Nonabelian differential cohomology in Street's descent theory

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Abstract

The general notion of cohomology, as formalized ∞-categorically by Ross Street, makes sense for coefficient objects which are ∞-category valued presheaves. For the special case that the coefficient object is just an ∞-category, the corresponding cocycles characterize higher fiber bundles. This is usually addressed as *nonabelian cohomology* [6, 32]. If instead the coefficient object is refined to presheaves of ∞-functors from ∞-*paths* to the given ∞-category, then one obtains the cocycles discussed in [4, 26, 27, 28] which characterize higher bundles *with connection* and hence live in what deserves to be addressed as nonabelian *differential cohomology* [18].

We concentrate here on ω-categorical models (strict globular ∞-categories [9, 13, 10, 11]) and discuss nonabelian differential cohomology with values in ω-groups obtained from integrating L(ie)-∞ algebras [16, 17].

Contents

1 Introduction

A principal G-bundle is given, with respect to a good cover by open sets of its base space, by a *trivial* G-bundle on each open subset, together with an isomorphism of trivial G-bundles on each double intersection, and an equation between these on each triple intersection. This is the archetypical example of what is called **descent data**, forming a **cocycle in nonabelian cohomology**. It can be vastly generalized by replacing the group G appearing here by some ∞-category. For each cocycle obtained this way there should be a corresponding ∞-bundle whose local trivialization it describes [34].

The crucial basic idea of [4, 26, 27, 28] is to describe ∞-bundles *with connection* by cocycles which have

- a (“transport”) functor from paths to G on each patch;
- an equivalence between such functors on double overlaps
- and so on.

The cocycles thus obtained deserve to be addressed as cocycles in **differential nonabelian cohomology**.

Forming the collection of ω-functors from paths in a patch to some codomain provides a functor from “spaces” to ω-categories: an ω-category valued presheaf.

In [29] Ross Street descibes a very general formalization for cohomology taking values in ω-category valued presheaves. We recall the basic ideas (subject to some slight modifications, a discussion of which is in 8) and describe how the differential cococycles of [4, 26, 27, 28] fit into that.

Of particular interest are differential cocycles which can be expressed differentially in terms of L(ie) ∞-algebras. Building on the discussion of [23] we give in 5 a definition (def. 14) of non-flat non-abelian differential cocycles and their characteristic classes.

There are two major approaches to general (nonabelian) ∞-cohomology:

- Ross Street in [29] explicitly writes down ∞-descent conditions for ∞-category valued presheaves.
- In the approach reviewed in [32, 19] instead simplicial set valued presheaves are used, and the descent condition is realized more implicitly, by passing to homotopy categories.

2 Descent and cohomology

Fix a topos C, whose objects we think of as

- the spaces whose cohomologies we want to understand;

or equivalently

- the spaces on which we want to understand the notion of higher fiber bundles and connections.

We work with ω-categories (strict globular ∞-categories) internal to C and write $\omega\mathrm{Cat}$ for the $(\omega\mathrm{Cat}, \otimes_{\mathrm{Gray}})$-category of all ω-categories internal to C (see [13] and section 9 of [29]).

The theory we are interested in is the theory of structures $P \in \mathrm{Bund}(X)$ on $X \in C$ for

$$\mathrm{Bund} : \; C^{\mathrm{op}} \longrightarrow \omega\mathrm{Cat}$$

some functorial assignment of structures to each object X in C, which have the property that when pulled back along a suitable regular surjection

$$\pi : Y \twoheadrightarrow X$$

in C they become equivalent to a structure

$$P_{\mathrm{triv}} \in \mathrm{TrivBund(Y)} \subset \mathrm{Bund}(Y)$$

from a chosen smaller collection $i : \; \mathrm{TrivBund} \hookrightarrow \mathrm{Bund}$:

$$\exists : \pi^* P \xrightarrow[\simeq]{t} P_{\mathrm{triv}} \; .$$

The equivalence t here is called the **local trivialization** of P relative to π and i. We speak of **π-local i-trivializations**.

The existence of this local trivialization implies that the existence of the structure P down on X is mirrored by the existence of P_{triv} up on Y *together* with various relations between the pullbacks of P_{triv} along the simplicial object

$$Y^\bullet := \mathrm{Ner}(\pi) : \ \Delta^{\mathrm{op}} \longrightarrow C$$

$$Y^\bullet = \left(\cdots Y \times_X Y \times_X Y \ \substack{\xrightarrow{\pi_{12}} \\ \xrightarrow{\pi_{23}} \\ \xrightarrow{\pi_{13}}} \ Y \times_X Y \ \substack{\xrightarrow{\pi_0} \\ \xrightarrow{\pi_1}} \ Y \right) .$$

The first of these relations says that there is an equivalence

$$\pi_1^* P_{\mathrm{triv}} \underset{\simeq}{\xrightarrow{g}} \pi_2^* P_{\mathrm{triv}}$$

between the two possible pullbacks of P_{triv} to $Y \times_X Y$. The second relation says that there is an equivalence

$$\begin{tikzcd}
 & \pi_2^* P_{\mathrm{triv}} \arrow[dr, "\pi_{23}^* g"] \arrow[d, phantom, "\simeq \Downarrow f"] & \\
\pi_1^* P_{\mathrm{triv}} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & {} & \pi_2^* P_{\mathrm{triv}}
\end{tikzcd}$$

between the three possible pullbacks of this equivalence to $Y \times_X \times Y \times_X Y$. The third relation says that there is an equivalence between the four possible pullbacks of this equivalence of equivalences. And so on.

These relations are variously known as the *transition data* or *gluing data* or **descent data**, since given a $P_{\mathrm{triv}} \in \mathrm{TrivBund}(Y)$, they ensure that P_{triv} may be "glued" along the fibers of Y such that result "descends" to a $P \in \mathrm{Bund}(X)$ down on X. Therefore descent is the converse to local trivialization:

$$\begin{array}{c}\text{trivial}\\ \text{structure}\\ \text{on } Y\end{array} \ \substack{\xmapsto{\ \text{descent}\ } \\ \xleftarrow[\text{local trivialization}]{}\!\!\!\shortmid} \ \begin{array}{c}\text{structure}\\ \text{on } X\end{array} .$$

The collections $(P_{\mathrm{triv},g,f,\cdots})$ consisting of a P_{triv} with its gluing data or descent data can hence usefully be regarded as a forming a kind of higher categorical coequalizer of the cosimplicial ω-category

$$\mathcal{E} : \Delta \xrightarrow{\mathrm{Ner}(\pi)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\mathrm{TrivBund}} \omega\mathrm{Cat}$$

$$\mathcal{E}^\bullet = \left(\cdots \mathrm{TrivBund}(Y \times_X Y \times_X Y) \ \substack{\xleftarrow{\pi_{01}^*} \\ \xleftarrow{\pi_{12}^*} \\ \xleftarrow{\pi_{23}^*}} \ \mathrm{TrivBund}(Y \times_X Y) \ \substack{\xleftarrow{\pi_0^*} \\ \xleftarrow{\pi_1^*}} \ \mathrm{TrivBund}(Y) \right) .$$

This coequalizer-like ω-category, whose objects are suitable collections $(P_{\mathrm{triv},g,f,\cdots})$ is the **descent category** $\mathrm{Desc}(\mathcal{E})$

$$\mathrm{Desc} : \omega\mathrm{Cat}^\Delta \longrightarrow \omega\mathrm{Cat} .$$

Its general definition for ω-categories was given in [29] (p. 32), based on [30]. A sketch of a more general definition for weak ∞-categories is given towards the end of [29].

2.1 ωCat-valued presheaves

The above considerations show that the objects of interest here are (pre)sheaves on C with values in ω-categories, corresponding to the presheaves with values in simplicial sets considered in the approach reviewed in [19, 32].

There is a standard construction to enrich $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$ over ωCat: for $X, Y \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$ write

$$\tilde{\hom}(X,Y) : \omega\mathrm{Cat} \to \mathrm{Set}$$

$$R \mapsto \omega\mathrm{Cat}(R \times X, Y)\,.$$

If this is representable, we identify the representing ω-category $\hom(X,Y) \in \omega\mathrm{Cat}$ with the ωCat-valued hom-object:

$$\tilde{\hom}(X,Y) \simeq \mathrm{Set}^{\omega\mathrm{Cat}^{\mathrm{op}}}(-, \hom(X,Y))\,.$$

So if C is such that this representing object exists, $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$ is ωCat-enriched and it makes sense to ask if our descent ω-category $\mathrm{Desc}(\mathcal{E})$ is actually co-representable in that there is $\Pi_0^Y(X) \in \omega\mathrm{Cat}$ such that

$$\mathrm{Desc}(\mathcal{E}) \simeq \omega\mathrm{Cat}^{C^{\mathrm{op}}}(\Pi_0^Y(X), \mathrm{TrivBund})\,,$$

where we are implicitly using the canonical embedding $\omega\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}}$. This $\Pi_0^Y(X)$ is the **codescent object**

$$\Pi_0^Y(X) := \mathrm{Codesc}(\mathrm{Ner}(\pi))$$

and the notation suggests that we shall later have use more generally for ω-catgories denoted $\Pi_n^Y(X)$ and $\Pi_\omega^Y(X)$: their k-morphisms are k-paths in Y combined with jumps in the fibers of Y [26, 28].

Notice that the map Desc from simplicial ω-categories to ω-categories is analogous (possibly even equivalent) to the *codiagonal* map from bisimplicial sets to simplicial sets.

We usually have that TrivBund is *representable*

$$\mathrm{TrivBund}(-) \simeq \omega\mathrm{Cat}^{C^{\mathrm{op}}}(-, \mathbf{A})$$

for some $\mathbf{A} \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$, where we are implicitly using the embedding $C \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}}$ which sends each object U to the ωCat-valued presheaf which sends each object V to the discrete ω-category over $C(V,U)$.

In that case we say that

Definition 1 ($\omega\mathrm{Cat}^{C^{\mathrm{op}}}$-valued cohomology) *For $X \in C$ and $\mathbf{A} \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$, the ω-category*

$$H(X,\mathbf{A}) := \mathrm{colim}_\pi \left(\mathrm{Desc} \left(\Delta \xrightarrow{\mathrm{Ner}(\pi)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\omega\mathrm{Cat}^{C^{\mathrm{op}}}(-,\mathbf{A})} \omega\mathrm{Cat} \right)\right)$$

is the **cocycle ω-category** *of X with coefficients in $\mathbf{A}$:*

- *objects are the* $\mathbf{A}$*-valued* **cocycles** *on* X*;*
- *(1-)morphisms are the* **coboundaries** *between these cocycles;*
- $(k > 1)$*-morphisms are the coboundaries of coboundaries;*
- *equivalence classes of objects are the* $\mathbf{A}$*-valued* **cohomology classes** *of* X*.*

The functor

$$H(-,\mathbf{A}) : C^{\mathrm{op}} \to \omega\mathrm{Cat}$$

is the **cohomology theory for coefficients** $\mathbf{A}$*.*

This is the general definition of cohomology that essentially appears in section 4 of [29].

2.2 ωCat-valued cohomology

The special case of cohomology with values in an ω-category – whose general idea goes back to [21] and is often addressed, somewhat loosely, as **nonabelian cohomology** – is obtained using the inclusion

$$\begin{aligned} \Pi_0^* \quad : \quad \omega\mathrm{Cat} &\hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}} \\ A &\mapsto \omega\mathrm{Cat}(\Pi_0(-), A) \ , \end{aligned}$$

where, in turn, $\Pi_0 : C \hookrightarrow \omega\mathrm{Cat}$ sends each object U to the discrete ω-category over it (which has U as its object of objects and no nontrivial morphisms.)

Hence

Definition 2 (ωCat-valued cohomology) *For* $A \in \omega\mathrm{Cat}$*, the cohomology theory with coefficients in* A *is*

$$H(-,A) := H(-,\Pi_0^*(A))\,.$$

Using the Yoneda-like argument on p. 12 of [29], which says that

$$\omega\mathrm{Cat}^{C^{\mathrm{op}}}(U,\Pi_0^*(A)) \simeq \Pi_0^*(A)(U) := \omega\mathrm{Cat}(\Pi_0(U),A)\,,$$

this becomes the theory considered on p. 3 of [29].

2.2.1 Cohomology classes for ω-bundles

Definition 3 (ω-groups) *Given any one-object ω-groupoid* Gr *we say that the* Hom*-thing* $G := \mathrm{Gr}(\bullet,\bullet)$ *is an ω-group and write*

$$\mathrm{Gr} := \mathbf{B}G$$

to indicate the property of Gr *of having one single object.*

Remark. The notation here is such that under taking realizations of nerves we have

$$|\mathbf{B}G| \simeq B|G|\,,$$

compare [2, 5]. We hence call $\mathbf{B}G$ the **classifying ω-groupoid** for the ω-group G.

Whichever way a **principal G-bundle** on X is defined [34], it must be such that its local trivializations are objects in $H(X, \mathbf{B}G)$ and, indeed, that the ω-category $G\mathrm{Bund}(X)$ they form is equivalent to $H(X, \mathbf{B}G)$

$$G\mathrm{Bund}(-) \simeq H(-, \mathbf{B}G)\,.$$

For $n = 2$ this is discussed in [34].

This is often thought of as saying that

$$G\text{-bundles are a geometric model for } H(-, \mathbf{B}G)\ \ .$$

One expects to revover the topologist's notion of classifying maps in the case that objects of C are topological spaces by using the corepresentation of $H(-, \mathbf{B}G)$ using the codescent object as

$$H(-, \mathbf{B}G) \simeq \omega\mathrm{Cat}(\Pi_0^Y(X), \mathbf{B}G)$$

for $\pi : (Y = \sqcup_i U_i) \to X$ a good cover of X. Upon applying the nerve functor one expects

$$|\omega\mathrm{Cat}(\Pi_0^Y(X), \mathbf{B}G)| \simeq [|\Pi_0^Y(X)|, |\mathbf{B}G|] \simeq [X, B|G|]\,.$$

For strict 2-groups G this was shown to be true in [5] if G is "well pointed". A more general argument for topological 2-categories is given in [2] though it is, while plausible, not obvious that the "concordances" used in [2] reproduce exactly the transformations that are the morphisms in $H(-, \mathbf{B}G)$.

2.2.2 Singular cohomology

Fact 1 *A direct consequence of a standard fact about Čech cohomology is that the ω-category $\mathbf{B}^n\mathbb{Z}$ exhibits ordinary singular cohomology as $\omega\mathrm{Cat}$-valued cohomology*

$$H^{n+1}_{\mathrm{singular}}(-, \mathbb{Z}) = H(-, \mathbf{B}^n U(1))_\sim = H(-, \mathbf{B}^{n+1}\mathbb{Z})_\sim$$

.

2.2.3 K-Theory

Fact 2 *A standard fact about K-theory says that*

$$K^0(-) \simeq H(-, (\mathbf{B}U)\mathbb{Z})_\sim\,.$$

2.3 $\omega\mathrm{Cat}$-valued differential cohomology

Recall that, with definition 2, we obtained ωCat-valued cohomology from the general $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$-valued cohomology by pulling back along an inclusion

$$\Pi_0^* : \omega\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}} .$$

But there are other such inclusions, which are no less natural. In particular, if the objects X of C are spaces that admit a notion of **path ω-groupoid** $\Pi(X)$,

$$\Pi : C \hookrightarrow \omega\mathrm{Cat}$$

then we can pull back along the corresponding

$$\begin{aligned} \Pi^* \quad : \quad & \omega\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}} \\ & A \longmapsto \omega\mathrm{Cat}(\Pi(-), A) \; . \end{aligned}$$

Definition 4 (ωCat-valued differential cohomology) *For a given notion of path ω-groupoid $\Pi : C \hookrightarrow \omega\mathrm{Cat}$ and a coefficient object $A \in \omega\mathrm{Cat}$ we address*

$$H^{\Pi}(-, A) := H(-, \Pi^*(A))$$

as Π-differential cohomology with values in A.

For $\Pi = \mathcal{P}_2$ and G a strict 2-group such cocycles in $H^{\mathcal{P}_2}(-, \mathbf{B}G)$ were first considered in [4, 28].

2.3.1 Cohomology classes for ω-bundles with connection

While our definition allows more general setups, usually one will want to interpret differential cohomology in the context of *smooth* spaces.

If this is so, one useful concrete choice for our ambient category is to take C to be the category of sheaves on the site S with

- $\mathrm{Obj}(S) = \mathbb{N}$;
- $S(n, m) = \{f : \mathbb{R}^n \to \mathbb{R}^m | f \text{ smooth}\}$.

Theorem 1 *Let G be an ordinary Lie group and let*

$$\Pi := \mathcal{P}_1 : C \to \omega\mathrm{Cat}$$

be the path 1-groupoid whose morphisms are thin-homotopy classes of paths. Then Π-differential cohomology with values in $\mathbf{B}G$ classifies principal G-bundles with connection

$$H^{\Pi}(-, \mathbf{B}G) = G\mathrm{Bund}_{\nabla}(-) \, .$$

This is the result of [26].

Theorem 2 *Let G be a strict Lie 2-group and let*

$$\Pi := \mathcal{P}_2 : C \to \omega\mathrm{Cat}$$

be the path 2-groupoid whose k-morphisms are thin-homotopy classes of k-paths. Then Π-differential cohomology with values in $\mathbf{B}G$ classifies fake-flat principal G-2-bundles with connection

$$H^{\Pi}(-, \mathbf{B}G) = G\mathrm{Bund}_{\nabla}(-)\,.$$

This is the result of [27] and [28].

In particular, for $G = \mathrm{AUT}(H)$, $H^{\Pi}(-, \mathbf{B}\mathrm{AUT}(H))$ classifies the fake-flat connections on H-gerbes studied in [7].

3 Codescent

Definition 5 (codescent) *Given a simplicial object $E : \Delta^{\mathrm{op}} \to C$, we say that*

$$\mathrm{Codesc}(E) \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$$

is, if it exists, the codescent $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$-object of E if it co-represents descent on E in the sense that

$$\mathrm{Desc}\left(\Delta \xrightarrow{E^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\omega\mathrm{Cat}^{C^{\mathrm{op}}}(-,\mathbf{A})} \omega\mathrm{Cat} \right) \simeq \omega\mathrm{Cat}^{C^{\mathrm{op}}}(\mathrm{Codesc}(E), \mathbf{A})$$

naturally for all coefficient objects $\mathbf{A} \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$.

If we let $\mathbf{A}$ just run over the image of $\omega\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}}$ we obtain the codescent object as an ω-category:

Definition 6 *Given a simplicial object $E : \Delta^{\mathrm{op}} \to C$, we say that*

$$\mathrm{Codesc}(E) \in \omega\mathrm{Cat}$$

is, if it exists, the codescent ω-category of E if it co-represents descent on E in the sense that

$$\mathrm{Desc}\left(\Delta \xrightarrow{E^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\omega\mathrm{Cat}(\Pi_0(-),A)} \omega\mathrm{Cat} \right) \simeq \omega\mathrm{Cat}(\mathrm{Codesc}(E), A)$$

naturally for all coefficient object $A \in \omega\mathrm{Cat}$.

For $E = Y^{\bullet} = \mathrm{Ner}(\pi : Y \to X)$, we have that $\mathrm{Codesc}(E) = \Pi_0^{\pi}(X)$ is nothing but the Čech groupoid of Y. In fact, as mentioned on p. 3 of [29], every category is the codescent object of its nerve.

We observe that, again, the above definition makes explicit use of an injection

$$\Pi : C \to \omega\mathrm{Cat}\,.$$

Hence we adapt the notion of codescent to the setup of differential cohomology as in 2.3:

Definition 7 (differential codescent) *Given a simplicial object $E : \Delta^{\mathrm{op}} \to C$, and an embedding*

$$\Pi : C \hookrightarrow \omega\mathrm{Cat}\,,$$

we say that

$$\mathrm{Codesc}^{\Pi}(E) \in \omega\mathrm{Cat}$$

is, if it exists, the codescent object of E if it co-represents descent on E in the sense that

$$\mathrm{Desc}\left(\Delta \xrightarrow{E^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\omega\mathrm{Cat}(\Pi(-),A)} \omega\mathrm{Cat} \right) \simeq \omega\mathrm{Cat}(\mathrm{Codesc}^{\Pi}(E), A)$$

naturally for all coefficient object $A \in \omega\mathrm{Cat}$.

Theorem 3 *For $\Pi := \mathcal{P}_1$ the path 1-groupoid, we have*

$$\mathrm{Codesc}^{\Pi}(Y^{\bullet}) = \mathcal{P}_1^{\pi}(X)\,,$$

where on the right we have the "path pushout" from [26].

Theorem 4 *For $\Pi := \mathcal{P}_2$ the path 2-groupoid, we have*

$$\mathrm{Codesc}^{\Pi}(Y^{\bullet}) = \mathcal{P}_2^{\pi}(X)\,,$$

where on the right we have the "bigon pushout" from [28].

3.1 Descent categories from codescent

We can use the codescent objects to express the corresponding descent ω-categories in a useful way:

as described on p. 5 of [29] every category is the codescent object of its own nerve. That means in particular that the codescent object of the nerve of an epimorphism $\pi : Y \twoheadrightarrow X$ is just the Čech groupoid $\mathrm{Codesc}(\mathrm{Ner}(\pi)) = X^{\pi}$.

3.1.1 Descent category for differential 1-cocycles

We consider $\pi : Y \twoheadrightarrow X$ a regular epimorphism and work out the descent category for differential 1-descent

$$\begin{aligned} &\mathrm{Desc}\left(\Delta \xrightarrow{\mathrm{Ner}(\pi)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\mathrm{Cat}(\mathcal{P}_1(-),\mathbf{B}G)} \mathrm{Cat} \right) \\ &= \mathrm{Desc}\left(\Delta \xrightarrow{\mathrm{Ner}(\pi)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\mathrm{Cat}^{C^{\mathrm{op}}}(-,\mathrm{Cat}(\mathcal{P}_1(-),\mathbf{B}G)} \mathrm{Cat} \right) \end{aligned}$$

as

$$= \mathrm{Cat}^{C^{\mathrm{op}}}(\mathrm{Cat}(\Pi_0(-), X^{\pi}), \mathrm{Cat}(\mathcal{P}_1(-), \mathbf{B}G))$$

Notice that for each test domain U the objects of $\mathrm{Cat}(\Pi_0(U), X^{\pi})$ are maps $U \to Y$ in C, while the morphisms are maps $U \to Y^{[2]}$.

The objects of $\mathrm{Cat}^{C^{\mathrm{op}}}(\mathrm{Cat}(\Pi_0(-), X^\pi), \mathrm{Cat}(\mathcal{P}_1(-), \mathbf{B}G))$ are over each test domain U functors

$$\mathrm{Cat}(\Pi_0(U), X^\pi) \to \mathrm{Cat}(\mathcal{P}_1(U), \mathbf{B}G)$$

natural in U. Such functors can be obtained from picking an object

$$\mathrm{triv} \in \mathrm{Cat}(\mathcal{P}_1(Y), \mathbf{B}G)$$

with an ismorphism

$$g : \pi_1^*\mathrm{triv} \to \pi_2^*\mathrm{triv}$$

such that

$$\begin{tikzcd} & \pi_2^*\mathrm{triv} \arrow[dr, "\pi_{23}^* g"] & \\ \pi_1^*\mathrm{triv} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & & \pi_3^*\mathrm{triv} \end{tikzcd}$$

and then sending $f : U \to Y$ to $f^*\mathrm{triv}$ $\hat{f} : U \to Y^{[2]}$ to $\hat{f}^* g$.

By the usual presheaf gymnastics, all such functors should arise this way. (** This must be true, but I need to check the precise argument**)

A natural transformation between two such functors is obtained from picking an isomorphism

$$h : \mathrm{triv} \to \mathrm{triv}'$$

making

$$\begin{tikzcd} \pi_1^*\mathrm{triv} \arrow[r, "g"] \arrow[d, "\pi_1^* h"'] & \pi_2^*\mathrm{triv} \arrow[d, "\pi_2^* h"] \\ \pi_1^*\mathrm{triv}' \arrow[r, "g'"'] & \pi_2^*\mathrm{triv}' \end{tikzcd}$$

commute, and then sending each object $f : U \to Y$ to the morphism f^*h.

By the usual presheaf gymnastics, all such functors should arise this way. (** This must be true, but I need to check the precise argument **)

The descent category found this way is the one given in [26].

3.1.2 Descent category for differential 2-cocycles

Analogously.

The descent category found this way is the one given in [28], the cocycles of which were also described in [4].

4 Local trivialization

If

$$\mathrm{Bund} : C^{\mathrm{op}} \to \omega\mathrm{Cat}$$

encodes structures on objects of C of certain kind and

$$i : \mathrm{TrivBund} \hookrightarrow \mathrm{Bund}$$

is a certain subcollection of these structures which we want to regard as being "trivial", and if

$$\pi : Y \twoheadrightarrow Y$$

is a regular epimorphism in C, then we say

Definition 8 *The pseudopullback* $\mathrm{Triv}(i,\pi)$

$$\begin{tikzcd}
\mathrm{Triv}(i,\pi) \arrow[r] \arrow[d] & \mathrm{TrivBund}(Y) \arrow[d, hook, "i"] \arrow[dl, Rightarrow, "\simeq"'] \\
\mathrm{Bund}(X) \arrow[r, "\pi^*"] & \mathrm{Bund}(Y)
\end{tikzcd}$$

is the ω-category of π-locally i-trivializable elements of $\mathrm{Bund}(X)$*, equipped with a chosen π-local i-trivialization.*

By forgetting the chosen local trivialization we obtain a factorization

$$\mathrm{Triv}(i,\pi) \twoheadrightarrow \mathrm{Trans}(i,\pi) \hookrightarrow \mathrm{Bund}(X)$$

where $\mathrm{Trans}(i,\pi)$ *is the ω-category of elements of* $\mathrm{Bund}(X)$ *which do* admit some *π-local i-trivialization.*

This is essentially the definition on p. 22 of [29], but with the notation following [26, 28] (so our $\mathrm{Triv}(i,\pi)$ is $Q(t;e)$ in [29] and our $\mathrm{Trans}(i,\pi)$ is $\mathrm{Loc}(t;e)$).

In [26, 28] we adopt a more concrete (less general) point of view on what counts as i-trivial: there we require that

$$\mathrm{Bund} := \omega\mathrm{Cat}(\Pi_n(-), T)$$

for T some **ω-category of fibers** and

$$\mathrm{TrivBund} := \omega\mathrm{Cat}(\Pi_n(-), \mathrm{Gr})$$

for Gr some **ω-category of *typical* fibers** and that the injection

$$i : \mathrm{TrivBund} \hookrightarrow \mathrm{Bund}$$

is postcomposition with a specified injection

$$i : \mathrm{Gr} \hookrightarrow T \ .$$

In that case an element $F \in \mathrm{Bund}(X)$ is π-locally i-triviall precisely if it fits into a square

$$\begin{tikzcd}
\Pi_n(Y) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{triv}"'] & \Pi_n(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\simeq"', "t"] \\
\mathrm{Gr} \arrow[r, "i"] & T
\end{tikzcd}$$

In [26, 28] we also see the need to strengthen the conditions on what counts as locally trivial: not only need the local trivialization t exist, it also may have to be itself locally trivial in some sense. We encode this by putting conditions on the descent data induced by t:

Observation 1 (extraction of descent data) *There is canonically a morphism*

$$\mathrm{Ex} : \mathrm{Triv}(i, \pi) \to H_\pi(X, \mathrm{Bund}) \, .$$

See [26, 28].

Definition 9 (π-local i-trivialization) *Given*

- $T \in \omega\mathrm{Cat}$*: thought of as an ω-category of fibers;*
- $\mathrm{Gr} \in \omega\mathrm{Grpd}$*: thought of as an ω-groupoid of* typical *fibers*
- *an inclusion*
$$i : \mathrm{Gr} \hookrightarrow T \ ;$$
- $\Pi : C \to \omega\mathrm{Cat}$*; a notion of path ω-groupoid;*
- *a factorization*

$$\begin{tikzcd}
\omega\mathrm{Cat}(\Pi(-), \mathrm{Gr}) \arrow[r, hook, "a"] \arrow[rr, bend right, "i_*"'] & \mathrm{TrivBund}^{\Pi}_{\mathrm{Gr}} \arrow[r, hook] & \omega\mathrm{Cat}(\Pi(-), T)
\end{tikzcd}$$

in $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$ with the map a surjective on objects and faithful, we say that

$$\mathrm{tra} \in \omega\mathrm{Cat}(\Pi(x), T)$$

is π-locally i-trivializable if there is an equivalence t

$$\begin{tikzcd} \Pi(Y) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{triv}"'] & \Pi(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\simeq"', "t"] \\ \mathrm{Gr} \arrow[r, "i"] & T \end{tikzcd}$$

such that the extracted cocylce has coefficients in $\mathrm{TrivBund}^{\Pi}_{\mathrm{Gr}}$

$$\mathrm{Ex}(t) \in H_\pi(X, \mathrm{TrivBund}^{\Pi}_{\mathrm{Gr}}) .$$

4.1 Local semi-trivialization: bundle gerbes

What is addressed as a "bundle gerbe" in the literature is really a cocycle. Bundle gerbes differ from cocycles with values in $\mathbf{B}G$ only in that the factorization appearing in definition 9 has a middle piece whose collection of morphisms is strictly larger than that of the left piece.

Theorem 5 *Line bundle gerbes [20] with connection ("and curving") are equal (meaning:* canonically *isomorphic) to cocycles with values in the subobject*

$$\mathrm{TrivBund} \subset \omega\mathrm{Cat}(\mathcal{P}_2(-), \mathbf{B}\mathrm{Vect})$$

on all objects which factor through

$$i : \mathbf{BB}U(1) \hookrightarrow \mathbf{B}\mathrm{Vect}$$

with morphisms those whose component maps are locally $(\mathbf{B}U(1) \hookrightarrow \mathrm{Vect})$*-trivializable.*

Details and proof in [28].

Analogous statements hold for other flavors of bundle gerbes, like higher bundle gerbes and nonabelian bundles gerbes[1].

5 Characteristic forms

5.1 L_∞-algebra valued differential forms

Definition 10 *For G an ω-group, we address*

$$\Omega^\bullet(-, \mathbf{B}G) := \omega\mathrm{Cat}(\Pi_\omega(-), \mathbf{B}G) : C^{\mathrm{op}} \to \omega\mathrm{Cat}$$

as flat G-valued differential forms.

For G a strict 2-group and Π_ω replaced with $\mathcal{P}_2$ this was studied in [27], see also [4]:

Theorem 6 *Objects of the 2-category*

$$2\mathrm{Cat}(\mathcal{P}_2(Y), \mathbf{B}G)$$

for G coming from a Lie crossed module $(t : H \to G)$ are pairs $(A, B) \in \Omega^1(Y, \mathrm{Lie}(G)) \times \Omega^2(Y, \mathrm{Lie}(H))$ satisfying $F_A + t_* \circ B = 0$.

This can be understood in terms of L_∞-algebra valued forms:

L_∞-algebras are to ordinary Lie algebras as ∞-groupoids are to ordinary groups. We can integrate L_∞-algebras to ω-groupoids internal to smooth spaces and use these as coefficients for nonabelian differential cohomology.

Definition 11 *A finite-dimensional L_∞-algebra is a finite dimensional $\mathbb{N}_+$-graded vector space $\mathfrak{g}$ together with a degree +1 differential on the graded symmetric tensor algebra over $\mathfrak{g}^*$*

$$d_{\mathfrak{g}} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

such that $d^2 = 0$. The resulting differential graded commutative algebra

$$\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d_{\mathfrak{g}})$$

is called the Chevalley-Eilenberg algebra of $\mathfrak{g}$.

There is a notion of mapping cone for L_∞-algebras and we write

$$\mathrm{W}(\mathfrak{g}) := \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) := \mathrm{CE}(\mathrm{Cone}(\mathfrak{g} \stackrel{\mathrm{Id}}{\to} \mathfrak{g}))\,.$$

We have a canonical sequence

$$\mathrm{CE}(\mathfrak{g}) \longleftarrow \mathrm{W}(\mathfrak{g}) \longleftarrow \mathrm{inv}(\mathfrak{g}) = \mathrm{W}(\mathfrak{g})_{\mathrm{basic}} \ . \qquad (1)$$

For more details on this and the following see [23].

There is a contravariant adjunction between smooth spaces and differential graded commutative algebras, induced by the ambimorphic deRham object $\Omega^\bullet(-) \in C$ which is a smooth space with the structure of a DGCA on it:

$$C \underset{S}{\overset{\Omega^\bullet(-)}{\rightleftarrows}} \mathrm{DGCA} \ .$$

Definition 12 (L_∞-algebra valued forms) *Given an L_∞-algebra $\mathfrak{g}$ and a smooth space Y, we address*

$$\Omega^\bullet(Y, \mathfrak{g}) := C(Y, S(\mathrm{W}(\mathfrak{g}))) \in C$$

as the space of $\mathfrak{g}$-valued forms, and

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) := C(Y, S(\mathrm{CE}(\mathfrak{g}))) \in C$$

as the space of flat $\mathfrak{g}$-valued forms.

This definition relates to the definition of G-valued differential forms when integrating L_∞-algebras to ω-groupoids. As noticed in [16] (see also [17] and [24]) this integration procedure is essentially nothing but the old construction in rational homotopy theory [31]: the ∞-group in question is that of k-paths/singular k-simplices in the space $S(\mathrm{CE}(\mathfrak{g}))$. Here we adopt this idea to the context of ω-categories internal to C:

Definition 13 (Integration of L_∞-algebras) *For $\mathfrak{g}$ any L_∞-algebra, we define the ω-groupoid $\mathbf{B}G$ integrating it, as well as ω-groupoids denoted $\mathbf{BE}G$ and $\mathbf{BB}G$ as the image under $\Pi \circ S$ of the sequence 1:*

$$\begin{array}{cccc} & \mathbf{B}G \longrightarrow \mathbf{BE}G \longrightarrow \mathbf{BB}G & \\ := \Pi_\omega \circ S(& \mathrm{CE}(\mathfrak{g}) \longleftarrow \mathrm{W}(\mathfrak{g}) \longleftarrow \mathrm{inv}(\mathfrak{g}) &) \end{array}$$

I believe that it should be true that morphisms of path ω-groupoids all come from push-forward along maps of the underlying spaces:

$$\omega\mathrm{Cat}(\Pi(X), \Pi(Y)) \simeq C(X, Y)\,.$$

If true, this would imply that for ω-groups G obtained from integration of L_∞-algebras $\mathfrak{g}$ by integration as above we have

$$\Omega^\bullet(Y, \mathbf{BG}) = \Omega^\bullet(Y, \mathfrak{g})\,.$$

5.2 Non-flat differential cocycles

Given an ω-group G and setting $\Pi := \Pi_\omega$ the ω-path groupoid, the differential cohomology $H^\Pi(-, \mathbf{B}G)$ classifies *flat* G-bundles with connection. It turns out that cocycles for non-flat G-bundles with connection are instead objects in $H^\Pi(-, \mathbf{BE}G)$. Here $\mathbf{BE}G := \mathrm{INN}_0(G) = \mathrm{Cone}(\mathbf{B}G \stackrel{\mathrm{Id}}{\to} \mathbf{B}G)$ as in [22]. These $\mathbf{BE}G$-cocycles are the curvature $(n+1)$-bundles of given n-bundles. But not every cocycle in there corresponds to such a curvature $(n+1)$-bundle. We need to identify a sub-ω-category

$$\bar{H}(-, \mathbf{B}G) \subset H^\Pi(-, \mathbf{BE}G)$$

exhibiting the non-flat differential cohomology with values in G.

This can be done if we have an ω-functor

$$p:\ \mathbf{BE}G \longrightarrow \mathbf{BB}G\ .$$

Definition 14 (Non-flat differential cocycles and characteristic forms)

We define $\bar{H}(-,\mathbf{B}G) \in \omega\mathrm{Cat}$ *as the pullback*

$$\begin{tikzcd}
 & \bar{H}(-,\mathbf{B}G) \arrow[dl] \arrow[d] \arrow[dr] & \\
H(-,\mathbf{B}G) \arrow[d] & H^{\Pi}(-,\mathbf{BE}G) \arrow[dl] \arrow[d] & \Omega^{\bullet}_{\mathrm{flat}}(-,\mathbf{BB}G) \arrow[dl] \\
H(-,\mathbf{BE}G) & H^{\Pi}(-,\mathbf{BB}G) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \text{G-bundle with connection and curvature} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{G-bundle} \arrow[d] & \text{G-valued connection and curvature} \arrow[dl] \arrow[d] & \text{characteristic forms} \arrow[dl] \\
{} & \text{characteristic classes} &
\end{tikzcd}$$

The various morphisms here are best understood in terms of the codescent objects $\Pi^Y(X)$ and $\Pi_0^Y(X)$:

the above pullback says that the cocycles in $\bar{H}(-,\mathbf{B}G)$ are the cocycles

$$\Pi^Y(X) \longrightarrow \mathbf{BE}G$$

in $H^{\Pi}(-,\mathbf{BE}G)$ which fit into a square

$$\begin{tikzcd}
\Pi_0^Y(X) \arrow[r] \arrow[d, hook, "i"] & \mathbf{B}G \arrow[d] & \text{G-cocycle} \\
\Pi^Y(X) \arrow[r] \arrow[d, "\pi"] & \mathbf{BE}G \arrow[d, hook] & \text{connection and curvature} \\
\Pi(X) \arrow[r] & \mathbf{BBG} & \text{characteristic forms}
\end{tikzcd} \quad . \tag{2}$$

The bottom morphism represents $\mathbf{BB}G$-valued forms. Precomposition of that with the lower left vertical arrow π is the map

$$\Omega^\bullet(-, \mathbf{BB}G) \to H^\Pi(-, \mathbf{BB}G)\,.$$

Postcomposition with the lower right vertical morphism is the map

$$H^\Pi(-, \mathbf{BE}G) \to H^\Pi(-, \mathbf{BB}G)\,.$$

Precomposition with the upper left vertical morphism i is the map

$$H^\Pi(-, \mathbf{BE}G) \to H(-, \mathbf{BE}G)\,.$$

Finally, postcomposition with the upper right vertical morphism is the map

$$H(-, \mathbf{B}G) \to H(-, \mathbf{BE}G)\,.$$

5.3 Line n-bundles with connection

The ω-groups obtained from the integration procedure definition 13 produces the analog of simply connected Lie groups. To get more general ω-groups one needs to quotient these.

The easiest example is $\mathbf{B}^n U(1)$, which is the ω-groupoid trivial everywhere except in degree n, where it has $U(1)$-worth of n-morphisms.

For all $n \in \mathbb{N}$ the sequence

$$\mathbf{B}^n U(1) \hookrightarrow \mathbf{BEB}^n U(1) \twoheadrightarrow \mathbf{BB^n} U(1)$$

of strict abelian ω-groups, which reads in terms of crossed modules for $n = 1$

$$(1 \to U(1)) \hookrightarrow (U(1) \to U(1)) \twoheadrightarrow (U(1) \to 1)$$

and analogously for higher n.

Our pullback diagram

$$\begin{tikzcd}
\bar{H}(-, \mathbf{B^n} U(1)) \arrow[r] \arrow[d] & \Omega^\bullet_{\mathrm{closed}}(-, b^{n+1}\mathfrak{u}(1)) \arrow[d] \\
H^\Pi(-, \mathbf{BEB}^{n-1} U(1)) \arrow[r] & H^\Pi(-, \mathbf{B}^{n+1} U(1))
\end{tikzcd}$$

says that a line n-bundle has a curvature $(n+1)$-form which can be regarded as the connection $(n+1)$-form on a flat trivial line $(n+1)$-bundle.

Notice that the commutativity of 2

$$
\begin{tikzcd}
\Pi^Y(X) \arrow[r] \arrow[d, two heads] & \mathbf{BEB}^{n-1}U(1) \arrow[d, two heads] \\
\Pi(X) \arrow[r] & \mathbf{BB}^n U(1)
\end{tikzcd}
$$

here says that the ω-functor

$$\Pi^Y(X) \to \mathbf{BEB}^{n-1}U(1)$$

assigns trivial values in the shifted copy of $\mathbf{B}^n U(1)$ to all k-morphisms in the kernel of $\Pi^Y(X) \twoheadrightarrow \Pi(X)$. But this says precisely that on vertical morphisms this $\mathbf{BE}G$-cocycle factors through a $\mathbf{B}G$-cocycle:

$$
\begin{tikzcd}
\Pi^Y_{\mathrm{vert}}(X) \arrow[r] \arrow[d, hook] & \mathbf{B}^n U(1) \arrow[d, hook] & G\text{-cocycle} \\
\Pi^Y(X) \arrow[r] \arrow[d, two heads] & \mathbf{BEB}^{n-1}U(1) \arrow[d, two heads] & \text{connection and curvature} \\
\Pi(X) \arrow[r] & \mathbf{BB}^n U(1) & \text{curvature}
\end{tikzcd}
$$

Theorem 7 *Our differential cohomology with values in $\mathbf{B}^n U(1)$ is isomorphic to the ordinary differential refinement $\bar{H}^{n+1}(-,\mathbb{Z})$ of integral singular cohomology:*

$$\bar{H}(-,\mathbf{B}^n U(1))_\sim = \bar{H}^{n+1}(-,\mathbb{Z})\,.$$

For $n = 1$ and $n = 2$ this is proven in [26, 28].

Notice that $\bar{H}^{n+1}(-,\mathbb{Z})$ is equivalently modelled by Deligne coholomolgy, Cheeger-Siomns differential characters and abelian $(n-1)$-gerbes with connection ("and curving"). See [18].

6 Equivariant cohomology

It is noteworthy that a cohomology theory in the present sense

$$H(-,\mathbf{A}) : C^{\mathrm{op}} \to \omega\mathrm{Cat}$$

is itself an ωCat-valued presheaf and hence does qualify itself as a coeffient object for cohomology.

Definition 15 (equivariant cohomology) *Let $A \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$ be a coefficient object and and let $E \in C^{\Delta^{\mathrm{op}}}$ be a simplical object in C, then E-equivariant cohomology with coefficients in $\mathbf{E}$ is*

$$\mathrm{Desc}(E, \mathbf{A})\,.$$

Equivariant bundles on a space X with an action by a group G are obtained by taking $E = \mathrm{Ner}(X//G)$, where $X//G$ denotes the action groupoid. Similarly for higher bundles.

Regarding line 2-bundles (abelian gerbes) not just as $\mathbf{B}U(1) = (U(1) \to 1)$-bundles, but more properly as $AUT(U(1)) = (U(1) \to \mathbb{Z}_2)$-bundles one finds that their equivariant structure includes the "Jandl structures" discussed in [25].

6.1 Structures on BG

Of particular interest is the simplicial space $\mathrm{Ner}(\mathbf{B}G)$ for G a Lie group. The Chern-Simons $\mathbf{B}^3U(1)$-bundle (2-gerbe) with connection on BG is given by an equivariant cocycle as in definition 14:

$$\begin{tikzcd}[column sep=tiny]
& & \bar{H}(\mathrm{Ner}(\mathbf{B}G), \mathbf{B}^2U(1)) \arrow[dll] \arrow[d] \arrow[drr] & & \\
\mathrm{Desc}(\mathrm{Ner}(\mathbf{B}G), \mathbf{B}^2U(1)) \arrow[dr] & & \mathrm{Desc}(\mathrm{Ner}(\mathbf{B}G), \Pi_3(-, \mathbf{BEB}U(1)) \arrow[dl] \arrow[dr] & & \mathrm{Desc}(\mathrm{Ner}(\mathbf{B}G), \Omega^\bullet_{\mathrm{flat}}(-, \mathbf{B}^3U(1)) \arrow[dl] \\
& \mathrm{Desc}(\mathrm{Ner}(\mathbf{B}G), \mathbf{BEB}U(1)) & & \mathrm{Desc}(\mathrm{Ner}(\mathbf{B}G), \mathbf{B}^3U(1)) &
\end{tikzcd}$$

On the far left we have the multiplicative gerbes on G [12]. On the far right the characteristic 4-form on BG. The middle item with its coefficients in $\mathbf{BEB}U(1)$ says that the connection on the multiplicative gerbe need not be equivariant on the nose, as discussed in [33].

7 Quantization of cocycles

Suppose C is the topos of sheaves on some site S. Given an ωCat-valued sheaf Bund on C it induces in particular a $0\mathrm{Cat} = \mathrm{Set}$-valued sheaf on S. This we can think of as the **classifiyng space for** the structures in Bund in that

$$C(X, \mathrm{Bund}) \simeq \mathrm{Bund}(X)\,.$$

Given such a morphism

$$X \xrightarrow{\nabla} \mathrm{Bund}(-)$$

and given any other object Σ, the **quantization** of ∇ over Σ is, it it exists, the pull-push of ∇ through the correspondence

$$
\begin{tikzcd}
 & \hom(\Sigma, X) \otimes \Sigma \arrow[dl, "\mathrm{ev}"'] \arrow[dr, "p_2"] & \\
X & & \Sigma
\end{tikzcd}
.
$$

$$
\nabla \xrightarrow{\int_{\hom(\Sigma,X)} \mathrm{ev}^*(-)} \int_{\hom(\Sigma,X)} \mathrm{ev}^*(\nabla)
$$

Here $\int_{\hom(\Sigma,X)}$ is supposed to denote the ω-functor adjoint to the pullback ω-functor along p_2, where we are making use of the $\omega\mathrm{Cat}$-enrichment of $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$ from 2.1.

$$
\begin{tikzcd}
\hom(\hom(\Sigma, X) \otimes \Sigma, \mathrm{Bund}(-)) \arrow[r, shift left, "\int_{\hom(\Sigma,X)}"] & \hom(\Sigma, \mathrm{Bund}(-)) \arrow[l, shift left, "p_2^*"]
\end{tikzcd}
$$

- ev^* followed by the Hom-adjunction is **transgression**.
- $\int_{\hom(\Sigma,X)} \mathrm{ev}^*(-)$ is **taking sections**.

Compare [14, 15].

7.1 Transgression

Given a differential form $\omega \in \Omega^\bullet(X)$ on a space X and another space Σ, the transgression of ω to the mapping space $C(\Sigma, X)$ is the image under

$$
\begin{tikzcd}
 & \Omega^\bullet(C(\Sigma, X)) \times \Sigma \arrow[dl, "\int_\Sigma"'] & \\
\Omega^\bullet(C(\Sigma, X)) & & \Omega^\bullet(X) \arrow[ul, "\mathrm{ev}^*"']
\end{tikzcd}
.
$$

In [27], following [4], it was shown that differential forms are equivalent to functors from paths

$$
2\mathrm{Cat}(\Pi_2(X), \mathbf{B}G) \simeq \Omega^\bullet(X, \mathfrak{g})
$$

and that under this equivalence transgression is nothing but the inner hom:

$$
\begin{tikzcd}
2\mathrm{Cat}(\Pi_2(X), \mathbf{B}G) \arrow[r, "\simeq"] \arrow[d, "2\mathrm{Cat}(\mathcal{P}_1(S^1), -)"] & \Omega^\bullet(X, \mathfrak{g}) \arrow[d, "\int_{S^1} \mathrm{ev}^*(-)"] \\
1\mathrm{Cat}(\Pi_2(LX), \Lambda\mathbf{B}G) \arrow[r, "\simeq"] & \Omega^\bullet(X, \Lambda\mathfrak{g})
\end{tikzcd}
.
$$

Hence for $P \in \mathrm{Trans}(i, \pi)$ a locally trivializable structure on X, we say that its transgression to $C(\Sigma, X)$ is

$$C(\Pi(\Sigma,)P)\,.$$

7.2 Sections

$$\begin{tikzcd}
 & V//G \arrow[r] \arrow[d] & T_{\mathrm{pt}}\mathrm{Set} \arrow[d] \\
Y^\bullet \arrow[r, "g"] \arrow[ur, dashed, "\sigma_u"] \arrow[urr, bend left, "\sigma_c"] \arrow[dr] & \mathbf{B}G \arrow[r, "\rho"] & \mathrm{Set} \\
 & \mathrm{pt} \arrow[ur] \arrow[u, Rightarrow, "\sigma_t"'] &
\end{tikzcd}$$

Figure 1: **Sections of cocycles**. Given a G-valued cocycle $g : Y^\bullet \to \mathbf{B}G$ and a representation $\rho : \mathbf{B}G \to \mathrm{Set}$ we are asking for the set of sections $\Gamma(\rho[g])$ of the corresponding ρ-associated G-bundle. Such a section is, equivalently, any one of the three morphisms carrying the symbol σ: The *t*ransformation σ_t from the terminal cocycle *into* our given cocycle is given in *c*omponents by the functor σ_c which in turn, by the *u*niversal property, is given by the morphism σ_u.

An action of an n-group $\mathbf{B}G$ on an $(n-1)$-category V is usually thought of as a morphism

$$\rho : \mathbf{B}G \to (n-1)\mathrm{Cat}$$

At least for $n = 1$ it is well understood that taking the weak colimit of this under the canonical embedding

$$j : (n-1)\mathrm{Cat} \hookrightarrow n\mathrm{Cat}$$

yields the **action n-groupoid**

$$V//G := \mathrm{colim}_{\mathbf{B}G}(j_*\rho)\,.$$

By the universal property of the colimit, this comes equipped with canonical morphisms

$$V \hookrightarrow V//G \xrightarrow{\tilde{\rho}} \mathbf{B}G\,,$$

where, at least for $n = 1$, the right functor is faithful.

Given such a situation, we obtain a morphism

$$H(-, V//G) \xrightarrow{\tilde{\rho}_*} H(-, \mathbf{B}G)$$

of cohomology theories. The fibers of this morphism over a given $\mathbf{B}G$-cocycle P are the **collections of sections** of the ρ-associated coycle corresponding to P.

$$\Gamma_\rho(P) := \tilde{\rho}_*^{-1}(P)\,.$$

For G an ordinary group and ρ an ordinary representation, this reproduces the ordinary notion of sections of associated G-bundles.

8 Technical remarks

8.1 The fundamental ω-path ω-groupoid

Fundamental ω-groupoids of homotopy classes of globular paths are considered in [8]. Our Π_ω is a slight modification of that, where only *thin*-homotopy is divided out. For $n = 2$ this is described in detail in [27]. The general definition is analogous.

I need to better understand if I am right with my expectation (see 5) that

$$\omega\mathrm{Cat}(\Pi(X), \Pi(Y)) \simeq C(X, Y)\,.$$

8.2 $\omega\mathrm{Cat}$-valued presheaves

While the discussion here follows [29] we have slightly modified it.

In [29] the morphisms $C^{\mathrm{op}} \to \mathrm{Cat}$ are allowed to be pseudo, i.e. to respect composition only weakly. This is familiar and necessary for examples such as on p. 23 of [29], where each space is sent to a category of bundles over it.

Here, however, we followed [26, 28] in that we perceive a bundle entirely in terms of its fiber-assigning functor. That makes pullback of bundles *strict*. Hence for us coefficient objects for cohomology are indeed 1-functors

$$C^{\mathrm{op}} \to \omega\mathrm{Cat}\,.$$

Our examples show that this is sufficient to capture all the desired nonabelian (differential) cohomology. While it excludes discussion of non-rectified n-stacks, it also shows that it is not necessary to consider these.

8.3 The main descent/local trivialization theorem

Theorem 6 in [29] corresponds to the main theorems in [26] and [28] which characterize global structures coming from descent as those admitting local trivialization.

It remains to be understood how this relates and how it generalizes to $\omega\mathrm{Cat}$. Notice that the definition of local trivialization in section 6 of [29] is essentially the one we gave, only that it makes explicit use of a "classifying space" for trivial structures (denoted T there), which so far we haven't seen the need to mention. This is closely related to the remarks in 8.2 and hence needs to be better understood.

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[34] 2-bundles (or "2-torsors") as such (meaning: as bundles $P \to X$ with 1-categorical fibers, as opposed to other models like gerbes (which are like sheaves of sections of proper 2-bundles) or bundle gerbes (which are actually cocycles in the present sense)) appear generally for instance

in the work of Glenn and Breen, also section 7 of [29], and have been highlighted in their concrete nature as higher fiber bundles more recently by Toby Bartels, *2-Bundles* [`arXiv:math/0410328`] and Igor Baković, *Biroupoid 2-Torsors* (PhD thesis), Christoph Wockel, *A global perspective to gerbes and their gauge stacks* [`arXiv:0803.3692`]. Notice that in [26, 28] the point is made that higher bundles are conveniently thought of not as fibrations $P \to X$ but as their fiber-assigning functors $X \to n\mathrm{Cat}$. In particular, this achieves a useful *rectification* of the n-stack of these bundles to a sheaf, a fact we are making use of above.