Transgression of n-transport and n-connections

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Abstract

After going through some ground work concerning generalized smooth spaces and their differential graded ommutative algebras of forms, I talk about the issue of transgression of transport ω-functors and of Lie ∞-valued connections to smooth mapping spaces.

I discuss how what we call transgression of ω-functors here is really the morphism part of an internal hom, and how that does reproduce the ordinary notion of transgression of differential forms under the relation between n-transport and differential forms.

Contents

1 Introduction

I want to better understand the

- general systematics

and the

- specific details

of what it means to *transgress*

- transport n-functors [1, 3, 4, 5]

and

- Lie n-algebra valued connections [6]

to mapping spaces.

This is essentially about understanding the pull-push operation of n-transport and n-connections on a "**tar**get space" tar from right to left through a span

$$\begin{tikzcd} & \mathrm{hom}(\mathrm{par}, \mathrm{tar}) \otimes \mathrm{par} \arrow[dl, "p_1"'] \arrow[dr, "\mathrm{ev}"] & \\ \mathrm{hom}(\mathrm{par}, \mathrm{tar}) & & \mathrm{tar} \end{tikzcd}$$

to obtain an n-transport and n-connection on the "**conf**iguration space" of maps

$$\mathrm{conf} := \mathrm{hom}(\mathrm{par}, \mathrm{tar})$$

from some "**par**ameter space" par to tar.

But in fact it turns out that the "good" answer does apparently not quite involve the naive push-forward along p_1, but a slight variant, which then amounts to simply defining the transgression of the n-transport or n-connection tra to be

$$\mathrm{hom}(\mathrm{Id}_{\mathrm{par}}, \mathrm{tra}) .$$

This difference to the naive definition of transgression as direct push-pull through the above span actually takes care of a fact neglected in standard discussions that do not make the n-categorical nature of n-transport manifest: namely that under transgression not only the domain, but also the *co*domain of n-transport and n-connections changes.

For instance, in the simplest kind of example, an ordinary abelian 2-connection is not really something taking values in $U(1)$, but rather something taking values in $\mathcal{B}U(1)$. Transgressing it to loop spaces by setting par $= S^1$ in the above turns it into a 1-connection with values in $\mathrm{hom}(S^1, \mathcal{B}U(1))$, which is indeed

$$\Lambda \mathcal{B}U(1) = U(1)$$

as it should be.

So this general notion of transgression is what shall be discussed here. I then recall the general relation between smooth n-functors and differential forms, and discuss how the above notion of transgression of n-functors does reproduce the ordinary notion of transgression of differential forms.

Before getting into the issue of transgression proper, I try to lay some necessary groundwork on the general concept of generalized smooth spaces and the differential graded-commutative algebras of differential forms on them.

Here I take "generalized smooth spaces" simply to be presheaves over manifolds. This is clearly the right ambient topos, in general, for any discussion of smooth parallel n-transport and smooth n-connections.

My tentative discussion of differential forms on such generalized smooth spaces, and the relation to general differential graded commutative algebras, is included here because I am not aware of a discussion of the necessary points in the literature. This may, however, well be – in parts or possibly even in total – just be due to my woeful ignorance.

Hopefully much of what I am trying to say concerning the general issue of smooth spaces versus differential graded algebras is actually well known, possibly in slightly different guise, in rational homtopy theory.

In any case, after having dealt to some extent with this groundwork, I'll define in more detail the problem of transgression to be discussed here, and then start looking at concrete questions and specific examples.

Acknowedgements. I am indebted to Todd Trimble for teaching me much of the correct abstract stuff appearing here and telling me about a bunch of facts. Of course all the incorrect abstract stuff and the remaining mistakes are mine.

Most of what I say here is aimed at, motivated by and draws from collaborative work with Konrad Waldorf [4] and with Hisham Sati and Jim Stasheff [6].

I benefitted some time ago from conversations with Simon Willerton about the general issue of transgression, for which I am grateful. With Bruce Bartlett I had talked in more detail about the idea of understanding the standard pull-push idea of transgression as an internal hom, which here constitutes 4.2.

2 Differential graded-commutative algebras

Well behaved differential graded commutative algebras correspond, by dualization, to codifferential graded co-commutative coalgebras, which in turn describe semistrict Lie ∞-algebras $\mathfrak{g}$, namely ω-categories internal to vector vector spaces and equipped with a skew-symmetric bracket functor which satisfies a Jacobi identity up to coherent equivalence. In straightforward generalization of the situation for ordinary Lie algebras, we say that the original DGCA is the *Chevalley-Eilenberg algebra* of the Lie ∞-algebra $\mathfrak{g}$.

In this way DGCAs describe the differential aspect of higher categegorical Lie groups, which are one-object ∞-groupoids (for instance modeled as suitably smooth Kan-complexes).

From a homotopy theorists point of view an ∞-groupoid is essentially "the same" as a topological space. A smooth ∞-groupoid is therefore something like a smooth space.

There is a also a more direct route leading from DGCAs to spaces: to some extent, every DGCA arises as (a sub-DGCA of) the algebra of differential forms on *some* smooth space (proposition 3) below.

Definition 1 *We write* DGCAs *for the category whose objects are differential graded commutative algebras of* non-negative *degree, and whose morphisms are linear maps which repsect both the algebra structure and the differential.*

Since they arise as Chevalley-Eilenberg algebras of Lie ∞-algebroids, we will by default denote differential graded commutative algebras by symbols "$\mathrm{CE}(\mathfrak{g}, V)$", where V is an associative algebra and $\mathfrak{g}$ an L_∞-algebra, satisfying certain properties. This will, however, not play any role in detail here.

3 Smooth spaces

In all of the following, fix once and for all S to be one of the following sites:

- $S = \mathrm{Conv}$ – the site of convex subsets of $\mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \cdots$ and all smooth maps between these;
- $S = \mathrm{Manifolds}$ – the site of all smooth manifolds and all smooth morphisms between these.

The objects of S are then used to "probe" smooth spaces. Or rather, a smooth space is now *defined* to by anything which may be "probed" by objects of S.

Definition 2 (smooth spaces) *A presheaf over S we will call a (generalized) smooth space. We write*

$$S^\infty := \mathrm{Set}^{S^{\mathrm{op}}}$$

for the category of set-valued presheaves on S.

The symbol "S^∞" is to remind us of "smooth *s*paces".

The set $X(U)$ which such a presheaf X assigns to an object U of S plays the role of the set of smooth maps from U into X. Precisely if X is itself *represented* by an object of S do these sets precisely coincide with the set of morphisms in S from U to X.

Sometime we will address the set $X(U)$ as the set of **plots** from U into X. This is standard terminology for the special kind of smooth spaces called *Chen smooth spaces* or *diffeological spaces*, discussed in 3.1.

Fact 1 *The category S^∞ is a presheaf topos, hence comes with a couple of nice properties.*

- *S^∞ is* **cartesian closed**.
  - *The tensor product is*
  $$X \times_{S^\infty} Y : U \mapsto X(U) \times_{\mathrm{Set}} Y(U)\,.$$
  - *The internal hom is*
  $$\mathrm{hom}_{S^\infty}(X,Y) : U \to \mathrm{Hom}_{S^\infty}(U \times_{S^\infty} X, Y)\,.$$

I'll generally write Hom for the external hom-set and hom for the internal hom object.

It is this internal hom which will play an important role in the discussion of transgression. Its existence makes the collection of maps from one smooth space to another itself into a smooth space. Trangression is about creating from a structure on a smooth space Y a corresponding structure on the smooth space $\mathrm{hom}_{S^\infty}(X,Y)$, for X any other smooth space.

3.1 Diffeological spaces as smooth spaces

A useful subclass of smooth spaces are those called

- diffeological spaces

or

- Chen-smooth spaces.

These are presheaves X on S which are "quasi representable" in the sense that the set of smooth maps $X(U)$, called **plots** in this case, which they assign to any object U of S, is required to be a subset of the set of set maps from U to some given set X_s,

$$X(U) \subset \mathrm{Hom}_{\mathrm{Set}}(U, X_s)\,.$$

A morphism $X \to Y$ of such diffeological or Chen-smooth spaces is taken to be a morphism of such presheaves which is induced by a map of sets $X_s \to Y_s$.

Definition 3 (diffeological smooth spaces) *The category* Diffeo *of diffeological spaces has as objects smooth spaces X for which there exists a set X_s such that for all $U \in \mathrm{Obj}(S)$ we have $X(U) \subset \mathrm{Hom}_{\mathrm{Set}}(U, X_s)$, and whose morphisms $f : X \to Y$ are morphisms of smooth spaces which come from maps of sets $f_s : X_s \to Y_s$, in that*

$$f(U) : \phi \in \mathrm{Hom}_{\mathrm{Set}}(U, X_s) \mapsto f_s \circ \phi \in \mathrm{Hom}_{\mathrm{Set}}(U, Y_s)$$

for all $U \in \mathrm{Obj}(S)$.

Proposition 1 *The category of diffeological spaces is itself already closed. The internal hom is given as follows: its underlying set is*

$$\mathrm{hom}_{\mathrm{Diffeo}}(X,Y)_s := \mathrm{Hom}_{\mathrm{Set}}(X_s, Y_s)$$

and its plots on any $U \in \mathrm{Obj}(X)$ are precisely those maps of sets $\phi : U \to \mathrm{Hom}_{\mathrm{Set}}(X,Y)$ for which the composite

$$U \times_{\mathrm{Set}} X_s \xrightarrow{\phi\times\mathrm{Id}} \mathrm{Hom}_{\mathrm{Set}}(X_s,Y_s) \times_{\mathrm{Set}} X_s \xrightarrow{\mathrm{ev}} Y_s$$

induces a map of diffeological spaces.

Proof. This follows by tracing everything back to the closedness of Set.

Let $f : X \times Y \to Z$ be a morphism of diffeological spaces, so that $f_s : X_s \times Y_s \to Z_s$ is a map of sets with the property that for any $\phi \in (X\times Y)(U)$ for any $U \in \mathrm{Obj}(S)$ we have

$$(f\circ\phi : \ U \xrightarrow{\phi} X_s \times Y_s \xrightarrow{f_s} Z_s\ ) \in Z_s(U)\,.$$

Then writing

$$\hat{f}_s : X_s \to \mathrm{Hom}_{\mathrm{Set}}(Y_s, Z_s)$$

for the transform of f_s in Set, we find that

$$\hat{f}_s \circ \rho : \ V \xrightarrow{\rho} X_s \xrightarrow{\hat{f}_s} \mathrm{Hom}_{\mathrm{Set}}(Y_s, Z_s)$$

is a plot of $\mathrm{hom}_{\mathrm{Diffeo}}(Y,Z)$ precisely if ρ is a plot of X_s by taking the product of everything with Y_s

$$\begin{tikzcd} \hat{f}_s \circ \rho : \ V \times Y_s \arrow[r, "\rho\times\mathrm{Id}"] & X_s \times Y_s \arrow[r, "\hat{f}_s"] \arrow[rr, bend right, "f_s"'] & \mathrm{Hom}_{\mathrm{Set}}(Y_s,Z_s)\times Y_s \arrow[r, "\mathrm{ev}"] & Z_s \end{tikzcd}\ .$$

Conversely, every diffeological map $\hat{f} : X \to \mathrm{hom}(Y,Z)$ gives rise to a diffeological map $f : X\times Y \to Z$ this way, using the fact that a plot of $X \times Y$ is a plot of X and a plot of Y. □

3.2 Differential forms on smooth spaces

Definition 4 (differential forms on smooth spaces) *We write $\Omega^\bullet$ for the smooth space of differential forms*

$$\Omega^\bullet : U \mapsto \Omega^\bullet(U)\,.$$

A differential form ω on a smooth space X is a morphism of smooth spaces

$$\omega : X \to \Omega^\bullet\,.$$

We write

$$\Omega^\bullet(X) := \mathrm{Hom}_{S^\infty}(X, \Omega^\bullet)$$

for the collection of differential forms on X.

Proposition 2 *For any smooth space X, the set $\Omega^\bullet(X)$ naturally inherits the structure of a graded-commutative differential algebra.*

This DGCA structure on $\Omega^\bullet(X)$ is induced by the dg-algebra structure of $\Omega^\bullet(U)$ for all objects U of S and the fact that exterior derivative and wedge product both commute with pullback of forms.

Definition 5 *We write*

$$\Omega^\bullet : S^\infty \to \mathrm{DGCAs}$$

for the contravariant functor from smooth spaces to differential non-negatively graded-commutative algebras.

Conversely, every differential graded-commutative algebra yields a smooth space.

Definition 6 *We write*

$$\mathrm{Hom}(-, \Omega^\bullet(-)) : \mathrm{DGCAs} \to S^\infty$$

for the contravariant functor which sends any DGCA $\mathrm{CE}(\mathfrak{g}, V)$ *to the smooth space* $X_{(\mathfrak{g},V)}$ *defined by setting*

$$X_{(\mathfrak{g},V)} : U \mapsto \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{CE}(\mathfrak{g}, V), \Omega^\bullet(U))$$

for all $U \in \mathrm{Obj}(S)$.

Proposition 3 *Notice that there is a canonical embedding of the original DGCA into the DGCA of differential forms on the corresponding space*

$$\mathrm{CE}(\mathfrak{g}, V) \hookrightarrow \Omega^\bullet(X_{\mathfrak{g},V}) \ .$$

Proof. This sends each elements k of $\mathrm{CE}(\mathfrak{g}, V)$ to the element of $\mathrm{Hom}_{S^\infty}(X_{(\mathfrak{g},V)}, \Omega^\bullet)$ which for any U sends any $f \in \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(U))$ to $f(k) \in \Omega^\bullet(U)$. □

Fact 2 *The contravariant functors*

$$\Omega^\bullet : \ S^\infty \longleftrightarrow \mathrm{DGCAs} \ : \mathrm{Hom}(-, \Omega^\bullet(-))$$

form an adjunction.

In other words for all smooth spaces X and DGCAs $\mathrm{CE}(\mathfrak{g}, V)$ we have isomorphisms of Hom-sets

$$\mathrm{Hom}_{S^\infty}(\mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{CE}(\mathfrak{g}, V), \Omega^\bullet(-)), X) \simeq \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{CE}(\mathfrak{g}, V), \Omega^\bullet(X))$$

natural in both arguments.

(** Urs: Thanks to Todd for this fact. I AM GUESSING that we should get more than an adjunction *in cohomology*, and possibly with some nice assumptions here and there, but I don't know. **)

3.3 Currents on smooth spaces

Using the construction of smooth spaces of maps, we can equip for any smooth space X the dg-algebra of differential forms, whose underlying set was defined to be

$$\Omega^\bullet(X) = \mathrm{Hom}_{S^\infty}(X, \Omega^\bullet)$$

with the structure of a smooth space, by setting, instead,

$$\Omega^\bullet(X) := \mathrm{hom}_{S^\infty}(X, \Omega^\bullet) \,.$$

Proposition 4 *The exterior differential*

$$d : \Omega^\bullet(X) \to \Omega^\bullet(X)$$

and the wedge product

$$\wedge : \Omega^\bullet(X) \times \Omega^\bullet(X) \to \Omega^\bullet(X)$$

are morphisms of smooth spaces.

Proof. I think this amounts to a triviality after just writing out what this means in detail, which is straightforward but a little lengthy. But I should check this again. $\square$

Using the smooth structure on $\Omega^\bullet(X)$ we can define *currents*, the generalization of distributions from 0-forms to arbitrary forms.

Definition 7 (currents) *For X any smooth space, and $p \in \mathbb{N}$, a degree p current c on X is a smooth linear map*

$$c : \Omega^p(X) \to \mathbb{R}\,.$$

The space of all currents

$$\mathbf{C}_\bullet(X) := \hom_{S^\infty}(\Omega^\bullet(X), \mathbb{R})$$

inherits the structure of a smooth chain complex.

3.3.1 Examples

Definition 8 (δ-currents) *For X any* compact orientable manifold *of dimension n and $f : X \to Y$ any smooth map, we obtain a current*

$$\delta_{X,f} \in \mathbf{C}(Y)_n$$

on Y defined on any $\omega \in \Omega^\bullet(Y)$ by

$$\delta_{X,f} : \omega \mapsto \int_X f^*\omega\,,$$

where we take the integral of an n-form over an m-dimensional compact manifold to be zero when $m \neq n$.

3.4 Differential forms on spaces of maps

A large class of differential forms on spaces of maps $\mathrm{maps}(X, Y)$ arises from pairs

$$(c, \omega) \in \mathbf{C}_\bullet(X) \otimes \Omega^\bullet(Y)$$

consisting of a current c on X and a differential form ω on Y.

Proposition 5 *For X and Y smooth spaces and for $(c, \omega) \in \mathbf{C}_n(X) \otimes \Omega^m(Y)$, the assignment*

$$f \in \mathrm{Hom}_{S^\infty}(U \otimes X, Y) \mapsto c(f^*\omega) \in \Omega^{m-n}(U)$$

for all $U \in \mathrm{Obj}(S)$ defines a differential form

$$\omega_c \in \Omega^{m-n}(\mathrm{maps}(X, Y))\,.$$

3.4.1 Examples

Let x be any point in X and write $\delta_x \in \mathbf{C}_0(X)$ for the corresponding delta-distribution 0-current which evaluates functions ar x. Let $F \in \Omega^0(Y)$ be any smooth function on Y, then we obtain a smooth function

$$F_{\delta_x} \in \Omega^0(\mathrm{maps}(X, Y))$$

whose value on any map $\gamma : X \to Y$ is $F(\gamma(x))$.

3.5 Transgression of differential forms on smooth spaces

Definition 9 (transgression of differential forms) *Let X be a* compact manifold *of dimension n. We obtain a smooth map of smooth DGCAs*

$$\mathrm{tg}_{X,Y} : \Omega^\bullet(Y) \longrightarrow \Omega^\bullet(\mathrm{hom}(X,Y))$$

by composing the pullback along

$$\mathrm{ev} : \mathrm{hom}(X,Y) \times X \to Y$$

with the δ-current from definition 8:

$$\begin{tikzcd} \mathrm{tg}_{X,Y} : \Omega^\bullet(Y) \arrow[rr, "\mathrm{tg}_{X,Y}"] \arrow[dr, "\mathrm{ev}^*"'] & & \Omega^\bullet(\mathrm{hom}(X,Y)) \\ & \Omega^\bullet(\mathrm{hom}(X,Y) \times X) \arrow[ur, "\delta_X"'] & \end{tikzcd}$$

Remark. This is usually interpreted as the pull-push operation from right to left through the span

$$\begin{tikzcd} & \mathrm{hom}(\mathrm{par}, \mathcal{P}) \otimes \mathrm{par} \arrow[dl, "p_1"] \arrow[dr, "\mathrm{ev}"] & \\ \mathrm{hom}(\mathrm{par}, \mathcal{P}) & & \mathcal{P} \end{tikzcd}$$

where the push-forward along p_1 corresponds to the "fiber integration" δ_X.

We will define a notion of trangression that makes sense more abstractly in 4, which will reproduce the above notion of transgression of differential forms using the relation between differential n-forms and $\mathcal{B}^n U(1)$-valued smooth ω-functors described in 9.

Proposition 6 *For X a closed manifold, the curvature of a transgressed form is the transgression of the original curvature: for Y any smooth space and $\omega \in \Omega^\bullet(Y)$ any differential form on Y, we have*

$$d(\mathrm{tg}_{X,Y}\omega) = \mathrm{tg}_{X,Y}(d\omega)\,.$$

Proof. The transgressed form is the presheaf morphism

$$\mathrm{tg}_{X,Y}\omega : \mathrm{hom}(X,Y) \to \Omega^{n-1}$$

given by

$$\mathrm{ev}_U \longmapsto \int_X \mathrm{ev}_U^* B \ \ .$$

$$U \qquad \mathrm{Hom}_{S^\infty}(U \times X, Y) \longrightarrow \Omega^\bullet(U)$$

Acting on this with the differential $d : \Omega^\bullet(\mathrm{hom}(X,Y)) \to \Omega^\bullet(\mathrm{hom}(X,Y))$ yields the presheaf morphism

$$\mathrm{ev}_U \mapsto d_U \int_X \mathrm{ev}_U^* B = \int_X d_U \mathrm{ev}_U^* B = \int_X (d_U + d_X)\mathrm{ev}_U^* B = \int_X \mathrm{ev}_U^* dB\,.$$

For all $U \in \mathrm{Obj}(S)$ we have

$$d_{X \times U} = d_X + d_U\,.$$

□

3.5.1 Examples

Let $X = S^1 \simeq \mathbb{R}/\mathbb{Z}$ be the circle. For any abject U of S, let $\frac{\partial}{\partial s} \in \Gamma T(U \times X)$ be the canonical vector field around the S^1 factor.

Given any differential form $B \in \Omega^n(Y)$, its transgression to $\mathrm{hom}(X, Y)$ is the morphism of presheaves

$$\mathrm{tg} : \mathrm{hom}(X, Y) \to \Omega^{n-1}$$

given by

$$\mathrm{ev}_U \longmapsto \int_{[0,1]} (\iota_{\frac{\partial}{\partial s}} \mathrm{ev}_u^* B)(s)\, ds \ .$$

$$U \qquad \mathrm{Hom}_{S^\infty}(U \times X, Y) \longrightarrow \Omega^{n-1}(U)$$

4 ω-Functor trangression

4.1 Smooth ω-functors and differential forms

For Y any smoth space, denote by $\mathcal{P}_n(Y)$ the smooth n-groupoid of thin-homotopy classes of n-paths in X.

Proposition 7 *Smooth n-functors*

$$\mathrm{tra} : \mathcal{P}_n(Y) \to \mathcal{B}^n U(1)$$

are in bijective correspondence with differential n-forms $\omega \in \Omega^n(X)$. In fact, these n-forms naturally live in an n-category $Z^n_{\mathrm{Id}}(Y)$ and we have a canonical isomorphism

$$n\mathrm{Funct}^\infty(Y, \mathcal{B}^n U(1)) \simeq Z^n_{\mathrm{Id}}(Y)$$

given by exponentiated intgration of forms.

Proof. For $n = 1$ this is shown in [3]. For $n = 2$ in [4]. The same kind of proof goes through for all n. □

4.2 The notion of transgression

We define what we will understand under the *transression* of an ω-functor, comment on how this fits into the general logic of transgression, and then discuss a couple of examples, proving in particular (proposition 10 and corollary 1 below) that under the identification of ω-functors with differential forms from proposition 7 our ω-functor transgression reproduces the ordinary notion of transgression of differential forms from definition 9.

For definiteness, consider ω-categories, living in the closed monoidal category $(\omega - \mathrm{Cat}, \otimes)$ described in [2].

Definition 10 (transgression for ω-functors) *Given an ω-functor*

$$\mathrm{tra} : \mathcal{P} \to T$$

and given an ω-category par*, we call the ω-functor*

$$\mathrm{hom}_{\omega\mathrm{Cat}}(\mathrm{par}, \mathrm{tra}) : \mathrm{hom}_{\omega\mathrm{Cat}}(\mathrm{par}, \mathcal{P}) \to \mathrm{hom}_{\omega\mathrm{Cat}}(\mathrm{par}, t)$$

the transgression of tra *to* par.

Throughout I write Hom for the external and hom for the internal hom.

Remark: Interpretation of transgression as a pull-push operation. Transgression of ω-functors as defined above is essentially the pull-push operation from right to left through the span

$$
\begin{tikzcd}
& \mathrm{hom}(\mathrm{par},\mathcal{P})\otimes \mathrm{par} \arrow[dl, "p_1"'] \arrow[dr, "\mathrm{ev}"] & \\
\mathrm{hom}(\mathrm{par},\mathcal{P}) & & \mathcal{P}
\end{tikzcd}
\,.
$$

Pulling back $\mathrm{tra} : \mathcal{P} \to T$ along the right leg of this span yields the ω-functor $\mathrm{ev}^*\mathrm{tra}$

$$
\begin{tikzcd}
& \mathrm{hom}(\mathrm{par},\mathcal{P})\otimes \mathrm{par} \arrow[dl] \arrow[dr, "\mathrm{ev}"] \arrow[drr, bend left, "\mathrm{ev}^*\mathrm{tra}"] & & \\
\mathrm{hom}(\mathrm{par},\mathcal{P}) & & \mathcal{P} \arrow[r, "\mathrm{tra}"] & T
\end{tikzcd}
\,.
$$

Then push-forward of $\mathrm{ev}^*\mathrm{tra}$ along the left leg amounts essentially to passing to the hom-adjunction transform image under

$$
k \;:\; \mathrm{Hom}(\mathrm{hom}(\mathrm{par},\mathcal{P})\otimes \mathrm{par}, T) \;\xrightarrow{\simeq}\; \mathrm{Hom}(\mathrm{hom}(\mathrm{par},\mathcal{P}), \mathrm{hom}(\mathrm{par}, T))\,.
$$

We have

$$
\mathrm{hom}(\mathrm{par},\mathrm{tra}) = k(\mathrm{ev}^*(\mathrm{tra}))
$$

as one checks in components. (This should follow immediately from trivial abstract nonsense, but I am not sure how to write that down.)

To see that k is essentially push-forward along p_1, let

$$
R : \mathrm{hom}(\mathrm{par},\mathcal{P})\otimes \mathrm{par} \to L
$$

$$
Q : \mathrm{hom}(\mathrm{par},\mathcal{P}) \to L
$$

be any 2-n-functors, and let

$$
r : \mathrm{Hom}(\mathrm{hom}(\mathrm{par},\mathcal{P}), L) \hookrightarrow \mathrm{Hom}(\mathrm{hom}(\mathrm{par},\mathcal{P}), \mathrm{hom}(\mathrm{par}, L))
$$

be the injection induced by the injection

$$
L \hookrightarrow \mathrm{hom}(\mathrm{par}, L)
$$

given by

$$
l \mapsto (\mathrm{par} \to \{\bullet\} \xrightarrow{l} L)\,.
$$

Then we have

$$
\mathrm{Hom}(R, p_1^* Q) \simeq \mathrm{Hom}(k(R), r(Q))
$$

by the hom-adjunction

$$
\begin{tikzcd}
\mathrm{hom}(\mathrm{par},\mathcal{P})\otimes \mathrm{par} \arrow[rr, bend left, "F"] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\Downarrow"] & L \\
& \mathrm{hom}(\mathrm{par},\mathcal{P}) \arrow[ur, "Q"'] &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
\mathrm{hom}(\mathrm{par},\mathcal{P}) \arrow[rr, bend left, "k(F)"] \arrow[rr, bend right, "r(Q)"'] \arrow[rr, phantom, "\Downarrow"] & & \mathrm{hom}(\mathrm{par}, L)
\end{tikzcd}
\,.
$$

4.3 Loop 2-groupoids

Let $G_{(2)}$ be a strict smooth 2-group coming from the crossed module

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

and $\mathcal{B}G_{(2)}$ the strict 1-object 2-groupoid it corresponds to. It is a smooth 2-groupoid, i.e. a strict 2-groupoid internal to S^∞.

Definition 11 (loop groupoid) *The 2-groupoid of strict 2-functors from the free abelian group on one generator to $G_{(2)}$*

$$\Lambda G_{(2)} := \mathrm{Hom}_{\mathrm{smooth2-groupoids}}(\mathcal{B}\mathbb{Z}, \mathcal{B}G_{(2)})$$

is called the loop groupoid of $G_{(2)}$.

The terminology here generalizes that introduced by Willerton, who considered loop 1-groupoids of ordinary 1-groups.

Proposition 8 *The loop 2-groupoid $\Lambda G_{(2)}$ is characterized in components as follows:*

- *Its space of objects is isomorphic to G.*
- *Its space of 1-morphisms is isomorphic to $(G \times G) \times (G \ltimes_\alpha H)$. Source and target maps are the projections on the first and second G-factor, respectively, and composition is given by*

  $$(s_2, t_2, (g_2, h_2)) \circ (s_1, t_1 = s_2, (g_1, h_1)) = (s_1, t_2, (g_2 g_1, h_2 \alpha(g_2)(h_1)))$$

  (the details of the appearance of semidirect product on the right depend on a pair of arbitrary conventions which define the choice of isomorphism between crossed modules and strict 2-groups.)

Proof. Consider this pasting diagram equation in $\mathcal{B}G_{(2)}$

$$\begin{tikzcd}
\bullet \arrow[r, "g_1"] \arrow[d, "s_1"] & \bullet \arrow[r, "g_2"] \arrow[d, "t_2 = s_2"'] \arrow[dl, Rightarrow, "h_1"] & \bullet \arrow[d, "t_2"] \arrow[dl, Rightarrow, "h_2"] & = & \bullet \arrow[r, "g_2 g_1"] \arrow[d, "s_1"] & \bullet \arrow[d, "t_2"] \arrow[dl, Rightarrow, "h_1"] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet & & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

□

4.3.1 Examples

Proposition 9 *The transgression of a $G_{(2)}$ 2-transport on X, truncated as degree 2, is a $\Lambda G_{(2)}$ 1-transort on loops in X. More precisley, if*

$$\mathrm{tra} : \mathcal{P}_2(X) \to T$$

is a locally $(\mathcal{B}G_{(2)} \hookrightarrow T)$-trivializable 2-transport on X, then

$$\mathrm{hom}(\mathcal{B}\mathbb{Z}, \mathrm{tra}) : (\Lambda \mathcal{P}_2(X))_1 \to (\Lambda T)_1$$

is a locally $(\Lambda G_{(2)} \hookrightarrow \Lambda T)$-trivializable 1-transport.

Proof. We simply pick any local trivialization

$$
\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, two heads, "\pi"] \arrow[d, "\mathrm{triv}"'] & \mathcal{P}_2(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\simeq"', "t"] \\
\mathcal{B}G_{(2)} \arrow[r, hook, "i"] & T
\end{tikzcd}
$$

of tra and hit the entire diagram with $\hom(\mathcal{B}\mathbb{Z}, -)$ to obtain a smooth local trivialization of $\hom(\mathcal{B}\mathbb{Z}, \mathrm{tra})$

$$
\begin{tikzcd}
\hom(\mathcal{B}\mathbb{Z}, \mathcal{P}_2(Y)) \arrow[r, two heads, "\hom(\mathcal{B}\mathbb{Z},\pi)"] \arrow[d, "\hom(\mathcal{B}\mathbb{Z},\mathrm{triv})"'] & \hom(\mathcal{B}\mathbb{Z}, \mathcal{P}_2(X)) \arrow[d, "\hom(\mathcal{B}\mathbb{Z},\mathrm{tra})"] \arrow[dl, Rightarrow, "\simeq"', "\hom(\mathcal{B}\mathbb{Z},t)"] \\
\Lambda G_{(2)} \arrow[r, hook, "i"] & \hom(\mathcal{B}\mathbb{Z}, T)
\end{tikzcd}
$$

$\square$

Proposition 10 (nonabelian Stokes theorem transgressed to loop space) *Let $G_{(2)}$ be a strict smooth 2-group and let $\mathfrak{g}_{(2)}$ be the corresponding Lie 2-algebra. Let*

$$\mathrm{tra} : \mathcal{P}_2(Y) \to \mathcal{B}G_{(2)}$$

be a smooth 2-functor which, by [4] and [6] corresponds bijectively to a $\mathfrak{g}_{(2)}$-valued form

$$(A, B) \in \Omega^\bullet(Y, \mathfrak{g}_{(2)})$$

with vanishing 2-form curvature

$$F_A + t_* B = 0$$

hence to a DGCA morphism

$$\Omega^\bullet(Y) \xleftarrow{((A,B),(F_A+t_*B=0,H=d_AB))} \mathrm{W}(\mathfrak{g})$$

then the transgressed 1-functor

$$\hom(\mathcal{B}\mathbb{Z}, \mathrm{tra}) : \Lambda\mathcal{P}_2(Y) \to \Lambda G_{(2)}$$

comes from a 1-form

$$\gamma \mapsto \int_{[0,1]} \alpha_{\mathrm{tra}(\gamma(s-))}(\delta_{s,\frac{d}{ds}} \mathrm{ev}^* B) ds$$

on loops in Y.

Proof. Once I fix the notation such as to actually make sense, this will follow from [4]. $\square$

Corollary 1 *Under the relation between smooth $\mathcal{B}^n U(1)$-valued n-functors and differential n-forms, our n-functor transgression from definition 10 reproduces the ordinary transgression of differential forms.*

5 ∞-Connection transgression

In [6] $\mathfrak{g}$-connections for $\mathfrak{g}$ any Lie ∞-algebra had been defined as follows.

Definition 12 *A $\mathfrak{g}$-valued form on the smooth space Y is a DGCA morphism*

$$\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \ .$$

A flat $\mathfrak{g}$-valued form is one which factors through $\mathrm{CE}(\mathfrak{g})$

$$\begin{tikzcd} \Omega^\bullet(Y) & & \mathrm{W}(\mathfrak{g}) \arrow[ll, "{(A,F_A=0)}"'] \arrow[dl, two heads] \\ & \mathrm{CE}(\mathfrak{g}) \arrow[ul] & \end{tikzcd} .$$

Definition 13 ($\mathfrak{g}$-connection) *Let $\mathfrak{g}$ be any Lie ∞-algebra. A $\mathfrak{g}$-connection* descent object *on a smooth space X is a structure consisting of a choice of surjective submersion*

$$\pi : Y \to X$$

playing the role of a cover of X, together with dg-algebra morphisms constituting a commuting diagram

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[u, two heads] \\ \Omega^\bullet(X) \arrow[u, hook, "\pi^*"] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{K_i\}"'] \arrow[u, hook] \end{tikzcd} .$$

Two such descent objects are taken to be equivalent if they are concordant in the obvious natural sense. An equivalence class of $\mathfrak{g}$-connection descent objects is a $\mathfrak{g}$-connection.

5.1 Universal $\mathfrak{g}$-connection and classifying spaces for n-bundles with connection

By hitting everything with the functor

$$\mathrm{Hom}(--, \Omega^\bullet(--)) : \mathrm{DGCA}s \to S^\infty$$

from definition 6, we obtain a definition of $\mathfrak{g}$-connections in terms of maps of smooth spaces

$$\begin{tikzcd} Y_{\mathrm{vert}} \arrow[r, "A^*_{\mathrm{vert}}"] \arrow[d, hook] & X_{\mathfrak{g}} \arrow[d, hook] \\ Y \arrow[r, "{(A^*,F^*_A)}"] \arrow[d, "\pi"] & X_{\mathrm{inn}(\mathfrak{g})} \arrow[d] \\ X \arrow[r, "\{K^*_i\}"] & BG \end{tikzcd} .$$

But care has to be exercised, since $\mathrm{Hom}(-, \Omega^\bullet(-))$ is not an equivalence, about what this does to the equivalence relations among such descent objects.

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[6] H. Sati, J. Stasheff, U.S., *Lie ∞-algebra connections and their application to String- and Chern-Simons n-Transport*