Curvature of Transport

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Abstract

We develop the functorial notion of curvature for parallel transport functors. We discuss how flat sections are morphisms into the transport functor, while general sections and their covariant derivative are morphisms into the corresponding curvature 2-functor.

Contents

1 Introduction

We have learned that parallel n-transport in an n-bundle with connection over a base space X is an n-functor

$$\mathrm{tra} : \mathcal{P}_n(X) \to T$$

from the n-path n-groupoid of X to some n-category of fibers.

With every notion of connection we expect to obtain notions of

- curvature
- Bianchi identity
- parallel sections

and

- covariant derivative.

There are natural functorial incarnations of these concepts. In fact, it turns out that the curvature of a transport functor is itself a transport 2-functor. More generally, the curvature of an n-transport is itself an $(n+1)$-transport. And the covariant derivative is encoded in trivializations of this curvature $(n+1)$-transport.

Curvature. Like $\mathcal{P}_n(X)$ is the path n-groupoid obtained by dividing out thin homotopy classes of n-paths, there is a path n-groupoid called $\Pi_n(X)$ which is obtained by dividing out ordinary homotopy classes of n-paths. This is known as the *fundamental n-groupoid* of X. For $n = 1$ and X connected it is equivalent to the fundamental group of X.

There is a canonical projection

$$\pi : \mathcal{P}_n(X) \to \Pi_n(X)\,.$$

Given any transport n-functor $\mathrm{tra} : \mathcal{P}_n(X) \to T$, we may therefore ask if it descends down to $\Pi_n(X)$, in that there is an n-functor $F : \Pi_n(X) \to T$ such that

$$\begin{tikzcd} \mathcal{P}_n(X) \arrow[r, "\pi"] \arrow[d, "\mathrm{tra}"'] \arrow[dr, phantom, "\swarrow\sim"] & \Pi_n(X) \arrow[d, "F"] \\ T \arrow[r, equal] & T \end{tikzcd}\,.$$

If such a descent exists, we say that tra is *flat*. The obstruction to this descent is an $(n+1)$-functor canonically associated with tra:

$$\mathrm{curv}_{\mathrm{tra}} : \mathcal{P}_{n+1}(X) \to T_{n+1}\,.$$

The functor tra is *flat* precisely if $\mathrm{curv}_{\mathrm{tra}}$ is trivial on all $(n+1)$-morphisms.

For the special case that $n = 1$ and $T = \Sigma G$, with $\mathrm{tra} : \mathcal{P}_1(X) \to \Sigma G$ coming from a 1-form A as in prop. **??**, one finds that $T_2 = \Sigma(\mathrm{INN}(G_2))$ is the inner automorphism 2-group of G and that the 2-functor $\mathrm{curv}_{\mathrm{tra}}$ comes from the curvature 2-form $F_A = dA + A \wedge A$ of A.

Bianchi identity. Since the curvature $\mathrm{curv}_{\mathrm{tra}}$ is itself an $(n+1)$-functor, it makes sense to in turn consider the corresponding $(n+2)$-curvature. One shows that this $(n+2)$-functor is necessarily trivial.

Again in the special case that $n = 1$ and $T = \Sigma G$, one finds that the 3-functor $\mathrm{curv}_{\mathrm{curv}_{\mathrm{tra}}}$ comes from the 3-form $d_A F_A$. The fact that this is trivial is the ordinary Bianchi identity.

Parallel sections. For any object $o \in \mathrm{Obj}(T)$, let

$$\mathrm{tra}_o : \mathcal{P}_n(X) \to T$$

be the transport n-functor that sends every morphisms to the identity on o. Then we say that the space of *flat n-sections* of a given transport functor tra is the space of morphisms

$$e : \mathrm{tra}_o \to \mathrm{tra}\,.$$

When $n = 1$, then for $T = G\mathrm{Tor}$ and $o = G$ this recovers the ordinary notion of flat sections of principal bundles; for $T = \mathrm{Vect}_{\mathbb{C}}$ and $o = \mathbb{C}$ this yields the ordinary notion of flat sections of vector bundles.

Covariant derivative. Similarly, one may consider flat $(n+1)$-sections

$$e : \mathrm{curv}_o \to \mathrm{curv}_{\mathrm{tra}}$$

of the curvature $(n+1)$-functor associated with tra. The component map of these is an n-transport

$$e : \mathcal{P}_n(X) \to T'$$

itself.

For $n = 1$, inspection of the transformation law shows that on points this describes an ordinary section $e|_{\mathrm{Obj}}$ of tra, while on morphisms this is the 1-functor coming from the 1-form

$$\nabla e|_{\mathrm{Obj}}\,,$$

where ∇ is the covariant derivative associated with the connection encoded by tra.

(In stating this, we are deliberately glossing over some further qualifications that are described below.)

2 Curvature

What is the *curvature* associated with an transport $\mathrm{tra} : \mathcal{P}_n(X) \to T$?

Whatever it is, it should vanish if our transport factors through the *fundamental* n-groupoid $\Pi_n(X)$ of X. As opposed to $\mathcal{P}_n(X)$, whose n-morphisms are *thin* homotopy classes of n-paths, the n-morphisms of Π_n are ordinary homotopy classes of n-paths.

Hence we have a canonical projection

$$\pi : \mathcal{P}_n(X) \to \Pi_n(X)$$

that sends any n-path to its homotopy class.

Definition 1 *We say that* tra *is* flat *precisely if there is an n-functor*

$$f : \Pi_n(X) \to T$$

such that

$$\begin{tikzcd} \mathcal{P}_n(X) \arrow[r, "\pi"] \arrow[d, "\mathrm{tra}"'] \arrow[dr, phantom, "\swArrow\sim"] & \Pi_n(X) \arrow[d, "f"] \\ T \arrow[r, equal] & T \end{tikzcd} .$$

Whatever the curvature of an n-functor is, it should be an obstruction for this construction.

Proposition 1 *Given any n-transport* $\mathrm{tra} : \mathcal{P}_n(X) \to T$*, we canonically obtain an $(n+1)$-category*

$$T_{n+1}$$

and an $(n+1)$-functor

$$\mathrm{curv}_{\mathrm{tra}} : \Pi_{n+1}(X) \to T_{n+1}$$

such that $\mathrm{curv}_{\mathrm{tra}}$ *is trivial on $(n+1)$-morphisms (sends every $(n+1)$-morphism to an identity $(n+1)$-morphism) if and only if* tra *is flat.*

The $(n+1)$-category T_{n+1} is defined by the pullback

$$\begin{tikzcd} \mathrm{Mor}(T_{n+1}) \arrow[r, "t"] \arrow[d, "s"'] & T \arrow[d, "(s,t)"] \\ T \arrow[r, "(s,t)"] & \mathrm{Mor}_{n-1}(T) \times \mathrm{Mor}_{n-1}(T) \end{tikzcd} ,$$

where $\mathrm{Mor}_{n-1}(T) \times \mathrm{Mor}_{n-1}(T)$ denotes the n-category with precisely one n-morphism between any ordered pair in the space $\mathrm{Mor}_{n-1}(T)$ of $(n-1)$-morphisms.

As stated, this is slightly imprecise as long as no model for weak n-categories has been specified. The following examples demonstrate the idea for low n and everything being strict.

As a simple but important corollary we get.

Corollary 1 (Bianchi identity) *Every curvature $(n+1)$-transport* $\mathrm{curv}_{\mathrm{tra}}$ *is itself flat.*

3 Examples

3.1 Curvature of 0-Transport

Let

$$\mathrm{tra} : \mathcal{P}_0(X) \to T$$

be a 0-functor.

Let $\Pi_0(X)$ be the homotopy 0-groupoid of X, i.e. the set of connected components. We have the canonical projection

$$\pi : \mathcal{P}_0(X) \to \Pi_0(X)\,.$$

We say tra is *flat* if there is

$$f : \Pi_0(X) \to T$$

such that

$$\begin{tikzcd} \mathcal{P}_0(X) \arrow[r, "\pi"] \arrow[d, "\mathrm{tra}"'] & \Pi_0(X) \arrow[d, "f"] \\ T \arrow[r, equal] & T \end{tikzcd}\,.$$

Flat 0-transport $\mathcal{P}_0(X) \to T$ is equivalent to functions $X \to T$ which are constant on each connected component.

If the 0-transport is *not* flat, we can measure its failure to be flat as follows.

Form the pullback

$$\begin{tikzcd} \mathcal{P}_0(X) \times_{\Pi_0(X)} \mathcal{P}_0(X) \arrow[r, "t"] \arrow[d, "s"'] & \mathcal{P}_0(X) \arrow[d, "\pi"] \\ \mathcal{P}_0(X) \arrow[r, "\pi"] & \Pi_0(X) \end{tikzcd}\,.$$

Like any such pullback, this canonically defines a groupoid with space of objects $\mathcal{P}_0(X)$, space of morphisms $\mathcal{P}_0(X) \times_{\Pi_0(X)} \mathcal{P}_0(X)$ and source and target map as indicated. Composition is then canonically defined by the pullback property.

Analogously, consider the pullback

$$\begin{tikzcd} T \times T \arrow[r, "t"] \arrow[d, "s"'] & T \arrow[d] \\ T \arrow[r] & \{\bullet\} \end{tikzcd}\,.$$

This defines the pair groupoid on T.

We then canonically obtain a morphism

$$\mathrm{curv}_{\mathrm{tra}} : X \times X \to T \times T$$

between these two groupoids as the universal morphism

$$\begin{tikzcd}
\mathcal{P}_0(X) \times_{\Pi_0(X)} \mathcal{P}_0(X) \arrow[r, "t"] \arrow[d, "s"'] \arrow[ddrr, bend left, "\mathrm{curv}_{\mathrm{tra}}"] & \mathcal{P}_0(X) \arrow[rr, "\mathrm{tra}"] \arrow[d, "\pi"] & & T \arrow[d, equal] \\
\mathcal{P}_0(X) \arrow[r, "\pi"] \arrow[dd, "\mathrm{tra}"'] & \Pi_0(X) & & T \arrow[dd] \\
& & T \times T \arrow[ur, "t"] \arrow[dl, "s"] & \\
T \arrow[r, equal] & T \arrow[rr] & & \{\bullet\}
\end{tikzcd} .$$

We say that $\mathrm{curv}_{\mathrm{tra}}$ is the *curvature 1-transport* of the 0-transport tra.

In the special case that tra is flat, we may, by definition, further fill this diagram to obtain

$$\begin{tikzcd}
\mathcal{P}_0(X) \times_{\Pi_0(X)} \mathcal{P}_0(X) \arrow[r, "t"] \arrow[d, "s"'] \arrow[ddrr, bend left, "\mathrm{tra}_{\mathrm{tra}}"] & \mathcal{P}_0(X) \arrow[rr, "\mathrm{tra}"] \arrow[d, "\pi"] & & T \arrow[d, equal] \\
\mathcal{P}_0(X) \arrow[r, "\pi"] \arrow[dd, "\mathrm{tra}"'] & \Pi_0(X) \arrow[rr, "f"] \arrow[dd, "f"'] & & T \arrow[dd] \\
& & T \times T \arrow[ur, "t"] \arrow[dl, "s"] & \\
T \arrow[r, equal] & T \arrow[rr] & & \{\bullet\}
\end{tikzcd} .$$

Then the universal property implies that $\mathrm{curv}_{\mathrm{tra}}$ factors through $\Pi_0(X)$

$$
\begin{tikzcd}
\mathcal{P}_0(X) \times_{\Pi_0(X)} \mathcal{P}_0(X) \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr] \arrow[ddrr, bend left, "\mathrm{curv}_{\mathrm{tra}}"] & \mathcal{P}_0(X) \arrow[rr, "\mathrm{tra}"] \arrow[d, "\pi"] & & T \arrow[d, equal] \\
\mathcal{P}_0(X) \arrow[r, "\pi"] \arrow[dd, "\mathrm{tra}"'] & \Pi_0(X) \arrow[rr, "f"] \arrow[dr] \arrow[dd, "f"'] & & T \arrow[dd] \\
& & T \times T \arrow[ur, "t"] \arrow[dl, "s"] & \\
T \arrow[r, equal] & T \arrow[rr] & & \{\bullet\}
\end{tikzcd}
.
$$

.

Example 1

Let $T = \mathbb{R}$ be the real numbers. Then the corresponding groupoid $T \times T$ may canonically be identified with the action groupoid of $\mathbb{R}$ on itself.

$$
T \times T \simeq \{\, s \xrightarrow{f} (s+f) \mid s, f \in \mathbb{R}\} \,.
$$

So the curvature $\mathrm{curv}_{\mathrm{tra}}$ of any 0-transport

$$
\mathrm{tra} : \mathcal{P}_0(X) \to \mathbb{R}
$$

assigns to each pair of points (x, y) in the same connected component of X the *difference*

$$
\mathrm{tra}(x) \xrightarrow{\mathrm{tra}(y) - \mathrm{tra}(x)} \mathrm{tra}(y)
$$

of the value of tra at these two points.

We may trivially think of $\mathrm{curv}_{\mathrm{tra}}$ as a functor on $\mathcal{P}_1(X)$, simply by pulling back

$$
\mathcal{P}_1(X) \xrightarrow{\pi} \Pi_1(X) \xrightarrow{\mathrm{curv}_{\mathrm{tra}}} \mathbb{R} \times \mathbb{R} \,.
$$

As such, if tra

$$
\mathrm{tra} : x \mapsto e^{f(x)}
$$

is smooth and since $\mathbb{R} \times \mathbb{R}$ is a locally trivial Lie groupoid equivalent to $\Sigma\mathbb{R}$, this 1-functor is given by a 1-form

$$
F_1 \in \Omega^1(X, \mathrm{Lie}(\mathbb{R})), .
$$

Here one finds that this 1-form is the differential of the original 0-functor, regarded as a 0-form:

$$
F_1 = df \,.
$$

The curresponding Bianchi identity, which says that the 1-functor

$$\mathrm{curv}_{\mathrm{tra}} : \mathcal{P}_1(X) \to \mathbb{R} \times \mathbb{R}$$

itself is flat is, at the level of differential forms, simply the statement that

$$dF_1 = d^2 f = 0\,.$$

3.2 Curvature of 1-Transport

This entire disucssion has an obvious categorification. Starting with a transport 1-functor and asking for flatness, we find a groupoid internal to Cat. This turns out to the the 2-groupoid whose objects are points of X, whose morphisms are paths, and whose 2-morphisms are homotopy classes of surfaces cobounding these paths.

So let now

$$\mathrm{tra} : \mathcal{P}_1(X) \to T$$

be a 1-functor.

Let $\Pi_1(X)$ be the fundamental groupoid of X. We have the canonical projection

$$\pi : \mathcal{P}_1(X) \to \Pi_1(X)$$

that sends each path to its homotopy class.

We say tra is *flat* if there is

$$f : \Pi_1(X) \to T$$

such that

$$\begin{tikzcd} \mathcal{P}_1(X) \arrow[r, "\pi"] \arrow[d, "\mathrm{tra}"'] \arrow[dr, phantom, "\swarrow\sim"] & \Pi_1(X) \arrow[d, "f"] \\ T \arrow[r, equal] & T \end{tikzcd}\,.$$

If the 1-transport is *not* flat, we can measure its failure to be flat as follows. Form the pullback

$$\begin{tikzcd} \mathcal{P}_1(X) \times_{\Pi_1(X)} \mathcal{P}_1(X) \arrow[r, "t"] \arrow[d, "s"'] & \mathcal{P}_1(X) \arrow[d, "\pi"] \\ \mathcal{P}_1(X) \arrow[r, "\pi"] & \Pi_1(X) \end{tikzcd}\,.$$

Like any such pullback, this canonically defines a groupoid with space of objects $\mathcal{P}_1(X)$, space of morphisms $\mathcal{P}_1(X) \times_{\Pi_1(X)} \mathcal{P}_1(X)$ and source and target map as indicated. Composition is then canonically defined by the pullback property.

This groupoid is evidently internal to Cat now. In fact, this is the 2-groupoid

$$\Pi_2(X)$$

whose objects are points in X, whose morphisms are thin homotopy classes of paths in X, and whose 2-morphisms are (notice the difference) homotopy classes of surfaces cobounding these paths.

Analogously, consider the pullback

$$\begin{tikzcd}
\mathrm{Mor}(T_2) \arrow[r, "t"] \arrow[d, "s"'] & T \arrow[d, "(s,t)"] \\
T \arrow[r, "(s,t)"] & \mathrm{Obj}(T) \times \mathrm{Obj}(T)
\end{tikzcd},$$

where $\mathrm{Obj}(T) \times \mathrm{Obj}(T)$ denotes the pair groupoid on the space of objects of T.

This, too, defines now a 2-groupoid, which we call T_2. Its space of objects is that of T, its space of 1-morphisms is the space of morphisms of T, and there is a unique 2-morphism between any pair of parallel 1-morphisms.

We then canonically obtain a morphism

$$\mathrm{curv}_{\mathrm{tra}} : \Pi_2(X) \to T_2$$

between these two 2-groupoids as the universal morphism

$$\begin{tikzcd}
\mathrm{Mor}(\Pi_2(X)) \arrow[r, "t"] \arrow[d, "s"'] \arrow[ddrr, bend left, "\mathrm{curv}_{\mathrm{tra}}"] & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d, "\pi"] & & T \arrow[d, equal] \\
\mathcal{P}_1(X) \arrow[r, "\pi"] \arrow[dd, "\mathrm{tra}"'] & \Pi_1(X) & & T \arrow[dd] \\
& & \mathrm{Mor}(T_2) \arrow[ur, "t"] \arrow[dl, "s"] & \\
T \arrow[r, equal] & T \arrow[rr] & & \mathrm{Obj}(T) \times \mathrm{Obj}(T)
\end{tikzcd}.$$

Notice that the fact that we happen to use $\mathrm{Obj}(T) \times \mathrm{Obj}(T)$ is

We say that $\mathrm{curv}_{\mathrm{tra}}$ is the *curvature 2-transport* of the 1-transport tra.

In the special case that tra is flat, we may, by definition, further fill this diagram to obtain

$$
\begin{tikzcd}
\mathrm{Mor}(\Pi_2(X)) \arrow[r, "t"] \arrow[d, "s"] \arrow[ddrr, bend left, "\mathrm{curv}_{\mathrm{tra}}"] & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d, "\pi"] \arrow[drr, phantom, "t^{-1}\ \Swarrow\sim"] & & T \arrow[d, equal] \\
\mathcal{P}_1(X) \arrow[r, "\pi"] \arrow[dd, "\mathrm{tra}"] \arrow[ddr, phantom, "t\ \Swarrow\sim"] & \Pi_1(X) \arrow[rr, "f"] \arrow[dd, "f"] & & T \arrow[dd] \\
& & \mathrm{Mor}(T_2) \arrow[ur, "t"] \arrow[dl, "s"] & \\
T \arrow[r, equal] & T \arrow[rr] & & \mathrm{Obj}(T)\times\mathrm{Obj}(T)
\end{tikzcd}
.
$$

Then the universal property implies that $\mathrm{curv}_{\mathrm{tra}}$ factors through $\Pi_1(X)$

$$
\begin{tikzcd}
\mathrm{Mor}(\Pi_2(X)) \arrow[r, "t"] \arrow[d, "s"] \arrow[dr] \arrow[ddrr, bend left, "\mathrm{curv}_{\mathrm{tra}}"] & \mathcal{P}_1(X) \arrow[rr, "\mathrm{tra}"] \arrow[d, "\pi"] \arrow[drr, phantom, "t^{-1}\ \Swarrow\sim"] & & T \arrow[d, equal] \\
\mathcal{P}_1(X) \arrow[r, "\pi"] \arrow[dd, "\mathrm{tra}"] \arrow[ddr, phantom, "t\ \Swarrow\sim"] & \Pi_1(X) \arrow[rr, "f"] \arrow[dd, "f"] \arrow[dr] & & T \arrow[dd] \\
& & \mathrm{Mor}(T_2) \arrow[ur, "t"] \arrow[dl, "s"] & \\
T \arrow[r, equal] & T \arrow[rr] & & \mathrm{Obj}(T)\times\mathrm{Obj}(T)
\end{tikzcd}
.
$$

This means that it sends every 2-morphism to an identity 2-morphism.

Example 2

Let $T = \Sigma G$ be a Lie group. Then we have a canonical identitfication

$$T_2 \simeq \Sigma(\mathrm{INN}(G))\,,$$

where $\mathrm{INN}(G)$ denotes the inner automorphism 2-group of G. This is the strict 2-group corresponding to the crossed module

$$(\ H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)\ )$$

of the form

$$(G \xrightarrow{\mathrm{Id}} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(H)) .$$

Notice that this means that the 2-morphism in $\Sigma(\mathrm{INN}(G_2))$ look like

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow g' g^{-1}"] & \bullet \end{tikzcd} ,$$

with the label of the 2-morphism completely specified by the source and target 1-morphism.

It follows that for a ΣG-valued 1-transport

$$\mathrm{tra} : \mathcal{P}_1(X) \to \Sigma G$$

the corresponding curvature 2-transport

$$\mathrm{curv}_{\mathrm{tra}} : \Pi_2(x) \to \Sigma(\mathrm{INN}(G))$$

sends any homotopy class of surfaces Σ to the 2-morphism

$$\mathrm{curv}_{\mathrm{tra}} \quad : \quad \begin{tikzcd} x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow \Sigma"] & y \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{tra}(\gamma)"] \arrow[r, bend right, "\mathrm{tra}(\gamma')"'] \arrow[r, phantom, "\Downarrow \mathrm{tra}(\partial\Sigma)"] & \bullet \end{tikzcd} ,$$

where we set

$$\partial \Sigma := \gamma' \circ \gamma^{-1} .$$

Using the canonical projection

$$\mathcal{P}_2(X) \to \Pi_2(X)$$

from thin homotopy classes of surfaces to ordinary homotopy classes, we may of course regard $\mathrm{curv}_{\mathrm{tra}}$ also as a 2-functor on $\mathcal{P}_2(X)$:

$$\mathrm{curv}_{\mathrm{tra}} : \mathcal{P}_2(X) \to \Pi_2(X) \to \Sigma(\mathrm{INN}(G)) .$$

As such, it is in bijection with a pair (A, B), where

$$A \in \Omega^1(X, \mathrm{Lie}(G))$$

and

$$B \in \Omega^2(X, \mathrm{Lie}(G))$$

such that

$$t_* \circ B + F_A = 0\,,$$

where F_A is the ordinary curvature 2-form of A. We find that A is nothing but the 1-form corresponding to tra itself, according to

$$\mathrm{tra}(\gamma) = \mathcal{P}\exp(\int_\gamma A)\,.$$

Hence using $t = \mathrm{Id}$ it follows that the 2-form defining the 2-functor

$$\mathrm{curv}_{\mathrm{tra}}$$

is already fixed to be the curvature 2-form

$$B = -F_A$$

(up to a sign) of A.

This explicitly shows that our general notion of curvature of transport is indeed a good generalization of the ordinary notion of curvature of a connection.

Analogously, we could now look at the 3-transport

$$\mathrm{curv}_{\mathrm{curv}_{\mathrm{tra}}}\,.$$

It is guaranteed to be trivial, but, again, by pulling it back to

$$\Pi_3(X)$$

we obtain a principal transport 3-functor from this for which we have an equivalence with certa p-form data for $1 \leq p \leq 3$. Under this equivalence, we find that the 3-form corrseponding to $\mathrm{curv}_{\mathrm{curv}_{\mathrm{tra}}}$ is

$$d_A F_A\,.$$

That this vanishes is indeed the ordinary Bianchi identity.

This explicitly shows that also our general notion of Bianchi identity for curvature of transport is indeed a good generalization of the ordinary notion of Bianchi identity.

Example 3

In order to understand curvature 2-functors of principal transport functors with values in $G\mathrm{Tor}$, consider the following 2-category.

Definition 2 *Let* $\mathrm{Grp} \subset \mathrm{Cat}$ *be the sub 2-category of* Cat *whose objects are all categories of the form* ΣG*, for* G *a group, and whose morphisms and 2-morphisms are all strictly invertible functors and transformations between these. This means that objects of* Grp *are groups, morphisms are group isomorphisms and two isomorphisms are connected by a 2-morphisms precisely if they differ by an inner automorphism of the target group. Let in turn*

$$\mathrm{Grp}_G \subset \mathrm{Grp}$$

be the full 2-category all whose objects are isomorphic to G*.*

Notice that we have a functor

$$j : G\mathrm{Tor} \to \mathrm{Grp}_G$$

which sends any (right) G-torsor P to the group

$$P \times_G G\,,$$

where the action of G on G which is divided out is that by conjugation.

This means that fixing any element $p \in P$ we get an isomorphism

$$p : (P \times_G G) \to G\,.$$

And any two such isomorphisms differ by an inner automorphism of G:

$$p' \circ p^{-1} = \mathrm{Ad}_g$$

for a unique $g \in G$.

Using this, one checks that the image of j

$$\mathrm{im}(j) \subset \mathrm{Grp}_G$$

is a (1-)category, and that there is a unique 2-morphism in Grp_G between any two 1-morphisms in $\mathrm{im}(j)$.

Hence $(\mathrm{im}(j))_2$ is the maximal sub 2-category in Grp_G that contains $\mathrm{im}(j)$.

One sees that (at the level of categories without extra structure), picking an element $p \in P$ for each G-torsor P, we have an equivalence

$$(\mathrm{im}(j))_2 \simeq \Sigma\mathrm{INN}(G)\,.$$

This way we find that if we start with the principal 1-transport

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$$

and form the *Ad-associated* transport

$$\tilde{\mathrm{tra}} : \; \mathcal{P}_1(X) \xrightarrow{\mathrm{tra}} G\mathrm{Tor} \xrightarrow{j} \mathrm{im}(j)$$

then the corresponding curvature 2-transport

$$\mathrm{curv}_{\mathrm{tra}} : \Pi_2(X) \to (\mathrm{im}(j))_2$$

actually has local $\Sigma(\mathrm{INN}(G))$-structure in that for a $\pi : Y \to X$ over which tra itself i-trivializes with respect to $i_G : \Sigma G \to G\mathrm{Tor}$, we have

$$\begin{tikzcd}
\Pi_2(Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{curv}_{\mathrm{triv}}"'] \arrow[dr, phantom, "\swarrow\sim"] & \Pi_2(X) \arrow[d, "\mathrm{curv}_{\mathrm{tra}}"] \\
\Sigma(\mathrm{INN}(G)) \arrow[r] & (\mathrm{im}(j))_2
\end{tikzcd} \quad .$$

(Well, I think this is clear now. But I should eventually write this out in detail.)

3.2.1 Groupoid Schreier Theory and the Atiyah Sequence

The relevance of Example 3 can also be understood as follows.

Schreier Theory for Groupoids. Consider extensions of a groupoid B by a groupoid K

$$K \to G \to B$$

where K is *skeletal*, meaning that all its morphisms have the same source and target. (Hence K is really a collection of groups).

Define the 2-groupoid

$$\mathrm{AUT}(K) \subset \mathrm{Grp}_2$$

as the smallest full sub 2-category of the 2-category Grp_2 from Definition 2 that contains all groups in K.

Then Schreier theory for groupoids says that groupoid extensions as above are classified by *pseudo*functors

$$\sigma : B \to \mathrm{AUT}(K)\,.$$

The integrated Atiyah Sequence of a Principal Bundle. As Charles Ehresmann has pointed out, any principal G-bundle $p : P \to X$ gives rise to an extension of Lie groupoids

$$\begin{tikzcd}
\mathrm{Ad}P \arrow[r] \arrow[d, equal] & \mathrm{Trans}(P) \arrow[r] \arrow[d, equal] & X \times X \arrow[d, equal] \\
P \times_G G \arrow[r] & P \times_G P \arrow[r] & X \times X
\end{tikzcd}$$

and such extensions classify principal G-bundles. All groupoids appearing here are Lie groupoids whose space of objects is X.

- the *pair groupoid* $X \times X$ has morphisms all pairs of points in X
- the *gauge groupoid* $\mathrm{Trans}(P)$ has morphisms all equivariant fiber isomorphisms of P
- the adjoint bundle $\mathrm{Ad}P$ is a bundle of groups, which we may regard as a skeletal groupoid.

In forming $P \times_G G$ we use the obvious G-action on P and the adjoint action on G, as before in the discussion of example 3.

Combining these two insights, we find that G-principal bundles on X are classified by *pseudo*functors

$$X \times X \to \mathrm{AUT}(\mathrm{Ad}P)\,.$$

So...

4 Covariant Derivative

4.1 Flat Sections

Before discussing the general definition of parallel sections of n-functors, consider the following to examples.

Let G be a Lie group.

Proposition 2 *Let* $\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$ *be a principal transport. Then flat sections of the corresponding principal bundle with connection are in bijection with morphisms*

$$e : \mathrm{tra}_0 \to \mathrm{tra}\,,$$

where tra_0 *is the parallel transport of the trivial bundle with trivial connection, i.e. that which sends every path to* Id_G*.*

Proof. The component maps of these transformations

$$e_x :\ G \xrightarrow{\sim} P_x$$

are in bijection with sections of the bundle since G-torsor morphisms $G \to P_x$ are in bijection with the elements of P_x (the image of the neutral element in G).

That this section is flat is follows from the naturality condition

$$\begin{tikzcd} G \arrow[r, "\mathrm{Id}"] \arrow[d, "e_x"'] & G \arrow[d, "e_y"] \\ P_x \arrow[r, "\mathrm{tra}(\gamma)"] & P_y \end{tikzcd}$$

□

Proposition 3 *Let* $\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$ *be a vector transport. Then flat sections of the corresponding vector bundle with connection are in bijection with morphisms*

$$e : \mathrm{tra}_0 \to \mathrm{tra}$$

where tra_0 *is the parallel transport of the trivial line bundle with trivial connection, i.e. that which sends every path to* $\mathrm{Id}_{\mathbb{C}}$*.*

Proof. We only need to notice that morphisms

$$e_x : \mathbb{C} \to V_x$$

are in bijection with vectors in V_x. Then the argument proceeds as in Prop. 2. □

This motivates the following definition.

Definition 3 *For any $o \in \mathrm{Obj}(T)$, let*

$$\mathrm{tra}_o : \mathcal{P}_n(X) \to T$$

the the n-functor that sends every n-path to Id_o*.*

Then a parallel section *of* tra *relative to o is a morphism*

$$e : \mathrm{tra}_o \to \mathrm{tra}\,.$$

We write

$$\Gamma_o(\mathrm{tra})$$

for the $(n-1)$-category of o-parallel sections of tra.

4.2 Non-flat Sections and their Covariant Derivative

Definition 4 *Given a parallel transport* $\mathrm{tra} : \mathcal{P}_n(X) \to T$ *with corresponding curvature* $\mathrm{curv}_{\mathrm{tra}}$*, we call*

$$\Gamma_o(\mathrm{curv}_{\mathrm{tra}})$$

the space of sections *of* tra*.*

Remark. In fact, $\Gamma_o(\mathrm{curv}_{\mathrm{tra}})$ has to be thought of as the space of pairs, consisting of a section and its *covariant derivative*. This is illustrated by the following examples.

Example 4

Consider a transport

$$\mathrm{tra} : \mathcal{P}_1(X) \to \Sigma G\,.$$

As we have seen in Example 2, the corresponding curvature 2-functor

$$\mathrm{curv}_{\mathrm{tra}} : \Pi_2(X) \to \Sigma(\mathrm{INN}(G))$$

acts as

$$\mathrm{curv}_{\mathrm{tra}} \;:\; \begin{tikzcd} x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow\Sigma"] & y \end{tikzcd} \;\mapsto\; \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{tra}(\gamma)"] \arrow[r, bend right, "\mathrm{tra}(\gamma')"'] \arrow[r, phantom, "\Downarrow\mathrm{tra}(\partial\Sigma)"] & \bullet \end{tikzcd}\,,$$

Since $\Sigma(\mathrm{INN}(G))$ has a *unique* 2-morphism between any pair of parallel 1-morphisms, it follows that any choice of component map

$$e_x : \bullet \xrightarrow{e_x} \bullet$$

defines a transformation

$$e : \mathrm{curv}_0 \to \mathrm{curv}_{\mathrm{tra}}\,,$$

where curv_0 is the 2-functor that sends everything to $\mathrm{Id}_{\mathrm{Id}_\bullet}$.

The corresponding component functor is

$$e : (x \xrightarrow{\gamma} y) \mapsto \begin{tikzcd} \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "e_x"'] \arrow[dr, phantom, "\swArrow e(\gamma)"] & \bullet \arrow[d, "e_y"] \\ \bullet \arrow[r, "\mathrm{tra}(\gamma)"'] & \bullet \end{tikzcd} ,$$

where $e(\gamma)$ is uniquely fixed by its source and target.

Hence e is a transport 1-functor itself, with values in a certain groupoid. The vertex group of that groupoid is G itself. Hence e comes, as any transport functor with values in a group, from a 1-form

$$A_e \in \Omega^1 X, \mathrm{Lie}(G) .$$

Proposition 4 *This 1-form is the covariant derivative of the section $e : x \mapsto e(x)$ with respect to the connection ∇ encoded in* tra*:*

$$A_e = \nabla e .$$

Example 5

Again, an analogous discussion can be made for covariant derivatives of sections in vector bundles. Component functors of sections of curvature 2-functors of vector bundles look like

$$e : (x \xrightarrow{\gamma} y) \mapsto \begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_x"'] \arrow[dr, phantom, "\swArrow e(\gamma)"] & \mathbb{C} \arrow[d, "e_y"] \\ V_x \arrow[r, "\mathrm{tra}(\gamma)"'] & V_y \end{tikzcd} .$$

The groupoid formed by the squares on the right, under horizontal composition, is equivalent to a groupoid of squares as above, whose filling 2-morphisms are labeled by the linear maps

$$(e_y - \mathrm{tra}(\gamma) \circ e_x) : \mathbb{C} \to V_y .$$

For any fixed $y \in X$, we have a smooth functor from this groupoid to the additive group $\mathrm{End}(V_y)$, which is an equivalence of categories at the level of non-smooth categories.

Proposition 5 *The 1-form A_e associated with 1-transport e with values in this group at this point y is precisely the covariant derivative of e:*

$$A_e = \nabla e .$$