On
Σ-models
and
nonabelian differential cohomology

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Abstract

A "Σ-model" can be thought of as a quantum field theory (QFT) which is determined by pulling back n-bundles with connection (aka $(n-1)$-gerbes with connection, aka nonabelian differential cocycles) along all possible maps (the "fields") from a "parameter space" to the given base space.

If formulated suitably, such Σ-models include gauge theories such as notably (higher) Chern-Simons theory. If the resulting QFT is considered as an "extended" QFT, it should itself be a nonabelian differential cocycle on parameter space whose parallel transport along pieces of parameter space encodes the QFT propagation and correlators.

We are after a conception of **nonabelian differential cocycles and their quantization** which captures this.

Our main motivation is the quantization of differential Chern-Simons cocycles to extended Chern-Simons QFT and its boundary conformal QFT, reproducing the cocycle structure implicit in [23].

- Classical
  - We conceive nonabelian differential cohomology in terms of cohomology with coefficients in ω-category-valued presheaves [48] of *parallel transport* ω-functors from ω-paths to a given structure ω-group [6, 44, 45, 46], discuss curvature and characteristic forms.
  - We describe Lie ∞-algeraic Cartan-Ehresmann connections [41] and integrate these, following [27, 29], to nonabelian differential cocycles whenever certain connectedness and integrality conditions are met.
  - For each transgressive L_∞-algebra cocycle there are Chern-Simons Lie ∞-connections arising as obstructions to lifts of L_∞-connections through String-like extensions of L_∞-algebras. Integrating these to differential cocycles yields general Chern-Simons n-bundles with connection, reproducing in particular the known cocycles for Pontryagin classes [17, 18].
- Quantum
  - Our aim is to *quantize* such differential cocycles.
  - We observe that for simple cases such as finite group and finite 2-group Chern-Simons theory (the Dijkgraaf-Witten and the Yetter model) the usual path integral is a decategorified categorical colimit over the given transport functor.
  - We interpret the holographic relation between Chern-Simons theory and its boundary conformal field theory as essentially being the hom-adjunction in ωCat applied to a morphism between the two chiral copies of the Chern-Simons transport. This allows to conceive the Reshetikhin-Turaev description of the Chern-Simons 3d TFT together with the corresponding Frobenius-algebraic description of the boundary conformal QFT [23] in terms of $\mathbf{B}\mathcal{C}$-valued differential cocycles, for $\mathcal{C}$ the corresponding modular tensor category.

Contents

1 Prelude

Acknowledgements. I am indebted for discussion and collaboration to plenty of people, which I will thank here when this document has achieved a more finalized status. At the moment I just want to express my thanks to Bruce Bartlett for discussion of the document at this early stage.

1.1 Σ-models

Quantum field theories on a space Σ which are obtained from performing a "path integral" over a space of maps γ from Σ into a *target space* X are known as Σ*-models*. The path integral is usually against a certain measure known as the *kinetic action*. The integrand itself, called the non-kinetic or *gauge coupling* action, is usually the holonomy of a (higher) bundle with connection on X over γ. This bundle is called the *background field*.

$$
\begin{tikzcd}
\begin{matrix}\text{parameter}\\ \text{space}\end{matrix} \arrow[r, "\overset{\text{fields}}{\in \text{ config. space}}"] & \begin{matrix}\text{target}\\ \text{space}\end{matrix} \arrow[r, "\begin{matrix}\text{background}\\ \text{field}\end{matrix}"] & \text{phases} \\
\Sigma \arrow[r, bend left=40, "\gamma_1"] \arrow[r, bend left=20, "\gamma_2"] \arrow[r, "\gamma_3"] \arrow[r, bend right=40, "\vdots"] & X \arrow[r, "\nabla"] & T
\end{tikzcd}
$$

Table 1: **Σ-models**

Another class of quantum field theories are *gauge theories*: these are obtained from a "path" integral over a space of fiber bundles with connection on Σ. If the notion of "space" and of "maps between spaces" is chosen suitably, then gauge theories are in fact special cases of Σ-models (this was maybe first observed in [10]): the target space is the classifying space for the given kind of bundles. One finds that for (higher) Chern-Simons theories, the action functional is again the holonomy of a higher bundle with connection on these classifying spaces.

$$
\begin{tikzcd}
\begin{matrix}\text{parameter}\\ \text{space}\end{matrix} \arrow[r, "\begin{matrix}\text{bundles with}\\ \text{connection}\end{matrix}"] & \begin{matrix}\text{classifying}\\ \text{space}\end{matrix} \arrow[r, "\begin{matrix}\text{Chern-Simons functional}\\ \text{etc.}\end{matrix}"] & \text{phases} \\
\Sigma \arrow[r, bend left=40, "\gamma 1"] \arrow[r, bend left=20, "\gamma_2"] \arrow[r, "\gamma_3"] \arrow[r, bend right=40, "\vdots"] & \text{“}BG\text{”} \arrow[r, "\nabla"] & T
\end{tikzcd}
$$

Table 2: **Gauge theories**

The various concepts and notions of Σ-models which we shall be concerned with are summarized in table **??**

It is sensible to distinguish *fundamental* from *non-fundamental* physical systems. Among fundamental physical systems we count

- charged n-particles (charged $(n-1)$-branes):
  - the ordinary particle propagating in spacetime, coupled to a vector bundle with connection;
  - the string propagating in spacetime, coupled to the Kalb-Ramond 2-bundle with connection;

X	target space
Σ	parameter space
T	category of fibers / space of phase
$X \xrightarrow{\nabla} T$	background field / gauge n-bundle with connection
$\mathrm{hom}(\Sigma, X)$	space of fields / configuration space / moduli space
$\mathrm{hom}(\Sigma, X) \xrightarrow{\mathrm{hom}(\Sigma,-)(\nabla)} \mathrm{hom}(\Sigma, T)$	background field transgressed to configuration space
$\mathrm{hom}(X, T) \xrightarrow{\int_{\mathrm{hom}(\Sigma,X)} \mathrm{ev}^*(-)} \mathrm{hom}(\Sigma, T)$	path integral
$\Sigma \xrightarrow{\int_{\mathrm{hom}(\Sigma),X} \mathrm{ev}^*(\nabla)} T$	quantum propagator

Table 3: **Notions in Σ-models.** All objects and morphisms here live in suitable category of "spaces". We will find it useful, convenient and sufficient to model spaces in terms of fundamental path ω-groupoids $\Pi_\omega(X)$ internal to the topos of sheaves on Euclidean spaces. We shall, at times, suppress the notation distinction between the sheaf X and its fundamental ω-groupoid $\Pi_\omega(X)$. For instance, later on we depict our background field mostly as an ω-functor $\nabla : \Pi_\omega(X) \to \Pi_\omega(S(\mathrm{CE}(\mathfrak{g})))$.

– the membrane propagating in spacetime, coupled to a Chern-Simons 3-bundle with connection;

- (higher) gauge theories:
  – Chern-Simons theories.

In other words: the fundamental physical systems of interest are Σ-models whose action functional is the holonomy of a higher bundle with connection on target space. Our aim is to give a systematic formalization of this concept.

And we notice that at the heart of this concept is the notion which is known equivalently as either of the following:

- higher gerbes with connection;
- higher bundles with connection;
- higher differential cocycles

all of which possibly non-abelian.

$$\begin{array}{ccc} \text{differential cocycle} & \xleftarrow{\text{quantization}} & \text{differential cocycle} \\ \text{on parameter space} & & \text{on target space} \end{array}$$

$$(\Sigma \xrightarrow{\int_{[\Sigma,X]} ev^*\nabla} T) \xleftarrow{?} (X \xrightarrow{\nabla} T)$$

Table 4: **Quantization of differential cocycles**: low-dimensional examples indicate that quantization of Σ-models is a procedure which sends differential cocycles on target space to differential cocycles on parameter space. We are after a conception of nonabelian differential cocycles and of the integration procedure indicated symbolically on the left which formalizes these examples.

Moreover, we shall now make an observation which suggests that quantization of Σ-models should be a natural operation which sends differential cocycles on target space to differential cocycles on parameter space. We mention two examples, the first rather trivial and familiar, the second rather non-trivial and in fact a main motivation for our discussion.

1.2 Examples for quantization sending differential cocycles to differential cocycles

fundamental object	**background field**
$(n-1)$-brane	$(n-1)$-gerbe
n-particle	n-bundle

Table 5: **The two schools of counting** higher dimensional structures. Here n is in $\mathbb{N} = \{0, 1, 2, \cdots\}$.

1.2.1 The charged particle

It is a familiar fact that the mathematical structure modelling the physics of a classical charged particle propagating on a space X is given by a fiber bundle with connection (P, ∇) on X.

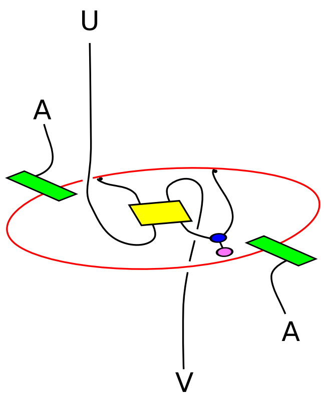
Even though it is not usually put this way, we may notice that the quantization of this setup yields another fiber bundle with connection: namely a bundle on the real line whose typical fiber is the space of sections $\Gamma(P)$ and whose connection is the Hamiltonian ∇^2.

Following Freed, we observe that the quantization arises entirely from forming certain *sums*:

- The connection ∇^2 comes from the path integral.
- The space of states $\Gamma(X)$ is the categorical sum (the coproduct) over the fiber-assigning functor $x \mapsto P_x$.

1.2.2 The charged 2-particle

The best description of the dynamics of the string, known as 2-dimensional conformal field theory, that exists is the description [23] of *rational* 2d CFT in terms of triangulations of worldsheets Σ colored by Frobenius algebra objects in modular tensor categories $\mathcal{C}$.



- the black line in the center plane is a transition line labelled by a Forbenius algebra A;
- U and V are the two chiral labels of a bulk field insertion;
- the yellow coupon is a homomorphism of induced A-A bimodules;
- the red line, encoding the boundary condition at the boundary of the disk (the "D-brane") is labelled by an A-module.

Figure 1: **The 2d CFT disk correlator** as a string diagram in $\mathcal{C}$ as described in [23].

This captures in particular the Σ-model known as Wess-Zumino-Witten theory, in which a string propagates on a compact Lie group G and is charged under a line 2-bundle with connection ∇ – a nonabelian differential 2-cocycle ∇ – on that Lie group.

Using the theory of higher descent described in **??** we can regard this decoration prescription as being the local data of the surface holonomy of a $\mathbf{B}\mathcal{C}$ descent object on Σ which arises from local trivialization with respect to the inclusions

$$\mathbf{B}\mathcal{C} \hookrightarrow \mathrm{Bimod}(\mathcal{C}) \longrightarrow 2\mathrm{Vect}(\mathcal{C}) \ .$$

Therefore the FFRS formulation of rational 2d CFT may be regarded as realizing the 2-dimensional quantum field theory as a nonabelian differential cocycle.

We want to understand how this 2-cocycle systematically arises from the background field ∇ encoding a 2-bundle with connection such that we may write it in terms of an ω-categorical "path integral" $\int_{\mathrm{hom}(\Sigma, G)} \mathrm{ev}^* \nabla$.

1.3 The program: formalization of Σ-models

In order to formalize the above examples and Σ-models in general, we want to find

- a suitable notion of space and morphisms between spaces (part 2);

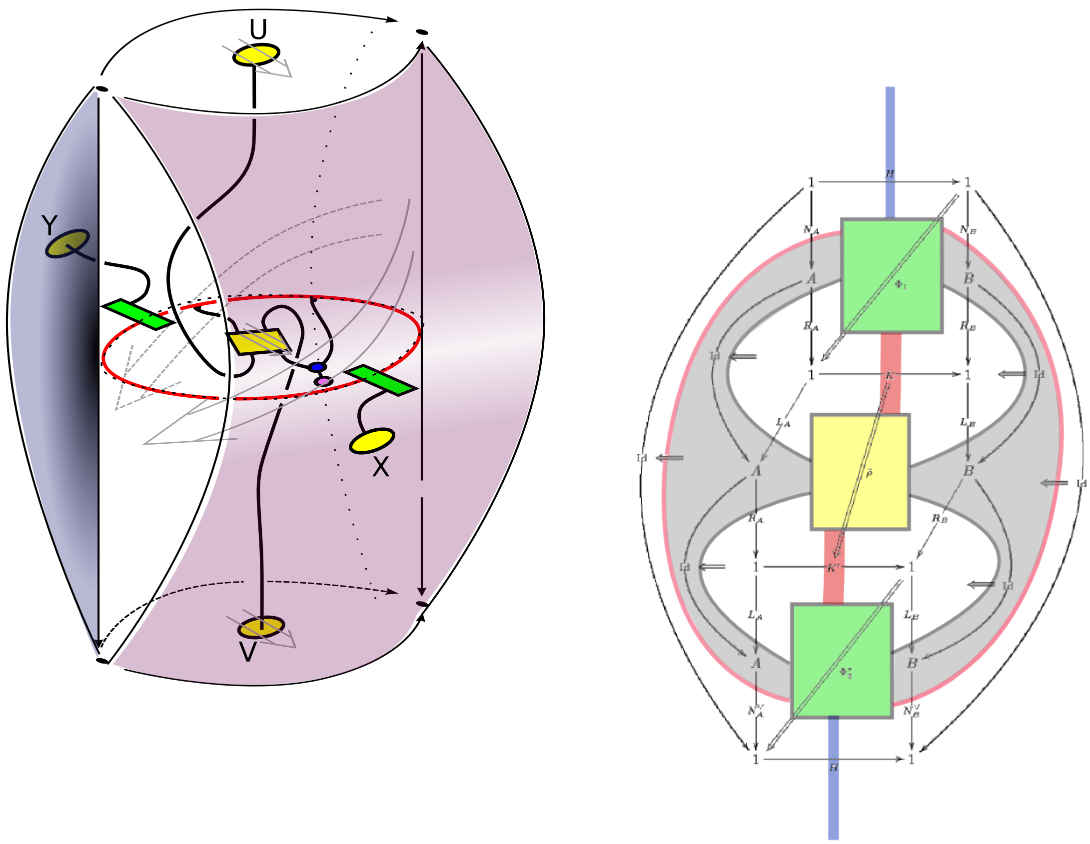


Figure 2: **The disk correlator as the holonomy of a differential 2-cocycle**. On the left, schematically as a pasting diagram of cylinders in $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$ which we may interpret as a pasting diagram of 2-morphism in $\mathrm{TwBimod}(\mathcal{C})$ by projecting the cylinders onto their equatorial plane. On the right as the locally $(\mathbf{B}\mathcal{C} \overset{i}{\hookrightarrow} \mathrm{TwBimod}(\mathcal{C}))$-trivialized holonomy of a $\mathrm{TwBim}(\mathcal{C})$-valued 2-functor. The thin black lines indicate the 2-morphisms in $\mathrm{TwBimod}(\mathcal{C})$. The coloring indicates the Poincaré-dual string diagram in $\mathcal{C}$, which reproduces the string diagram shown in figure 1.

- a suitable notion of nonabelian differential cocycles (part 4) encoded by morphisms of spaces

$$X \xrightarrow{\nabla} T \;;$$

- a suitable way to obtain the bottom morphism in the diagram

$$\begin{tikzcd}[column sep=small]
 & \mathrm{hom}(\Sigma, X) \times \Sigma \arrow[dl, "p_2"'] \arrow[dr, "\mathrm{ev}"] & & & & & \mathrm{hom}(\Sigma, X) \arrow[dl] \arrow[dr, "\mathrm{Id}"] & & \\
\Sigma \arrow[rrr, dashed, bend right, "\int_{\mathrm{hom}(\Sigma,X)} \mathrm{ev}^*\nabla"'] & & X \arrow[r, "\nabla"] & T & \leftrightsquigarrow & \mathrm{pt} \arrow[rrr, dashed, bend right, "\int_{\mathrm{hom}(\Sigma,X)} \mathrm{hom}(\Sigma,\nabla)"'] & & \mathrm{hom}(\Sigma, X) \arrow[r, "\mathrm{hom}(\Sigma,\nabla)"] & \mathrm{hom}(\Sigma, T)
\end{tikzcd} \quad , \qquad (1)$$

where the two versions are related under the *hom*-adjunction.

We will show that $\hom(\Sigma, \nabla)$ is the differential cocycle on the "space of paths" to be addressed as the *transgression* of ∇ to $\hom(\Sigma, X)$.

We shall also argue that, when all morphisms are modeled as ω-functors between ω-categories and a major subtlety about the correct definition of ∇ is taken into account, described in **??**, the dashed morphism $\int\limits_{\hom(\Sigma,X)} \hom(\Sigma, \nabla)$ should be nothing but the *categorical integral* going by precisely the same symbol: the *coend* or *colimit* of the functor $\hom(\Sigma, \nabla)$ over the domain $\hom(\Sigma, X)$.

This is motivated by the following two examples.

1.4 Quantization and categorical coends

The best understood Σ-models that we are interested in are Chern-Simons theory for finite groups and finite 2-groups. These are known as the Dijkgraaf-Witten and the Yetter model, respectively.

We review these theories briefly and describe how the data provided by them fits into the pattern of diagram 1.

There is one additional piece of data which people specify when considering these theories: a measure $d\mu$ on the configuration space of fields $\hom(\Sigma, X)$ against which the path integral (which is just a finite sum in these simple cases) is being evaluated. These measures have been found "by hand" by requiring invariance of the resulting sum under certain choices, and compatibility of this sum with gluing of parameter spaces.

We observe that these measures actually are the canonical measures on the given configuration space groupoids when regarded as the decategorification of a categorical *coend* or *colimit* [33].

This suggests that the dashed morphism in diagram 1 indeed arises from categorical coends, *if* the background field ∇ is increased in categorical dimension by one compared to its naive dimension.

Such a **shift in dimension** is also what we shall find in **??** to be necessary for the description of nonabelian differential cocycles.

1.4.1 The Leinster measure on n-categories

We are used to integrating functions with values in numbers. But natural numbers are *really* cardinalities of finite sets (0-categories). Moreover, rational number are cardinalities of finite groupoids [2] and in fact of finite categories in general.

Leinster measure on 1-categories. If A is a finite category, a functor

$$F : A \to \mathrm{Set}$$

may therefore be regarded as a categorified function, whose value at $a \in A$ is the cardinality $|F(a)|$ of the set $F(a)$.

Any function on $\mathrm{Obj}(A)$ is a sum of delta-functions which take the value 0 everywhere except at one point. If A is the discrete category over its finite set of objects (i.e. if it has no nontrivial morphisms) then the delta-functions correspond precisely to the *representable* functors on A.

Let us assume, therefore, that, generally, the functors F which we consider are sums of representable functors. Then [35] relates the colimit over these functors to an integral, a weighted sum, of the corresponding function over $\mathrm{Obj}(A)$ with respect to a certain measure, or weighting, which is given entirely by the morphism structure of the category A:

a function

$$d\mu : \mathrm{Obj}(A) \to \mathbb{Q}$$

is called a *weighting* for A if for all $a \in \mathrm{Obj}(A)$ we have

$$\int_A |A(a,-)| \, d\mu := \sum_{b \in \mathrm{Obj}(A)} |A(a,b)| \, d\mu(b) = 1 \tag{2}$$

For instance, if A is a connected finite groupoid there is $n \in \mathbb{N}$ such that there are n morphisms emanating from each object, and the constant function

$$d\mu : a \mapsto \frac{1}{n}$$

is a weighting on A.

The cardiality of the colimit over the sum of representables F is (proposition 3.1 of [35])

$$\left| \int^{a \in A} F(a) \right| = \left| \lim_{\to} F \right| = \int_A |F(-)| \, d\mu = \sum_{a \in \mathrm{Obj}(a)} |F(a)| \, d\mu(a) \,. \tag{3}$$

The integral sign on the very left denotes the categorical coend. For our ordinary functor F this is trivially equal to the colimit, but we include it to emphasize the remarkable notational coherence obtained this way:

the cardinality of the catorical integral over A is the ordinary integral of the cardinalities over $\mathrm{Obj}(A)$ *with respect to a certain measure $d\mu$.*

For example if A is a finite groupoid and F a sum of representables, then

$$\left| \int^{a \in A} F(a) \right| = \sum_{a \in A} |F(a)| \, \frac{1}{\oplus_b |A(a,b)|} = \sum_{[a] \in A_\sim} |F(a)| \, \frac{1}{|\mathrm{Aut}(a)|} \,,$$

where the last sum is over isomorphism classes of objects in A.

What is remarkable about this from the point of view of applications to quantum theory is the appearance of the singled-out measure $d\mu$: much of the subtlety of quantum theory is due to the fact that for each quantum system one usually knows the configuration space and a function on it which one wishes to integrate, but one has problems with finding the proper measure to use in this integral. Below we will show that for simple cases of quantum field theories the measures which were found "by hand" actually are nothing but the Leinster measures $d\mu$, as above, on the given configuration groupoids.

Crucial use of these Leinster mesaures in a context closely related to quantum theory has also been made in [55].

One can use the Leinster measure also to assign a "size", a cardinality to a finite category A. Given a weighting $d\mu$ on A the cardinality or *Euher characteristic* of A (definition 2.2 of [35]) is

$$|A| := \int_A d\mu \,.$$

This is well defined if a weighting exists also on A^{op}, which is in particular the case for all groupoids.

For instance a groupoid A with n_0 objects and n_1 morphisms emanating from each object has weighting $d\mu : a \mapsto \frac{1}{n_1}$ and cardinality

$$|A| = (n_0)^1 (n_1)^{-1} \,. \tag{4}$$

Leinster measure on 2-categories. The defining condition 2 on a weighting on a catgeory A manifestly makes sense for $\mathcal{V}$-enriched categories when there is a cardinality operation on the objects of $\mathcal{V}$:

$$|\cdot| : (\mathcal{V}, \oplus) \to (\mathbb{Q}, +) \,.$$

In particular, this allows to define cardinalities for strict n-categories inductively by regarding these as $(\mathcal{V} = (n-1)\mathrm{Cat})$-enriched categories.

Consider, for example, a finite strict 2-groupoid A which we can assume to be connected (otherwise we apply the following argument to each connected component) with

- n_0 objects
- n_1 1-morphisms starting at each object

- n_2 2-morphisms starting at each 1-morphism.

There are then n_1/n_0 1-morphisms between any two objects a, b and therefore the Hom-groupoids $A(a,b)$ have n_1/n_0 objects and n_2 morphisms emanating from each of these. Therefore by equation 4 the cardinality of each Hom-groupoid $A(a,b)$ is

$$|A(a,b)| = (n_0)^{-1}(n_1)^1(n_2)^{-1}\,.$$

A weighting on A in this situation is given by the constant function

$$d\mu : a \mapsto (n_1)^{-1}(n_2)^1\,. \tag{5}$$

This formula is shown in 1.4.3 to reproduce the path integral measure for the finite 2-group Chern-Simons theory called the Yetter model.

The cardinality of this 2-groupoid A is

$$|A| := \int_{a\in \mathrm{Obj}(A)} d\mu = (n_0)^1(n_1)^{-1}(n_2)^1\,.$$

1.4.2 Finite group Chern-Simons: Dijkgraaf-Witten

A useful comprehensive account of Dijkgraaf-Witten quantum field theory, originally due to [21], is in [26]. A nice review is in [9]. We slightly reformulate it in more category-theoretic terms as follows.

Pick a finite group G and write

$$\mathbf{B}G := \left\{ \bullet \xrightarrow{g} \bullet \,|g \in G\right\}$$

for the corresponding one-object groupoid. This category is the *target space* for Dijkgraaf-Witten theory.

The (naive, see below) *background field* is a 3-bundle with connection on this target space given by a pseudofunctor

$$\mathbf{B}G \xrightarrow{\nabla} \mathbf{B}^3U(1)\,,$$

which is precisely a $U(1)$-valued group 3-cocycle on G.

Given a manifold Σ, let $\Pi_1(\Sigma)$ be its fundamental groupoid. Dijkgraaf-Witten theory is the Σ-model encoded by the diagram

$$\begin{tikzcd}
 & \hom(\Pi_1(\Sigma),\mathbf{B}G)\times\Pi_1(\Sigma) \arrow[dl, "p_2"'] \arrow[dr, "\mathrm{ev}"] & \\
\Pi_1(\Sigma) & & \mathbf{B}G \arrow[r, "\nabla"] & \mathbf{B}^3U(1)
\end{tikzcd} \quad .$$

Here the *configuration space* or *space of fields* is the groupoid $\hom(\Pi_1(\Sigma),\mathbf{B}G)$.

To make this technically more easily tractable without losing any information, we may choose a point in each connected component of Σ and let $\Pi_1(\Sigma)$ be instead the full subgroupoid of the fundamental groupoid on these chosen points. That makes the parameter space groupoid $\Pi_1(\Sigma)$ and hence also the *configuration space* groupoid $\hom(\Pi_1(\Sigma),\mathbf{B}G)$ a finite groupoid.

Consider the *transgression* of the background field ∇ to configuration space, which diagram 1 claims is just its image under $\hom(\Pi_1(\Sigma),-)$:

$$\hom(\Pi_1(\Sigma),\mathbf{B}G) \xrightarrow{\hom(\Pi_1(\Sigma),\nabla)} \hom(\Pi_1(\Sigma),\mathbf{B}^3U(1)) \ \ .$$

Let's assume for simplicity that parameter space is the 3-sphere, $\Sigma = S^3$. Then as we pass to equivalence classes we get

$$\hom(\Pi_1(\Sigma),\mathbf{B}^3U(1))_\sim \simeq U(1)\,.$$

This makes the image of the transgressed background field under taking equivalence classes into a $U(1)$-valued functions on equivalence classes of field configurations. This function we address as the *holonomy* of the background field around Σ, or as the *integrated action* on Σ and write it as

$$\mathrm{hol}_\Sigma(\nabla) = \exp(2\pi i S_\Sigma(\nabla)) : \mathrm{hom}(\Pi_1(\Sigma), \mathbf{B}^3 U(1))_\sim \to U(1)\,.$$

According to the standard prescription [26] (see [9] for a nice review) the path integral in Dijkgraaf-Witten theory is supposed to be the sum

$$Z_\nabla(\Sigma) := \int\limits_{[P]\in \mathrm{hom}(\Pi_1(\Sigma),\mathbf{B}G)_\sim} \mathrm{hol}_\Sigma(P^*\nabla)\, d\mu_{[P]}$$

- over all isomorphism classes $[P]$ in $\mathrm{hom}(\Pi_1(\Sigma), \mathbf{B}G)$;
- of the "holonomy" of $\Pi_1(\Sigma) \xrightarrow{P} \mathbf{B}G \xrightarrow{\nabla} \mathbf{B^3}U(1)$ regarded as an element
$$\exp(2\pi i S_\Sigma(P)) := \mathrm{hol}_\Sigma(P^*\nabla) \in U(1)\,;$$
- against the measure
$$d\mu : [P] \mapsto \frac{1}{|\mathrm{Aut}(P)|}\,.$$

In the usual discussion, this measure $d\mu_{[P]}$ is introduced "by hand", justifying itself by the fact that it induces invariance of the resulting sum under the choices made and compatibility of the sum under gluing of parameter spaces.

We may notice, however, that this measure happens to have a deep categorical meaning: it is the *Leinster measure* [35] on $\mathrm{hom}(\Pi_1(\Sigma), \mathbf{B}G)$. The path integral of Dijkgraaf-Witten theory for trivial background field ∇

$$Z(\Sigma) := \int\limits_{[P]\in \mathrm{hom}(\Pi_1(\Sigma),\mathbf{B}G)_\sim} d\mu_{[P]}$$

is the Baez-Dolan *groupoid cardinality* [2] or equivalently Leinster's *Euler characteristic* of the configuration space groupoid $\mathrm{hom}(\Pi_1(\Sigma), \mathbf{B}G)$.

	$\left\lvert \int^{P\in \mathrm{hom}(\Pi_1(\Sigma),\mathbf{B}G)} F(P) \right\rvert$	cardinality of categorical coend of F over $\mathrm{hom}(\Pi_1(\Sigma), \mathbf{B}G)$
$=$	$\left\lvert \lim_{\to} F \right\rvert$	cardinality of categorical colimit of F over $\mathrm{hom}(\Pi_1(\Sigma), \mathbf{B}G)$
$=$	$\sum_{[P]\in \mathrm{hom}(\Pi_1(\Sigma),\mathbf{B}G)_\sim} \lvert F(P)\rvert \frac{1}{\lvert\mathrm{Aut}(P)\rvert}$	avarage with respect to the Leinster measure of the observable $\lvert F(-)\rvert : \mathrm{hom}(\Pi_1(\Sigma), \mathbf{B}G) \to \mathbb{R}$
$=$	$\int_{[P]\in \mathrm{hom}(\Pi_1(\Sigma),\mathbf{B}G)_\sim} \lvert F(P)\rvert\, d\mu_{[P]}$	Dijkgraaf-Witten expectation value of the observable $\lvert F(-)\rvert : \mathrm{hom}(\Pi_1(\Sigma), \mathbf{B}G) \to \mathbb{R}$

It may be worthwhile to observe that the canonical categorical Leinster measure achieves precisely what in more sophisticated quantum field theories the *BRST-BV-formalism* [28] is supposed to achieve:

- restrict the path integral to gauge orbits in configuration space;
- divide out the automorphisms of every given configuration.

1.4.3 Finite 2-group Chern-Simons: Yetter model

As discussed in [41], Chern-Simons theories can be considered for any transgressive cocycle on any Lie n-algebra. For $n = 1$ and a semisimple Lie algebra $\mathfrak{g}$, the ordinary Chern-Simons theory is that coming from the canonical 3-cycles on $\mathfrak{g}$. For $n = 2$ and $\mathfrak{g}$ a strict Lie 2-algebra coming from a differential crossed module, one obtains a higher version of Chern-Simons theory.

Again there is accordingly a finite group version of this, where target space is taken to come from a strict 2-group [4]

$$G_{(2)} = (H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H))$$

coming from some crossed module of finite groups, indicated on the right.

The one-object 2-groupoid $\mathbf{B}G_{(2)}$ G as its space of 1-morphisms and H as its space of 2-morphisms starting at the identity 1-morphism:

$$\mathbf{B}G_{(2)} = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g' := t(h)g"'] \arrow[r, phantom, "\Downarrow (g,h)"] & \bullet \end{tikzcd} \middle| \; g \in G, h \in H \right\}.$$

Horizontal composition is the product in the semidirect product group $G \ltimes_\alpha H$, vertical composition is just the product in H. The two conditions on α and t in a crossed module are precisely equivalent to this composition law yielding a 2-groupoid.

This QFT with target $\mathbf{B}G_{(2)}$ is known as the *Yetter model* [53, 54]. Nontrivial background fields on $\mathbf{B}G_2$ were first considered in [36], which also provides the proof that the path integral measure to be dsicussed below has the right properties.

For Σ some manifold we take now $\Pi_2(\Sigma)$ to be a strict 2-groupoid obtained from picking any triangulation of Σ and letting $\Pi_2(\Sigma)$ be generated from the v_0 many vertices, the v_1 many edges and the v_2 many faces, modulo the requirement that all tetrahedra which can be formed 2-commute.

Then the Σ-model diagram for the Yetter model is

$$\begin{tikzcd} & \hom(\Pi_2(\Sigma), \mathbf{B}G_{(2)}) \times \Pi_2(\Sigma) \arrow[dl, "p_2"'] \arrow[dr, "\mathrm{ev}"] & & \\ \Pi_2(\Sigma) & & \mathbf{B}G_{(2)} \arrow[r, "\nabla"] & \mathbf{B}^n U(1) \end{tikzcd} .$$

We can form again the transgressed background field

$$\hom(\Pi_2(\Sigma), \mathbf{B}G_{(2)}) \xrightarrow{\hom(\Pi_2(\Sigma),\nabla)} \hom(\Pi_2(\Sigma), \mathbf{B}^n U(1))$$

and ask for its "path integration".

To understand the configuration space 2-groupoid

$$\hom(\Pi_2(\Sigma), \mathbf{B}G_{(2)})$$

notice that morphisms η

$$\begin{tikzcd} \Pi_2(\Sigma) \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathbf{B}G_{(2)} \end{tikzcd}$$

between two "fields", namely two $\mathbf{B}G_{(2)}$-valued functors, being a pseudonatural transformation, are in bijection to $G^{v_0} \times H^{v_1}$ since their component map is a 1-functor

$$\eta : \mathrm{Mor}_1(\Pi_2(X)) \to (\mathbf{B}G_{(2)})^I$$

given by

$$\eta : \left(x \xrightarrow{\gamma} y \right) \mapsto \begin{tikzcd} \bullet \arrow[r, "\phi(\gamma)"] \arrow[d, "\eta(x)"'] & \bullet \arrow[d, "\eta(y)"] \arrow[dl, Rightarrow, "\eta(\gamma)"] \\ \bullet \arrow[r, "\phi'(\gamma)"'] & \bullet \end{tikzcd}$$

and that the label of the bottom morphism on the right is fixed, by the rules for strict 2-groups which we mentioned above, when the rest of the labels are given. The pseudo-naturality condition for η then fixes F' in terms of F and the component map of η.

Analogously, one sees that morphisms between morphisms of fields (gauge of gauge transformations)

$$\eta \xRightarrow{\lambda} \eta'$$

are in bijection to H^{v_0}: they are given by modifications of pseudonatural transformations whose component map

$$\lambda : \mathrm{Obj}(\Pi_2(\Sigma)) \to (\mathbf{B}G_{(2)})^D$$

is fixed by an element in H:

$$\lambda : x \mapsto \begin{tikzcd} \bullet \arrow[d, bend right=60, "\eta'(x)"'] \arrow[d, bend left=60, "\eta(x)"] \arrow[d, phantom, "\Leftarrow\lambda(x)"] \\ \bullet \end{tikzcd}$$

and again the target η' here is fixed once η and the component map of λ are given.

Therefore the 2-groupoid of fields $\hom(\Pi_2(\Sigma), \mathbf{B}G_{(2)})$ has

- $|G|^{v_0}|H|^{v_1}$ 1-morphisms emanating at each object;
- $|H|^{v_0}$ 2-morphisms emanating at each 1-morphism.

By formula 5 the categorical measure on the configuration 2-groupoid is therefore the constant function

$$d\mu : P \mapsto |G|^{-v_0}|H|^{v_0 - v_1} \, .$$

And indeed, this is the measure used in the literature for the Yetter model, which has been introduced and justified as the measure which makes the path integral independent of the chosen triangulation [36].

1.5 Higher Chern-Simons differential cocycles

The Dijkgraaf-Witten and Yetter model discussed above are finite group versions of the two first of an infinite class of Σ-models: higher Chern-Simons theory.

Just as a Lie group has a Lie algebra, higher Lie groupd have Lie ∞-algebras, known as L_∞-algebras. As in ordinary Lie theory, we can define cocycles, invariant polynomials and transgression elements for any L_∞-algebra.

We find [41]:

- For each Lie ∞-algebra cocycle μ of degree $(n+1)$ which is in transgression with an invariant polynomial P on the L_∞-algebra $\mathfrak{g}$ we obtain an extension $\mathfrak{g}_\mu$ of $\mathfrak{g}$ by $b^{n-1}\mathfrak{u}(1)$

  $$0 \longrightarrow b^{n-1}\mathfrak{u}(1) \longrightarrow \mathfrak{g}_\mu \longrightarrow \mathfrak{g} \longrightarrow 0$$

  called a *string-like extension*.

- For every L_∞-algebra $\mathfrak{g}$ there is a notion of L_∞-algebra valued connection generalizing the notion of Cartan-Ehresmann connections.
- For every string-like extension as above there is a Chern-Simons L_∞-algebra $\mathrm{cs}_P(\mathfrak{g})$ with the special property that differential forms with values in it come from a $\mathfrak{g}$-connection and the corresponding Chern-Simons form.
- The obstruction to lifting a $\mathfrak{g}$-connection through a string-like extension to $\mathfrak{g}_\mu$ is a $b^n\mathfrak{u}(1)$ connection, called the Chern-Simons connection, whose characteristic class is that corresponding to the invariant polynomial P of the original $\mathfrak{g}$-connection.

If certain connectedness and intragrality conditions are met, L_∞-connections may be intgerated to non-abelian differential cocycles.

Locally this is an ω-functor from ω-paths to some structure ω-group

$$\nabla_{\mathrm{loc}} : \Pi_\omega(Y) \to \mathbf{B}G$$

if it is flat, or to

$$\nabla_{\mathrm{loc}} : \Pi_\omega(Y) \to \mathbf{BE}G$$

if it is non-flat such that certain gluing conditions are satisfied which make this functor *descent* to a globally defined parallel transport functor

$$\nabla : \Pi_\omega(X) \to T\,.$$

For G an ω-group which admits a sequence $\mathbf{B}G \to \mathbf{BE}G \to \mathbf{BB}G$, which we will discuss, we write

- $H(X, \mathbf{B}G)$ for the ω-category of G-bundles on X;
- $H^\Pi(X, \mathbf{B}G)$ for the ω-category of flat G-bundles with connection on X;
- $\Omega^\bullet(X, \mathbf{BB}G)$ for the ω-category of G-characteristic forms;
- $\bar{H}(X, \mathbf{B}G)$ for the ω-category of non-flat G-bundles with connection.

2 Space and quantity

We want to talk about *differential* cohomology and therefore need a good notion of *smooth spaces*, general enough to admit smooth differential forms and at the same have nice structural properties as in particular the existence of inner homs, which guarantees that the space of maps between two smooth spaces is again a smooth space. We also want smooth classifying spaces for smooth higher bundles with connection.

We find it convenient and useful to consider smooth spaces to be sheaves on the site S whose

- objects are the integers $\mathrm{Obj}(S) = \mathbb{N}$;
- morphisms are smooth maps between Euclidean spaces

$$S(n, m) = \mathrm{Hom}_{\text{smooth manifolds}}(\mathbb{R}^n, \mathbb{R}^m)\,.$$

Let that category of sheaves be denoted by C. This is a topos.

All of the following directly generalizes to the *super*case if this category of Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$ is replaced by that of super Euclidean spaces $\mathbb{R}^{n|m}$, $n, m \in \mathbb{N}$.

As Lawvere teaches [34], once a site S is fixed, the notions of *space and quantity* and the duality between them is captured by the adjunction called Isbell conjugation

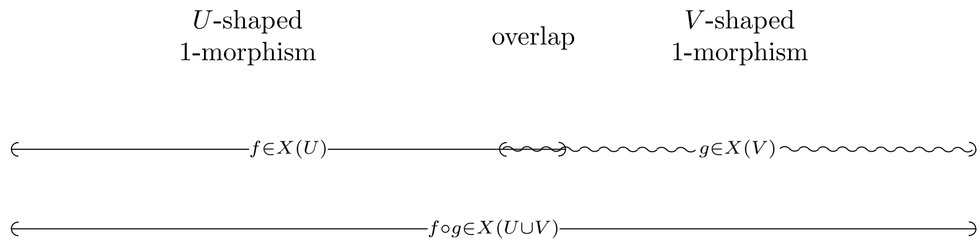


Figure 3: **Spaces and ∞-groupoids.** A sheaf X on open subsets of $\mathbb{R}^n$ behaves not entirely unlike a presheaf on Δ (a simplicial set) satisfying the Kan condition: for each object $U \subset \mathbb{R}^k$ there is a collection $X(U)$ of "U-shaped k-morphisms" and the sheaf condition says that whenever these overlap with V-shaped k-morphisms, there is a (unique) composite $(U \cup V)$-shaped k-morphism. We see that this is more than a faint analogy when discussing integration of L_∞-algebras in 2.3.

2.1 Duality of space and quantity

Of particular importance are *ambimorphic* sheaves [52]: sheaves which carry also a compatible structure as objects of another category.

Examples of **ambimorphic presheaves** of relevance for us are

- algebra-valued sheaves
  - $C(-,-) : S^{\mathrm{op}} \to C^\infty\mathrm{Algebras}$ – this canonical ambimorphic presheaf send each test domain to a monoidal Set-valued functor on S. Such functors are known as C^∞-algebras [37];
  - $C(-,\mathbb{R}) = C^\infty(-)$ – the presheaf of ordinary function algebras;
  - $\Omega^\bullet(-) : C^{\mathrm{op}} \to \mathrm{DGCAs}$ — the presheaf of differential forms, which carries itself the structure of a differental $\mathbb{N}$-graded commutative algebra;
- ω-category valued co-presheaves (compare [13, 44, 45])
  - $\mathcal{P}_n(-) : C \to \omega\mathrm{Cat}$ – the n-groupoid whose $k \leq n$-morphisms are thin homotopy classes of globukar k-paths;
  - $\Pi_n(-) : C \to \omega\mathrm{Cat}$ – the n-groupoid whose $k < n$-morphisms are thin homotopy classes of globukar k-paths; and whose n-morphisms are full homotopy classes of such paths;
  - $\Pi_\omega(-) : C \to \omega\mathrm{Cat}$ — for the ω-groupoid whose k-morphisms are thin homotopy classes of globular k-paths.

  The ambimorphicity of the path ω-groupoid valued sheaves is the fact that these may be regarded as ω-stacks on C^{op}. But as stacks they are rather special in that they are *rectified*: they respect composition in C strictly.

From each ambimorphic (co-)presheaf A which is also an object of a category D we obtain a (co- or contravariant) adjunction by homming into it

$$\begin{tikzcd} C \arrow[r, shift left, "C(-,A)"] & D \arrow[l, shift left, "D(A,-)"] \end{tikzcd} .$$

space $\qquad$ A-quantity

For the important case of $A = \Omega^\bullet$ we write

$$\Omega^\bullet : \begin{tikzcd} C \arrow[r, shift left, "C(-,\Omega^\bullet(-))"] & \mathrm{DGCA} \arrow[l, shift left, "\mathrm{Hom}_{\mathrm{DGCA}}(\Omega^\bullet(-),-)"] \end{tikzcd} : S .$$

This duality will translate for us between the world of Lie ω-groups and their infinitesimal approximations, Lie ∞-algebras (L_∞-algebras).

2.2 Lie ∞-algebras

A finite dimensional L_∞-algebra is a $\mathbb{N}_+$-graded vector space $\mathfrak{g}^*$ together with a graded differential

$$d_{\mathfrak{g}} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

which is of degree +1 and squares to 0.

The corresponding differential graded commutative algebra

$$\mathrm{CE}(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^*, d_{\mathrm{CE}(\mathfrak{g})})$$

we call the *Chevalley-Eilenberg algebra* of $\mathfrak{g}$.

The DGCA mapping cone on the identity of this is the *Weil algebra*

$$\mathrm{W}(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^* \oplus \mathfrak{g}^*[1], d_{\mathrm{W}(\mathfrak{g})} = \begin{pmatrix} d_{\mathrm{CE}(\mathfrak{g})} & 0 \\ [1] & [1] \circ d_{\mathrm{CE}(\mathfrak{g})} \circ [\text{-}1] \end{pmatrix})$$

2.3 Integration of Lie ∞-algebras

From any L_∞-algebra $\mathfrak{g}$ we obtain an ω-group

$$\mathbf{B}G := \Pi_\omega(S(\mathrm{CE}(\mathfrak{g})))\,.$$

This we address as the strict, fully simply connected ∞-group integrating $\mathfrak{g}$.

(Doing the same while not dividing out thin homotopy of k-paths and then passing to ω-nerves leads to the integration setup considered in [27, 29]).

Applying this procedure to an ordinary Lie algebra yields the simply connected Lie group integrating it. Analogously, the ω-groups obtained this way are higher simply connected. Other ω-groups are obtained from quotienting out discrete ω-groups.

For instance the ω-group integrating $b^n\mathfrak{u}(1)$ is

$$\mathbf{B}^n\mathbb{R} = \Pi_\omega(S(\mathrm{CE}(b^n\mathfrak{u}(1))))\,.$$

To obtain $\mathbf{B}^nU(1)$ one forms the homotopy quotient of

$$\mathbf{B}^nU(1) \simeq \mathrm{Cone}(\mathbf{B}^n\mathbb{Z} \to \mathbf{B}^n\mathbb{R})\,.$$

Examples. Let $\mathfrak{g}$ be an ordinary Lie algebra and $\Pi_1(X)$ the strict fundamental 1-groupoid of a space X (morphisms are homotopy classes of paths). Let G be the simply connected Lie group integrating $\mathfrak{g}$. Then

$$\Pi_1(S(\mathrm{CE}(\mathfrak{g}))) = \mathbf{B}G\,,$$

where the right hand side denotes the strict one object 1-groupoid obtained from G.

Now let $\mathfrak{g}$ be an ordinary Lie algebra with a bilinear invariant form on it and let μ be the associated canonical Lie algebra 3-cocycle. The corresponding String Lie 2-algebra is $\mathfrak{g}_\mu$. Let $\Pi_2(X)$ be the strict fundamental 2-groupoid of a space X: morphisms are *thin* homotopy classes of paths and 2-morphisms are homotopy classes of paths [**?**].

Then, I am claiming, the 2-group G_μ defined by

$$\mathbf{B}G_2 := \Pi_2(S(\mathrm{CE}(\mathfrak{g}_\mu)))$$

is essentially the strict version of the String Lie 2-group presented in [7], only that the horizontal composition of paths is not pointwise multiplication, but concatenation. This is, I am claiming, the strict 2-group secretly underlying the discussion in [17, **?**].

Forming instead $\Pi_\infty^{\mathrm{wk}}(S(\mathrm{CE}(\mathfrak{g})))$ leads to the integration discussed in [29].

3 Descent

The general notion of cohomology, as formalized ∞-categorically by Ross Street [48], makes sense for coefficient objects which are ∞-category valued presheaves. For the special case that the coefficient object is just an ∞-category, the corresponding cocycles characterize higher fiber bundles. This is usually addressed as *nonabelian cohomology* [11, 51]. If instead the coefficient object is refined to presheaves of ∞-functors from ∞-*paths* to the given ∞-category, then one obtains the cocycles discussed in [6, 44, 45, 46] which characterize higher bundles *with connection* and hence live in what deserves to be addressed as nonabelian *differential cohomology* [31].

We concentrate here on ω-categorical models (strict globular ∞-categories [14, 20, 15, 16]) and discuss nonabelian differential cohomology with values in ω-groups obtained from integrating L(ie)-∞ algebras [27, 29].

total space fibration	$\begin{tikzcd} P \arrow[d, "p"] \\ X \end{tikzcd}$	pullback by universal property	$\begin{tikzcd} f^*P \arrow[r] \arrow[d] & P \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$	weak respect for composition of pullback
fiber-assigning functor	$\begin{tikzcd} G\mathrm{Tor} \\ X \arrow[u] \end{tikzcd}$	pullback by precomposition	$\begin{tikzcd} & G\mathrm{Tor} \\ Y \arrow[r, "f"] & X \arrow[u] \end{tikzcd}$	strict respect for composition

Table 6: **The rectification of ∞-stacks of ∞-bundles in terms of fiber-assigning functors.** The crucial method which allows us to work entirely within ω-category valued sheaves without having to deal with "∞-prestacks" is that we conceive all n-bundles (P, ∇) with connection not as fibrations of their total spaces $p : P \to X$ but entirely in terms of their fiber-assigning and parallel transport-assigning functors. The equivalence of fiber-assigning functors with the total space perspective of bundles is established in [44, 46].

3.1 Introduction

A principal G-bundle is given, with respect to a good cover by open sets of its base space, by a *trivial* G-bundle on each open subset, together with an isomorphism of trivial G-bundles on each double intersection, and an equation between these on each triple intersection. This is the archetypical example of what is called **descent data**, forming a **cocycle in nonabelian cohomology**. It can be vastly generalized by replacing the group G appearing here by some ∞-category. For each cocycle obtained this way there should be a corresponding ∞-bundle whose local trivialization it describes [57].

The crucial basic idea of [6, 44, 45, 46] is to describe ∞-bundles *with connection* by cocycles which have

- a ("transport") functor from paths to G on each patch;
- an equivalence between such functors on double overlaps
- and so on.

The cocycles thus obtained deserve to be addressed as cocycles in **differential nonabelian cohomology**.

Forming the collection of ω-functors from paths in a patch to some codomain provides a functor from "spaces" to ω-categories: an ω-category valued presheaf.

In [48] Ross Street descibes a very general formalization for cohomology taking values in ω-category valued presheaves. We recall the basic ideas (subject to some slight modifications, a discussion of which is in **??**) and describe how the differential cocoycles of [6, 44, 45, 46] fit into that.

Of particular interest are differential cocycles which can be expressed differentially in terms of L(ie) ∞-algebras. Building on the discussion of [41] we give in 3.5 a definition (def. **??**) of non-flat non-abelian differential cocycles and their characteristic classes.

There are two major approaches to general (nonabelian) ∞-cohomology:

- Ross Street in [48] explicitly writes down ∞-descent conditions for ∞-category valued presheaves.
- In the approach reviewed in [51, 32] instead simplicial set valued presheaves are used, and the descent condition is realized more implicitly, by passing to homotopy categories.

3.2 Descent and cohomology

Fix a topos C, whose objects we think of as

- the spaces whose cohomologies we want to understand;

or equivalently

- the spaces on which we want to understand the notion of higher fiber bundles and connections.

We work with ω-categories (strict globular ∞-categories) internal to C and write $\omega\mathrm{Cat}$ for the $(\omega\mathrm{Cat}, \otimes_{\mathrm{Gray}})$-category of all ω-categories internal to C (see [20] and section 9 of [48]).

The theory we are interested in is the theory of structures $P \in \mathrm{Bund}(X)$ on $X \in C$ for

$$\mathrm{Bund} : \; C^{\mathrm{op}} \longrightarrow \omega\mathrm{Cat}$$

some functorial assignment of structures to each object X in C, which have the property that when pulled back along a suitable regular surjection

$$\pi : Y \twoheadrightarrow X$$

in C they become equivalent to a structure

$$P_{\mathrm{triv}} \in \mathrm{TrivBund(Y)} \subset \mathrm{Bund}(Y)$$

from a chosen smaller collection $i : \; \mathrm{TrivBund} \hookrightarrow \mathrm{Bund}$:

$$\exists : \pi^* P \xrightarrow[\simeq]{t} P_{\mathrm{triv}} \; .$$

The equivalence t here is called the **local trivialization** of P relative to π and i. We speak of **π-local i-trivializations**.

The existence of this local trivialization implies that the existence of the structure P down on X is mirrored by the existence of P_{triv} up on Y *together* with various relations between the pullbacks of P_{triv} along the simplicial object

$$Y^\bullet := \mathrm{Ner}(\pi) : \; \Delta^{\mathrm{op}} \longrightarrow C$$

$$Y^\bullet = \left(\cdots Y \times_X Y \times_X Y \underset{\pi_{13}}{\overset{\pi_{12}}{\underset{\pi_{23}}{\rightrightarrows}}}\!\!\to Y \times_X Y \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} Y \right) .$$

The first of these relations says that there is an equivalence

$$\pi_1^* P_{\mathrm{triv}} \xrightarrow[\simeq]{g} \pi_2^* P_{\mathrm{triv}}$$

between the two possible pullbacks of P_{triv} to $Y \times_X Y$. The second relation says that there is an equivalence

$$\begin{tikzcd} & \pi_2^* P_{\mathrm{triv}} \arrow[dr, "\pi_{23}^* g"] \arrow[d, phantom, "\simeq \Downarrow f"] & \\ \pi_1^* P_{\mathrm{triv}} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & & \pi_2^* P_{\mathrm{triv}} \end{tikzcd}$$

between the three possible pullbacks of this equivalence to $Y \times_X \times Y \times_X Y$. The third relation says that there is an equivalence between the four possible pullbacks of this equivalence of equivalences. And so on.

These relations are variously known as the *transition data* or *gluing data* or **descent data**, since given a $P_{\mathrm{triv}} \in \mathrm{TrivBund}(Y)$, they ensure that P_{triv} may be "glued" along the fibers of Y such that result "descends" to a $P \in \mathrm{Bund}(X)$ down on X. Therefore descent is the converse to local trivialization:

$$\begin{array}{c}\text{trivial}\\ \text{structure}\\ \text{on } Y\end{array} \;\underset{\text{local trivialization}}{\overset{\text{descent}}{\rightleftarrows}}\; \begin{array}{c}\text{structure}\\ \text{on } X\end{array} .$$

The collections $(P_{\mathrm{triv},g,f,\cdots})$ consisting of a P_{triv} with its gluing data or descent data can hence usefully be regarded as a forming a kind of higher categorical coequalizer of the cosimplicial ω-category

$$\mathcal{E} : \Delta \xrightarrow{\mathrm{Ner}(\pi)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\mathrm{TrivBund}} \omega\mathrm{Cat}$$

$$\mathcal{E}^\bullet = \left(\cdots \mathrm{TrivBund}(Y \times_X Y \times_X Y) \begin{array}{c}\xleftarrow{\pi_{01}^*}\\ \xleftarrow{\pi_{12}^*}\\ \xleftarrow{\pi_{23}^*}\end{array} \mathrm{TrivBund}(Y \times_X Y) \begin{array}{c}\xleftarrow{\pi_0^*}\\ \xleftarrow{\pi_1^*}\end{array} \mathrm{TrivBund}(Y) \right) .$$

This coequalizer-like ω-category, whose objects are suitable collections $(P_{\mathrm{triv},g,f,\cdots})$ is the **descent category** $\mathrm{Desc}(\mathcal{E})$

$$\mathrm{Desc} : \omega\mathrm{Cat}^\Delta \longrightarrow \omega\mathrm{Cat} .$$

Its general definition for ω-categories was given in [48] (p. 32), based on [49]. A sketch of a more general definition for weak ∞-categories is given towards the end of [48].

3.2.1 ωCat-valued presheaves

The above considerations show that the objects of interest here are (pre)sheaves on C with values in ω-categories, corresponding to the presheaves with values in simplicial sets considered in the approach reviewed in [32, 51].

There is a standard construction to enrich $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$ over ωCat:
for $X, Y \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$ write

$$\tilde{\mathrm{hom}}(X,Y) : \omega\mathrm{Cat} \to \mathrm{Set}$$

$$R \mapsto \omega\mathrm{Cat}(R \times X, Y) .$$

If this is representable, we identify the representing ω-category $\mathrm{hom}(X,Y) \in \omega\mathrm{Cat}$ with the ωCat-valued hom-object:

$$\tilde{\mathrm{hom}}(X,Y) \simeq \mathrm{Set}^{\omega\mathrm{Cat}^{\mathrm{op}}}(-, \mathrm{hom}(X,Y)) .$$

So if C is such that this representing object exists, $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$ is ωCat-enriched and it makes sense to ask if our descent ω-category $\mathrm{Desc}(\mathcal{E})$ is actually co-representable in that there is $\Pi_0^Y(X) \in \omega\mathrm{Cat}$ such that

$$\mathrm{Desc}(\mathcal{E}) \simeq \omega\mathrm{Cat}^{C^{\mathrm{op}}}(\Pi_0^Y(X), \mathrm{TrivBund}) ,$$

where we are implicitly using the canonical embedding $\omega\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}}$. This $\Pi_0^Y(X)$ is the **codescent object**

$$\Pi_0^Y(X) := \mathrm{Codesc}(\mathrm{Ner}(\pi))$$

and the notation suggests that we shall later have use more generally for ω-catgories denoted $\Pi_n^Y(X)$ and $\Pi_\omega^Y(X)$: their k-morphisms are k-paths in Y combined with jumps in the fibers of Y [44, 46].

Notice that the map Desc from simplicial ω-categories to ω-categories is analogous (possibly even equivalent) to the *codiagonal* map from bisimplicial sets to simplicial sets.

We usually have that TrivBund is *representable*

$$\mathrm{TrivBund}(-) \simeq \omega\mathrm{Cat}^{C^{\mathrm{op}}}(-, \mathbf{A})$$

for some $\mathbf{A} \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$, where we are implicitly using the embedding $C \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}}$ which sends each object U to the ωCat-valued presheaf which sends each object V to the discrete ω-category over $C(V, U)$.

In that case we say that

Definition 1 ($\omega\mathrm{Cat}^{C^{\mathrm{op}}}$-valued cohomology) *For $X \in C$ and $\mathbf{A} \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$, the ω-category*

$$H(X, \mathbf{A}) := \mathrm{colim}_\pi \left(\mathrm{Desc} \left(\Delta \xrightarrow{\mathrm{Ner}(\pi)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\omega\mathrm{Cat}^{C^{\mathrm{op}}}(-,\mathbf{A})} \omega\mathrm{Cat} \right) \right)$$

is the **cocycle ω-category** *of X with coefficients in $\mathbf{A}$:*

- *objects are the $\mathbf{A}$-valued* **cocycles** *on X;*
- *(1-)morphisms are the* **coboundaries** *between these cocycles;*
- *$(k > 1)$-morphisms are the coboundaries of coboundaries;*
- *equivalence classes of objects are the $\mathbf{A}$-valued* **cohomology classes** *of X.*

The functor

$$H(-, \mathbf{A}) : C^{\mathrm{op}} \to \omega\mathrm{Cat}$$

is the **cohomology theory for coefficients $\mathbf{A}$**.

This is the general definition of cohomology that essentially appears in section 4 of [48].

3.2.2 ωCat-valued cohomology

The special case of cohomology with values in an ω-category – whose general idea goes back to [39] and is often addressed, somewhat loosely, as **nonabelian cohomology** – is obtained using the inclusion

$$\begin{aligned} \Pi_0^* \quad : \quad \omega\mathrm{Cat} &\hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}} \\ A &\longmapsto \omega\mathrm{Cat}(\Pi_0(-), A) \ , \end{aligned}$$

where, in turn, $\Pi_0 : C \hookrightarrow \omega\mathrm{Cat}$ sends each object U to the discrete ω-category over it (which has U as its object of objects and no nontrivial morphisms.)

Hence

Definition 2 (ωCat-valued cohomology) *For $A \in \omega\mathrm{Cat}$, the cohomology theory with coefficients in A is*

$$H(-, A) := H(-, \Pi_0^*(A)) \, .$$

Using the Yoneda-like argument on p. 12 of [48], which says that

$$\omega\mathrm{Cat}^{C^{\mathrm{op}}}(U, \Pi_0^*(A)) \simeq \Pi_0^*(A)(U) := \omega\mathrm{Cat}(\Pi_0(U), A) \, ,$$

this becomes the theory considered on p. 3 of [48].

3.2.3 ωCat-valued differential cohomology

Recall that, with definition 2, we obtained ωCat-valued cohomology from the general $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$-valued cohomology by pulling back along an inclusion

$$\Pi_0^* : \ \omega\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}} \ .$$

But there are other such inclusions, which are no less natural. In particular, if the objects X of C are spaces that admit a notion of **path ω-groupoid** $\Pi(X)$,

$$\Pi : \ C \hookrightarrow \omega\mathrm{Cat}$$

then we can pull back along the corresponding

$$\begin{aligned} \Pi^* \quad : \quad & \omega\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}} \\ & A \longmapsto \omega\mathrm{Cat}(\Pi(-), A) \ . \end{aligned}$$

Definition 3 (ωCat-valued differential cohomology) *For a given notion of path ω-groupoid $\Pi : C \hookrightarrow \omega\mathrm{Cat}$ and a coefficient object $A \in \omega\mathrm{Cat}$ we address*

$$H^{\Pi}(-, A) := H(-, \Pi^*(A))$$

as Π-differential cohomology with values in A.

For $\Pi = \mathcal{P}_2$ and G a strict 2-group such cocycles in $H^{\mathcal{P}_2}(-, \mathbf{B}G)$ were first considered in [6, 46].

3.2.4 Examples

Cohomology classes for ω-bundles

Definition 4 (ω-groups) *Given any one-object ω-groupoid* Gr *we say that the* Hom*-thing* $G := \mathrm{Gr}(\bullet, \bullet)$ *is an ω-group and write*

$$\mathrm{Gr} := \mathbf{B}G$$

to indicate the property of Gr *of having one single object.*

Remark. The notation here is such that under taking realizations of nerves we have

$$|\mathbf{B}G| \simeq B|G| \, ,$$

compare [3, 8]. We hence call $\mathbf{B}G$ the **classifying ω-groupoid** for the ω-group G.

Whichever way a **principal G-bundle** on X is defined [57], it must be such that its local trivializations are objects in $H(X, \mathbf{B}G)$ and, indeed, that the ω-category $G\mathrm{Bund}(X)$ they form is equivalent to $H(X, \mathbf{B}G)$

$$G\mathrm{Bund}(-) \simeq H(-, \mathbf{B}G) \, .$$

For $n = 2$ this is discussed in [57].

This is often thought of as saying that

$$G\text{-bundles are a geometric model for } H(-, \mathbf{B}G) \ .$$

One expects to revover the topologist's notion of classifying maps in the case that objects of C are topological spaces by using the corepresentation of $H(-, \mathbf{B}G)$ using the codescent object as

$$H(-, \mathbf{B}G) \simeq \omega\mathrm{Cat}(\Pi_0^Y(X), \mathbf{B}G)$$

for $\pi : (Y = \sqcup_i U_i) \to X$ a good cover of X. Upon applying the nerve functor one expects

$$|\omega\mathrm{Cat}(\Pi_0^Y(X), \mathbf{B}G)| \simeq [|\Pi_0^Y(X)|, |\mathbf{B}G|] \simeq [X, B|G|] \, .$$

For strict 2-groups G this was shown to be true in [8] if G is "well pointed". A more general argument for topological 2-categories is given in [3] though it is, while plausible, not obvious that the "concordances" used in [3] reproduce exactly the transformations that are the morphisms in $H(-, \mathbf{B}G)$.

Singular cohomology

Fact 1 *A direct consequence of a standard fact about Čech cohomology is that the ω-category $\mathbf{B}^n\mathbb{Z}$ exhibits ordinary singular cohomology as $\omega\mathrm{Cat}$-valued cohomology*

$$H^{n+1}_{\mathrm{singular}}(-,\mathbb{Z}) = H(-,\mathbf{B}^n U(1))_\sim = H(-,\mathbf{B}^{n+1}\mathbb{Z})_\sim$$

.

K-Theory

Fact 2 *A standard fact about K-theory says that*

$$K^0(-) \simeq H(-,(\mathbf{B}U)\mathbb{Z})_\sim\,.$$

Cohomology classes for ω-bundles with connection While our definition allows more general setups, usually one will want to interpret differential cohomology in the context of *smooth* spaces.

If this is so, one useful concrete choice for our ambient category is to take C to be the category of sheaves on the site S with

- $\mathrm{Obj}(S) = \mathbb{N}$;
- $S(n,m) = \{f : \mathbb{R}^n \to \mathbb{R}^m | f \text{ smooth}\}$.

Theorem 1 *Let G be an ordinary Lie group and let*

$$\Pi := \mathcal{P}_1 : C \to \omega\mathrm{Cat}$$

be the path 1-groupoid whose morphisms are thin-homotopy classes of paths. Then Π-differential cohomology with values in $\mathbf{B}G$ classifies principal G-bundles with connection

$$H^\Pi(-,\mathbf{B}G) = G\mathrm{Bund}_\nabla(-)\,.$$

This is the result of [44].

Theorem 2 *Let G be a strict Lie 2-group and let*

$$\Pi := \mathcal{P}_2 : C \to \omega\mathrm{Cat}$$

be the path 2-groupoid whose k-morphisms are thin-homotopy classes of k-paths. Then Π-differential cohomology with values in $\mathbf{B}G$ classifies fake-flat principal G-2-bundles with connection

$$H^\Pi(-,\mathbf{B}G) = G\mathrm{Bund}_\nabla(-)\,.$$

This is the result of [45] and [46].

In particular, for $G = \mathrm{AUT}(H)$, $H^\Pi(-,\mathbf{B}\mathrm{AUT}(H))$ classifies the fake-flat connections on H-gerbes studied in [12].

3.3 Codescent

Definition 5 (codescent) *Given a simplicial object $E : \Delta^{\mathrm{op}} \to C$, we say that*

$$\mathrm{Codesc}(E) \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$$

is, if it exists, the codescent $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$-object of E if it co-represents descent on E in the sense that

$$\mathrm{Desc}\left(\Delta \xrightarrow{E^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\omega\mathrm{Cat}^{C^{\mathrm{op}}}(-,\mathbf{A})} \omega\mathrm{Cat}\right) \simeq \omega\mathrm{Cat}^{C^{\mathrm{op}}}(\mathrm{Codesc}(E),\mathbf{A})$$

naturally for all coefficient objects $\mathbf{A} \in \omega\mathrm{Cat}^{C^{\mathrm{op}}}$.

If we let $\mathbf{A}$ just run over the image of $\omega\mathrm{Cat} \hookrightarrow \omega\mathrm{Cat}^{C^{\mathrm{op}}}$ we obtain the codescent object as an ω-category:

Definition 6 *Given a simplicial object $E : \Delta^{\mathrm{op}} \to C$, we say that*

$$\mathrm{Codesc}(E) \in \omega\mathrm{Cat}$$

is, if it exists, the codescent ω-category of E if it co-represents descent on E in the sense that

$$\mathrm{Desc}\left(\Delta \xrightarrow{E^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\omega\mathrm{Cat}(\Pi_0(-),A)} \omega\mathrm{Cat} \right) \simeq \omega\mathrm{Cat}(\mathrm{Codesc}(E), A)$$

naturally for all coefficient object $A \in \omega\mathrm{Cat}$.

For $E = Y^\bullet = \mathrm{Ner}(\pi : Y \to X)$, we have that $\mathrm{Codesc}(E) = \Pi_0^\pi(X)$ is nothing but the Čech groupoid of Y. In fact, as mentioned on p. 3 of [48], every category is the codescent object of its nerve.

We observe that, again, the above definition makes explicit use of an injection

$$\Pi : C \to \omega\mathrm{Cat}\,.$$

Hence we adapt the notion of codescent to the setup of differential cohomology as in 3.2.3:

Definition 7 (differential codescent) *Given a simplicial object $E : \Delta^{\mathrm{op}} \to C$, and an embedding*

$$\Pi : C \hookrightarrow \omega\mathrm{Cat}\,,$$

we say that

$$\mathrm{Codesc}^\Pi(E) \in \omega\mathrm{Cat}$$

is, if it exists, the codescent object of E if it co-represents descent on E in the sense that

$$\mathrm{Desc}\left(\Delta \xrightarrow{E^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\omega\mathrm{Cat}(\Pi(-),A)} \omega\mathrm{Cat} \right) \simeq \omega\mathrm{Cat}(\mathrm{Codesc}^\Pi(E), A)$$

naturally for all coefficient object $A \in \omega\mathrm{Cat}$.

Theorem 3 *For $\Pi := \mathcal{P}_1$ the path 1-groupoid, we have*

$$\mathrm{Codesc}^\Pi(Y^\bullet) = \mathcal{P}_1^\pi(X)\,,$$

where on the right we have the "path pushout" from [44].

Theorem 4 *For $\Pi := \mathcal{P}_2$ the path 2-groupoid, we have*

$$\mathrm{Codesc}^\Pi(Y^\bullet) = \mathcal{P}_2^\pi(X)\,,$$

where on the right we have the "bigon pushout" from [46].

3.3.1 Descent categories from codescent

We can use the codescent objects to express the corresponding descent ω-categories in a useful way:

as described on p. 5 of [48] every category is the codescent object of its own nerve. That means in particular that the codescent object of the nerve of an epimorphism $\pi : Y \twoheadrightarrow X$ is just the Čech groupoid $\mathrm{Codesc}(\mathrm{Ner}(\pi)) = X^\pi$.

3.3.2 Examples

Descent category for differential 1-cocycles We consider $\pi : Y \twoheadrightarrow X$ a regular epimorphism and work out the descent category for differential 1-descent

$$\begin{aligned} &\mathrm{Desc}\left(\Delta \xrightarrow{\mathrm{Ner}(\pi)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\mathrm{Cat}(\mathcal{P}_1(-),\mathbf{B}G)} \mathrm{Cat} \right) \\ &= \mathrm{Desc}\left(\Delta \xrightarrow{\mathrm{Ner}(\pi)^{\mathrm{op}}} C^{\mathrm{op}} \xrightarrow{\mathrm{Cat}^{C^{\mathrm{op}}}(-,\mathrm{Cat}(\mathcal{P}_1(-),\mathbf{B}G)} \mathrm{Cat} \right) \end{aligned}$$

as

$$= \mathrm{Cat}^{C^{\mathrm{op}}}(\mathrm{Cat}(\Pi_0(-), X^\pi), \mathrm{Cat}(\mathcal{P}_1(-), \mathbf{B}G))$$

Notice that for each test domain U the objects of $\mathrm{Cat}(\Pi_0(U), X^\pi)$ are maps $U \to Y$ in C, while the morphisms are maps $U \to Y^{[2]}$.

The objects of $\mathrm{Cat}^{C^{\mathrm{op}}}(\mathrm{Cat}(\Pi_0(-), X^\pi), \mathrm{Cat}(\mathcal{P}_1(-), \mathbf{B}G))$ are over each test domain U functors

$$\mathrm{Cat}(\Pi_0(U), X^\pi) \to \mathrm{Cat}(\mathcal{P}_1(U), \mathbf{B}G)$$

natural in U. Such functors can be obtained from picking an object

$$\mathrm{triv} \in \mathrm{Cat}(\mathcal{P}_1(Y), \mathbf{B}G)$$

with an ismorphism

$$g : \pi_1^*\mathrm{triv} \to \pi_2^*\mathrm{triv}$$

such that

$$\begin{tikzcd} & \pi_2^*\mathrm{triv} \arrow[dr, "\pi_{23}^* g"] & \\ \pi_1^*\mathrm{triv} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & & \pi_3^*\mathrm{triv} \end{tikzcd}$$

and then sending $f : U \to Y$ to $f^*\mathrm{triv}$ $\hat{f} : U \to Y^{[2]}$ to $\hat{f}^* g$.

By the usual presheaf gymnastics, all such functors should arise this way.

A natural transformation between two such functors is obtained from picking an isomorphism

$$h : \mathrm{triv} \to \mathrm{triv}'$$

making

$$\begin{tikzcd} \pi_1^*\mathrm{triv} \arrow[r, "g"] \arrow[d, "\pi_1^* h"'] & \pi_2^*\mathrm{triv} \arrow[d, "\pi_2^* h"] \\ \pi_1^*\mathrm{triv}' \arrow[r, "g'"'] & \pi_2^*\mathrm{triv}' \end{tikzcd}$$

commute, and then sending each object $f : U \to Y$ to the morphism f^*h.

By the usual presheaf gymnastics, all such functors should arise this way.

The descent category found this way is the one given in [44].

Descent category for differential 2-cocycles Analogously.

The descent category found this way is the one given in [46], the cocycles of which were also described in [6].

3.4 Local trivialization

If

$$\mathrm{Bund} : C^{\mathrm{op}} \to \omega\mathrm{Cat}$$

encodes structures on objects of C of certain kind and

$$i : \mathrm{TrivBund} \hookrightarrow \mathrm{Bund}$$

is a certain subcollection of these structures which we want to regard as being "trivial", and if

$$\pi : Y \twoheadrightarrow Y$$

is a regular epimorphism in C, then we say

Definition 8 *The pseudopullback* $\mathrm{Triv}(i, \pi)$

$$\begin{tikzcd}
\mathrm{Triv}(i,\pi) \arrow[r] \arrow[d] & \mathrm{TrivBund}(Y) \arrow[d, hook, "i"] \arrow[dl, Rightarrow, "\simeq"] \\
\mathrm{Bund}(X) \arrow[r, "\pi^*"] & \mathrm{Bund}(Y)
\end{tikzcd}$$

is the ω-category of π-locally i-trivializable elements of $\mathrm{Bund}(X)$*, equipped with a chosen π-local i-trivialization.*

By forgetting the chosen local trivialization we obtain a factorization

$$\mathrm{Triv}(i,\pi) \twoheadrightarrow \mathrm{Trans}(i,\pi) \hookrightarrow \mathrm{Bund}(X)$$

where $\mathrm{Trans}(i,\pi)$ *is the ω-category of elements of* $\mathrm{Bund}(X)$ *which do* admit some *π-local i-trivialization.*

This is essentially the definition on p. 22 of [48], but with the notation following [44, 46] (so our $\mathrm{Triv}(i,\pi)$ is $Q(t;e)$ in [48] and our $\mathrm{Trans}(i,\pi)$ is $\mathrm{Loc}(t;e)$).

In [44, 46] we adopt a more concrete (less general) point of view on what counts as i-trivial: there we require that

$$\mathrm{Bund} := \omega\mathrm{Cat}(\Pi_n(-), T)$$

for T some **ω-category of fibers** and

$$\mathrm{TrivBund} := \omega\mathrm{Cat}(\Pi_n(-), \mathrm{Gr})$$

for Gr some **ω-category of *typical* fibers** and that the injection

$$i : \mathrm{TrivBund} \hookrightarrow \mathrm{Bund}$$

is postcomposition with a specified injection

$$i : \mathrm{Gr} \hookrightarrow T\ .$$

In that case an element $F \in \mathrm{Bund}(X)$ is π-locally i-triviall precisely if it fits into a square

$$\begin{tikzcd}
\Pi_n(Y) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{triv}"'] & \Pi_n(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\simeq"', "t"] \\
\mathrm{Gr} \arrow[r, "i"] & T
\end{tikzcd}$$

In [44, 46] we also see the need to strengthen the conditions on what counts as locally trivial: not only need the local trivialization t exist, it also may have to be itself locally trivial in some sense. We encode this by putting conditions on the descent data induced by t:

Observation 1 (extraction of descent data) *There is canonically a morphism*

$$\mathrm{Ex} : \mathrm{Triv}(i, \pi) \to H_\pi(X, \mathrm{Bund})\,.$$

See [44, 46].

Definition 9 (π-local i-trivialization) *Given*

- $T \in \omega\mathrm{Cat}$*: thought of as an* ω*-category of fibers;*
- $\mathrm{Gr} \in \omega\mathrm{Grpd}$*: thought of as an* ω*-groupoid of* typical *fibers*
- *an inclusion*

$$i : \mathrm{Gr} \hookrightarrow T\ ;$$

- $\Pi : C \to \omega\mathrm{Cat}$*; a notion of path* ω*-groupoid;*
- *a factorization*

$$\begin{tikzcd} \omega\mathrm{Cat}(\Pi(-), \mathrm{Gr}) \arrow[r, hook, "a"] \arrow[rr, bend right, "i_*"'] & \mathrm{TrivBund}^{\Pi}_{\mathrm{Gr}} \arrow[r, hook] & \omega\mathrm{Cat}(\Pi(-), T) \end{tikzcd}$$

in $\omega\mathrm{Cat}^{C^{\mathrm{op}}}$ *with the map* a *surjective on objects and faithful, we say that*

$$\mathrm{tra} \in \omega\mathrm{Cat}(\Pi(x), T)$$

is π*-locally* i*-trivializable if there is an equivalence* t

$$\begin{tikzcd} \Pi(Y) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{triv}"'] & \Pi(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\simeq"', "t"] \\ \mathrm{Gr} \arrow[r, "i"] & T \end{tikzcd}$$

such that the extracted cocylce has coefficients in $\mathrm{TrivBund}^{\Pi}_{\mathrm{Gr}}$

$$\mathrm{Ex}(t) \in H_\pi(X, \mathrm{TrivBund}^{\Pi}_{\mathrm{Gr}})\,.$$

3.4.1 Local semi-trivialization: bundle gerbes

What is addressed as a "bundle gerbe" in the literature is really a cocycle. Bundle gerbes differ from cocycles with values in $\mathbf{B}G$ only in that the factorization appearing in definition 9 has a middle piece whose collection of morphisms is strictly larger than that of the left piece.

Theorem 5 *Line bundle gerbes [38] with connection ("and curving") are equal (meaning:* canonically *isomorphic) to cocycles with values in the subobject*

$$\mathrm{TrivBund} \subset \omega\mathrm{Cat}(\mathcal{P}_2(-), \mathbf{B}\mathrm{Vect})$$

on all objects which factor through

$$i : \mathbf{B}\mathbf{B}U(1) \hookrightarrow \mathbf{B}\mathrm{Vect}$$

with morphisms those whose component maps are locally $(\mathbf{B}U(1) \hookrightarrow \mathrm{Vect})$*-trivializable.*

Details and proof in [46].

Analogous statements hold for other flavors of bundle gerbes, like higher bundle gerbes and nonabelian bundles gerbes[1].

3.5 Characteristic forms

3.5.1 L_∞-algebra valued differential forms

Definition 10 *For G an ω-group, we address*

$$\Omega^\bullet(-,\mathbf{B}G) := \omega\mathrm{Cat}(\Pi_\omega(-),\mathbf{B}G) : C^{\mathrm{op}} \to \omega\mathrm{Cat}$$

as flat G-valued differential forms.

For G a strict 2-group and Π_ω replaced with $\mathcal{P}_2$ this was studied in [45], see also [6]:

Theorem 6 *Objects of the 2-category*

$$2\mathrm{Cat}(\mathcal{P}_2(Y),\mathbf{B}G)$$

for G coming from a Lie crossed module $(t : H \to G)$ are pairs $(A,B) \in \Omega^1(Y,\mathrm{Lie}(G)) \times \Omega^2(Y,\mathrm{Lie}(H))$ satisfying $F_A + t_* \circ B = 0$.

This can be understood in terms of L_∞-algebra valued forms:

L_∞-algebras are to ordinary Lie algebras as ∞-groupoids are to ordinary groups. We can integrate L_∞-algebras to ω-groupoids internal to smooth spaces and use these as coefficients for nonabelian differential cohomology.

Definition 11 *A finite-dimensional L_∞-algebra is a finite dimensional $\mathbb{N}_+$-graded vector space $\mathfrak{g}$ together with a degree +1 differential on the graded symmetric tensor algebra over $\mathfrak{g}^*$*

$$d_{\mathfrak{g}} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

such that $d^2 = 0$. The resulting differential graded commutative algebra

$$\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d_{\mathfrak{g}})$$

is called the Chevalley-Eilenberg algebra of $\mathfrak{g}$.

There is a notion of mapping cone for L_∞-algebras and we write

$$\mathrm{W}(\mathfrak{g}) := \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) := \mathrm{CE}(\mathrm{Cone}(\mathfrak{g} \stackrel{\mathrm{Id}}{\to} \mathfrak{g}))\,.$$

We have a canonical sequence

$$\mathrm{CE}(\mathfrak{g}) \longleftarrow \mathrm{W}(\mathfrak{g}) \longleftarrow \mathrm{inv}(\mathfrak{g}) = \mathrm{W}(\mathfrak{g})_{\mathrm{basic}}\ . \qquad (6)$$

For more details on this and the following see [41].

There is a contravariant adjunction between smooth spaces and differential graded commutative algebras, induced by the ambimorphic deRham object $\Omega^\bullet(-) \in C$ which is a smooth space with the structure of a DGCA on it:

$$C \underset{S}{\overset{\Omega^\bullet(-)}{\rightleftarrows}} \mathrm{DGCA}\ .$$

Definition 12 (L_∞-algebra valued forms) *Given an L_∞-algebra $\mathfrak{g}$ and a smooth space Y, we address*

$$\Omega^\bullet(Y,\mathfrak{g}) := C(Y, S(\mathrm{W}(\mathfrak{g}))) \in C$$

as the space of $\mathfrak{g}$-valued forms, and

$$\Omega^\bullet_{\mathrm{flat}}(Y,\mathfrak{g}) := C(Y, S(\mathrm{CE}(\mathfrak{g}))) \in C$$

as the space of flat $\mathfrak{g}$-valued forms.

This definition relates to the definition of G-valued differential forms when integrating L_∞-algebras to ω-groupoids. As noticed in [27] (see also [29] and [42]) this integration procedure is essentially nothing but the old construction in rational homotopy theory [50]: the ∞-group in question is that of k-paths/singular k-simplices in the space $S(\mathrm{CE}(\mathfrak{g}))$. Here we adopt this idea to the context of ω-categories internal to C:

Definition 13 (Integration of L_∞-algebras) *For $\mathfrak{g}$ any L_∞-algebra, we define the ω-groupoid $\mathbf{B}G$ integrating it, as well as ω-groupoids denoted $\mathbf{BE}G$ and $\mathbf{BB}G$ as the image under $\Pi \circ S$ of the sequence 6:*

$$
\begin{array}{rcccccl}
 & \mathbf{B}G & \longrightarrow & \mathbf{BE}G & \longrightarrow & \mathbf{BB}G & \\
:= \Pi_\omega \circ S(& \mathrm{CE}(\mathfrak{g}) & \longleftarrow & \mathrm{W}(\mathfrak{g}) & \longleftarrow & \mathrm{inv}(\mathfrak{g}) &)
\end{array}
$$

I believe that it should be true that morphisms of path ω-groupoids all come from push-forward along maps of the underlying spaces:

$$
\omega\mathrm{Cat}(\Pi(X), \Pi(Y)) \simeq C(X, Y)\,.
$$

If true, this would imply that for ω-groups G obtained from integration of L_∞-algebras $\mathfrak{g}$ by integration as above we have

$$
\Omega^\bullet(Y, \mathbf{BG}) = \Omega^\bullet(Y, \mathfrak{g})\,.
$$

4 Nonabelian differential cohomology

Given the motivation provided by the context of Σ-models as described in 1.3, we want to conceive higher bundles with connection in terms of their parallel transport ω-functors

$$
\nabla :\ \Pi_\omega(X) \longrightarrow T\ .
$$

The value of these functors over objects, points $x \in X$, is the fiber P_x of the bundle P in question. The value over 1-paths $\gamma : x \to y$ is the parallel transport along this path. The value on higher path is the higher dimensional analog of parallel transport.

$$
\begin{tikzcd}
T \\
\Pi_\omega(X) \arrow[u, "\nabla"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & P_y \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
P_x \arrow[ur] \arrow[rr] & y \arrow[dr] \arrow[d, phantom, "\Downarrow"] & P_z \\
x \arrow[u, mapsto] \arrow[ur] \arrow[rr] & & z \arrow[u, mapsto]
\end{tikzcd}
$$

Among all functors $\Pi_\omega(X) \to T$ those which qualify as parallel transport in higher fiber bundles need to be characterized.

For G an ω-group we say that such an ω-functor is a parallel transport functor with local structure $\mathbf{B}G$ if it is *locally i-trivial*

$$
\exists(\ \pi : Y \twoheadrightarrow X\ )\ :\quad
\begin{tikzcd}
\Pi_\omega(Y) \arrow[r, "\pi_*"] \arrow[d, "\nabla_{\mathrm{loc}}"'] & \Pi_\omega(X) \arrow[d, "\nabla"] \arrow[dl, Rightarrow, "t"', "\simeq"] \\
\mathbf{B}G \arrow[r, "i"] & \omega\mathrm{Cat}
\end{tikzcd}
$$

for a given representation

$$
i :\ \mathbf{B}G \xrightarrow{i} T
$$

and if the local trivialization t induces $\omega\mathrm{Cat}(\Pi_\omega(-), \mathbf{B}G)$-descent on $\mathrm{Ner}(\pi)$.

Conceiving higher bundles with connection this way in terms of their fiber-assigning and parallel transport functors in particular makes them form *sheaves* instead of higher n-stacks. This is technically convenient and useful. For example it allows us use of the descent theory for cosimplicial ω-categories of [48].

4.1 Flat and non-flat nonabelian differential cocycles

For each n-group G we get ω-groupoid valued presheaves (sheaves, actually)

- $\omega\mathrm{Cat}(\Pi_0(-), \mathbf{B}G)$ – trivial G-bundles
- $\omega\mathrm{Cat}(\Pi_n(-), \mathbf{B}G)$ – trivial G-bundles with fake-flat connection
- $\omega\mathrm{Cat}(\Pi_\omega(-), \mathbf{B}G)$ – trivial G-bundles with flat connection.

We say that

- $$H(-, \mathbf{B}G) := H(-, \omega\mathrm{Cat}(\Pi_0(-), \mathbf{B}G))$$
  is nonabelian cohomology with values in G;
- $$H^{\mathcal{P}_n}(-, \mathbf{B}G) := H(-, \omega\mathrm{Cat}(\mathcal{P}_n(-), \mathbf{B}G))$$
  is fake-flat nonabelian cohomology with values in G;
- $$H^{\Pi}(-, \mathbf{B}G) := H(-, \omega\mathrm{Cat}(\Pi_\omega(-), \mathbf{B}G))$$
  is flat nonabelian cohomology with values in G.

The non-obvious part is the definition of non-flat and non-fake flat differential cohomology. For some ω-groups G we can form

$$\mathbf{B}G \hookrightarrow \mathbf{BE}G \longrightarrow \mathbf{BB}G\ ,$$

large classes of explicit examples will be constructed in terms of integrated L_∞-algebras below.

In such a case we define

- nonabelian differential cohomology with values in G

to be the joint pullback

$$\begin{tikzcd}
 & & \bar{H}(-, \mathbf{B}G) \arrow[dll] \arrow[d] \arrow[drr] & & \\
H(-, \mathbf{B}G) \arrow[dr] & & H^{\Pi}(-, \mathbf{BE}G) \arrow[dl] \arrow[dr] & & \Omega^\bullet_{\mathrm{flat}}(-, \mathbf{BB}G) \arrow[dl] \\
 & H(-, \mathbf{BE}G) & & H^{\Pi}(-, \mathbf{BB}G) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \begin{array}{c}G\text{-bundle}\\ \text{with connection}\\ \text{and curvature}\end{array} \arrow[dll] \arrow[d] \arrow[drr] & & \\
G\text{-bundle} \arrow[dr] & & \begin{array}{c}G\text{-valued}\\ \text{connection}\\ \text{and curvature}\end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c}\text{characteristic}\\ \text{forms}\end{array} \arrow[dl] \\
 & & & \begin{array}{c}\text{characteristic}\\ \text{classes}\end{array} &
\end{tikzcd}$$

The various morphisms here are best understood in terms of the codescent objects $\Pi^Y(X)$ and $\Pi_0^Y(X)$: the above pullback says that the cocycles in $\bar{H}(-, \mathbf{B}G)$ are the cocycles

$$\Pi^Y(X) \longrightarrow \mathbf{BE}G$$

in $H^\Pi(-, \mathbf{BE}G)$ which fit into a square

$$\begin{tikzcd}
\Pi_0^Y(X) \arrow[r] \arrow[d, hook, "i"] & \mathbf{B}G \arrow[d] & G\text{-cocycle} \\
\Pi^Y(X) \arrow[r] \arrow[d, two heads, "\pi"] & \mathbf{BE}G \arrow[d, hook] & \text{connection and curvature} \\
\Pi(X) \arrow[r] & \mathbf{BBG} & \text{characteristic forms}
\end{tikzcd} \quad . \tag{7}$$

The bottom morphism represents $\mathbf{BB}G$-valued forms. Precomposition of that with the lower left vertical arrow π is the map

$$\Omega^\bullet(-, \mathbf{BB}G) \to H^\Pi(-, \mathbf{BB}G)\,.$$

Postcomposition with the lower right vertical morphism is the map

$$H^\Pi(-, \mathbf{BE}G) \to H^\Pi(-, \mathbf{BB}G)\,.$$

Precomposition with the upper left vertical morphism i is the map

$$H^\Pi(-, \mathbf{BE}G) \to H(-, \mathbf{BE}G)\,.$$

Finally, postcomposition with the upper right vertical morphism is the map

$$H(-, \mathbf{B}G) \to H(-, \mathbf{BE}G)\,.$$

4.2 Representations, sections and holography

4.2.1 For Lie ω-groups

A representation of a 1-group G is a morphism

$$\rho : \mathbf{B}G \to \mathrm{Set}\,.$$

Pulling back the universal Set-bundle

$$T_{\mathrm{pt}}\mathrm{Set} \to \mathrm{Set}$$

along this representation yields the action groupoid

$$V \longrightarrow V//G \longrightarrow \mathbf{B}G \quad . \tag{8}$$

Under the hom-adjunction

$$\mathrm{Hom}(Y^\bullet \otimes I, T) \simeq \mathrm{Hom}(Y^\bullet, \mathrm{hom}(I, T))$$

$$\begin{tikzcd}
 & V//G \arrow[r] \arrow[d] & T_{\mathrm{pt}}\mathrm{Set} \arrow[d] \\
Y^\bullet \arrow[r, "g"] \arrow[ur, dashed, "\sigma_u"] \arrow[urr, bend left, "\sigma_c"] \arrow[dr] & \mathbf{B}G \arrow[r, "\rho"] & \mathrm{Set} \\
 & \mathrm{pt} \arrow[ur] \arrow[u, Rightarrow, "\sigma_t"'] &
\end{tikzcd}$$

Figure 4: **Sections of cocycles**. Given a G-valued cocycle $g : Y^\bullet \to \mathbf{B}G$ and a representation $\rho : \mathbf{B}G \to \mathrm{Set}$ we are asking for the set of sections $\Gamma(\rho[g])$ of the corresponding ρ-associated G-bundle. Such a section is, equivalently, any one of the three morphisms carrying the symbol σ: The *t*ransformation σ_t from the terminal cocycle *into* our given cocycle is given in *c*omponents by the functor σ_c which in turn, by the *u*niversal property, is given by the morphism σ_u.

morphism between transport functors

$$\begin{tikzcd}
 & \mathbf{B}G \arrow[dr, "\rho"] \arrow[dd, Rightarrow, "\phi"] & \\
Y^\bullet \arrow[ur, "g"] \arrow[dr, "h"'] & & T \\
 & \mathbf{B}G' \arrow[ur, "\rho'"'] &
\end{tikzcd}$$

Correspond to morphism

$$\phi : \; Y^\bullet \longrightarrow T^I$$

which are themselves transport functors. This is essentially holography.

Such morphisms correspond to bi-section

$$\begin{tikzcd}
 & V//G \times G' \arrow[r] \arrow[d] & T^I \arrow[d] \\
Y^\bullet \arrow[r, "g\times h"'] \arrow[ur, dashed, "\sigma"] & \mathbf{B}(G \times G') \arrow[r, "\rho\times\rho'"] & T
\end{tikzcd} \; .$$

The twisted n-bundles appearing in String theory, corresponding to the Green-Schwarz and the Freed-Witten anomaly cancellation are of this origin.

4.2.2 For L_∞-algebras

Following the structure of action groupoids of ω-group representations 8, we *define* a representation of an L_∞-algebra $\mathfrak{g}$ on an ∞-vector space in terms of a cochain complex V as an extension of $\mathfrak{g}$ by V:

$$\wedge^\bullet V^* \twoheadleftarrow \mathrm{CE}(\mathfrak{g}, V) \hookleftarrow \mathrm{CE}(\mathfrak{g}) \; .$$

Every L_∞-algebra $\mathfrak{g}$ has an *adjoint representation* on itself: let $\sigma^{-1} : \mathfrak{g}^* \to \mathfrak{g}^*[-1]$ be the canonical isomorphism extended as a degree -1 derivation to all of $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[-1])$. Then define

$$\mathrm{CE}(\mathfrak{g}, \mathfrak{g}) := (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[-1]), d_{\mathrm{ad}})$$

with

$$d_{\mathrm{ad}} a := d_{\mathrm{CE}(\mathfrak{g})} a$$

and

$$d_{\mathrm{ad}} \sigma^{-1} a = -\sigma^{-1} d_{\mathrm{CE}(\mathfrak{g})} a$$

for all $a \in \mathfrak{g}$. This is nilpotent due to

$$d_{\mathrm{ad}}^2 \sigma^{-1} a = -d_{\mathrm{ad}} \sigma^{-1} d_{\mathrm{CE}(\mathfrak{g})} a = -[d_{\mathrm{ad}}, \sigma^{-1}] d_{\mathrm{CE}(\mathfrak{g})} a = 0\,.$$

For every Ł$_\infty$-representation ρ of $\mathfrak{g}$ on V with the special property that

$$dV^* \subset V^* \otimes \wedge^\bullet(\mathfrak{g}^* \oplus V^*)$$

there is a *dual representation* ρ^* of $\mathfrak{g}$ on V^*.

We can generalize the definition of L_∞-algebras and their representations from monoids in cochain complexes of vector spaces to monoids of cochain complexes of modules over any commutative algebra. These arise automatically when we take the L_∞-connections discussed in 4.3 and transgress them to mapping spaces.

The BRST complex of an L_∞ algebra $\mathfrak{g}$ acting on a submanifold conf_S of configuration space conf is then seen to be the representation of $\mathfrak{g}$ on the resolution of conf_S in conf.

4.3 Differential cocycles from Cartan-Ehresmann L_∞-connections

In [41] the Lie ∞-algebraic analog of diagram 7 was considered, termed a higher Cartan-Ehresmann- or L_∞-connection.

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, twoheadrightarrow] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[u, twoheadrightarrow] \\
\Omega^\bullet(X) \arrow[u, twoheadrightarrow] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "{\{P_i(F_A)\}}"'] \arrow[u, twoheadrightarrow]
\end{tikzcd}$$

Applying

$$\Pi_\omega \circ S : \mathrm{DGCA}s \to \omega\mathrm{Cat}$$

to the entire diagram turns it into a diagram in ωCat. Assume that either the fibers of Y are n-connected (for G an n-group) or that we can pull back to a Y whose fibers are. Then it may happen that we can smoothly *choose* 1-paths between all pairs of points in the fibers, 2-paths between triangles of such 1-paths, and so on. Using the fact that A_{vert} is *flat*, the restriction to those chosen paths yields a differential G-cocycle.

5 Differential Chern-Simons n-cocycles

5.1 L_∞-algebra cohomology and String-like extensions

5.2 Chern-Simons cocycles from obstructions to String-like lifts

6 Differential cocycles and local nets of algebras

Nets of operator algebras are functors from subcategories of open subsets of pseudo-Riemannian spaces to a poset of subalgebras of some ambient algebra, usually that of bounded operators on some Hilbert space.

Using the pseudo-Riemannian structure, one singles out those pairs of subsets which are spacelike separated. A net of operators is called *local* if the subalgebras assigned to spacelike separated subsets commute with each other.

Nets of local operator algebras have been introduced in order to formalize the concept of the algebra of observables in quantum field theory.

Out of the study of these structures a large subfield of mathematical physics has developed, which is equivalently addressed as *algebraic quantum field theory*, or as *axiomatic quantum field theory* or as *local quantum field theory*, but usually abbreviated as **AQFT**.

To my mind, all three of these terms as such would equally well describe also what is probably the main alternative parallel development, as endeavours towards giving quantum field theory a good axiomatic framework: the study of representations of cobordism categories.

While this approach did apparently not receive a canonical name so far, I am used to referring to it as *functorial quantum field theory*. Here I shall abbreviate that as **FQFT**.

An obvious question is: **What is the precise relation between AQFT and FQFT?**

I am not aware of any explicit attempt to answer this.

To a large extent, developments in AQFT and FQFT have been, in the past, rather disconnected.

The most successful – strikingly successful – application of AQFT has actually been to chiral 2-dimensional conformal field theory. Here AQFT has provided a rich collection of results, notably important classification results. (On the other hand it is still unclear, as far as I am aware, how to realize the main motivating example, 4-dimensional gauge theory of Yang-Mills type, in the language of AQFT.)

The most well known application of FQFT is to topological quantum field theory (TQFT): the theory of representations of categories of cobordisms up to diffeomorphism. This goes so far that some people have expressed the believe that FQFT = TQFT instead of FQFT $\supset$ TQFT. While this is actually not the case – since whenever we have a category of cobordisms with extra structure (conformal, Riemannian) the notion of FQFT on it makes sense – it is true that the tractability of FQFT away from the topological realm drops sharply.

But progress is visible. The closest possible point of contact between AQFT and FQFT obtained so far is possible the description of *full* 2-dimensional conformal field theory in terms of a topological QFT internal to the representation category of a chiral net, as given by Fuchs, Runkel and Schweigert (FRS).

I have indicated elsewhere how at least parts of the topological aspect of the FRS description arises from a "local trivialization" of an ($n = 3$) extended FQFT transport 3-functor. The discussion to follow can be regarded as providing also the connection between this n-functor and the chiral nets. But many details remain to be better understood.

In any case, it is clear that both AQFT and FQFT are relevant for understanding what quantum field theory really is. Since there is just one reality, there should be a way to relate them systematically.

Here I shall try to present evidence which suggests that the situation is as indicated by the following slogan:

AQFT is to FQFT like the Heisenberg picture of quantum mechanics is to the Schrödinger picture. Forming endomorphism algebras provides a systematic map from FQFT to AQFT.

For this to work, it is important to understand FQFT as what is sometimes called *extended functorial QFT*: originally FQFT was conceived as being about functors from cobordisms categories to some category of vector spaces. But later it was realized that, more generally, one wants to model *extended* cobordsisms, which live in higher categories. Hence an extended n-dimensional functorial quantum field theory is an n-functor.

6.1 The situation for 1-dimensional QFT

To put the following construction into perspective, it is useful to indicate what the transition from FQFT to AQFT that we are after looks like for the simple case where we are dealing with 1-dimensional quantum field theory, also known as quantum mechanics.

Therefore let $X = \mathbb{R}$ be the real line, thought of as the *worldline* of a particle. Let $P_1(X)$ be the category of paths in X. Here this is simply the category whose objects are the points of X and which has a unique morphism from x to y whenever $x \leq y$. In other words, $P_1(X)$ is $\mathbb{R}$ regarded as a poset.

The FQFT description of a specific realization of such a 1-dimensional QFT is a functor

$$Z : P_1(X) \to \mathrm{Vect}\,.$$

Here we take Vect to be the category whose objects are vector space and whose emorphisms are linear *iso*morphisms.

Depending on the precise details, this functor is usually demanded to factor through vector spaces with suitable extra structure. Topological vector spaces and Hilbert spaces are common choices.

The value of Z over any point $x \in X$ is called the *space of states* at (worldline) time x. The value of Z over any morphism $x \to y$ is called the *time propagator* or *time evolution operator* from x to y.

Given such a functor, we can form for each point $x \in X$ the *endomorphism algebra* of the vector space, by sending

$$x \mapsto \mathrm{End}(Z(x))\,.$$

In the case that there is extra structure on our vector spaces we would demand suitable endomorphisms. In the case of Hilbert spaces one usually demands all endomorphisms to be *bounded* operators.

The endomorphism algebras thus obtained is known often as the *algebra of observables*. In the present case, we would be tempted to associate this algebra at time x with the entire future of x.

So let $S(X)$ be the category whose objects are open sets $O_x := \{x' \in X | x' > x\}$ and whose morphisms are inclusions $O_x \subset O_y$ of open subsets.

Of course, due to the simplicity of the present setup, $S(X)$ is canonically isomorphic to the opposite of $P_1(X)$ itself, hence is itself just the opposite catgeory of $\mathbb{R}$ regarded as a poset. But for the discussions to follow it is useful to think of $S(X)$ as a category of open subsets of X.

The crucial point now is that sending spaces of states to their algebras of endomorphisms sends the functor

$$Z : P_1(X) \to \mathrm{Vect}$$

to a functor

$$A_Z : S(X) \to \mathrm{Algebras}\,.$$

The functor A_Z sends open subsets in $S(X)$ to the algebras of endomorphisms of the spaces of states sitting over their boundary, and it sends inclusions of open subsets to the inclusion of the algebras which is induced from using conjugation with the propagator that is assigned to the path connecting the respective boundaries. More precisely

$$A : (O_y \subset O_x) \mapsto (\ \mathrm{End}(Z(y)) \xhookrightarrow{\mathrm{Ad}_{Z(x\to y)}} \mathrm{End}(Z(x))\ )\,.$$

Of course this means that all inclusions of algebras here are actually isomorphisms. This is again due to the simplicity of the one-dimensional example.

It is this simple situation which we want to generalize from 1- to 2-dimensional QFT.

6.2 Nets of local monoids

We start by considering a simple version of the relevant axioms. Various refinements and generalizations are possible.

Let $X = \mathbb{R}^2$ and let g be the standard Minkowski metric on X.

By a causal subset of X I shall mean as usual an open subset which is the interior of a rectangle all whose sides are lightlike.

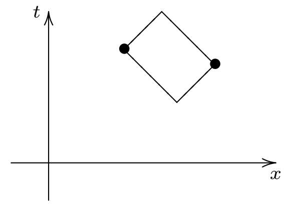


Figure 5: A "causal subset" of 2-dimensional Minkwoski space is the interior of a rectangle all whose sides are lightlike. Such subsets are entirely fixed in particular by their left and right corners.

Definition 14 *We denote by $S(X)$ the category whose objects are open causal subsets $V \subset X$ of X and whose morphisms are inclusions $V \subset V'$.*

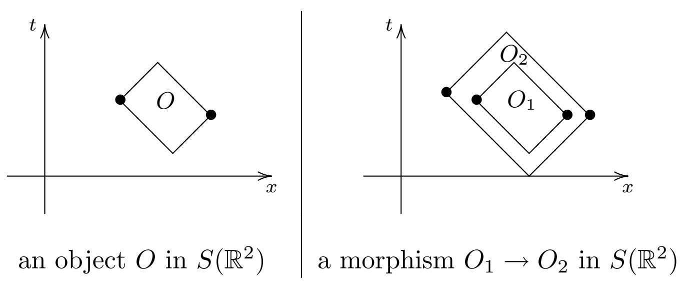


Figure 6: The category $S(\mathbb{R}^2)$ of causal subsets of 2-dimensional Minkowski space. Objects are causal subsets, morphisms are inclusions of these.

Nets of local operators algebras are usually formulated in the context of von Neumann algebras. Before getting into the peculiarities of these special kinds of algebras, I would like to clarify just the underlying structure. Therefore I shall now talk about mere nets of local *monoids*.

Definition 15 *Two objects O_1, O_2 in $S(X)$ are called spacelike separated if all pairs of points $(x_1, x_2) \in O_1 \times O_2$ are spacelike separated.*

Definition 16 *A* **net of monoids** *on 2-dimensional Minkwoski space is a functor*

$$A : S(\mathbb{R}^2) \to \text{Monoids}\,.$$

This is a **net of local monoids** *when*

$$\forall O_1 \subset O_3 \text{ spacelike to } O_2 \subset O_3 \;:\; [\mathrm{im}(A(O_1 \subset O_3)), \mathrm{im}(A(O_2 \subset O_3))] = 0\,.$$

Here the expression on the right says that all pairs (a_1, a_2) in the image of $A(O_1) \times A(O_2)$ under the embedding $A(- \subset O_3)$ commute, $a_1 a_2 = a_2 a_1$.

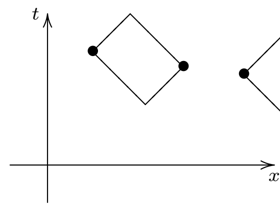
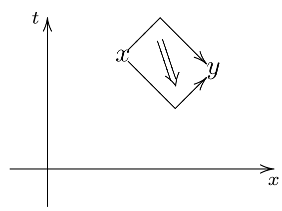


Figure 7: Two spacelike separated causal subsets of $\mathbb{R}^2$.

6.3 Extended 2-dimensional FQFT

Instead of regarding causal subsets as a category under inclusion of subsets, we can think of them as living in a 2-category under *composition* (gluing).

Definition 17 *Let $P_2(\mathbb{R}^2)$ be the 2-category whose objects are the points of $\mathbb{R}^2$, whose morphisms are piecewise lightlike paths in $\mathbb{R}^2$ and whose 2-morphisms are* generated *from the causal bigons*



under gluing along pieces of joint boundary. Composition is by gluing.

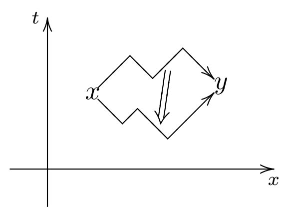


Figure 8: A typical 2-morphism in $P_2(\mathbb{R}^2)$

An extended 2-dimensional FQFT is a 2-functor

$$Z : P_2(\mathbb{R}2) \to C$$

from $P_2(X)$ to some suitable 2-category C of 2-vector spaces, as for instance those whose objects are von Neumann algebras, whose morphisms are bimodules over these with composition being fusion of von Neumann bimodules, and whose 2-morphisms are bimodule homomorphisms.

Here I do not want to get into the technical details of the codomain 2-category of an extended 2-dimensional FQFT. Instead, all I want is to point out how for any choice of such codomain, we obtain a map from 2-dimensional extended FQFTs to nets of local monoids on $\mathbb{R}^2$. The only necessary requirement needed is that the 2-morphisms all be invertible and that horizontal composition by the images of the 1-morphisms under Z is injective.

So let, from now on, C be any 2-category with all 2-morphisms being isomorphisms.

6.4 The main point: AQFT from extended FQFT

We define the map from FQFTs to AQFTs and demonstrate that it indeed sends 2-functors to local nets of monoids.

Definition 18 *Given any extended 2-dimensional FQFT, i.e. a 2-functor*

$$Z : P_2(\mathbb{R}^2) \to C$$

we define a local net of monoids

$$A_Z : S(\mathbb{R}^2) \to \mathrm{Monoids}$$

by defining it on objects as

$$A_Z : \left(\begin{tikzcd} & {} \arrow[dr, dash] & \\ x \arrow[ur, dash] \arrow[dr, dash] & & y \\ & {} \arrow[ur, dash] & \end{tikzcd} \right) \mapsto \mathrm{End}_C \left(Z \left(\begin{tikzcd} x \arrow[dr, dash] & & y \\ & {} \arrow[ur] & \end{tikzcd} \right) \right)$$

and on morphisms as follows.

For any inclusion $O_{x',y'} \subset O_{x,y}$

$$\begin{tikzcd} & & & 1 \arrow[drrr, dash] & & & \\ x \arrow[urrr, dash] \arrow[ddr, dash] & & & 2 \arrow[dl, dash] \arrow[dr, dash] & & & y \\ & & x' \arrow[dr, dash] & & y' \arrow[dl, dash] & & \\ & 3 \arrow[drr, dash] & & 4 & & 5 \arrow[uur, dash] & \\ & & & 6 \arrow[urr, dash] & & & \end{tikzcd}$$

we form the pasting diagram

$$\begin{tikzcd} & & & 1 \arrow[drrr] & & & \\ x \arrow[urrr] \arrow[ddr] & & & 2 \arrow[dr] & & & y \\ & & x' \arrow[ur] \arrow[dr] & & y' \arrow[dr] & & \\ & 3 \arrow[ur] \arrow[drr] & & 4 \arrow[ur] \arrow[d, Rightarrow, "f"] & & 5 \arrow[uur] & \\ & & & 6 \arrow[urr] & & & \end{tikzcd}$$

in $P_2(X)$. Let f' be the 2-morphism obtained by whiskering the indicated 2-morphism f with the 1-morphisms $(x,3)$ and $(5,y)$.

$$f' := \begin{tikzcd}
x \arrow[ddrr] & & & & & & & & y \\
& & & x' \arrow[dr] & & y' \arrow[dr] & & & \\
& & 3 \arrow[ur] \arrow[drr] & & 4 \arrow[ur] \arrow[d, Rightarrow, "f"] & & 5 \arrow[uurr] & & \\
& & & & 6 \arrow[urr] & & & &
\end{tikzcd}$$

For any $a \in \mathrm{End}_C Z(x',4,y')$, let a' be the corresponding re-whiskering by $Z(x,3,x')$ and $Z(y',5,y)$.

Then we obtain in injection

$$\mathrm{End}_C(Z(x',8,y')) \hookrightarrow \mathrm{End}_C(Z(x,3,9,5,y))$$

by setting

$$a \mapsto \mathrm{Ad}_{Z(f)'}(a')\,.$$

Remark. Notice that this prescription is essentially just the one we described already for the 1-dimensional case: to open subsets we assign the endomorphism algebra of the space of states assigned to one part of their boundary. To an inclusion of open subsets we then assign the inclusion of such algebras obtained by *parallel transporting* the algebra of the inner set into the algebra of the outer set using conjugation with the propagators that the 2-functor assigns to 2-morphisms in $P_2(\mathbb{R}^2)$. The difference to the 1-dimensional case here is that this conjugation operation involves some (the obvious) rewhiskering. We will see that it is essentially this rewhiskering which leads to the locality of the net of monoids obtained this way.

Now we come to our main point.

Proposition 1 *The above construction does indeed yield a net of local monoids.*

Proof. We need to demonstrate two things

- that the above assignment is functorial;
- that the above assignment satisfies the locality axiom.

Both properties turn out to be a direct consequence of 2-functoriality of Z and the exchange law in 2-categories.

To see functoriality, consider a chain of inclusions

$$O_{x'',y''} \subset O_{x',y'} \subset O_{x,y}$$

in $S(\mathbb{R}^2)$ and the corresponding pasting diagram

$$\begin{tikzcd}
x \arrow[ddr] & & & & & & 1 \arrow[drrrr] & & & & & & y \\
& & x' \arrow[urrrr] \arrow[ddrr] \arrow[dddr, Rightarrow, "f_l"] & & & & 2 \arrow[dr] & & & & y' \arrow[dr] \arrow[dddl, Rightarrow, "f_r"'] & & \\
& 3 \arrow[ur] \arrow[ddrr] & & & & x'' \arrow[ur] \arrow[dr] & & y'' \arrow[dr] & & & & 4 \arrow[uur] & \\
& & & & 5 \arrow[ur] \arrow[drr] & & 6 \arrow[ur] \arrow[d, Rightarrow, "f'"] & & 7 \arrow[uurr] \arrow[dr] & & & & \\
& & & 8 \arrow[ur] \arrow[drrr] & & & 9 \arrow[urr] \arrow[d, Rightarrow, "f_c"] & & & 10 \arrow[uurr] & & & \\
& & & & & & 11 \arrow[urrr] & & & & & &
\end{tikzcd}$$

in $P_2(\mathbb{R}^2)$.

The direct inclusion

$$\mathrm{End}_C(Z(x'', 6, y'')) \hookrightarrow \mathrm{End}_C(Z(x, 3, 8, 11, 10, 4, y))$$

sends $a \in \mathrm{End}_C(Z(x'', 6, y''))$ to the endomorphism

$$\begin{tikzcd}
Z(x, 3, 8, 11, 10, 4, y) \arrow[d, "Z(f_c \circ f')^{-1}"] \\
Z(x, 3, 8, 5, x'', 6, y'', 7, 10, 4, y) \arrow[d, "a"] \\
Z(x, 3, 8, 5, x'', 6, y'', 7, 10, 4, y) \arrow[d, "Z(f_c \circ f')"] \\
Z(x, 3, 8, 11, 10, 4, y)
\end{tikzcd} .$$

All necessary re-whiskering is notationally suppressed here. No confusion can arise.

On the other hand, the composite inclusion

$$\mathrm{End}_C(Z(x'', 6, y'')) \hookrightarrow \mathrm{End}_C(Z(x', 5, 9, 7, y')) \hookrightarrow \mathrm{End}_C(Z(x, 3, 8, 11, 10, 4, y))$$

sends a to

$$\begin{tikzcd}
Z(x,3,8,11,10,4,y) \arrow[d, "Z(f_l \cdot f_c \cdot f_r)^{-1}"] \\
Z(x,3,x',5,9,7,y',4,y) \arrow[d, "Z(f')^{-1}"] \\
Z(x,3,x',5,x'',6,y'',7,y',4,y) \arrow[d, "a"] \\
Z(x,3,x',5,x'',6,y'',7,y',4,y) \arrow[d, "Z(f')"] \\
Z(x,3,x',5,9,7,y',4,y) \arrow[d, "Z(f_l \cdot f_c \cdot f_r)"] \\
Z(x,3,8,11,10,4,y)
\end{tikzcd} .$$

But by using the exchange law in this pasting diagram, one sees that the contributions by f_l and f_r drop out and both expressions are indeed equal.

To see locality, let $O_{2,4}$ and $O(2',4')$ be two spacelike separated causal subsets inside $O_{(3,5')}$. The relevant pasting diagram in $P_2(\mathbb{R}^2)$ is

$$\begin{tikzcd}
 & & 7 \arrow[dr] & & & & 7' \arrow[dr] & & \\
 & 2 \arrow[ur] \arrow[dr] & & 4 \arrow[dr] & & 2' \arrow[ur] \arrow[dr] & & 4' \arrow[dr] & \\
3 \arrow[ur] \arrow[drr] & & 8 \arrow[ur] \arrow[d, Rightarrow, "f_1"] & & 5 \arrow[ur] \arrow[drr] \arrow[dd, Rightarrow, "f_0"] & & 8' \arrow[ur] \arrow[d, Rightarrow, "f_2"] & & 5' \\
 & & 9 \arrow[urr] \arrow[drr] & & & & 9' \arrow[urr] & & \\
 & & & & 10 \arrow[urr] & & & &
\end{tikzcd}$$

An endomorphism a of $Z(2,8,4)$ induces an endomorphism

$$\begin{tikzcd}
Z(3,9,10,9',5') \arrow[d, "Z(f_0)^{-1}"] \\
Z(3,9,5,9',5') \arrow[d, "Z(f_1)^{-1}"] \\
Z(3,2,8,4,5,9',5') \arrow[d, "a"] \\
Z(3,2,8,4,5,9',5') \arrow[d, "Z(d_1)"] \\
Z(3,9,5,9',5') \arrow[d, "Z(f_0)"] \\
Z(3,9,10,9',5')
\end{tikzcd}$$

of $Z(3,9,10,9',5')$. All necessary re-whiskering is notationally suppressed.

An endomorphism a' of $Z(2',8',4')$ induces an endomorphism

$$\begin{tikzcd}
Z(3,9,10,9',5') \arrow[d, "Z(f_0)^{-1}"] \\
Z(3,9,5,9',5') \arrow[d, "Z(f_2)^{-1}"] \\
Z(3,9,5,2',8',4',5') \arrow[d, "a'"] \\
Z(3,9,5,2',8',4',5') \arrow[d, "Z(f_2)"] \\
Z(3,9,5,9',5') \arrow[d, "Z(f_0)"] \\
Z(3,9,10,9',5')
\end{tikzcd}$$

of $Z(3,9,10,9',5')$. By using the exchange law, one finds that both these endomorphisms of $Z(3,9,10,9',5')$

commute. Their composition in both bossible directions yields the endomorphism

$$\begin{tikzcd}
Z(3,9,10,9',5') \arrow[d, "Z(f_0)^{-1}"] \\
Z(3,9,5,9',5') \arrow[d, "Z(f_1\cdot f_2)^{-1}"] \\
Z(3,2,8,4,5,2',8',4',5') \arrow[d, "a\cdot a'"] \\
Z(3,2,8,4,5,2',8',4',5') \arrow[d, "Z(f_1\cdot f_2)"] \\
Z(3,9,5,9',5') \arrow[d, "Z(f_0)"] \\
Z(3,9,10,9',5')
\end{tikzcd},$$

where $f_1 \cdot f_2$ and $a \cdot a'$ denotes the obvious horizontal composition.

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[57] 2-bundles (or "2-torsors") as such (meaning: as bundles $P \to X$ with 1-categorical fibers, as opposed to other models like gerbes (which are like sheaves of sections of proper 2-bundles) or bundle gerbes (which are actually cocycles in the present sense)) appear generally for instance in the work of Glenn and Breen, also section 7 of [48], and have been highlighted in their concrete nature as higher fiber bundles more recently by Toby Bartels, *2-Bundles* [`arXiv:math/0410328`] and Igor Baković, *Biroupoid 2-Torsors* (PhD thesis), Christoph Wockel, *A global perspective to gerbes and their gauge stacks*

[arXiv:0803.3692]. Notice that in [44, 46] the point is made that higher bundles are conveniently thought of not as fibrations $P \to X$ but as their fiber-assigning functors $X \to n\mathrm{Cat}$. In particular, this achieves a useful *rectification* of the n-stack of these bundles to a sheaf, a fact we are making use of above.

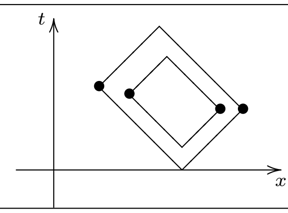
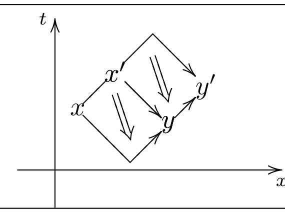
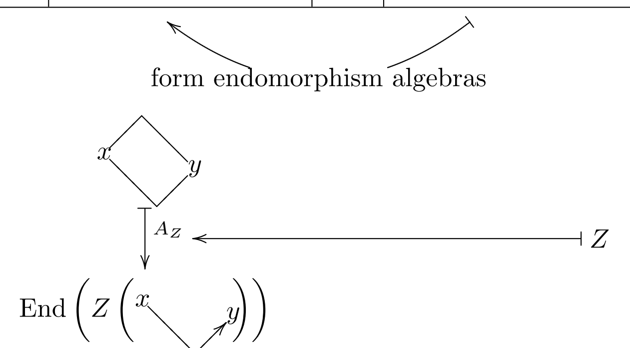
names	algebraic QFT axiomatic QFT local QFT	functorial QFT
abbreviations	**AQFT**	**FQFT**
idea	assign algebras (of observables) to patches, compatible with inclusion	assign (time evolution) operators to patches, compatible with composition (gluing)
axioms due to	Haag, Kastler	Atiyah, Segal
aspect of QFT	Heisenberg picture	Schrödinger picture
formal structure	cosheaf	transport n-functor
cartoon of domain structure		
relation		
main existing general theorems	spin-statistics theorem, PCT theorem	results about topological invariants
main existing nontrivial examples	chiral 2-d CFT	topological QFTs

Table 7: The two approaches to the axiomatization of quantum field theory together with their interpretation and relation as discussed here. I demonstrate below how the map $Z \mapsto A_Z$ indeed produces a local net A_Z from an n-functor Z.