Sections and covariant derivatives of L_∞-algebra connections

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Abstract

For every L_∞-algebra $\mathfrak{g}$ there is a notion of $\mathfrak{g}$-bundles with connection, according to [5]. Here I discuss how to describe

- associated $\mathfrak{g}$-bundles;
- their spaces of sections;
- and the corresponding covariant derivatives

in this context.

Contents

1 Introduction

Representations of n-groups are usually thought of as n-functors from the n-group into the n-category of representing objects. In the program [1] one sees that possibly a more fundamental perspective on representations is in terms of the corresponding action groupoids sitting *over* the given group.

This is the perspective I will adopt here and find to be fruitful.

The definition of L_∞-modules which I proposed in [6] can be seen to actually comply with this perspective. Here I further develop this by showing that this perspective also helps to understand associated L_∞-connections, their sections and covariant derivatives.

This is the picture to be developed here:

Sections A **background field** on a smooth space X is a morphism $\nabla : \Pi_{n+1}(X) \to \mathbf{T}$ fitting into a diagram

$$\begin{tikzcd}
\Pi_{n+1}^{\mathrm{vert}}(Y) \arrow[r, "g"] \arrow[d, hook] & \mathbf{B}G_{(n)} \arrow[d, hook] \\
\Pi_{n+1}(Y) \arrow[r, "\nabla_{\mathrm{loc}}"] \arrow[d] & \mathbf{BE}G_{(n)} \arrow[d] \arrow[dl, Rightarrow, "\simeq"] \\
\Pi_{n+1}(X) \arrow[r, "\nabla"] & \mathbf{T}
\end{tikzcd}$$

of smooth $(n+1)$-funtors, or alternatively a morphism

$$\Omega^\bullet(X) \xleftarrow{\nabla} \mathrm{inv}(\mathfrak{g})$$

of DGCAs, fitting into a diagram

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "\nabla_{\mathrm{loc}}"'] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\nabla"'] \arrow[u, hook]
\end{tikzcd}$$

A **representation** of the structure group is given by a sequence

$$F_{(n-1)} \to \hat{G} \to G_{(n)} \,,$$

whose fiber has categorical degree at least one less that the structure n-group, alternatively a sequence

$$\mathrm{CE}(\mathfrak{f}) \leftarrow \mathrm{CE}(\hat{\mathfrak{g}}) \leftarrow \mathrm{CE}(\mathfrak{g})\,.$$

For instance, for $\mathfrak{g}$ an ordinary Lie algebra we get the ordinary notion of a vector bundle associated by a representation ρ of $\mathfrak{g}$ on V by using

$$V^* \leftarrow \mathrm{CE}_\rho(\hat{\mathfrak{g}}, V) \leftarrow \mathrm{CE}(\mathfrak{g})\,.$$

But crucially, we also get such sequences from weak cokernels ($\simeq$ homotopy quotients)

$$\hat{G} \to (B^{n-1}U(1) \to \hat{G}) \to B^n U(1)\,.$$

We say that a **section** of ∇ with respect to such a sequence is simply a lift of ∇ through such a sequence sequence:

$$\begin{tikzcd} & \mathbf{BE}\hat{G}_{(n)} \arrow[dr] & \\ \Pi_{n+1}(Y) \arrow[ur, dotted, "\sigma"] \arrow[rr, "\nabla_{\mathrm{loc}}"] & & \mathbf{BE}G_{(n)} \end{tikzcd}$$

and extended accordingly to the full diagram.

We show in examples how this does indeed capture the ordiary notion of sections. For instance it is easy to see that

$$\begin{tikzcd} & \mathrm{CE}_\rho(\mathfrak{g}) \arrow[dl, dotted, "\sigma"'] & \\ \Omega^\bullet(Y)_{vert} & & \mathrm{CE}(\mathfrak{g}) \arrow[ul] \arrow[ll, "A_{\mathrm{vert}}"'] \end{tikzcd}$$

encodes an ordinary section on the ordinary associated $\mathfrak{g}$-bundle encoded by A_{vert}.

Comparison with the two rightmost columns of figure 12 in [5] shows that "twisted n-bundles" are the same things as sections of $b^n\mathfrak{u}(1)$ $(n+)$-bundles. This is the result we found earlier, using the description of representations in terms of representation n-functors.

Σ-model QFTs Our goal is to find the representations of cobordisms categories which are induced from a given background field. In physics terms, this means that we are interested in the **quantization of Σ-models**.

The Σ-model associated with background field ∇ should associate to a given manifold Σ a collection of sections of the result obtained by transgressing ∇ to the space of maps from Σ into X.

As described elsewhere, the process ordinarily addressed as transgression is nothing but applying the inner hom

$$\hom(\Pi_{n+1}(\Sigma), -)$$

to the entire diagram giving the background field.

It remains to be understood how then taking sections yields a representation of cobordism categories.

I am not completely sure yet about all details, but it should probably work along the following lines:

I think of our cobordisms as spans

$$\begin{tikzcd} & \Sigma & \\ \Sigma_1 \arrow[ur] & & \Sigma_2 \arrow[ul] \end{tikzcd}$$

and require these to behave analogously to a simplicial space, in that in addition to the co-face maps, there are co-degeneracy maps, at least after we take path-groupoids:

$$\begin{tikzcd} & \Pi(\Sigma) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \Pi(\Sigma_1) & & \Pi(\Sigma_2) \end{tikzcd}$$

For instance the fundamental groupoid of the pair-of-pants, $\simeq$

$$\begin{tikzcd} & \bullet \arrow[dr, "c_2"] & \\ \bullet \arrow[ur, "c_1"] \arrow[rr, "c_3"] & & \bullet \end{tikzcd}$$

has the obvious injections of the circle $\mathbf{B}\mathbb{Z}$ and projection onto it (the latter each regarding one of the cycles c_i as degenerate).

Given such a span and co-span, we can apply $\mathrm{Hom}(--,\Pi_{n+1}(X))$ to it to obtain a (co)span

$$\begin{tikzcd} & \mathrm{Hom}(\Pi(\Sigma),\Pi_{n+1}(X)) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \mathrm{Hom}(\Pi(\Sigma_1),\Pi_{n+1}(X)) & & \mathrm{Hom}(\Pi(\Sigma_2),\Pi_{n+1}(X)) \end{tikzcd}$$

of configuration spaces. Similarly for the rest of the diagram defining the background field.

Then forming sections as above yields a span of section n-groupoids

$$\begin{tikzcd} & \mathrm{sect}(\mathrm{tg}_\Sigma\nabla) \arrow[dl] \arrow[dr] & \\ \mathrm{sect}(\mathrm{tg}_{\Sigma_1}\nabla) & & \mathrm{sect}(\mathrm{tg}_{\Sigma_2}\nabla) \end{tikzcd}$$

by using the maps going back and forth between Σ and Σ_i:

$$\begin{tikzcd} \mathrm{hom}(\Pi(\Sigma),\Pi(X)) \arrow[r] & \mathrm{hom}(\Pi(\Sigma),\mathbf{T}) \arrow[d, "\text{pullback along injection}"] \\ \mathrm{hom}(\Pi(\Sigma_i),\Pi(X)) \arrow[u, "\text{pushforward along surjection}"] \arrow[r, "\text{defined as the above composite}"'] & \mathrm{hom}(\Pi(\Sigma_i),\mathbf{T}) \end{tikzcd}$$

It looks tempting to read such spans of section n-groupoids in the context of Baez-Dolan-Trimble groupoidification as the combinatorial version of the linear maps that one expects to see.

2 Representations

2.1 Action groupoids

For a group G, with $\mathbf{B}G$ denoting the corresponding 1-object groupoid, a representation of G on objects in the category S is a functor

$$\rho : \mathbf{B}G \to S$$

$$(\bullet \xrightarrow{g} \bullet) \mapsto (V \xrightarrow{\rho(g)} V)$$

For instance setting $S = \mathrm{Vect}$ produces linear represenations. For any such representation we obtain the corresponding **action groupoid**

$$V//_\rho G$$

which is the weak quotient of V by the action of G. This comes canonically with a faithful (injective on each Hom-space) functor back to $\mathbf{B}G$

$$V//_\rho G \longrightarrow \mathbf{B}G\,;$$

and in fact every groupoid C with a faithful functor to $\mathbf{B}G$

$$C \longrightarrow \mathbf{B}G$$

is the action groupoid of some representatiton of G.

The underlying object V that G is acting on sits inside this action groupoid

$$V \hookrightarrow C$$

and the inclusion is surjective on morphsims.

In fact, this way we obtain a sequence of groupoids

$$V \hookrightarrow C \longrightarrow \mathbf{B}G\ , \tag{1}$$

where V is regarded as a groupoid with only identity morphisms.

This perspective on action groupoids will be useful in the following.

2.1.1 Examples

The archetypical example is the action of G on itself by right (or left) multiplication: the regular representation of G. The corresponding action groupoid is $G//G = \mathrm{INN}(G) = \mathrm{Codisc}(G)$, the groupoid with one object per element of G and precisely one morphism between any pair of objects.

In this case our sequence

$$G \hookrightarrow G//G \longrightarrow \mathbf{B}G$$

is actually the groupoid version of the universal G-bundle, in that

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, mapsto, "|\cdot|"] & G//G \arrow[r] \arrow[d, mapsto, "|\cdot|"] & \mathbf{B}G \arrow[d, mapsto, "|\cdot|"] \\ G \arrow[r, hook] & EG \arrow[r] & BG \end{tikzcd} ,$$

where $|\cdot|$ denotes forming nerves and the geometric realizations. See [4].

2.2 Action Lie ∞-algebroids

Let A be some commutative associative algebra, to be thought of as the algebra of functions on some space V, as above.

Modeled on 1, we now define the *action* of an L_∞-algebra on a cochain complex of A-modules in non-negative degree,

$$V^* = (\ V_0^* \xleftarrow{d_V} V_1^* \xleftarrow{d_V} \cdots\ )\,.$$

Given any (finite dimensional) L_∞-algebra $\mathfrak{g}$, we write $\mathfrak{g} \otimes A$ for the free A-module generated by it.

Definition 1 (L_∞-representation) *For $\mathfrak{g}$ any L_∞-algebra and V a chain complex of A-modules as above, we say that an action ρ of $\mathfrak{g}$ on V is an extension of the graded-commutative algebra*

$$\wedge_A^\bullet(V^* \oplus (\mathfrak{g}^* \otimes A))$$

to a differential *graded commutative algebra*

$$\mathrm{CE}_\rho(\mathfrak{g}, V) := (\wedge_A^\bullet(V^* \oplus (\mathfrak{g}^* \otimes A)), d_\rho)$$

together with a DGCA morphisms

$$\mathrm{CE}_\rho(\mathfrak{g}, V) \hookleftarrow \mathrm{CE}(\mathfrak{g})$$

and a morphism of complexes

$$V^* \twoheadleftarrow \mathrm{CE}_\rho(\mathfrak{g}, V)$$

such that in the category of complexes

$$\begin{tikzcd} V^* & \mathrm{CE}_\rho(\mathfrak{g}, V) \arrow[l, two heads] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[ll, bend left, "0"] \end{tikzcd} .$$

Definition 2 (adjoint L_∞-representation) *Let $\mathbf{i}$ be the DGCA on $\mathbb{R} \oplus \mathbb{R}[1]$ with the product being tensor product over $\mathbb{R}$ and the differential being trivial. For $\mathfrak{g}$ any L_∞-algebra, we denote by*

$$\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}, \mathfrak{g}) := \mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \mathbf{i})$$

the Chevalley-Eilenberg algebra of the adjoint action of $\mathfrak{g}$ on itself

Proposition 1 *This is indeed a representation, in that we do have a sequence*

$$\begin{tikzcd} \mathfrak{g}^*[-1] & \mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}, \mathfrak{g}) \arrow[l, two heads] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[ll, bend left, "0"] \end{tikzcd} .$$

Here $\mathfrak{g}^*[-1]$ on the left is the cochain complex underlying $\mathrm{CE}(\mathfrak{g})$ after restriction to generators (and recall that in our conventions $\mathfrak{g}$ is concentrated in degrees $1 \leq d \leq \infty$):

$$\mathfrak{g}^*[-1]_0 \xrightarrow{p_\mathfrak{g} \circ d_{\mathrm{CE}(\mathfrak{g})}} \mathfrak{g}^*[-1]_1 \xrightarrow{p_\mathfrak{g} \circ d_{\mathrm{CE}(\mathfrak{g})}} \cdots ,$$

where $p_\mathfrak{g} : \wedge^\bullet \mathfrak{g}^* \to \mathfrak{g}^*$ is the canonical projection onto the first wedge power.

Proof. Over any test domain U, a given morphism $\mathrm{CE}(\mathfrak{g}) \to \mathbf{i} \otimes \Omega^\bullet(U)$ comes from a homomorphism ω and a corresponding derivation λ in the familiar way, such that for each $a \in \mathrm{CE}(\mathfrak{g})$ we have

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"'] & \omega(a) + t \otimes \lambda(x) \arrow[d, mapsto, "d_U"'] \\ d_{\mathrm{CE}(\mathfrak{g})} a \arrow[r, mapsto] & \substack{d\omega(a) + t \otimes d\lambda(a)) \\ = \omega(d_{\mathrm{CE}(\mathfrak{g})}) + t \otimes \lambda(d_{\mathrm{CE}(\mathfrak{g})} a)} \end{tikzcd} ,$$

where $\{t\}$ denotes the canonical basis of $\mathbb{R}[1]$. So the GCA underlying $\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}, \mathfrak{g})$ is $\wedge^\bullet(\mathfrak{g}^*[-1] \oplus \mathfrak{g}^*)$ and restricted to $\wedge^\bullet \mathfrak{g}^*$ the differential is that of $\mathrm{CE}(\mathfrak{g})$. Hence we have a canonical inclusion

$$\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}, \mathfrak{g}) \hookleftarrow \mathrm{CE}(\mathfrak{g}) .$$

Moreover, if $a \in \mathfrak{g}^*$ is a generator then so is $\lambda(a)$ and the restriction of $\lambda(d_{\mathrm{CE}}a)$ on generators is the image under λ of $d_{\mathrm{CE}}a$ restricted to generators. Therefore we do have canonically a morphism of complexes

$$\mathfrak{g}^*[-1] \twoheadleftarrow \mathrm{CE}_{\mathrm{ad}}(\mathfrak{g},\mathfrak{g})$$

and clearly the composite vanishes. □

2.2.1 Examples

Ordinary Lie modules. For $\mathfrak{g}$ an ordinary (finite dimensional) Lie algebra and V^* an ordinary Lie module, we let $\mathrm{CE}_\rho(\mathfrak{g},V)$ be the ordinary Chevalley-Eilenberg algebra.

In this case A is just the ground field and the differential d_ρ acts as

$$d_\rho|_{V^*} : v \mapsto \rho(\cdot)(v) \in \mathfrak{g}^* \otimes V^*$$

$$d_\rho|_{\mathfrak{g}^*} = [\cdot,\cdot]^* \,.$$

Proposition 2 *For $\mathfrak{g}$ an ordinary Lie algebra, $\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g},\mathfrak{g})$ coincides with the Chevalley-Eilenberg algebra of the ordinary adjoint representation of $\mathfrak{g}$ on itself.*

The Weil algebra of an ordinary Chevalley-Eilenberg algebra. Notice that for ordinary Lie algebra modules we have the Weil algebra (the mapping cone of the identity on $\mathrm{CE}_\rho(\mathfrak{g},V)$)

$$\mathrm{W}_\rho(\mathfrak{g},V) = (\wedge^\bullet(V^* \oplus \mathfrak{g}^* \oplus V^*[1] \oplus \mathfrak{g}^*[1]))$$

whose differential is given by

$$d_{\mathrm{W}}|_{V^*} : v \mapsto \rho(\cdot)(v) + \sigma v$$

$$d_{\mathrm{W}}|_{\mathfrak{g}^*} : a \mapsto [\cdot,\cdot]^*(a) + \sigma a$$

etc.

Examples for higher adjoint representation.

Proposition 3 *For $\mathfrak{g}$ a semisiple Lie 1-algebra and μ the canonical 3-cocycle, the adjoint representation of the String Lie 2-algebra $\mathfrak{g}_\mu$ looks as follows:*

$$\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}_\mu,\mathfrak{g}_\mu) = ((\mathfrak{g}^*[-1] \oplus \mathbb{R}[1]) \oplus (\mathfrak{g}^* \oplus \mathbb{R}[2]), d_{\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}_\mu)})$$

with the differential acting on the shifted copy as

$$d_{\mathrm{ad}}|_{\mathfrak{g}^*[-1]} = d_{\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g})}$$

and

$$d_{\mathrm{ad}}|_{\mathbb{R}[1]} : a \mapsto \sigma\mu \,,$$

where σ is the canonical isomorphism $\mathfrak{g}^* \to \mathfrak{g}^*[-1]$ extended as a derivation.

Recall again that in our conventions $\mathfrak{g}$ is in degree $1 \leq d \leq \infty$.

The nature of $\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}_\mu)$ becomes most vivid when we consider sections of adjoint String 2-bundles in 3.2.1.

2.3 Loop groupoids and loop Lie algebroids

In [9] Simon Willerton observes that for G a finite group and $\mathbf{B}G$ the one-object groupoid induced by it, the groupoid of functors,

$$\Lambda\mathbf{B}G := \mathrm{Hom}_{Cat}(\mathbf{B}\mathbb{Z}, \mathbf{B}G)\,,$$

called the *loop groupoid* of G, is, in some sense, to G like the loop group of a Lie group is to that Lie group.

But notice that $\Lambda\mathbf{B}G$ is in fact also the action groupoid

$$G \hookrightarrow \Lambda\mathbf{B}G \twoheadrightarrow \mathbf{B}G$$

of the adjoint action of G on itself.

We can see more clearly what is going on by considering the analogous functorial construction for Lie groups. So let G be Lie and let $\Pi_1(S^1)$ be the fundamental Lie groupoid of the circle. Then, according to [8], the groupoid of smooth functors

$$\Lambda\mathbf{B}G := \mathrm{Funct}^\infty(\Pi_1(S^1), \mathbf{B}G)$$

has as objects smooth $\mathfrak{g}$-valued 1-forms $A \in \Omega^1(S^1, \mathfrak{g})$, and as morphisms

$$A \xrightarrow{f} A'$$

smooth G-valued functions $f \in \Omega^0(G)$, such that $A' = \mathrm{Ad}_f A + f df^{-1}$.

Since compositon of morphisms is given by pointwise multiplication in G, there is a canonical *faithful* functor

$$\Lambda\mathbf{B}G \twoheadrightarrow \mathbf{B}(\Omega G)$$

from the loop groupoid to the one-object groupoid coming from the loop group of G.

The kernel of this is the discrete category over the smooth space of $\mathfrak{g}$-valued 1-forms on the circle

$$\Omega^1(S^1, \mathfrak{g}) \hookrightarrow \Lambda\mathbf{B}G \twoheadrightarrow \mathbf{B}(\Omega G)\ .$$

Hence $\Lambda\mathbf{B}G$ is the *action groupoid* of ΩG acting on $\mathfrak{g}$-valued forms on S^1.

It will be useful for us to repeat this discussion at the level of L_∞-algebras:

Definition 3 (loop Lie algebroid) *For $\mathfrak{g}$ any Lie ∞-algebra we write*

$$\mathrm{CE}(\Lambda\mathfrak{g}) := \mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(S^1))$$

and address this is the Chevally-Eilenberg algebra of the loop Lie ∞-algebroid of $\mathfrak{g}$.

2.3.1 Examples

Loop Lie algebras of ordinary Lie algebras.

Proposition 4 *For $\mathfrak{g}$ an ordinary Lie algebra we have*

$$\Omega^0(S^1, \mathfrak{g}) \twoheadleftarrow \mathrm{CE}(\Lambda\mathfrak{g}) \hookleftarrow \mathrm{CE}(\Omega\mathfrak{g})\ .$$

Hence $\Lambda\mathfrak{g}$ is the action Lie algebroid of an action of the loop Lie algebra $\Omega\mathfrak{g}$ on $\mathfrak{g}$-valued functions on S^1.

Proof. We compute forms on the space of maps as usual by considering DGCA morphisms

$$\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(S^1) \otimes \Omega^\bullet(U)$$

over each test domain U

$$
\begin{tikzcd}
t^a \arrow[r, mapsto] \arrow[d, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"] & \lambda^a \theta + \omega^a \arrow[d, mapsto, "d_{S^1} + d_U"] \\
-\frac{1}{2} C^a{}_{bc} t^b \wedge t^c \arrow[r, mapsto] & {\scriptstyle (d_U \lambda^a) \wedge \theta + d_U \omega^a + \theta \wedge \frac{\partial}{\partial \sigma} \omega^a \atop \scriptstyle = -\frac{1}{2} C^a{}_{bc} \omega^b \wedge \omega^c + \theta C^a{}_{bc} \omega^b \wedge \lambda^c}
\end{tikzcd}
$$

and then using elements in $\mathrm{CE}(\mathfrak{g})$ and currents on $\Omega^\bullet(S^1)$ to extract the generators

$$\lambda^a(\sigma) \in \Omega^0(\mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(S^1))$$

$$\omega^a(\sigma) \in \Omega^1(\mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(S^1))$$

for all a running over a chosen basis $\{t_a\}$ of $\mathfrak{g}$ and for all $\sigma \in S^1$, and relations

$$d\omega^a(\sigma) = -\frac{1}{2} C^a{}_{bc} \omega^b(\sigma) \wedge \omega^c(\sigma)$$

$$\frac{\partial}{\partial \sigma} \omega^a(\sigma) = d\lambda^a(\sigma) + C^a{}_{bc} \omega^b(\sigma) \wedge \lambda^c(\sigma)\,.$$

The first of these equations is the Chevalley-Eilenberg algebra of $\Omega\mathfrak{g}$, the second one is the action of $\Omega\mathfrak{g}$ on $\Omega^0(S^1, \mathfrak{g})$. $\square$

Loop Lie algebras of String Lie 2-algebras: Kac-Moody central extensions. Recall that the Kac-Moody central extension of the loop Lie algebra $\mathfrak{g}$ coming from a degree 2 invariant polynomial P on $\mathfrak{g}$ is the central extension coming from the 2-cocycle on $\Omega\mathfrak{g}$ given by

$$\mathrm{tg}_{S^1} \mu : f \otimes g \mapsto \int_{S^1} P_{ab}(\frac{\partial}{\partial \sigma} f^a(\sigma)) g^b(\sigma)\, d\sigma$$

Proposition 5 *The loop Lie algebroid of the String Lie 2-algebra $\mathfrak{g}_\mu$ is the action Lie 2-algebroid of the Kac-Moody central extension $\hat{\Omega}\mathfrak{g}$*

$$
\begin{tikzcd}
\textit{something} & \mathrm{CE}(\Lambda \mathfrak{g}_\mu) \arrow[l, two heads] & \mathrm{CE}(\hat{\Omega}\mathfrak{g}) \arrow[l, hook']
\end{tikzcd}\,.
$$

Proof.

I am getting a little too tired. Here are some fomulas one runs into while doing the computation, following the same principle for computing $\mathrm{maps}(A, B)$ as always.

$$
\begin{tikzcd}
t^a \arrow[r, mapsto] \arrow[d, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"] & \lambda^a \theta + \omega^a \arrow[d, mapsto, "d_{S^1} + d_U"] \\
-\frac{1}{2} C^a{}_{bc} t^b \wedge t^c \arrow[r, mapsto] & {\scriptstyle (d_U \lambda^a) \wedge \theta + d_U \omega^a + \theta \wedge \frac{\partial}{\partial \sigma} \omega^a \atop \scriptstyle = -\frac{1}{2} C^a{}_{bc} \omega^b \wedge \omega^c + \theta C^a{}_{bc} \omega^b \wedge \lambda^c}
\end{tikzcd}
$$

$$d\omega^a(\sigma) = -\frac{1}{2} C^a{}_{bc} \omega^b(\sigma) \wedge \omega^c(\sigma)$$

$$\frac{\partial}{\partial\sigma}\omega^a(\sigma) = d\lambda^a(\sigma) + C^a{}_{bc}\omega^b(\sigma)\wedge\lambda^c(\sigma)$$

$$\begin{tikzcd}
b \arrow[rr, mapsto] \arrow[d, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"] & & \theta\wedge\rho+\kappa \arrow[d, mapsto, "d_{S^1}+d_U"] \\
-C_{abc}t^a\wedge t^b\wedge t^c \arrow[r, mapsto] & {} & {\substack{-\theta\wedge d_U\rho+\theta\wedge\frac{\partial}{\partial\sigma}\kappa+d_U\kappa \\ -C_{abc}\omega^a\wedge\omega^b\wedge\omega^c+-3\theta\wedge C_{abc}\lambda^a\wedge\omega^b\wedge\omega^c}}
\end{tikzcd}$$

$$d\rho(\sigma) = \frac{\partial}{\partial\sigma}\kappa(\sigma) - 3C_{abc}\lambda^a(\sigma)\wedge\omega^b(\sigma)\wedge\omega^c(\sigma)$$

$$d\kappa(\sigma) = -C_{abc}\omega^a(\sigma)\wedge\omega^b(\sigma)\wedge\omega^c(\sigma)$$

$$d\int_{S^1}\rho(\sigma)\,d\sigma = -3\int_{S^1}C_{abc}\lambda^a(\sigma)\wedge\omega^b(\sigma)\wedge\omega^c(\sigma)\,d\sigma$$

$$d\int_{S^1}P_{ab}\lambda^a(\sigma)\wedge\omega^b(\sigma)\,d\sigma = \underbrace{\int_{S^1}(\frac{\partial}{\partial\sigma}\omega^a(\sigma))\wedge\omega^b(\sigma)\,d\sigma}_{=\mathrm{tg}_{S^1}\mu} + \frac{1}{2}\int_{S^1}C_{abc}\lambda^a(\sigma)\wedge\omega^b(\sigma)\wedge\omega^c(\sigma)\,d\sigma$$

□

3 Associated L_∞-connections

For $\mathfrak{g}$ any (finite dimensional) L_∞-algebra and X some smooth space, recall that a $\mathfrak{g}$-connection descent object (a $\mathfrak{g}$-bundle with connection) is a diagram

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{K_i\}"'] \arrow[u, hook]
\end{tikzcd} .$$

The topmost horizontal morphism specifies the underlying $\mathfrak{g}$-bundle. The middle one specifies the $\mathfrak{g}$-connection on that. The bottom horizontal morphism picks up the corresponding characteristic classes.

3.1 Associated L_∞-bundles and sections

Definition 4 *Given a $\mathfrak{g}$-descent object*

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \end{tikzcd}$$

as above and a $\mathfrak{g}$-representation ρ on V as in definition 1, the ρ-associated descent object with chosen section σ is a completion of

$$\begin{tikzcd} & \mathrm{CE}_\rho(\mathfrak{g}, V) \\ \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"] \arrow[u, hook] \end{tikzcd}$$

to

$$\begin{tikzcd} & \mathrm{CE}_\rho(\mathfrak{g}, V) \arrow[dl, "(\sigma, A'_{\mathrm{vert}})"'] \\ \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"] \arrow[u, hook] \end{tikzcd} .$$

Here σ denotes the component on the chosen morphism on V^*, while A'_{vert} denotes the component on $\mathfrak{g}^*$.

3.1.1 Examples.

The situation in the integral picture. In order to appreciate this definition, it may be helpful to consider the analogous situation in the world of groupoids.

So let G be an ordinary group, and ρ a representation of that on a space V.

Fix some base space X and consider a surjective submersion $\pi : Y \to X$. Then a G-bundle descent object with respect to Y is a functor

$$g : Y^{[2]} \to \mathbf{B}G$$

$$\begin{tikzcd} & \pi_2(y) \arrow[dr] & \\ \pi_1(y) \arrow[ur] \arrow[rr] & & \pi_3(y) \end{tikzcd} \mapsto \begin{tikzcd} & \bullet \arrow[dr, "\pi_{23}^* g(y)"] & \\ \bullet \arrow[ur, "\pi_{12}^* g(y)"] \arrow[rr, "\pi_{13}^* g(y)"'] & & \bullet \end{tikzcd}$$

for all $y \in Y^{[3]}$, whereas a $V//_\rho G$ descent object (local data for a principal groupoid bundle) is a functor

$$g_\rho : Y^{[2]} \to V//_\rho G$$

$$\begin{tikzcd} & \pi_2(y) \arrow[dr] & \\ \pi_1(y) \arrow[ur] \arrow[rr] & & \pi_3(y) \end{tikzcd} \mapsto \begin{tikzcd} & \pi_2^*\sigma(y) \arrow[dr, "\rho(\pi_{23}^* g(y))"] & \\ \pi_1^*\sigma(y) \arrow[ur, "\rho(\pi_{12}^* g(y))"] \arrow[rr, "\rho(\pi_{13}^* g(y))"'] & & \pi_3^*\sigma(y) \end{tikzcd} .$$

The new datum here is a map

$$\sigma : Y \to V$$

which, by the above cocycle condition, is required to glue properly with respect to the given transition function g. Hence it is indeed a section of the bundle which is ρ-associated to that for which g is the local data.

Our Lie ∞-algebraic formulation is just the differential version of this phenomenon.

Sections of ordinary G-bundles. For $\mathfrak{g}$ the ordinary Lie algebra of the simply connected Lie group G and V^* an ordinary Lie module, for $Y = P$ a principal G-bundle and and for

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$$

the corresponding $\mathfrak{g}$-connection descent object, one finds that the section σ is a V^*-valued function

$$\sigma : P \to V$$

on P such that

$$(d\sigma + A(\sigma))_{\mathrm{vert}} = 0\,.$$

This says indeed that σ is a section of $P \times_\rho V$.

3.2 Covariant derivatives

Extending a ρ-associated $\mathfrak{g}$-descent to a connection descent object computes the covariant derivative $\nabla_A\sigma$ of the chosen section σ with respect to the $\mathfrak{g}$-connection A

$$\begin{tikzcd}
 & \mathrm{CE}_\rho(\mathfrak{g},V) \arrow[dl, "{(\sigma,A_{\mathrm{vert}})}"'] & \\
\Omega^\bullet_{\mathrm{vert}}(Y) & & \mathrm{CE}(\mathfrak{g}) \arrow[ll, "A_{\mathrm{vert}}"'] \arrow[ul, hook] \\
 & \mathrm{W}_\rho(\mathfrak{g},V) \arrow[uu, two heads] \arrow[dl, "{(\sigma,\nabla_A\sigma,A,F_A)}"'] & \\
\Omega^\bullet(Y) \arrow[uu, two heads] & & \mathrm{W}(\mathfrak{g}) \arrow[ll, "{(A,F_A)}"] \arrow[ul, hook] \arrow[uu, two heads]
\end{tikzcd}$$

as the component of

$$\begin{tikzcd}
 & \mathrm{W}_\rho(\mathfrak{g},V) \arrow[dl, "{(\sigma,\nabla_A\sigma,A,F_A)}"'] \\
\Omega^\bullet(Y) &
\end{tikzcd}$$

on $V^*[1]$.

3.2.1 Examples.

Ordinary covariant derivatives. In our example of an ordinary $\mathfrak{g}$-connection A, we find that

$$\nabla_A\sigma = d\sigma + A(\sigma)\,,$$

as one expects.

Adjoint String 2-bundles. For $\mathfrak{g}_\mu$ the String Lie 2-algebra, let $\Omega^\bullet(Y) \xleftarrow{((A,B),(F_A,F_B))} \mathrm{W}(\mathfrak{g}_\mu)$ be a String connection coming from a $\mathfrak{g}$-avlued 1-form A and a 2-form B with 3-form curvature

$$F_B = 3\langle A \wedge F_A\rangle\,.$$

(Recall that, while this is not quite the Chern-Simons 3-form, the latter is the curvature when we pass along the isomorphism $\mathrm{W}(\mathfrak{g}_\mu) \simeq \mathrm{cs}(\mathfrak{g})$).

Lifting this through the adjoint representation of the String Lie 2-algebra on itself

$$\begin{tikzcd} & \mathrm{W}_{\mathrm{ad}}(\mathfrak{g}_\mu) \arrow[dl, "{((\sigma_0,\sigma_1),(\nabla\sigma_0,\nabla\sigma_1),(A,B,F_A,F_B))}"'] \\ \Omega^\bullet(Y) & \mathrm{W}(\mathfrak{g}_\mu) \arrow[l, "{((A,B),(F_A,F_B))}"'] \arrow[u, hook] \end{tikzcd}$$

shows that a section of that adjoint String 2-bundle is

- a $\mathfrak{g}$-valued 0-form $\sigma_0 \in \Omega^\bullet(Y, \mathfrak{g})$;
- a 1-form $\sigma_1 \in \Omega^1(Y)$

whose covariant derivatives are

$$\nabla\sigma_0 = d\sigma_0 + A \wedge (\sigma_0)\,,$$
$$\nabla\sigma_1 = d\sigma_1 + 3\mu(A \wedge A \wedge \sigma_0)\,.$$

So if $\{t_a\}$ is a chosen basis of $\mathfrak{g}$, then

$$\nabla\sigma_1 = d\sigma_1 + 3\sigma_0^c \mu_{abc} A^a \wedge A^b\,.$$

4 The transgression/suspension of $(4k + 1)$-Brane structures to loop spaces

It is well known that String structures on a space X can be conceived

- either in terms of a 3-bundle on X classified by a four class on X obstructing the lift of a 1-bundle on X to a 2-bundle;
- or in terms of a 2-bundle on LX classified by a 3-class on LX obstructing the lift of a 1-bundle on X to another 1-bundle.

In the second case, one is dealing with the transgression/suspension of the first case to loop space.

The relation between the two points of views is carefully described in [?]. Essentially, the result is that *rationally* both obstructions are equivalent.

Remark. Unfortunately, there is no universal agreement on the convention of the direction of the operation called transgression. Both possible conventions are used in the iterature relevant for our purpose here. For instance [?] say transgression for what [?] calls the inverse of transgression (which, in turn, should be called suspension).

We will now demonstrate in the context of L_∞-algebra connections how Lie algebra $(n + 1)$-cocycles related to p-brane structures on X transgress/suspend to loop Lie algebra n-cocycles on loop space.

So let $\mathfrak{g}$ be an ordinary Lie algebra, μ an $(n+1)$-cocycle on it in transgression with an invariant polynomial P, where the transgression is mediated by the transgression element cs as described in **??**.

Recall from **??** that the corresponding universal obstruction structure is the $b^n\mathfrak{u}(1)$-connection

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l, "\mu"'] \\
\mathrm{W}(\mathfrak{g}) \arrow[u, two heads] & \mathrm{W}(b^n\mathfrak{u}(1)) \arrow[l, "(\mathrm{cs},P)"'] \arrow[u, two heads] \\
\mathrm{inv}(\mathfrak{g}) \arrow[u, hook] & \mathrm{inv}(b^n\mathfrak{u}(1)) = \mathrm{CE}(b^{n+1}\mathfrak{u}(1)) \arrow[l, "\{P\}"'] \arrow[u, hook]
\end{tikzcd}$$

to be thought of as the universal higher Chern-Simons $(n+1)$-bundle with connection on the classifying space of the simply connected Lie group integrating $\mathfrak{g}$.

We transgress/suspend this to loops by applying the functor $\mathrm{maps}(-, \Omega^\bullet(S^1))$ to it, which can be thought of as computing for all DGC algebras the DGC algebra of differential forms on the space of maps from the circle into the space that the original DGCA was the algebra of differential forms of:

$$\begin{tikzcd}
\mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(S^1)) & \mathrm{maps}(\mathrm{CE}(b^n\mathfrak{u}(1)), \Omega^\bullet(S^1)) \arrow[l, "\mathrm{tg}_{S^1}\mu"'] \\
\mathrm{maps}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(S^1)) \arrow[u, two heads] & \mathrm{maps}(\mathrm{W}(b^n\mathfrak{u}(1)), \Omega^\bullet(S^1)) \arrow[l, "\mathrm{tg}_{S^1}(\mathrm{cs},P)"'] \arrow[u, two heads] \\
\mathrm{maps}(\mathrm{inv}(\mathfrak{g}), \Omega^\bullet(S^1)) \arrow[u, hook] & \mathrm{maps}(\mathrm{CE}(b^{n+1}\mathfrak{u}(1)), \Omega^\bullet(S^1)) \arrow[l, "\{\mathrm{tg}_{S^1}P\}"'] \arrow[u, hook]
\end{tikzcd} .$$

We want to think of the result as a $b^{n-1}\mathfrak{u}(1)$-bundle. This we can achieve by pulling back along the inclusion

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \hookrightarrow \mathrm{maps}(\mathrm{CE}(b^n\mathfrak{u}(1)), \Omega^\bullet(S^1))$$

which comes from the integration current $\int_{S^1}$ on $\Omega^\bullet(S^1)$ according to proposition **??**.

(This restriction to the integration current can be understood from looking at the basic forms of the loop bundle descent object, which induces *integration without integration* essentially in the sense of [3]. But this we shall not further go into here.)

We now show that the transgressed cocycles $\mathrm{tg}_{S^1}\mu$ are the familiar cocycle on loop algebras, as appearing for instance in Lemma 1 of [**?**]. For simplicity of exposition, we shall consider explicitly just the case where $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ is the canonical 3-cocycle on a Lie algebra with bilinear invariant form $\langle \cdot, \cdot \rangle$.

Proposition 6 *The transgressed cocycle in this case is the 2-cocycle of the Kac-Moody central extension of the loop Lie algebra* $\Omega\mathfrak{g}$

$$\mathrm{tg}_{S^1}\mu : (f, g) \mapsto \int_{S^1} \langle f(\sigma), g'(\sigma) \rangle d\sigma \,.$$

Proof. We compute maps$(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(S^1))$ as before from proposition **??**: for $\{t_a\}$ a basis of $\mathfrak{g}$ and U any test domain, a DGCA homomorphism

$$\phi : \mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(S^1) \otimes \Omega^\bullet(U)$$

sends

$$\begin{tikzcd}
t^a \arrow[rr, mapsto, "\phi"] \arrow[d, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"] && c^a + A^a\theta \arrow[d, mapsto, "d_{S^1}+d_U"] \\
-\frac{1}{2}C^a{}_{bc}t^b \wedge t^c \arrow[rr, mapsto, "\phi"] && \begin{array}{c} \theta\wedge(\frac{\partial}{\partial\sigma}c^a)+d_U c^a + d_U A^a\wedge\theta \\ = -\frac{1}{2}C^a{}_{bc}c^b\wedge c^c - C^a{}_{bc}c^b\wedge A^b\wedge\theta \end{array}
\end{tikzcd} .$$

Here $\theta = d\sigma \in \Omega^1(S^1)$ is the canonical 1-form on S^1 and $\frac{\partial}{\partial\sigma}$ the canonical vector field; moreover $c^a \in \Omega^0(S^1)\otimes\Omega^1(U)$ and $A^a\theta \in \Omega^1(S^1)\otimes\Omega^0(U)$.

By contracting with δ-currents on S^1 we get 1-forms $c^a(\sigma)$, $\frac{\partial}{\partial\sigma}c^a(\sigma)$ and 0-forms $A^a(\sigma)$ for all $\sigma \in S^1$ on maps$(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(S^1))$ satisfying

$$d_{\mathrm{maps}(\cdots)}c^a(\sigma) + \frac{1}{2}C^a{}_{bc}c^b(\sigma)\wedge c^c(\sigma) = 0 \tag{2}$$

and

$$d_{\mathrm{maps}(\cdots)}A^a(\sigma) - C^a{}_{bc}A^b(\sigma)\wedge c^c(\sigma) = \frac{\partial}{\partial\sigma}c^a(\sigma) . \tag{3}$$

So $A^a(\sigma)$ (a “field”) is the function on (necessarily flat) $\mathfrak{g}$-valued 1-forms on S^1 which sends each such 1-form for its t^a-component along θ at σ, while $c^a(\sigma)$ (a “ghost”) is the 1-form which sends each tangent vector field to the space of flat $\mathfrak{g}$-valued forms to the gauge transformation in t^a direction which it induces on the given 1-form at $\sigma \in S^1$.

Notice that the transgression of our 3-cocycle

$$\mu = \mu_{abc}t^a\wedge t^b\wedge t^c = C_{abc}t^a\wedge t^b\wedge t^c \in H^3(\mathrm{CE}(\mathfrak{g}))$$

is

$$\mathrm{tg}_{S^1}\mu = \int_{S^1} C_{abc}A^a(\sigma)c^b(\sigma)\wedge c^c(\sigma)\, d\sigma \;\;\in\;\; \Omega^2(\Omega^1_{\mathrm{flat}}(S^1,\mathfrak{g}) .$$

We can rewrite this using the identity

$$d_{\mathrm{maps}(\cdots)}\left(\int_{S^1} P_{ab}A^a(\sigma)c^b(\sigma)d\sigma\right) = \int_{S^1} P_{ab}\left(\partial_\sigma c^a(\sigma)\right)\wedge c^b(\sigma) + \frac{1}{2}\int_{S^1} C_{abc}A^a(\sigma)c^b(\sigma)\wedge c^c(\sigma) , \tag{4}$$

which follows from 5 and 6, as

$$\mathrm{tg}_{S^1}\mu = \int_{S^1} P_{ab}\left(\partial_\sigma c^a(\sigma)\right)\wedge c^b(\sigma) + d_{\mathrm{maps}(\cdots)}(\cdots) .$$

Then notice that

- equation 5 is the Chevalley-Eilenberg algebra of the loop algebra $\Omega\mathfrak{g}$;
- the term $\int_{S^1} P_{ab}(\partial_\sigma c^a(\sigma))\wedge c^b(\sigma)$ is the familiar 2-cocycle on the loop algebra obtained from transgression of the 3-cocycle $\mu = \mu_{abc}t^a\wedge t^b\wedge t^c = C_{abc}t^a\wedge t^b\wedge t^c$.

5 States of Chern-Simons theory

In [5] a Lie ∞-algebraic model of the Chern-Simons 2-gerbe (line 3-bundle) with connection over BG was given in terms of the generalized $\mathfrak{g}$-connection descent object

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[l, "\mu"'] \\
\mathrm{W}(\mathfrak{g}) \arrow[u, two heads] & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[l, "(\mathrm{cs},P)"'] \arrow[u, two heads] \\
\mathrm{inv}(\mathfrak{g}) \arrow[u, hook] & \mathrm{inv}(b^2\mathfrak{u}(1)) = \mathrm{CE}(b^3\mathfrak{u}(1)) \arrow[l, "\{P\}"'] \arrow[u, hook]
\end{tikzcd} ;$$

Here we describe how to use this to comput the "spaces" of states which Chern-Simons theory assigns to 2-dimensional surfaces and to 1-dimensional circles.

5.1 States over a circle

Let parameter space be the circle

$$\mathrm{par} = S^1 ,$$

and then proceed as follows:

- start with the universal Chern-Simons $b^2\mathfrak{u}(1)$-3-bundle with connection over the classifying space of a simply connected group G, which in the context of [5] is given by the $b^2\mathfrak{u}(1)$-connection descent object

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[l, "\mu"'] \\
\mathrm{W}(\mathfrak{g}) \arrow[u, two heads] & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[l, "(\mathrm{cs},P)"'] \arrow[u, two heads] \\
\mathrm{inv}(\mathfrak{g}) \arrow[u, hook] & \mathrm{inv}(b^2\mathfrak{u}(1)) = \mathrm{CE}(b^3\mathfrak{u}(1)) \arrow[l, "\{P\}"'] \arrow[u, hook]
\end{tikzcd} ;$$

- transgress that from target space $\mathrm{inv}(\mathfrak{g})$ to configuration space $\Omega^\bullet(\mathrm{inv}(\mathfrak{g}), \Omega^\bullet(S^1))$ by applying the functor maps$(--,\Omega^\bullet(S^1))$ to everything, this way obtaining the transgressed object

$$
\begin{tikzcd}
\mathrm{maps}(\mathrm{CE}(\mathfrak{g}),\Omega^\bullet(S^1)) & \mathrm{maps}(\mathrm{CE}(b^2\mathfrak{u}(1)),\Omega^\bullet(S^1)) \arrow[l, "\mathrm{tg}_{S^1}\mu"'] \\
\mathrm{maps}(\mathrm{W}(\mathfrak{g}),\Omega^\bullet(S^1)) \arrow[u, two heads] & \mathrm{maps}(\mathrm{W}(b^2\mathfrak{u}(1)),\Omega^\bullet(S^1)) \arrow[l, "\mathrm{tg}_{S^1}(\mathrm{cs},P)"'] \arrow[u, two heads] \\
\mathrm{maps}(\mathrm{inv}(\mathfrak{g}),\Omega^\bullet(S^1)) \arrow[u, hook] & \mathrm{maps}(\mathrm{CE}(b^3\mathfrak{u}(1)),\Omega^\bullet(S^1)) \arrow[l, "\{\mathrm{tg}_{S^1}P\}"'] \arrow[u, hook]
\end{tikzcd}
\;;
$$

- we pick a representation, form the associated bundle and compute the space of its sections.

Unwrapping this. Recall from [5] what maps$(\mathrm{CE}(\mathfrak{g}),\Omega^\bullet(S^1))$ looks like: for $\{t_a\}$ a basis of $\mathfrak{g}$, a plot of this space over the test domain U, i.e. a morphism

$$\phi : \mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(S^1)\otimes\Omega^\bullet(U)$$

sends

$$
\begin{tikzcd}
t^a \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"] & c^a + A^a\theta \arrow[d, mapsto, "d_{S^1}+d_U"] \\
-\frac{1}{2}C^a{}_{bc}t^b\wedge t^c \arrow[r, mapsto, "\phi"] & {\substack{\theta\wedge(\frac{\partial}{\partial\sigma}c^a)+d_U c^a+d_U A^a\wedge\theta \\ =-\frac{1}{2}C^a{}_{bc}c^b\wedge c^c - C^a{}_{bc}c^b\wedge A^b\wedge\theta}}
\end{tikzcd}
\;.
$$

Here $\theta = d\sigma \in \Omega^1(S^1)$ is the canonical 1-form on S^1 and $\frac{\partial}{\partial\sigma}$ the canonical vector field; moreover $c^a \in \Omega^0(S^1)\otimes\Omega^1(U)$ and $A^a\theta \in \Omega^1(S^1)\otimes\Omega^0(U)$.

By contracting with δ-currents on S^1 we get 1-forms $c^a(\sigma)$, $\frac{\partial}{\partial\sigma}c^a(\sigma)$ and 0-forms $A^a(\sigma)$ for all $\sigma\in S^1$ on maps$(\mathrm{CE}(\mathfrak{g}),\Omega^\bullet(S^1))$ satisfying

$$d_{\mathrm{maps}(\cdots)}c^a(\sigma) + \frac{1}{2}C^a{}_{bc}c^b(\sigma)\wedge c^c(\sigma) = 0 \tag{5}$$

and

$$d_{\mathrm{maps}(\cdots)}A^a(\sigma) - C^a{}_{bc}A^b(\sigma)\wedge c^c(\sigma) = \frac{\partial}{\partial\sigma}c^a(\sigma)\,. \tag{6}$$

So $A^a(\sigma)$ (a "field") is the function on (necessarily flat) $\mathfrak{g}$-valued 1-forms on S^1 which sends each such 1-form for its t^a-component along θ at σ, while $c^a(\sigma)$ (a "ghost") is the 1-form which sends each tangent vector field to the space of flat $\mathfrak{g}$-valued forms to the gauge transformation in t^a direction which it induces on the given 1-form at $\sigma\in S^1$.

Notice that the transgression of our 3-cocycle

$$\mu = \mu_{abc}t^a\wedge t^b\wedge t^c = C_{abc}t^a\wedge t^b\wedge t^c \in H^3(\mathrm{CE}(\mathfrak{g}))$$

is

$$\mathrm{tg}_{S^1}\mu = \int_{S^1} C_{abc}A^a(\sigma)c^b(\sigma)\wedge c^c(\sigma)\,d\sigma \;\;\in\;\; \Omega^2(\Omega^1_{\mathrm{flat}}(S^1,\mathfrak{g})\,.$$

We can rewrite this using the identity

$$d_{\mathrm{maps}(\cdots)}\left(\int_{S^1} P_{ab}A^a(\sigma)c^b(\sigma)d\sigma\right) = \int_{S^1} P_{ab}\left(\partial_\sigma c^a(\sigma)\right)\wedge c^b(\sigma) + \frac{1}{2}\int_{S^1} C_{abc}A^a(\sigma)c^b(\sigma)\wedge c^c(\sigma)\,, \qquad (7)$$

which follows from 5 and 6, as

$$\mathrm{tg}_{S^1}\mu = \int_{S^1} P_{ab}\left(\partial_\sigma c^a(\sigma)\right)\wedge c^b(\sigma) + d_{\mathrm{maps}(\cdots)}(\cdots)\,.$$

Then notice that

- equation 5 is the Chevalley-Eilenberg algebra of the loop algebra $\Omega\mathfrak{g}$;
- the term $\int_{S^1} P_{ab}(\partial_\sigma c^a(\sigma))\wedge c^b(\sigma)$ is the familiar 2-cocycle on the loop algebra obtained from transgression of the 3-cocycle $\mu = \mu_{abc}t^a\wedge t^b\wedge t^c = C_{abc}t^a\wedge t^b\wedge t^c$.

So we find that sections of the Chern-Simons 3-bundle transgressed to a 2-bundle over the circle come from bundles of representations of the centrally extended loop algebra $\hat{\Omega}\mathfrak{g}$ over the space of $\mathfrak{g}$-holonomies over the circle:

let $(V,\hat{\rho})$ be a representation of the centrally extended loop Lie algebra $\hat{\Omega}\mathfrak{g}$ and

$$\mathrm{CE}(\hat{\Omega}\mathfrak{g},V)_{\mathrm{tg}_{S^1}\mu}$$

the corresponding Chevalley-Eilenberg algebra obtained from forming its String-like extension with the transgressed cocycle, hence

$$\mathrm{CE}(\hat{\Omega}\mathfrak{g},V)_{\mathrm{tg}_{S^1}\mu} = \left(\wedge^\bullet(V\oplus\hat{\Omega}\mathfrak{g}^*\oplus\mathbb{R}[1]), d_{\mathrm{CE}(\hat{\Omega}\mathfrak{g},V)_{\mathrm{tg}_{S^1}\mu}}\right)$$

with

$$d_{\mathrm{CE}(\hat{\Omega}\mathfrak{g},V)_{\mathrm{tg}_{S^1}\mu}}|_{\mathbb{R}[1]} : b\mapsto \mathrm{tg}_{S^1}\mu\,,$$

for $\{b\}$ the canonical basis of $\mathbb{R}[1]$.

Notice that$\mathrm{CE}(\hat{\Omega}\mathfrak{g},V)_{\mathrm{tg}_{S^1}\mu}$ is a puffed-up version of the CE algebra of a non-extended loop representation.

Then the corresponding sections of our transgressed Chern-Simons 3-bundle are morphisms

$$\begin{tikzcd}
 & \mathrm{CE}(\hat{\Omega}\mathfrak{g},V)_{\mathrm{tg}_{S^1}\mu} \arrow[dl] \\
\mathrm{maps}(\mathrm{CE}(\mathfrak{g}),\Omega^\bullet(S^1)) & \mathrm{maps}(\mathrm{CE}(b^2\mathfrak{u}(1)),\Omega^\bullet(S^1)) \arrow[l, "\mathrm{tg}_{S^1}\mu"'] \arrow[u, hook]
\end{tikzcd}$$

which hence define bundle of loop group representations over the space of $\mathfrak{g}$-connections on the circle.

5.2 States over a 2-dimensional surface

For a given parameter space

$$\mathrm{par} = \Sigma\,,$$

a manifold of dimension 2 (the "membrane"), this means, according to [5] that

- we start with the universal Chern-Simons $b^2\mathfrak{u}(1)$-3-bundle with connection over the classifying space of a simply connected group G, which in the context of [5] is given by the $b^2\mathfrak{u}(1)$-connection descent object

$$
\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) & & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[ll, "\mu"'] \\
\mathrm{W}(\mathfrak{g}) \arrow[u, two heads] & & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[ll, "(\mathrm{cs}, P)"'] \arrow[u, two heads] \\
\mathrm{inv}(\mathfrak{g}) \arrow[u, hook] & & \mathrm{inv}(b^2\mathfrak{u}(1)) = \mathrm{CE}(b^3\mathfrak{u}(1)) \arrow[ll, "\{P\}"'] \arrow[u, hook]
\end{tikzcd}
\quad ;
$$

- we transgress that from target space $\mathrm{inv}(\mathfrak{g})$ to configuration space $\Omega^\bullet(\mathrm{inv}(\mathfrak{g}), \Omega^\bullet(\Sigma))$ by applying the functor $\mathrm{maps}(--, \Omega^\bullet(\Sigma))$ to everything, this way obtaining the transgressed object

$$
\begin{tikzcd}
\mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(\Sigma)) & \mathrm{maps}(\mathrm{CE}(b^2\mathfrak{u}(1)), \Omega^\bullet(\Sigma)) \arrow[l, "\mathrm{tg}_\Sigma \mu"'] \\
\mathrm{maps}\mathrm{W}(\mathfrak{g}), \Omega^\bullet(\Sigma)) \arrow[u, two heads] & \mathrm{maps}(\mathrm{W}(b^2\mathfrak{u}(1)), \Omega^\bullet(\Sigma)) \arrow[l, "\mathrm{tg}_\Sigma(\mathrm{cs}, P)"'] \arrow[u, two heads] \\
\mathrm{maps}(\mathrm{inv}(\mathfrak{g}), \Omega^\bullet(\Sigma)) \arrow[u, hook] & \mathrm{maps}(\mathrm{CE}(b^3\mathfrak{u}(1)), \Omega^\bullet(\Sigma)) \arrow[l, "\{\mathrm{tg}_\Sigma P\}"'] \arrow[u, hook]
\end{tikzcd}
\quad ;
$$

- we pick a representation, form the associated bundle and compute the space of its sections.

Using the considerations in [7] we restrict attention on the space $\mathrm{maps}(\mathrm{CE}(b^2\mathfrak{u}(1)), \Omega^\bullet(\Sigma))$ of all 3-forms on Σ (of course there is not a single nontrivial 3-form on the 2-dimensional Σ, so this space has a single point, but it still has nontrivial 1-forms on it) to those 1-forms which come from the current on $\Omega^2(\Sigma)$ which integrates any 2-form over Σ, we get a morphism

$$
\begin{tikzcd}
 & \mathrm{CE}_\rho(\mathfrak{u}(1), \mathbb{C}) & \\
\mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(\Sigma)) & & \mathrm{CE}(\mathfrak{u}(1)) \arrow[ll, "\mathrm{tg}_\Sigma \mu"] \arrow[ul, hook]
\end{tikzcd}
$$

and the space of sections σ that we are after is then the space of completions of this diagram

$$
\begin{tikzcd}
 & \mathrm{CE}_\rho(\mathfrak{u}(1), \mathbb{C}) \arrow[dl, "(\sigma, \mathrm{tg}_\Sigma \mu)"'] & \\
\mathrm{maps}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(\Sigma)) & & \mathrm{CE}(\mathfrak{u}(1)) \arrow[ll, "\mathrm{tg}_\Sigma \mu"] \arrow[ul, hook]
\end{tikzcd}
\quad .
$$

Such a σ is hence a function on the space of flat $\mathfrak{g}$-valued 1-forms on Σ, with the property that

$$d\sigma + (\mathrm{tg}_\Sigma \mu)\sigma = 0\,,$$

where d is the differential on the space of flat $\mathfrak{g}$-valued 1-form on Σ, and where $\mathrm{tg}_\Sigma\mu$ is the 1-form on that space which comes from the transgression of $\mathrm{CE}(\mathfrak{g}) \xleftarrow{\mu} \mathrm{CE}(b^2\mathfrak{u}(1))$.

This requires some unwrapping.

Unwrapping this. Recall from [5] what maps$(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(\Sigma))$ looks like, computed for the simple case that we take $\Sigma = \mathbb{R}^2$: for $\{t_a\}$ a basis of $\mathfrak{g}$, a plot of this space over the test domain U, i.e. a morphism

$$\phi : \mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(\Sigma) \otimes \Omega^\bullet(U)$$

sends

$$\begin{tikzcd}
t^a \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"] & c^a + A^a_\mu dx^\mu \arrow[d, mapsto, "d_\Sigma + d_U"] \\
-\frac{1}{2}C^a{}_{bc} t^b \wedge t^c \arrow[r, mapsto, "\phi"] & \begin{array}{l} = (\frac{\partial}{\partial x^\mu} c^a)\wedge dx^\mu + d_U c^a + ((d_\Sigma + d_U A^a_\mu))\wedge dx^\mu \\ -\frac{1}{2}C^a{}_{bc} c^b \wedge c^c - C^a{}_{bc} c^b \wedge A^b_\mu \wedge dx^\mu - \frac{1}{2} C^a{}_{bc} A^a_\mu \wedge A^b_\nu \wedge dx^\mu \wedge dx^\nu \end{array}
\end{tikzcd}\,.$$

So that we get 1-forms $c^a(x)$ and 0-forms $A^a_\mu(x)$ and $\partial_{[\mu}\lambda^a_{\nu]}$ on maps$(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(\Sigma))$ for all $\mu \in \{1,2\}$ and $x \in \Sigma$ satisfying

$$d_{\mathrm{maps}(\cdots)} c^a(x) + \frac{1}{2} C^a{}_{bc} c^b(x) \wedge c^c(x) = 0$$

and

$$\partial_{[\mu} A^a_{\nu]}(x) + \frac{1}{2} C^a{}_{bc} A^b_{[\mu}(x) A^c_{\nu]}(x) = 0$$

and

$$d_{\mathrm{maps}(\cdots)} A^a_\mu(x) + C^a{}_{bc} A^b_\mu(x) \wedge c^c(x) + \partial_\mu c^a(x) = 0\,. \tag{8}$$

So $A^a_\mu(x)$ (a "field") is the function on flat $\mathfrak{g}$-valued 1-forms on Σ which sends each such 1-form for its t^a-component along dx^μ at x, while $c^a(x)$ (a "ghost") is the 1-form which sends each tangent vector field to the space of flat $\mathfrak{g}$-valued forms to the gauge transformation in t^a direction which it induces on the given 1-form at $x \in \Sigma$.

We have in particular the transgression of the 3-form which specified the Chern-Simons 3-bundle as a descent object, which is

$$\mathrm{tg}_\Sigma \mu = (\int_\Sigma \mu_{abc} A^a_\mu(x) \wedge A^b_\nu(x) dx^\mu \wedge dx^\nu \wedge c^c(x)) \in \Omega^1(\Omega^1_{\mathrm{flat}}(\Sigma, \mathfrak{g}))\,,$$

where we keep the notation with $\{x^\mu\}$ the canonical coordinates on Σ assumed to be $\mathbb{R}^2$, just because otherwise this will become rather intransparent. The point is that the 1-form $\mathrm{tg}_\Sigma\mu$ is obtained by pulling back the degree 3 element μ to the space of flat $\mathfrak{g}$-valued 1-forms on Σ and then doing the fiber integral over Σ.

So a section of the transgressed Chern-Simons bundle is a function

$$\sigma = \psi(A^a_\mu(x))$$

of the fields $\{A^a_\mu(x) | x \in \Sigma, a \in \{1, \cdots, \dim\mathfrak{g}\}, \mu \in \{1,2\}\}$ such that

$$d\psi(A^a_\mu(x)) + (\int_\Sigma \mu_{abc} A^a_\mu(x) \wedge A^b_\nu(x) dx^\mu \wedge dx^\nu \wedge c^a(x))\psi(A^a_\mu(x)) = 0\,.$$

To get a sense for what this equation says it may be helpful, (not for people who cannot but think like pure mathematicians, though) to write this in the fashion common in physics, where it says

$$\left((\frac{\partial}{\partial A^a_\mu(x)}\psi) C^a{}_{bc} A^b_\mu(x) + \mu_{abc} A^a_\mu(x) \wedge A^b_\nu(x) \right) \psi c^c(x) = 0$$

for all $x \in \Sigma$.

Now recalling that our cocycle μ is actually given by $\mu_{abc} = C_{abc} := P_{ad} C^d{}_{bc}$ with $\{P_{ab}\}$ the components of the invariant polynomial P (the Killing form), we can turn that into

$$C_{abc} \left(A^b_\mu(x) \frac{\partial}{\partial A_{a\mu}(x)} + A^a_\mu(x) \wedge A^b_\nu(x) \right) \psi = 0$$

for all $x \in \Sigma$ and $c \in \{1, \cdots, \dim(\mathfrak{g})\}$.

Using 8 we can rewrite this equivalently as

$$\left(\partial_{[\mu} A^a_{\nu]} + C^a{}_{bc} A^b_\mu(x) \frac{\partial}{\partial A_{c\mu}(x)} \right) \psi = 0\,.$$

This is beginning to look not unlike equations like (2.3) in [2]. But in order to reproduce that exactly, I need to introduce a complex structure into the game first. I haven't yet fully figured out how that arises naturally from the present point of view.

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