G \arrow[r, "\rho"] & F
\end{tikzcd}$$

as the F-bundle ρ-associated to the universal G-bundle. Correspondingly the further pullback along a g-cocycle

$$\begin{tikzcd}
g^*\rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[r] \arrow[d, two heads] \arrow[dd, bend right, two heads] & \rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[r] \arrow[d, two heads] & \mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\
\hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & \mathbf{B}G \arrow[r, "\rho"] & F \\
X & &
\end{tikzcd}$$

is the F-bundle ρ-associated to the specific G-principal bundle $g^*\mathbf{E}G$.

3.3 Sections and homotopies

One way to think of a section of an ω-bundle is as a morphism from a certain trivial ω-bundle into it. The following formalizes this and then provides reformulations of this notion which are useful later on in section 4.

Definition 3.9 (section) *A section σ of an F-cocycle $\hat{X} \xrightarrow{\nabla} F$ is a directed homotopy from the trivial F-cocycle with fiber pt_F into ∇.*

$$\Gamma(\nabla) := \left\{ \begin{tikzcd} & \hat{X} \arrow[dl] \arrow[dr, "\mathrm{Id}"] & \\ \mathrm{pt} \arrow[rr, Rightarrow, dashed, "\sigma"] \arrow[dr, "\mathrm{pt}_F"'] & & \hat{X} \arrow[dl, "\nabla"] \\ & F & \end{tikzcd} \right\}$$

Proposition 3.10 *If the F-cocycle ∇ is ρ-associated to a G-principal cocycle $\hat{X} \xrightarrow{g} \mathbf{B}G$ then sections of ∇ are equivalently lifts of g through $\rho^*\mathbf{E}_{\mathrm{pt}}F \twoheadrightarrow \mathbf{B}G$*

$$\Gamma(\nabla) \simeq \left\{ \begin{tikzcd} & \rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\ \hat{X} \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \right\}$$

Proof. First rewrite

$$\left\{ \begin{tikzcd} & \hat{X} \arrow[dl] \arrow[d, "g"] \\ \mathrm{pt} \arrow[r, Rightarrow, "\sigma"'] \arrow[dr, "\mathrm{pt}_F"'] & \mathbf{B}G \arrow[d, "\rho"] \\ & F \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & \mathrm{pt} \arrow[r, "\mathrm{pt}_F"] & F \\ \hat{X} \arrow[ur] \arrow[rr, dashed, "\sigma"] \arrow[dr, "g"'] & & F^{\mathcal{I}} \arrow[u, "d_0"'] \arrow[d, "d_1"'] \\ & \mathbf{B}G \arrow[r, "\rho"] & F \end{tikzcd} \right\}$$

using the characterization of right (directed) homotopies by the (directed) path object $F^{\mathcal{I}}$. Using the universal property of $\mathbf{E}_{\mathrm{pt}}F$ as a pullback this yields

$$\cdots \simeq \left\{ \begin{tikzcd} & & \mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\ \hat{X} \arrow[urr, dashed, "\sigma"] \arrow[r, "g"'] & \mathbf{B}G \arrow[r, "\rho"'] & F \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & \rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\ \hat{X} \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \right\}.$$

□

Remark. One advantage of the second formulation is that it involves only ω-groupoids, even if F is an ω-category which is not an ω-groupoid. This is particularly useful in the smooth context, where it allows to for,mulate the notion of sections of ω-bundles not only in terms of smooth ω-groupoids but in terms of their linear approximations, smooth L_∞-algebroids [25, 27].

A third way to think about sections comes from observing that since a directed homotopy between two cocycles $$\begin{tikzcd} \hat{X} \arrow[r, bend left=50, "g_1", ""{name=U, below}] \arrow[r, bend right=50, "g_2"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "\eta"] & F \end{tikzcd}$$ is given by a morphism $$\begin{tikzcd} \hat{X} \arrow[r, "\eta"] \arrow[d, two heads, "\simeq"] & F^{\mathcal{I}} \\ X & \end{tikzcd}$$ it can itself be regarded as an $F^{\mathcal{I}}$-cocycle.

Definition 3.11 (universal $F^{\mathcal{I}}$-bundle) *$F^{\mathcal{I}}$ is naturally equipped with the point $\mathrm{pt}_{F^{\mathcal{I}}}$ defined by*

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "\mathrm{pt}_F"] \arrow[rr, bend right, "\mathrm{pt}_{F^{\mathcal{I}}}"'] & F \arrow[r] & F^{\mathcal{I}} \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathrm{pt} \arrow[r, "\mathrm{pt}_{F^{\mathcal{I}}}"] \arrow[rr, bend right, "\mathrm{pt}_F \times \mathrm{pt}_F"'] & F^{\mathcal{I}} \arrow[r, "d_0 \times d_1"] & F \times F \end{tikzcd} \; .$$

We write $\mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \twoheadrightarrow F^{\mathcal{I}}$ *for the corresponding universal $F^{\mathcal{I}}$-bundle according to definition 3.6.*

Notice the commutativity of the diagram

$$\begin{tikzcd} \mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[r] \arrow[d] & (F^{\mathcal{I}})^{\mathcal{I}} \arrow[r, "{d_i}^{\mathcal{I}}"] \arrow[d, "\mathrm{d}_0"] & F^{\mathcal{I}} \arrow[d, "d_0"] \\ \mathrm{pt} \arrow[r, "\mathrm{pt}_{F^{\mathcal{I}}}"] \arrow[rr, bend right, "\mathrm{pt}_F"'] & F^{\mathcal{I}} \arrow[r, "d_i"] & F \end{tikzcd}$$

for $i = 0$ and $i = 1$. The right square commutes by the functoriality of the internal hom, the left square and the bottom triangle by definition 3.11 of the universal $F^{\mathcal{I}}$-bundle.

Definition 3.12 *Let* $\mathbf{E}_{\mathrm{pt}}F \xleftarrow{\mathbf{E}d_0} \mathbf{E}(F^{\mathcal{I}}) \xrightarrow{\mathbf{E}d_1} \mathbf{E}_{\mathrm{pt}}F$ *be the universal morphisms induced from the commutativity of the outermost rectangle of the above diagram in view of the universal property of $\mathbf{E}F$ as a pullback.*

Proposition 3.13 *The morphism $\mathbf{E}d_0 \times \mathbf{E}d_1$ covers the morphism* $F^{\mathcal{I}} \xrightarrow{d_0 \times d_1} F \times F$ *of base spaces:*

$$\begin{tikzcd} \mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] & \mathbf{E}(F^{\mathcal{I}}) \arrow[l, "\mathbf{E}d_0"'] \arrow[r, "\mathbf{E}d_1"] \arrow[d, two heads] & \mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\ F & F^{\mathcal{I}} \arrow[l, "d_0"'] \arrow[r, "d_1"] & F \end{tikzcd} \; .$$

Proof. By inspection of the above commuting diagram. □
This is sometimes called a <u>concordance</u> of ω-bundles.

In total this yields for every homotopy of F-cocycles a span of the corresponding ω-bundles.

Definition 3.14 (associated span of ω-bundles) *For* $\begin{tikzcd}[row sep=small, column sep=small] & \hat{X} \arrow[dr, "\nabla_1"] \arrow[dd, Rightarrow] & \\ \hat{Q} \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & & F \\ & \hat{Y} \arrow[ur, "\nabla_2"'] & \end{tikzcd}$ *a directed homotopy of F-cocycles,* *we say that the rear of the joint pullback diagram*

$$\begin{tikzcd}
 & & \eta^*\mathbf{E}_{\mathrm{pt}}F \arrow[dll] \arrow[ddd] \arrow[dr, two heads] \arrow[drr] & & & \\
\nabla_1^*\mathbf{E}_{\mathrm{pt}}F \arrow[ddd] \arrow[dr, two heads] & & & \hat{Q} \arrow[dll, "\iota"] \arrow[drr, "\iota'"] \arrow[ddd, "\eta"] & \nabla_2^*\mathbf{E}_{\mathrm{pt}}F \arrow[ddd] \arrow[dr, two heads] & \\
 & \hat{X} \arrow[ddd, "\nabla_1"] & & & & \hat{Y} \arrow[ddd, "\nabla_2"] \\
 & & \mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[dll] \arrow[dr, two heads] \arrow[drr] & & & \\
\mathbf{E}_{\mathrm{pt}}F \arrow[dr, two heads] & & & F^{\mathcal{I}} \arrow[dll, "d_0"] \arrow[drr, "d_1"] & \mathbf{E}_{\mathrm{pt}}F \arrow[dr, two heads] & \\
 & F & & & & F
\end{tikzcd}$$

is the associated span of ω-bundles.

From definition 3.7 notice the following observation:

Lemma 3.15 *In the case that the homotopy in question is a section* $\begin{tikzcd} & \hat{X} \arrow[dl] \arrow[dr, "\mathrm{Id}"] & \\ \mathrm{pt} \arrow[dr, "\mathrm{pt}_F"'] \arrow[rr, phantom, "\overset{\sigma}{\Rightarrow}"] & & \hat{X} \arrow[dl, "\nabla"] \\ & F & \end{tikzcd}$ *the associated span of ω-bundles is of the form* $\begin{tikzcd} & \sigma^*\mathbf{E}_{\mathrm{pt}} \arrow[dl] \arrow[dr] & \\ K & & g^*\mathbf{E}_{\mathrm{pt}}F \end{tikzcd}$ *with K the* ground ω-monoid *from definition 3.7.*

Remark on groupoidification. This realizes a section of an associated ω-bundle as an ω-groupoid over the total space ω-groupoid of the associated ω-bundle, and equipped with a map to to the ground ω-monoid. Since this is the description of vectors in the context of *groupoidification* [1, 2] it motivates the following definition.

Definition 3.16 (generalized sections of associated ω-bundles) *Given an F-bundle $V := \nabla^*\mathbf{E}_{\mathrm{pt}}F$, its generalized sections are spans of ω-groupoids* $|\Psi\rangle :=$ $\begin{tikzcd} & \Psi \arrow[dl] \arrow[dr] & \\ K & & V \end{tikzcd}$ *and its generalized co-sections are spans* $\langle\Psi| :=$ $\begin{tikzcd} & \Psi \arrow[dl] \arrow[dr] & \\ V & & K \end{tikzcd}$ *. We write $\mathcal{H}(V)$ for the collection of all generalized sections of V.*

4 Quantization of ω-bundles to σ-models

We want to think of a ρ-associated F-cocycle $\hat{X} \xrightarrow{\nabla} F$ and the corresponding ω-bundle $\nabla^*\mathbf{E}_{\mathrm{pt}}F \twoheadrightarrow X$ as a *background field* (a generalization of an electromagnetic field) on X to which a higher dimensional *fundamental brane* – such as a *particle*, a *string* or a *membrane* – propagating on X may *couple.*

We now propose a formalization in the context of ω-functorial cohomology of what it means to *quantize* such a background field to obtain the corresponding σ-model quantum field theory as a functortial QFT (as described in [28] and references given there). Our constructions are motivated by and supposed to implement and generalize the considerations of [13, 34] and make contact with [20].

We define a notion of *parameter space* or *worldvolume* category and a notion of *background field* over a *target space* coming from an ω-bundle. This pair of data we call a *σ-model*, We show how this data induces a functor from the parameter space category to spans in ω-groupoids. Using methods from *groupoidification* [1, 2] we show that these spans represent linear maps which deserve to be addressed as *propagation* in the quantum field theory induced by the σ-model.

4.1 σ-Models

Definition 4.1 (background structure) *A background structure for a σ-model is*

- *an ω-groupoid X called target space;*
- *an ω-group G, called the gauge group;*
- *a representation*

$$\begin{tikzcd}
\rho^*\mathbf{E}G \arrow[r] \arrow[d] & \mathbf{E}_{\mathrm{pt}}F \arrow[d] & \\
\mathbf{B}G \arrow[r, "\rho"] & F & \mathrm{pt} \arrow[l, "\mathrm{pt}_F"']
\end{tikzcd}$$

  called the matter content;
- *an F-cocycle*

$$\begin{tikzcd}
\hat{X} \arrow[r, "\nabla"] \arrow[d, two heads, "\simeq"] & F \\
X &
\end{tikzcd}$$

  ρ-associated to a G-principal cocycle on X, $\nabla : \hat{X} \xrightarrow{g} \mathbf{B}G \xrightarrow{\rho} F$, called the background field.

For brevity we shall indicate a background structure just as ($\hat{X} \xrightarrow{\nabla} F$), leaving the choice of representation and the target space X weakly equivalent to the hypercover $\hat{X}$ implicit.

Definition 4.2 (parameter space category) *A parameter space ω-category* Cob *is a sub ω-category of* Cospans(ωGroupoids).

Definition 4.3 (σ-model) *A σ-model is a pair consisting of a parameter space category and a background structure for a σ-model.*

Definition 4.4 *Given a σ-model with background structure $\nabla : \hat{X} \xrightarrow{\nabla} F$ and with parameter space* Cob *for every object $\Sigma \in$* Cob *we say that*

- *$C(\Sigma) := [\Sigma, \hat{X}]$ is the space of fields over Σ;*
- *The $[\Sigma, F]$-cocycle $[\Sigma, \nabla] : [\Sigma, \hat{X}] \to [\Sigma, F]$ on the space of fields over Σ is the action functional over Σ.*

Remark. One can identify $[\Sigma, \nabla]$ with the *transgression* of the cocycle g to the mapping space $[\Sigma, X]$. Examples showing that this canonical operation indeed reproduces the ordinary notion of transgression of cocycles are in [29, 30] and in our section 5.

In section 4.3 we construct for every σ-model its corresponding quantum field theory. This involves the notion of *bibranes* discussed in section 4.2.

4.2 Branes and bibranes

From the second part of definition 3.9 one sees that spaces spaces of sections of ω-bundles are given by certain morphisms between background fields pulled back to spans/correspondences of target spaces. From the diagrammtics this has an immediate generalization, which leads to the notion of *branes* and *bibranes*.

Definition 4.5 (branes and bibranes) *A* <u>*brane*</u> *for a background structure* $(\hat{X} \xrightarrow{\nabla} F)$ *is a morphism* $\iota : \hat{Q} \to \hat{X}$ *equipped with a section of the background field pulled back to* $\hat{Q}$*, i.e. a transformation*

$$\begin{tikzcd} & \hat{Q} \arrow[dl] \arrow[dr, "\iota"] & \\ \mathrm{pt} \arrow[rr, Rightarrow, "V"] \arrow[dr, "\mathrm{pt}_F"'] & & \hat{X} \arrow[dl, "g"] \\ & F & \end{tikzcd} .$$

More generally, given two background structures $(\hat{X} \xrightarrow{\nabla} F)$ *and* $(\hat{X}' \xrightarrow{\nabla'} F)$*, a* <u>*bibrane*</u> *between them is a span* $\begin{tikzcd} & \hat{Q} \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\ \hat{X} & & \hat{X}' \end{tikzcd}$ *equipped with a transformation* $\begin{tikzcd} & \hat{Q} \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\ \hat{X} \arrow[rr, Rightarrow, "V"] \arrow[dr, "\nabla"'] & & \hat{X}' \arrow[dl, "\nabla'"] \\ & F & \end{tikzcd}$.

Bibranes may be composed –"fused" – along common background structures $(\hat{X} \xrightarrow{\nabla} F)$: the composite or *fusion* of a bibrane V_1 on $\hat{Q}$ with a bibrane V_2 on $\hat{Q}'$ is the bibrane $V_1 \cdot V_2$ given by the diagram

$$\begin{tikzcd} & \hat{Q} \times_{\hat{X}'} \hat{Q} \arrow[dl] \arrow[dr] & \\ \hat{X}_1 \arrow[rr, Rightarrow, "V_1 \cdot V_2"] \arrow[dr, "\nabla_1"'] & & \hat{X}_3 \arrow[dl, "\nabla_3"] \\ & F & \end{tikzcd} := \begin{tikzcd} & & \hat{Q} \times_{X'} \hat{Q}' \arrow[dl, "s"'] \arrow[dr, "t"] & & \\ & \hat{Q} \arrow[dl] \arrow[dr] & & \hat{Q}' \arrow[dl] \arrow[dr] & \\ \hat{X}_1 \arrow[rr, Rightarrow, "V"] \arrow[drr, "\nabla_1"'] & & \hat{X}_2 \arrow[rr, Rightarrow, "V'"] \arrow[d, "\nabla_2"] & & \hat{X}_3 \arrow[dll, "\nabla_3"] \\ & & F & & \end{tikzcd}$$

Proposition 4.6 (composition of associated spans from fusion of bibranes) *The associated span of ω-groupoids corresponding, according to definition 3.14, to the fusion of two bibranes is the composition of* *the spans associated with each bibrane:*

$$\begin{tikzcd}
 & & & & (\eta_1 \cdot' \eta_2)^*\mathbf{E}_{\mathrm{pt}}(F^{I\sqcup_{\mathrm{pt}}I}) \arrow[dll] \arrow[dr] \arrow[drr] & & & & & . \\
 & & \eta_1^*\mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[dll] \arrow[dddd] \arrow[dr] \arrow[drr] & & & \hat{Q} \times_{\hat{Y}} \hat{Q}' \arrow[dll, bend right] \arrow[drr] \arrow[ddddddd, dashed, "\eta_1 \cdot' \eta_2"] & \eta_2^*\mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[dll] \arrow[dr] \arrow[drr] \arrow[ddd] & & & \\
\nabla_1^*\mathbf{E}_{\mathrm{pt}}F \arrow[dr] \arrow[ddd] & & & \hat{Q} \arrow[dll, "\iota_1"'] \arrow[drr, "\iota_2"] \arrow[ddd, "\eta_1"] & \nabla_2^*\mathbf{E}_{\mathrm{pt}}F \arrow[dr] \arrow[ddd] & & & \hat{Q}' \arrow[dll, "\iota_3"'] \arrow[drr, "\iota_4"] \arrow[ddd, "\eta_2"] & \nabla_3^*\mathbf{E}_{\mathrm{pt}}F \arrow[dr] \arrow[ddd] & \\
 & \hat{X} \arrow[ddd, "\nabla_1"] & & & & \hat{Y} \arrow[ddd, "\nabla_2"] & & & & \hat{Z} \arrow[ddd, "\nabla_3"] \\
 & & \mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[dll] \arrow[dr, two heads] \arrow[drr] & & & & \mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[dll] \arrow[dr, two heads] \arrow[drr] & & & \\
\mathbf{E}_{\mathrm{pt}}F \arrow[dr] & & & F^{\mathcal{I}} \arrow[dll, "d_0"'] \arrow[drr, "d_1"] & \mathbf{E}_{\mathrm{pt}}F \arrow[dr] & & & F^{\mathcal{I}} \arrow[dll, "d_0"'] \arrow[drr, "d_1"] & \mathbf{E}_{\mathrm{pt}}F \arrow[dr] & \\
 & F & & & & F & & & & F \\
 & & & & \mathbf{E}_{\mathrm{pt}}(F^{I\sqcup_{\mathrm{pt}}I}) \arrow[uuull] \arrow[uuurr] \arrow[dr, two heads] & & & & & \\
 & & & & & F^{I\sqcup_{\mathrm{pt}}I} \arrow[uuull] \arrow[uuurr] \arrow[d] & & & & \\
 & & & & & F^{\mathcal{I}} & & & &
\end{tikzcd}$$

Proof. By commutativity of pullbacks. ☐

Remark on groupoidification. Comparing with the remark above definition 3.16 we find that fusion of bibranes corresponds to composition of groupoidified linear maps.

If $\hat{Q}$ carries further structure, the fused bibrane on $\hat{Q} \times_{\hat{Y}} \hat{Q}$ may be pushed down again to $\hat{Q}$.

Definition 4.7 *Let B be a category enriched in the bicategory $\mathcal{V} := \mathsf{Spans}(\omega\mathsf{Categories})$ of spans in $\omega\mathsf{Categories}$ and let F be an ω-category. Then the category of bibranes relative to B and F is given by:*

- *objects are background structures $\hat{X} \xrightarrow{\nabla} F$ for $\hat{X}$ an object of B;*
- *morphisms are bibranes on morphisms of B;*
- *composition of morphisms is given by bibrane fusion followed by push-forward along the composition map in B.*

A simple special case is a category $\hat{Q} \underset{t}{\overset{s}{\rightrightarrows}} \hat{X}$ *internal* to ω-groupoids, equivalently a monad in the bicategory of spans internal to $\omega\mathsf{Groupoids}$, with composition operation the morphism of spans

$$\begin{tikzcd}
 & & \hat{Q} \times_{t,s} \hat{Q} \arrow[dl] \arrow[dr] \arrow[ddd, bend left, "\mathrm{comp}"] & & \\
 & \hat{Q} \arrow[dl, "s"'] \arrow[dr, "t"] & & \hat{Q} \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\hat{X} & & \hat{X} & & \hat{X} \\
 & & \hat{Q} \arrow[ull, "s"] \arrow[urr, "t"'] & &
\end{tikzcd} \quad .$$

Definition 4.8 (monoidal structure on bibranes) *Given an internal category as above, and given a background structure* $\nabla : \hat{X} \to F$*, the composite of two bibranes* $\begin{tikzcd} & \hat{Q} \arrow[dl] \arrow[dr] & \\ \hat{X} \arrow[rr, Rightarrow, "V{,}W"] \arrow[dr, "\nabla"'] & & \hat{X} \arrow[dl, "\nabla"] \\ & \hat{F} & \end{tikzcd}$ *on* $\hat{Q}$ *is the result of first forming their composite bibrane on on* $\hat{Q} \times_{t,s} \hat{Q}$ *and then pushing that forward along* comp*:*

$$V \star W := \int_{\mathrm{comp}} (s^* V) \cdot (t^* W)\,.$$

Here for finite cases, which we concentrate on, push-forward is taken to be the right adjoint to the pullback in a proper context.

Remarks. Notice that branes are special cases of bibranes and that bibrane composition restricts to an action of bibranes on branes. Also recall that the sections of a cocycle on X are the same as the branes of this cocycle for $\iota = \mathrm{Id}_X$.

The idea of bibranes was first formulated in [15] in the language of modules for bundle gerbes. We show in section 5.1.4 how this is reproduced within the present formulation. In its smooth L_∞-algebraic version the idea also appears in [26].

4.3 Quantum propagation

Every σ-model with parameter space Cob and background structure $\hat{X} \xrightarrow{\nabla} F$ induces a functor

$$\exp(\int \nabla) : \mathrm{Cob} \to \mathrm{Spans}(\omega\mathsf{Groupoids})$$

which sends

$$\exp(\int \nabla) : \begin{tikzcd} & \Sigma & \\ \Sigma_{\mathrm{in}} \arrow[ur, "\iota"] & & \Sigma_{\mathrm{out}} \arrow[ul, "\tau"'] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & {[\Sigma, \nabla]^* \mathbf{E}_{\mathrm{pt}} [\Sigma, F]} \arrow[dl] \arrow[dr] & \\ {[\Sigma_{\mathrm{in}}, \nabla]^* \mathbf{E}_{\mathrm{pt}} [\Sigma_{\mathrm{in}}, \nabla]} & & {[\Sigma_{\mathrm{out}}, \nabla]^* \mathbf{E}_{\mathrm{pt}} \Sigma_{\mathrm{out}}, F]} \end{tikzcd}$$

a morphism $\Sigma : \Sigma_{\mathrm{in}} \to \Sigma_{\mathrm{out}}$ in Cob to the span of ω-bundles associated to, definition 3.14, the bibrane on the span

$$\begin{tikzcd} & {[\Sigma, \hat{X}]} \arrow[dl, "\iota^*"'] \arrow[dr, "\tau_*"] & \\ {[\Sigma, \hat{X}]} & & {[\Sigma, \hat{X}]} \end{tikzcd}$$

which is induced by transgression of the background field:

$$\begin{tikzcd}
 & & {[\Sigma,\nabla]^*\mathbf{E}_{\mathrm{pt}}[\Sigma,F]} \arrow[dll] \arrow[ddd] \arrow[dr, two heads] \arrow[drr] & & & \\
{[\Sigma_{\mathrm{in}},\nabla]^*\mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{in}},F]} \arrow[ddd] \arrow[dr, two heads] & & & {[\Sigma,\hat{X}]} \arrow[dll, "\iota"] \arrow[ddd, "{[\Sigma,\nabla]}"] \arrow[drr, "\iota'"] & {[\Sigma_{\mathrm{out}},\nabla]^*\mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{out}},F]} \arrow[ddd] \arrow[dr, two heads] & \\
 & {[\Sigma_{\mathrm{in}},\hat{X}]} \arrow[ddd, "{[\Sigma_{\mathrm{in}},\nabla]}"] & & & & {[\Sigma_{\mathrm{out}},\hat{X}]} \arrow[ddd, "{[\Sigma_{\mathrm{out}},\nabla]}"] \\
 & & {\mathbf{E}_{\mathrm{pt}}[\Sigma,F]} \arrow[dll] \arrow[dr, two heads] \arrow[drr] & & & \\
{\mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{in}},F]} \arrow[dr, two heads] & & & F^{\Sigma} \arrow[dll, "d_0"] \arrow[drr, "d_1"] & {\mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{out}},F]} \arrow[dr, two heads] & \\
 & {[\Sigma_{\mathrm{in}},F]} & & & & {[\Sigma_{\mathrm{out}},F]}
\end{tikzcd}$$

A state of the σ-model over Σ is a generalized section, definition 3.16 of $[\Sigma_a,\nabla]$ in $\mathcal{H}_a := \mathcal{H}([\Sigma_a,\nabla])$ and the propagation of along a morphism Σ is the map

$$\int_{\hom(\Sigma,X)} \exp(\int \nabla) : \mathcal{H}_{\Sigma_{\mathrm{in}}} \longrightarrow \mathcal{H}_{\Sigma_{\mathrm{out}}}$$

induced by pull-push through the span $\exp(\int \nabla)(\Sigma)$.

An example is spelled out in section 5.1.6.

5 Examples and applications

We start with some simple applications to illustrate the formalism and then exhibit some useful constructions in the context of finite group quantum field theory.

5.1 General examples

5.1.1 Ordinary vector bundles

Let G be an orinary group, hence a 1-group, and denote by $F :=$ Vect the 1-category of vector spaces over some chosen ground field k. A linear representation ρ of G on a vector space V is indeed the same thing as a functor $\rho : \mathbf{B}G \to \mathrm{Vect}$ which sends the single object of $\mathbf{B}G$ to V.

The canonical choice of point $\mathrm{pt}_F : \mathrm{pt} \to \mathrm{Vect}$ is the ground field k, regarded as the canonical 1-dimensional vector space over itself. Using this we find

- from definition 3.7 that the *ground ω-monoid* in this case is just the ground field itself, $K = k$,
- from definition 3.6 that the *universal* Vect-*bundle* is $\mathbf{E}_{\mathrm{pt}}\mathrm{Vect} = \mathrm{Vect}_*$, the category of *pointed* vector spaces with $\mathrm{Vect}_* \twoheadrightarrow \mathrm{Vect}$ the canonical forgetful functor;
- from definition 3.8 that the ρ-associated vector bundle to the universal G-bundle is $V//G \twoheadrightarrow \mathbf{B}G$, where $V//G := (\ V \times G \underset{\rho}{\overset{p_1}{\rightrightarrows}} V\ )$ is the *action groupoid* of G acting on V, the weak quotient of V by G;

- From definition 3.9 that for $g : X \overset{g}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbf{B}G$ a cocycle describing a G-principal bundle and for V the corresponding ρ-associated vector bundle according to definition 3.8, that sections $\sigma \in \Gamma(V)$ are precisely sections of V in the ordinary sense.

5.1.2 Group algebras and category algebras from bibrane monoids

In its simplest version the notion of monoidal bibranes from section 4.2 reproduces the notion of *category algebra* $k[C]$ of a category C, hence also that of a *group algebra* $k[G]$ of a group G. Recall that the category algebra $k[C]$ of C is defined to have as underlying vector space the span of C_1, $k[C] = \mathrm{span}_k(C_1)$, where the product is given on generating elements $f, g \in C_1$ by

$$f \cdot g = \begin{cases} g \circ f & \text{if the composite exists} \\ 0 & \text{otherwise} \end{cases}$$

To reproduce this as a monoid of bibranes in the sense of section 4.2, take the category of fibers in the sense of section 3.2 to be $F = \mathrm{Vect}$ as in section 5.1.1. Consider on the space (set) of objects, C_0, the trivial line bundle given as an F-cocycle by $i : C_0 \longrightarrow \mathrm{pt} \xrightarrow{\mathrm{pt}_k} \mathrm{Vect}$. An element in the monoid of bibranes for this trivial line bundle on the span given by the source and target map

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd}$$

is a transformation of the form

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 \arrow[dr, "i"'] \arrow[rr, Rightarrow, "V"] & & C_0 \arrow[dl, "i"] \\ & \mathrm{Vect} & \end{tikzcd}$$

. In terms of its components this is canonically identified with a function $V : C_1 \to k$ from the space (set) of morphisms to the ground field and every such function gives such a transformation. This identifies the C-bibranes with functions on C_1.

Given two such bibranes V, W, their product as bibranes is, according to definition 4.8, the push-forward along the composition map on C of the function on the space (set) of composable morphisms

$$C_1 \times_{t,s} C_1 \to k$$

$$(\xrightarrow{f}\xrightarrow{g}) \mapsto V(f) \cdot W(g)\,.$$

This push-forward is indeed the product operation on the category algebra.

5.1.3 Monoidal categories of graded vector spaces from bibrane monoids

The straightforward categorification of the discussion of group algebras in section 5.1.2 leads to bibrane monoids equivalent to monoidal categories of graded vector spaces.

Let now $F := 2\mathrm{Vect}$ be a model for the 2-category of 2-vector spaces. For our purposes and for simplicity, it is sufficient to take $F := \mathbf{B}\mathrm{Vect} \hookrightarrow 2\mathrm{Vect}$, the 2-category with a single object, vector spaces as morphims with composition being the tensor product, and linear maps as 2-morphisms. This can be regarded as the full sub-2-category of 2Vect on 1-dimensional 2-vector spaces. And we can assume $\mathbf{B}\mathrm{Vect}$ to be strictified.

Notice from definition 3.7 that the ground ω-monoid in this case is the monoidal category $K = \mathrm{Vect}$.

Then bibranes over G for the trivial 2-vector bundle on the point, i.e. transformations of the form

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ \mathrm{pt} \arrow[dr] \arrow[rr, Rightarrow] & & \mathrm{pt} \arrow[dl] \\ & \mathbf{B}\mathrm{Vect} & \end{tikzcd}$$

canonically form the category Vect^G of G-graded vector spaces. The fusion of such bibranes reproduces the standard monoidal structure on Vect^G.

5.1.4 Twisted vector bundles

The ordinary notion of a brane in string theory is: for an abelian gerbe $\mathcal{G}$ on target space X a map $\iota : Q \to X$ and a PU(n)-principal bundle on Q whose lifting gerbe for a lift to a $U(n)$-bundle is the pulled back gerbe $\iota^*\mathcal{G}$. Equivalently: a twisted $U(n)$-bundle on Q whose twist is $\iota^*\mathcal{G}$. Equivalently: a gerbe module for $\iota^*\mathcal{G}$.

We show how this is reproduced as a special case of the general notion of branes from definition 4.5, see also [31].

The bundle gerbe on X is given by a cocycle $g :\ X \longrightarrow\!\!\triangleright \mathbf{BB}U(1)$. The coefficient group has a canonical representation $\rho : \mathbf{B}^2U(1) \to F := \mathbf{B}\mathrm{Vect} \hookrightarrow 2\mathrm{Vect}$ on 2-vector spaces (as in section 5.1.3) given by

$$\begin{tikzcd}
\rho : & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow c\in U(1)"] & \bullet & \mapsto & \bullet \arrow[r, bend left=50, "\mathbb{C}"] \arrow[r, bend right=50, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow \cdot c"] & \bullet & .
\end{tikzcd}$$

See also [31, 28].

By inspection one indeed finds that branes in the sense of diagrams $\begin{tikzcd} & Q \arrow[dl] \arrow[dr, "\iota"] & \\ \mathrm{pt} \arrow[rr, Rightarrow, "V"] \arrow[dr, "\mathrm{pt}_F"'] & & X \arrow[dl, "\rho\circ g"] \\ & \mathbf{B}\mathrm{Vect} & \end{tikzcd}$ are canonically identified with twisted vector bundles on Q with twist given by the ι^*g: the naturality condition satisfied by the components of V is

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, Rightarrow, "\mathrm{Id}"] & & & & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[dd, "(\pi_2^*E)_y"] \arrow[dddl, Rightarrow, "\pi_{12}^*g_{\mathrm{tw}}(y)"] & & \\
\mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] \arrow[dd, "(\pi_1^*E)_y"'] & {} & \mathbb{C} \arrow[dd, "\pi_3^*E_y"] \arrow[ddll, Rightarrow, "\pi_{13}^*g_{\mathrm{tw}}(y)"] & = & \mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[dd, "\pi_1^*E_y"'] & & \mathbb{C} \arrow[dd, "(\pi_3^*E_y)"] \arrow[dl, Rightarrow, "\pi_{23}^*g_{\mathrm{tw}}(y)"] & , \\
 & & & & & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, Rightarrow, "\cdot g(y)"] & & \\
\mathbb{C} \arrow[rr, "\mathbb{C}"] & & \mathbb{C} & & \mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] & {} & \mathbb{C} &
\end{tikzcd}$$

for all $y \in Y \times_X Y \times_X Y \times_X Y$ in the triple fiber product of a local-sections admitting map $\pi : Y \to X$ whose simplicial nerve $Y^\bullet$, regarded as an ω-category, provides the cover for the ω-anafunctor $X \overset{\simeq}{\twoheadleftarrow} Y^\bullet \xrightarrow{g} \mathbf{B}^2U(1)$ representing the gerbe. See [31] for details. $E \to Y$ is the vector bundle on the cover encoded by the transformation V. The above naturality diagram says that its transition function g_{tw} satisfies the usual cocycle condition for a bundle only up to the twist given by the gerbe g: if $Y \to X$ is a cover by open subsets $Y = \sqcup_i U_i$, then the above diagram is equivalent to the familiar equation

$$(g_{\mathrm{tw}})_{ij}(g_{\mathrm{tw}})_{jk} = (g_{\mathrm{tw}})_{ik} \cdot g_{ijk} \, .$$

In this functorial cocyclic form twisted bundles on branes were described in [22, 31].

5.1.5 2-Hilbert spaces

Let B be the category internal to spans in ω-categories given by all product spans in Sets with composition morphisms the canonical morphisms. Let $F = \mathbf{B}\mathrm{Vect}$ as before. Then the 2-category of (B, F)-bibranes is the 2-category of 2-Hilbert spaces as in [5].

5.1.6 The path integral

We unwrap the notion of propagation in a σ-model form section 4.3 for the case that the background field is an ordinary vector bundle (with connection), i.e. for the case $F = \mathrm{Vect}$. This can be regarded in terms of the quantization of the charged 1-particle as well as, after transgression, as the top-dimensional propagation in higher dimensional theories. We shall re-encounter this example in the discussion of Dijkgraaf-Witten theory in section 5.2.

Let for the present example the parameter space Cob consist just of a single edge

$$\mathrm{Cob} = \left\{ \begin{tikzcd} & \Sigma := \{a \to b\} & \\ \Sigma_{\mathrm{in}} := \{a\} \arrow[ur] & & \Sigma_{\mathrm{in}} := \{b\} \arrow[ul] \end{tikzcd} \right\} .$$

Recall from section 5.1.1 that for $F = \mathrm{Vect}$ and $\rho : \mathbf{B}G \to \mathrm{Vect}$ a linear representation, we have $\rho^* \mathbf{E}_{\mathrm{pt}} F = V//G$ is the action groupoid of G acting on the representation space V.

Write $\nabla := \rho \circ g$ for the background field. It follows that the ω-bundle over X is given by the groupoid $\nabla^* \mathbf{E}_{\mathrm{pt}} F$ with morphisms

$$(\nabla^* \mathbf{E}_{\mathrm{pt}} F)_1 = \left\{ (x_1, v_1) \xrightarrow{\gamma} (x_2, v_2) \,|(\, x \xrightarrow{\gamma} y \,) \in X, v_1, v_2 \in V, v_2 = \rho(g(\gamma)) \right\}$$

with the obvious composition operation.

So a *state* in $\mathcal{H}_{\Sigma_a}$, a groupoid $v : \Psi \to \nabla^* V//G$ over $\nabla^* V//G$, is over each point $x \in X$ a groupoid over V. By the yoga of groupoid cardinality [1, 2] we can hence identify a state $v : \Psi \to \nabla^* V//G$ with a V-valued function on $\mathrm{Obj}(X)$.

The objects of the transgressed background bundle $(\nabla^{\Sigma})^* \mathbf{E}_{\mathrm{pt}}(F^{\Sigma})$ are the morphisms of $\nabla^* \mathbf{E}_{\mathrm{pt}} F$.

The pull-push propagation map

$$\int_{\hom(\Sigma, X)} \exp(\int \nabla) : \mathcal{H}_a \to \mathcal{H}_b$$

reproduces the path integral in this setup as described in [23].

5.2 Dijkgraaf-Witten model: target space $\mathbf{B}G_1$

Dijkgraaf-Witten theory [14] is the σ-model which in our terms is specified by the data

- target space $X = \mathbf{B}G$, the one-object groupoid corresponding to an ordinary 1-group G;
- background field $\alpha : \mathbf{B}G \to \mathbf{B}^3 U(1)$, a group 3-cocycle on G.

5.2.1 The 3-cocycle

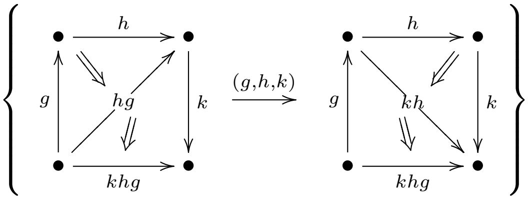
Indeed, we can understand group cocycles precisely as ω-anafunctors $\begin{tikzcd} \mathbf{B}G & Y \arrow[l, two heads, "\simeq"'] \arrow[r, "\alpha"] & \mathbf{B}^n U(1) \end{tikzcd}$. This is described in [7]. Here it is convenient to take Y to be essentially the free ω-category on the nerve of $\mathbf{B}G$, i.e. $Y := F(N(\mathbf{B}G))$, but with a few formal inverses thrown in to ensure that we have an acyclic fibration to $\mathbf{B}G$:

the 1-morphisms of Y are given by finite sequences of elements of G, its 2-morphisms are freely generated as pasting diagrams from 2-morphisms of the form

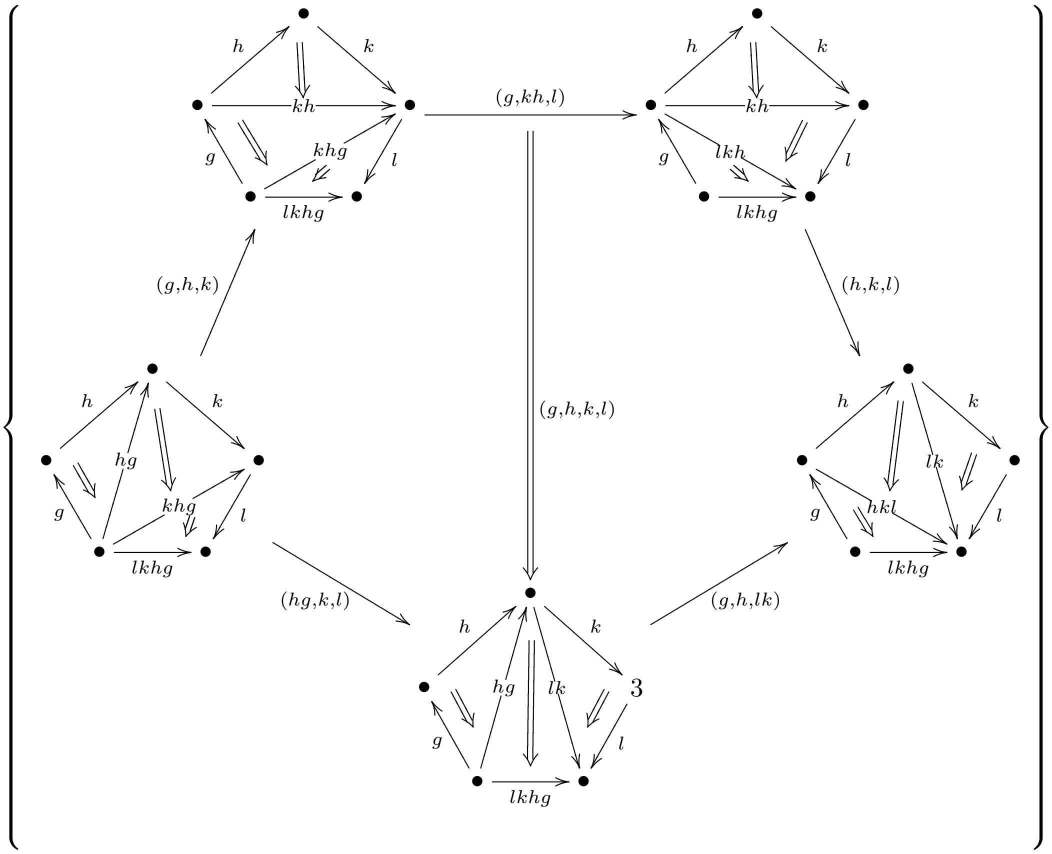
$$\left\{ \begin{tikzcd} & \bullet \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur, "g"] \arrow[rr, "hg"'] & {} & \bullet \end{tikzcd} \right\}$$

together with their formal

inverses. Its 3-morphisms are freely generated as pasting diagrams from 3-morphisms of the form



together with their formal inverses. Its 4-morphisms are freely generated from pasting diagrams of 4-morphisms of the form



together with their formal inverses.

The ω-functor $\alpha : Y \to \mathbf{B}^3U(1)$ has to send the generating 3-morphisms (g, h, k) to a 3-morphism in $\mathbf{B}^3U(1)$, which is an element $\alpha(g, h, k) \in U(1)$. In addition, it has to map the generating 4-morphisms between pasting diagrams of these 3-morphisms to 4-morphisms in $\mathbf{B}^3U(1)$. Since there are only identity 4-morphisms in $\mathbf{B}^3U(1)$ and since composition of 3-morphisms in $\mathbf{B}^3U(1)$ is just the product in $U(1)$, this says that α has to satisfy the equations

$$\forall g, h, k, l \in G : \ \alpha(g, h, k)\alpha(g, kh, l)\alpha(h, k, l) = \alpha(hg, k, l)\alpha(g, h, lk)$$

in $U(1)$. This identifies the ω-functor α with a group 3-cocycle on G. Conversely, every group 3-cocycle gives rise to such an ω-functor and one can check that coboundaries of group cocycles correspond precisely to transformations between these ω-functors. Notice that α uniquely extends to the additional formal inverses of cells in Y which ensure that $Y \xrightarrow{\simeq} \mathbf{B}G$ is indeed an acyclic fibration. For instance the 3-cell

$$\left\{ \begin{tikzcd} \bullet \arrow[r, "h"] \arrow[dr, phantom, "\Nwarrow"] & \bullet \arrow[d, "k"] \\ \bullet \arrow[u, "g"] \arrow[r, "khg"'] \arrow[ur, "hg"] \arrow[r, phantom, bend left=30, "\Uparrow"] & \bullet \end{tikzcd} \xrightarrow{(g,h,k)'} \begin{tikzcd} \bullet \arrow[r, "h"] \arrow[dr, no head, "kh"] \arrow[dr, phantom, bend left=30, "\Nearrow"] & \bullet \arrow[d, "k"] \\ \bullet \arrow[u, "g"] \arrow[r, "khg"'] \arrow[ur, phantom, bend right=30, "\Uparrow"] & \bullet \end{tikzcd} \right\}$$

has to go to $\alpha(g,h,k)^{-1}$.

5.2.2 Chern-Simons theory

In this article we do not want to get into details of the discussion of ω-categories internal to smooth spaces, but in light of the previous section 5.2 it should be noted that in terms of nonabelian cocycles the appearance of Chern-Simons theory is formally essentially the same as that of Dijkgraaf-Witten theory:

if we take BG to be a smooth model of the classifying space of G-principal bundles, then a smooth cocycle $BG \longrightarrow \mathbf{B}^3U(1)$, i.e. an ω-anafunctor internal to (suitably generalized) smooth spaces is precisely the cocycle for a 2-gerbe, i.e. a line 3-bundle. In nonabelian cohomology, the difference between group cocycles and higher bundles is no longer a conceptual difference, but just a matter of choice of target "space" ω-groupoid.

5.2.3 Transgression of DW theory to loop space

Proposition 5.1 *The background field α of Dijkgraaf-Witten theory transgressed according to definition 4.4 to the mapping space of parameter space $\Sigma := \mathbf{B}\mathbb{Z}$ – a combinatorial model of the circle –*

$$\tau_{\mathbf{B}\mathbb{Z}}\alpha := \hom(\mathbf{B}\mathbb{Z}, \alpha)_1 : \Lambda G \to \mathbf{B}^2U(1)$$

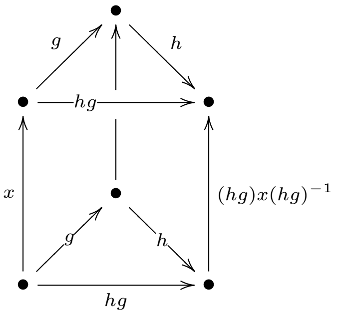
is the groupoid 2-cocycle known as the twist of the Drinfeld double, as recalled for instance on the first page of [34]:

$$(\tau_{\mathbf{B}\mathbb{Z}}\alpha) : (\, x \xrightarrow{g} gxg^{-1} \xrightarrow{h} (hg)x(hg)^{-1} \,) \mapsto \frac{\alpha(x,g,h)\ \alpha(g,h,(hg)x(hg)^{-1})}{\alpha(h, gxg^{-1}, g)} .$$

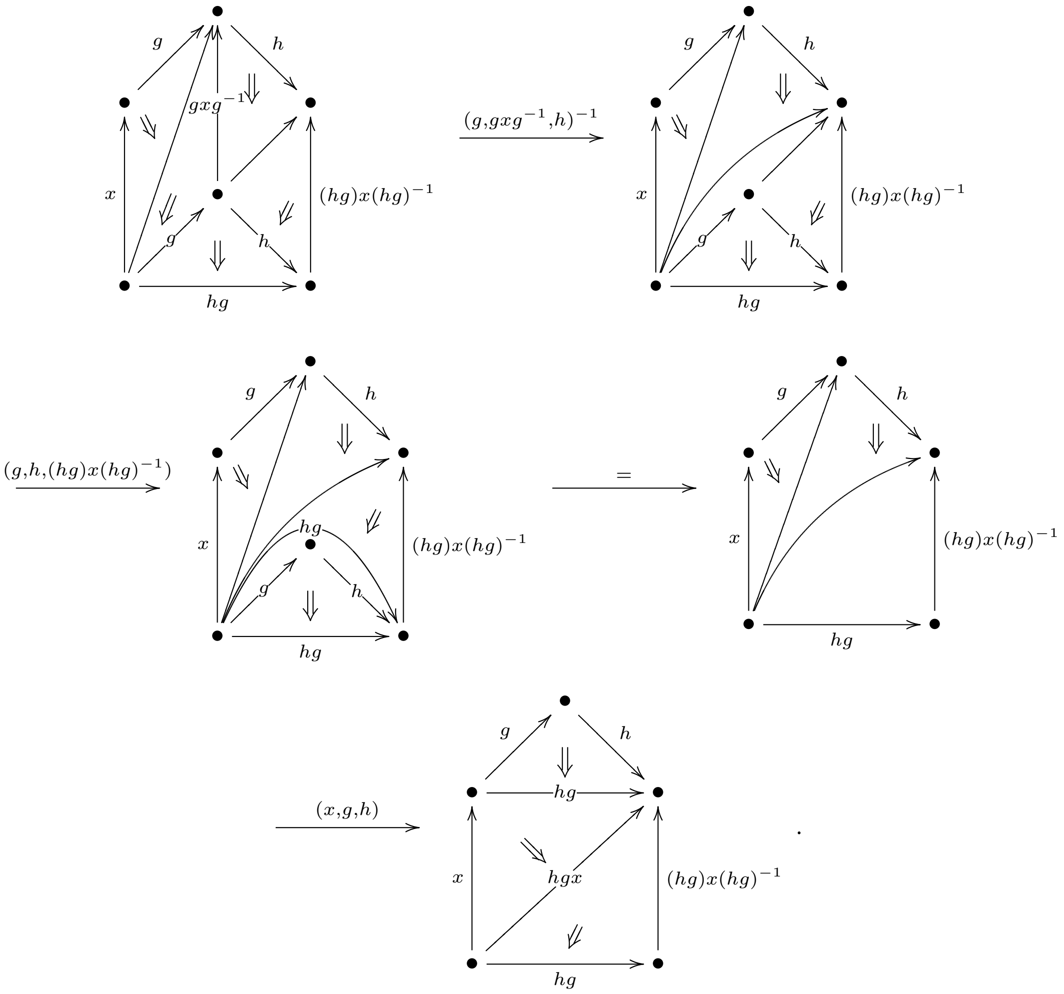
Proof. According to definition 2.13 the transgressed functor is obtained on 2-cells as the composition of ω-anafunctors $\mathbf{B}\mathbb{Z} \xrightarrow{(x,g,)} \mathbf{B}G \xrightarrow{\alpha} \mathbf{B}^3U(1)$, given by

$$\begin{tikzcd} (x,g,h)^*Y \arrow[r] \arrow[d, two heads, "\simeq"] & Y \arrow[r, "\alpha"] \arrow[d, two heads, "\simeq"] & \mathbf{B}^3U(1) \\ \mathbf{B}\mathbb{Z} \otimes O([2]) \arrow[r, "(x,g,h)"] & \mathbf{B}G & \end{tikzcd} ,$$

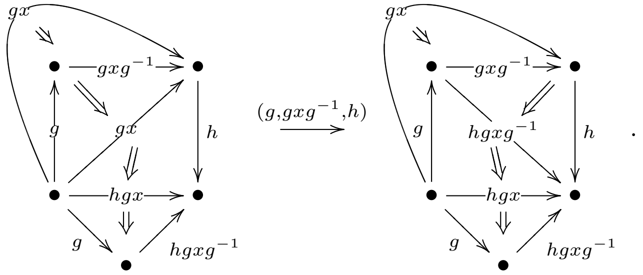
where (x, g, h) denotes a 2-cell in ΛG
$$\begin{tikzcd} & gxg^{-1} \arrow[dr, "h"] & \\ x \arrow[ur, "g"] \arrow[rr, "hg"'] & & (hg)x(hg)^{-1} \end{tikzcd}$$
which comes from a prism



in $\mathbf{B}G$. The 2-cocycle $\tau_{\mathbf{B}\mathbb{Z}}\alpha$ evidently sends this to the evaluation of α on a 3-morphism in the cover Y filling this prism. One representation of such a 3-morphism, going from the back and rear to the top and front of this prism, is



Here the first step follows by 2-dimensional whiskering of the standard 3-morphism:



This manifestly yields the cocycle as claimed. ☐

5.2.4 The Drinfeld double modular tensor category from DW bibranes

Let again $\rho : \mathbf{B}^2U(1) \to 2\mathrm{Vect}$ be the representation of $\mathbf{B}U(1)$ from section 5.1.3 and let $\tau_{\mathbf{B}\mathbb{Z}}\alpha : \Lambda G \to \mathbf{B}^2U(1)$ be the 2-cocycle obtained in section 5.2.3 from transgression of a Dijkgraaf-Witten line 3-bundle on $\mathbf{B}G$ and consider the the ρ-associated 2-vector bundle $\rho \circ \tau_{\mathbf{B}\mathbb{Z}}\alpha$ corresponding to that. Its sections according to definition 3.9 form a category $\Gamma(\tau_{\mathbf{B}\mathbb{Z}}\alpha)$.

Corollary 5.2 *The category $\Gamma(\tau_{\mathbf{B}\mathbb{Z}}\alpha)$ is canonically isomorphic to the representation category of the α-twisted Drinfeld double of G.*

Proof. Follows by inspection of our definition of sections applied to this case and using the relation established in 5.2.3 between nonabelian cocycles and the ordinary appearance of the Drinfeld double in the literature. ☐

In the case that α is trivial, the representation category of the twisted Drinfeld double is well known to be a modular tensor category. We now show how the fusion tensor product on this category is reproduced from a monoid of bibranes on ΛG.

Consider any 2-group $\mathbf{B}G_2 := (\, G \ltimes H \overset{p_1}{\underset{(\mathrm{Id}\cdot\delta)}{\rightrightarrows}} G \longrightarrow \mathrm{pt}\,)$.

Pullback to the single object of $\mathbf{BE}\mathbb{Z}$ yields a canonical morphism from the *disk-space* $DG_2 := \mathrm{hom}(\mathbf{BE}\mathbb{Z}, \mathbf{B}G_2)$ to $\mathbf{B}G$, $p : DG_2 \to \mathbf{B}G$ which inherits from the 2-group the structure of a category internal to groupoids in that on the span

$$\begin{tikzcd} & DG_2 \arrow[dl, "p"'] \arrow[dr, "p"] & \\ \mathbf{B}G & & \mathbf{B}G \end{tikzcd}$$

there is induced the structure of a monad from the horizontal composition in G_2. Notice that DG_2 is very similar to but in general slightly different from the action groupoid $H//G$ obtained from the canonical action of G on H in a 2-group. Both coinide in the special case that $G_2 = \mathbf{E}G$, so that $H = G$. In this case the morphism p exhibits DG_2 as the action groupoid (as in section 5.1.1) of G acting on itself by the adjoint action.

For $\mathbf{B}G \to 2\mathrm{Vect}$ the trivial gerbe, the transformations

$$\begin{tikzcd} & DG_2 \arrow[dl] \arrow[dr] & \\ \mathbf{B}G \arrow[rr, Rightarrow] \arrow[dr] & & \mathbf{B}G \arrow[dl] \\ & 2\mathrm{Vect} & \end{tikzcd}$$

are representations of DG_2 on vector spaces. In the case that $H = G$ and the boundary map is the identity we have $DG_2 = \Lambda G$, so that, by the above, bibranes on DG_2 become representations of ΛG.

One checks in this case that the fusion product of bibranes using the internal category structure on DG_2 according to 4.8 does reproduce the familiar fusion tensor product on representations of ΛG, hence of the Drinfeld double.

5.2.5 The DW path integral

Let S_{in} and S_{out} be two oriented surfaces and let V be an oriented 3-manifold with boundary $\partial V = S_{\mathrm{in}} \sqcup \bar{S}_{\mathrm{out}}$.

Forming fundamental groupoids yields a co-span

$$\begin{tikzcd}
& \Sigma := \Pi_1(V) & & . \\
\Sigma_{\mathrm{in}} := \Pi_1(S_{\mathrm{in}}) \arrow[ur] & & \Sigma_{\mathrm{out}} := \Pi_1(S_{\mathrm{out}}) \arrow[ul] &
\end{tikzcd}$$

Notice that the space of fields on V in DW theory $\mathrm{hom}(\Sigma, \mathbf{B}G)$ is equivalent to the groupoid of G-principal bundles on V. This implies that pull-push quantum propagation in the sense of section 4.3 reproduces the right DW path integral.

5.3 Yetter-Martins-Porter model: target space $\mathbf{B}G_2$

The Yetter-Martins-Porter model is a σ-model with target space $X = \mathbf{B}G$ for G a 2-group.

Here, too, our quantization reproduces the right combinatorial path integral factor [**?**].

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