Homotopy, Concordance and Natural Transformation

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November 23, 2007

Abstract

Transformations between (ω-)functors are like homotopies or concordances between maps of topological spaces. This statement can be given a precise meaning using the closed structure of ωCat in terms of the extension of the Gray tensor product from 2-categories to ω-categories given by Sjoerd Crans. The analogous construction is familar in homological algebra from categories of chain complexes.

After recalling the basics, we turn to "anafunctors" (certain spans of functors) and highlight how the general relation between transformations, homotopies and concordances appears in the study of nonabelian n-cocycles classifying n-bundles.

Contents

1 Introduction

One way to state the question that this document is concerned with is this:

> Is there a definition of ana-ω-functors such that the biclosed structure of ωCat extends to a biclosed structure of $\omega\mathrm{Cat}_{\mathrm{ana}}$, where in the latter we take morphisms not to be ω-functors but ana-ω-functor?

A more gentle way to state what this document is concerned with is this:

A *homotopy* is a map

$$h : X \times I \to Y$$

which restricts on the two ends of the interval I to two fixed maps.

A *natural transformation* on the other hand is a functor

$$\tilde{h} : C \to \mathrm{hom}(I, D)\,,$$

where now I is the category with one single nontrivial morphisms.

Using adjointness of the internal hom with a tensor product, both pictures coincide:

$$\mathrm{Hom}(C, \mathrm{hom}(I, D)) \simeq \mathrm{Hom}(C \otimes I, D)\,.$$

For this to be true for higher categories, $\otimes$ must not be the cartesian product, $\times$, but the Gray tensor product [1] $\otimes_{\mathbf{Gray}}$ and its generalization to ω-categories given by Sjord Crans [2]: like the product of an n-dimensional space with an m-dimensional space is an $(n+m)$-dimensional space, the $\otimes$-product of an n-category with an m-category is an $(n+m)$-category.

Often, morphisms from X to Y, need to be taken as "generalized morphisms", namely spans

$$\begin{tikzcd} Y \arrow[d, "\simeq"'] \arrow[drr, "f"] & & \\ C & & D \end{tikzcd}$$

where f starts not at C itself, but on a cover of C.

Such morphisms have been termed *anafunctors* by Makkai [4], being closely related to *profunctors*, *Morita morphisms* and the like. An archetypical example for an anafunctor is a G-cocycle classifying a principal G-bundle, for G some n-group, to which we come in moment.

A homotopy between two anafunctors has to take the choice of cover into account. A *concordance* is a span

$$\begin{tikzcd} \hat{Y} \arrow[d, "\simeq"'] \arrow[drr, "f"] & & \\ C \otimes I & & D \end{tikzcd}$$

which restricts to two given anafunctors over the endpoints of I.

Here I want to eventually talk about how concordance of anafunctors relates to natural transformations of anafunctors, as given originally by Makkai:

$$\begin{tikzcd} & Y \arrow[dd, Rightarrow, "h"] \arrow[dr, "f"] & \\ Y \times_C Y' \arrow[ur] \arrow[dr] & & B \\ & Y' \arrow[ur, "f'"'] & \end{tikzcd} .$$

One motivation is to clarify how the notion of concordance of 2-bundles used in [8] relates precisely to other natural notions of morphisms of 2-bundles, like those used in [9, 12, 13]:

given a space X and given an n-category S, a ("nonabelian") S-cocycle on X is (this observation, which feels tautologous nowadays, has apparently first been made long ago by [6]) a choice of regular epimorphism

$$\pi : Y \to X$$

together with an n-functor

$$g : Y^\bullet \to S \,,$$

where $Y^\bullet$ is the obvious n-groupoid obtained from p.

For instance when $S = \mathbf{B}G = \left\{ \bullet \xrightarrow{g} \bullet \,|g \in G \right\}$ is the one-object groupoid obtained from a group G, then a functor

$$g : Y^\bullet \to \mathbf{B}G$$

is precisely a labelling of points in $Y \times_X Y$ by elements in G, such that the cocycle condition

$$\pi_{23}^* g \cdot \pi_{12}^* g = \pi_{13}^* g$$

familiar from principal G-bundles is satisfied.

Analogously, by letting $S = \mathbf{B}\mathrm{AUT}(G)$, for $\mathrm{AUT}(G)$ the automorphism *2-group* of an ordinary group G, one obtains the 2-cocycle classifiying a G-gerbe [9, 10, 11].

In these cases, there is a *global* notion of the structure being classified by the cocycle. The cocycle itself, including the choice of cover it involves, is part of the *descent data* [5] which describes the descent of a trivial n-bundle on the cover down to a possibly nontrivial n-bundle on the base.

Correspondingingly, there is then little choice for the right notion of morphisms of n-cocycles: whatever these are, they need to reproduce the morphisms between the global objects that they come from.

Still, there is quite some leeway in making the details precise, as n increases above $n = 1$. For some time this issue had found attention mainly in the context of *bundle gerbes*:

a line bundle gerbe is, in our language used here, a smooth 2-functor

$$g : Y^\bullet \to \mathbf{B}1d\mathrm{Vect}\,.$$

Here $\mathbf{B}1d\mathrm{Vect} \subset 2\mathrm{Vect}$ is the 2-category with a single object such that the End-category of that object is the category of 1-dimensional vector spaces, with horizontal composition being the tensor product over the ground field.

While it is in principle clear how morphisms between bundle gerbes over different covers Y should behave (see the review of anafunctors in section 3.1), there are some subtleties involved in spelling this out [14, 15].

Seeing exactly how the notion of concordance fits into this picture, which has been done to great effect in [**?**], should be helpful.

By the above considerations, something like the following should be true:

Concordance is what becomes of transformations in the joint context of anafunctors and the closed structure on ωCat.

Since, at the time of this writing, it is Friday evening and I need to catch a train in a moment, the following is unfinished. But I guess the main point should already be visible.

2 Homotopy and Transformation

To set the scene, I'll dare to bore the reader with recalling the definition of homotopies and of natural transformations.

- For X and Y topological spaces and

  $$f, g : \; X \longrightarrow Y$$

  two continuous maps between them, a **homotopy** h from f to g

  $$\begin{tikzcd} X \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow h"] & & Y \end{tikzcd}$$

  is a continuous map

  $$h : \; X \times I \longrightarrow Y \;\;,$$

  where

  $$I = [0, 1]$$

  is the interval with

  $$s, t : \; \{\bullet\} \longrightarrow I$$

the injection of its two endpoints, such that

$$(\mathrm{Id}_X \times s)^* h = f$$

and

$$(\mathrm{Id}_X \times t)^* h = g\,,$$

i.e. such that

$$\begin{tikzcd} X \times \{\bullet\} \arrow[r, "\mathrm{Id}\times s"] \arrow[rr, bend right, "f"'] & X \times I \arrow[r, "h"] \arrow[d, phantom, "\|"] & Y \\ & {} & \end{tikzcd} \qquad \begin{tikzcd} X \times \{\bullet\} \arrow[r, "\mathrm{Id}\times t"] \arrow[rr, bend right, "g"'] & X \times I \arrow[r, "h"] \arrow[d, phantom, "\|"] & Y \\ & {} & \end{tikzcd} \,.$$

- For C and D categories and

$$f, g :\ X \longrightarrow Y$$

two functors between them, a **natural transformation** h from f to g

$$\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow h"] & D \end{tikzcd}$$

is a functor

$$h :\ X \times I \longrightarrow Y\ ,$$

where

$$I = \{\ \bullet \longrightarrow \circ\ \}$$

is the catgegory with one nontrivial morphism, with

$$s, t :\ \{\bullet\} \longrightarrow I$$

the injection of its two endpoints, such that

$$(\mathrm{Id}_C \times s)^* h = f$$

and

$$(\mathrm{Id}_C \times t)^* h = g\,,$$

i.e. such that

$$\begin{tikzcd} C \times \{\bullet\} \arrow[r, "\mathrm{Id}\times s"] \arrow[rr, bend right, "f"'] & C \times I \arrow[r, "h"] \arrow[d, phantom, "\|"] & D \\ & {} & \end{tikzcd} \qquad \begin{tikzcd} C \times \{\bullet\} \arrow[r, "\mathrm{Id}\times t"] \arrow[rr, bend right, "g"'] & C \times I \arrow[r, "h"] \arrow[d, phantom, "\|"] & D \\ & {} & \end{tikzcd} \,.$$

Notice how the definition of natural transformation here turns into the maybe more familiar one in terms of commuting naturality squares by unwrapping the definition of the product $A \times B$ of two categories A and B:

Definition 1 *For A and B any two categories, the product category $A \times B$ has as objects the cartesian product of the collection of objects of A and B*

$$(A \times B)_0 = A_0 \times B_0$$

and its morphisms are generated *from those of A and B*

$$\forall (\ a \xrightarrow{f} a'\ ) \in A_1,\ b \in B_0 : (\ (a,b) \xrightarrow{(f,\mathrm{Id}_b)} (a',b)\ ) \in (A \times B)_1$$

$$\forall (\ b \xrightarrow{g} b'\ ) \in B_1,\ a \in A_0 : (\ (a,b) \xrightarrow{(\mathrm{Id}_a,g)} (a,b')\ ) \in (A \times B)_1$$

modulo the relations

$$\forall f,g \in A_1 \times B_1 : \quad \begin{tikzcd} (a,b) \arrow[r, "(\mathrm{Id}_a,g)"] \arrow[d, "(f,\mathrm{Id}_b)"'] & (a,b') \arrow[d, "(f,\mathrm{Id}_{b'})"] \\ (a',b) \arrow[r, "(\mathrm{Id}_{a'},g)"] & (a',b') \end{tikzcd} .$$

Therefore defining a functor

$$h : C \times I \to D$$

is precisely the same as defining two functors $f, g : C \to D$ with a natural transformation between them:

$$\forall \left(\begin{tikzcd} c \arrow[d, "\gamma"] \\ c' \end{tikzcd} \right) \in C_1 : \ h : \left(\begin{tikzcd} (c,\bullet) \arrow[r, "(\mathrm{Id}_c,\to)"] \arrow[d, "(\gamma,\mathrm{Id}_\bullet)"'] & (c,\circ) \arrow[d, "\gamma{,}\mathrm{Id}_\circ)"] \\ (c',\bullet) \arrow[r, "(\mathrm{Id}_{c'},\to)"] & (c',\circ) \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} f(c) \arrow[r, "h(c)"] \arrow[d, "f(\gamma)"'] & g(c) \arrow[d, "g(\gamma)"] \\ f(c') \arrow[r, "h(c')"] & g(c') \end{tikzcd} \right)$$

The commutativity of the ("naturality") square on the right is just the respect of h for the relations in $C \times I$.

By reading the square on the right from top to bottom, we obtain the more common conception of a natural transformation as a map that sends objects to morphisms: functors $h : C \times I \to D$ are in bijection with functors

$$\tilde{h} : C \to \mathrm{Funct}(I, D)$$

under the identitfication

$$\forall c \in C_0 : \ \tilde{h} : c \mapsto \Big((\bullet \longrightarrow \circ) \mapsto (f(c) \xrightarrow{h(c)} g(c)) \Big)$$

and

$$\forall \left(\begin{tikzcd} c \arrow[d, "\gamma"] \\ c' \end{tikzcd} \right) \in C_1 : \ \tilde{h} : c \mapsto \left((\bullet \longrightarrow \circ) \mapsto \left(\begin{tikzcd} f(c) \arrow[r, "h(c)"] \arrow[d, "f(\gamma)"'] & g(c) \arrow[d, "g(\gamma)"] \\ f(c') \arrow[r, "h(c')"] & g(c') \end{tikzcd} \right) \right) .$$

What we are spelling out here in possibly superfluous detail is nothing but a special case of the general statement

Fact 1 *The category* Cat *of categories is closed, with the tensor product* $\otimes := \times$ *being the cartesian product from above and with the internal hom coming from the categories of functors*

$$\mathrm{hom}(A, B) = \mathrm{Funct}(A, B) .$$

The above bijection between "functorial homotopies" $h : C \times I \to D$ and natural transformations $\tilde{h} : C \to \mathrm{Funct}(I, D)$ is just a special case of the internal hom being adjoint to the tensor product

$\mathrm{Hom}(C \times I, D)$	$\simeq$	$\mathrm{Hom}(C, \mathrm{hom}(I, D))$
homotopies		nat. transformations

.

2.1 Products of spaces and the biclosed structure on ω-categories

While true, this may not appear to be overly exciting. But it is remarkable for the following reason:

we have seen that the interval

$$I = [0, 1]$$

is categorically represented by the one-morphism 1-category

$$2 = \{ \bullet \longrightarrow \circ \} ,$$

which above we called, by abuse of notation, by the same name as the interval.

But the natural kind of product of the interval with itself produces the square

$$I \times I = [0,1]^2 \,,$$

which is 2-dimensional, while the cartesian product of 2 with itself is still just a 1-category:

$$\{ \bullet \longrightarrow \circ \} \times \{ \bullet \longrightarrow \circ \} \;=\; \left\{ \begin{tikzcd} (\bullet,\bullet) \arrow[r] \arrow[d] & (\bullet,\circ) \arrow[d] \arrow[dl, equal] \\ (\circ,\bullet) \arrow[r] & (\circ,\circ) \end{tikzcd} \right\} .$$

Consider the category of strict 2-categories and strict 2-functors between these. The *cartesian* product $\times$ of categories used above generalizes straightforwardly to this category of 2-categories. However, the internal hom functor adjoint to this product

$$(\mathrm{hom}_{2\mathrm{Cat},\times}(A,-)) \vdash (- \times A)$$

is not the one that is usually needed: because the above argument continues to run through literally at the level of 1-morphisms, one finds that

Fact 2 *The internal hom* $\mathrm{hom}_{2\mathrm{Cat},\times}$ *with respect to the cartesian product of strict 2-categories sends 2-categories A, B to the 2-category* $\mathrm{hom}_{2\mathrm{Cat},\times}(A,B)$ *whose objects are strict 2-functors and whose 1-morphisms are just ordinary natural transformations.*

Here "ordinary natural transformations" refers to the fact that the naturality squares still do commute. But more generally, the "right" notion of transformation of 2-functors involves naturality squares that are *filled* with a 2-morphism, either an arbitrary one – in this case we have "lax-" or "oplax-" natural transformations, or by an invertible one – in which case we have "pseudonatural" transformations.

It does not matter much which of these cases we concentrate on. But the pseudo case is singled out both conceptually (we want the naturality squares to "commute weakly" which should reallz mean that they commute up to an *iso*morphism) and by the fact that it seems to be the relevant one for most applications. Therefore we choose to concentrate from now on on all transformations being pseudonatural transformations.

So then let, for A and B two strict 2-categories,

$$\mathrm{Funct}_2(A,B)$$

be the strict 2-category whose objects are functors, whose morphisms are *pseudo*natural transformations and whose 2-morphisms are transformations.

Following standard convention, we will therefore suppress the term "pseudo-natural transformation" entirely and speak, equivalently, of just "transformations".

One may then look for the tensor product adjoint to $\mathrm{Funct}_2(A, -)$. The tensor product one finds this way was originally given by Gray [1], and since then carries its name.

Definition 2 (Gray tensor product for 2-categories) *The* **Gray tensor product** *for 2-categories*

$$\otimes_{\mathbf{Gray}} : 2\mathrm{Cat} \times 2\mathrm{Cat} \to 2\mathrm{Cat}$$

is defined by the fact that $(- \otimes A)$ *is adjoint to* $\mathrm{Funct}_2(A, -)$.

Following our above argument, it is easy to see what the crucial aspect of the Gray tensor product must be: the image under the Gray tensor product of two 1-morphisms is not a *commuting* ("thin") square , but a *filled* square:

$$\{\, a \xrightarrow{f} a' \,\} \otimes_{\mathbf{Gray}} \{\, b \xrightarrow{g} b' \,\} \;=\; \left\{ \begin{tikzcd} (a,b) \arrow[r, "(\mathrm{Id}_a, g)"] \arrow[d, "(f, \mathrm{Id}_b)"'] & (a,b') \arrow[d, "(f, \mathrm{Id}_{b'})"] \arrow[dl, Rightarrow, "\eta_{f,g}"] \\ (a',b) \arrow[r, "(\mathrm{Id}_{a'}, g)"'] & (a',b') \end{tikzcd} \right\} .$$

So in particular, the Gray tensor product of two 1-categories is no longer a 1-category, but a 2-category. The extra 2-morphisms appearing here, denoted η in the above diagram, satisfy some essentially obvious coherence laws.

It was Sjoerd Crans who realized that this just a special case of a general principle: the category $\omega\mathrm{Cat}$ of ω-categories (strict higher categories of unbounded categorical degree) carries a generalization of the Gray tensor product such that $(\omega\mathrm{Cat}, \otimes)$ is closed. And this tensor product $\otimes$ has the property that the product of an n-category with an m-category is an $(n + m)$-category.

Sjoerd Crans very nicely points out the relevance of this fact in the the introduction of [3], which I find important enough to quote at length here:

> The difference between the cartesian product and Gray's tensor product, and between 2-natural and lax- (or pseudo-) natural transformations, abd between 3-categories and **Gray**-categories, is not just the difference between a commuting square and a square commuting up to a 2-arrow. [...]

> No, the conceptual difference lies in the treatment of dimension. The cartesian produc of 2-categories, and of ω-categories, is basically set-theoretical, [...]. The tensor product of 2-categories, and of ω-categories [2] is basically topological. [...]
>
> This difference in viewpoint has profound implications. Firstly, lax-natural transformations and modifications of 2-categories, and, more generally, lax-q-transformations of ω-categories [2], are 2- or ω-functors $\mathbb{C} \otimes 2_q \to \mathbb{D}$, where 2_q denotes the ω-category free on one q-dimensional element [modelling the q-dimensional ball $\simeq [0,1]^q$, U.S.]. Because of the dimension-raising aspect of the tensor product, they become maps $\mathbb{C} \to \mathbb{D}$ sending a p-arrow to a $(p+q)$-arrow [...]. This is very much like in topology, where degree q maps between chain complexes satisfying some condition with respect to the boundary are known as *q-homotopies*. In fact, there is a precise correspondence between q-homotopies and lax q-transformations, the latter being the directed, functorial form of the former [7].

([3], pages 13 and 14)

Sjoerd Crans makes further remarks on the relation between the dimension-raising of the tensor product categories and topological phenomena, such as Whitehead products. The interested reader should have a look at his article.

With the dimension-raising tensor product $\otimes$ thus replacing the cartesian tensor product $\times$, we obtain a generalization of our relation between homotopies and transformations for arbitrary ω-categories:

$\mathrm{Hom}(C \otimes I, D)$	$\simeq$	$\mathrm{Hom}(C, \mathrm{hom}(I, D))$
homotopies		transformations

.

Here our $I = \{\, \bullet \longrightarrow \circ \,\}$ is what Sjoerd Crans denotes 2_1.

For our discussion of transformatinons and concordances of 2-bundles the Gray tensor product on 2-categories will already suffice. But we should in principle be able to define and discuss the notion of transformations of anafunctors, and hence of concordance, inside all of ωCat. Eventually.

3 Anafunctors and descent data

We now look in detail at ana-n-functors ("n-cocycles"), in order to prepare the ground for the discussion of transformations and concordance of these.

In [12] we indicate how the category of descent data for funcors is canonically isomorphic to that of the corresponding anafunctors.

Definition 3 *A* **descent object** *(see [5]) for a functor $F : C \to D$ is essentially an epimorphism*

$$\pi : Y \to C$$

(recall that an epimorphism in Cat *is a functor whose image* generates *the codomain: every morphism in the codomain is the composite of morphisms in the image, but it need not be in the image itself!) together with a functor*

$$\mathrm{tr} : Y \to S$$

and a choice of natural isomorphism

$$g : \pi_1^*\mathrm{tr} \to \pi_2^*\mathrm{tr}$$

satisfying the gluing condition/cocycle condition/descent condition

$$\begin{tikzcd} & \pi_2^*\mathrm{tr} \arrow[dr, "\pi_{23}^* g"] & \\ \pi_1^*\mathrm{tr} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & & \pi_3^*\mathrm{tr} \end{tikzcd} .$$

Here we are using the obvious simplicial structure induced by π

$$\cdots \begin{tikzcd} {} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & Y^{[3]} \arrow[r, shift left=2, "\pi_{12}"] \arrow[r, "\pi_{13}" description] \arrow[r, shift right=2, "\pi_{23}"'] & Y^{[2]} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & Y \arrow[r, "\pi"] & X \end{tikzcd}$$

where $Y^{[n]} := \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n \text{ factors}}$.

Such a descent object is precisely what one obtains from a choice of **local trivialization** of a given globally defined functor F.

$$\begin{tikzcd} Y \arrow[r, two heads, "\pi"] \arrow[d, "\mathrm{tr}"'] & X \arrow[d, "F"] \arrow[dl, Rightarrow] \\ S \arrow[r, "="'] & S \end{tikzcd}$$

by setting $g := \pi_2^* t^{-1} \circ \pi_1^* t$.

One obtains from this the corresponding anafunctor by forming the weak pushout

$$\begin{tikzcd} Y^{[2]} \arrow[r] \arrow[d] & Y \\ Y & \end{tikzcd} .$$

This weak pushout comes equipped with a *surjective equivalence* down to X, and this is what the anafunctor is built from. This is described in section 3.2 below (and in more detail in [12]).

So as a slogan:

> An anafunctor is a concise repackaging of a descent object for a functor in a single functor.

The main interest in this and related statements is that it prepares the ground for climbing up the categorical ladder.

Example [Cocycles for an G-bundles]. When we are talking about cocycles for G-bundles (without connection, in particular) the domain categories that we are considering are "discrete categories" in the categorical meaning of that term (*not* in the topological one).

A discrete category is just one that has only identity morphisms. (But it may still have a non-discrete topology on its set of objects!).

For X a (topological, smooth, etc.) space, we use X also to denote the discrete (topological, smooth, etc.) category over that space. Then

Fact 3 *A G-cocycle for a G-bundle is the same thing as*

- *descent data for a functor $X \to \mathbf{B}G$*
- *an anafunctor $X \to \mathbf{B}G$.*

In fact, we have an equivalence of categories

$$G\mathrm{Bun}(X) \simeq \mathrm{Desc}(X, \mathbf{B}G) \simeq \mathrm{AnaFun}(X, \mathbf{B}G)\,.$$

That's essentially just a standard fact reformulated in more or less non-standard (depending on what your standards are) language. But it is that language which we'll need later on.

3.1 Anafunctors

In his article on anafunctors [4], M. Makkai presents almost everything in terms of two equivalent definitions. The one exception is the composition of morphisms of anafunctors, which is not presented in the otherwise more elegant definition in terms of spans.

Here I want to recall the definition of anafunctors in terms of spans and write down the composition of their morphisms in that form.

In fact, Toby Bartels does exactly that, even internally, in his thesis. Just for my own benefit, I want to see the relevant structure stripped off the complexity introduced by writing down everything internalized.

Definition 4 (Makkai) *Given two categories A and B, an* **anafunctor**

$$F : A \to B$$

is a span

$$\begin{tikzcd} {|F|} \arrow[r, "F_1"] \arrow[d, "F_0"'] & B \\ A \end{tikzcd}$$

such that F_0 is surjective on objects and on morphisms and such that every morphism in A has at most one lift with given source and target.

Example 1

Let

$$A = \mathcal{P}_1(X)$$

be paths in a smooth space X, let

$$Y \to X$$

be a surjective submersion, let $Y^\bullet$ be the associated groupoid and

$$\mathcal{P}_1(Y^\bullet)$$

be the category of paths in $Y^\bullet$. Then the canonical projection

$$p : \mathcal{P}_1(Y^\bullet) \to \mathcal{P}_1(X)$$

is surjective on objects and morphisms and every path in X has a unique lift for given lift of its endpoints.

A smooth functor

$$(\mathrm{tra}_Y, g) : \mathcal{P}_1(Y^\bullet) \to \Sigma(G)$$

for G any Lie group is precisely the cocycle data of a locally trivialized G-bundle with connection on X.

Definition 5 (Makkai) *A* **morphism of anafunctors**

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow h"] & B \end{tikzcd}$$

is a natural transformation

$$\begin{tikzcd} & {|F|} \arrow[dr, "F_1"] & \\ {|F| \times_A |G|} \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow h_1"] & & B \\ & {|G|} \arrow[ur, "G_1"'] & \end{tikzcd} \quad .$$

The next definition is supposed to be equivalent to what Makkai defines in other terms.

Definition 6 *The composition*

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow h"] \arrow[r, phantom, shift right=3, "\Downarrow k"] & B
\end{tikzcd}
$$

of morphisms of anafunctors is the morphism given by the natural transformation

$$
\begin{tikzcd}
 & & & {|F|} \arrow[ddr, "F_1"] \arrow[dd, phantom, "\Downarrow h_1"] & \\
 & & {|F| \times_A |G|} \arrow[ur] \arrow[dr] & & \\
{|F| \times_A |H|} \arrow[r, "t"] & {|F| \times_A |G| \times_A |H|} \arrow[ur] \arrow[dr] & & {|G|} \arrow[r, "G_1"] \arrow[dd, phantom, "\Downarrow k_1"] & B \\
 & & {|G| \times_A |H|} \arrow[ur] \arrow[dr] & & \\
 & & & {|H|} \arrow[uur, "H_1"'] &
\end{tikzcd},
$$

where t is

$$
t:\ |F| \times_A |H| \xrightarrow{\simeq} |F| \times_A A \times_A |H| \xrightarrow{\mathrm{Id}\times s\times\mathrm{Id}} |F| \times_A |G| \times_A |H|\mathrm{Id}
$$

for any *lift* $s : A \to |G|$.

The crucial point which makes this work is that a s always exists and, crucially, that the above natural transformation is completely independent of the choice of s.

To see this notice first that all choices of s are isomorphic. Then use the rules for horizontal composition of natural transformations to see that

$$
\begin{tikzcd}
 & & & {|F|} \arrow[ddr, "F_1"] \arrow[dd, phantom, "\Downarrow h_1"] & \\
 & & {|F| \times_A |G|} \arrow[ur] \arrow[dr] & & \\
{|F| \times_A |H|} \arrow[r, bend left, "t"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow\sim"] & {|F| \times_A |G| \times_A |H|} \arrow[ur] \arrow[dr] & & {|G|} \arrow[r, "G_1"] \arrow[dd, phantom, "\Downarrow k_1"] & B \\
 & & {|G| \times_A |H|} \arrow[ur] \arrow[dr] & & \\
 & & & {|H|} \arrow[uur, "H_1"'] &
\end{tikzcd}
$$

still equals the expression in the above definition, because everything involving G is projected out by the 1-morphisms bounding this diagram.

3.2 Anafunctors and Transitions

We recall the definition of an anafunctor [4] and of the transition data of a functor [12]. Then we want to show that both are equivalent. The connection is made by the **universal transition**. (** this thing is called the *path pushout* in [12] **) We use this to propose a notion of higher anafunctors.

Definition 7 (Makkai) *Given two categories A and Q, an* **anafunctor**

$$\mathbf{F} : A \to Q$$

is a span

$$
\begin{tikzcd}
{|F|} \arrow[r, "F_1"] \arrow[d, "F_0"'] & Q \\
A &
\end{tikzcd}
$$

such that F_0 is surjective on objects and on morphisms and such that every morphism in A has at most one lift with given source and target.

I would like to reformulate this slightly.

Definition 8 *For A any category, a* **cover** *of A is a morphism*

$$p : K \to A$$

such that the image of p generates *A.*

Example 2

Let $A = \mathcal{P}_1(X)$ be the category of paths in a space X. Let $U \to X$ be an ordinary cover at the level of objects and let $K = P_1(U)$ be the category of paths in the cover. The obvious projection $p : \mathcal{P}_1(U) \to \mathcal{P}_1(X)$ hits all paths that remain within one patch of the cover. Under composition, these generate all paths in X.

We write $K^{[n]}$ for the n-fold strict pullback of K along itself. For instance $K^{[2]}$ is the universal category making

$$\begin{tikzcd} K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] & K \arrow[d, "p"] \\ K \arrow[r, "p"'] & A \end{tikzcd}$$

commute.

In our example, $K^{[2]} = \mathcal{P}_1(U^{[2]})$ is the category of paths in double intersections of the given cover of X.

Neither of the $K^{[n]} \to A$ is, in general, epi. But we can throw in *gluing morphisms* into $K^{[2]}$ such that we do get a surjection in a universal way by forming a certain weak pushout.

Definition 9 *Given a cover $K \to A$, denote by $K^\bullet$ the object sitting in a diagram*

$$\begin{tikzcd} K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{v}{\swarrow}\sim"] & K \arrow[d, "i"] \\ K \arrow[r, "i"'] & K^\bullet \end{tikzcd}$$

satisfying

$$\begin{tikzcd} & p_2^* i \arrow[dr, "p_{23}^* v"] \arrow[d, phantom, "\mathrm{Id}\Downarrow"] & \\ p_1^* i \arrow[ur, "p_{12}^* v"] \arrow[rr, "p_{13}^* v"'] & & p_3^* i \end{tikzcd}$$

on $K^{[3]}$, that is (strictly) universal *in the sense that for any other*

$$\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[d] \\
K \arrow[r] & Q
\end{tikzcd}$$

satisfying a triangle law we have

$$\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[ddr] & & & K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{v}{\swarrow}\sim"] & K^{[2]} \arrow[d, "i"] \arrow[ddr] & \\
K \arrow[drr] & & & = & K^{[2]} \arrow[r, "i"] \arrow[drr] & K^{\bullet} \arrow[dr, dashed] & \\
& & Q & & & & Q
\end{tikzcd}$$

for a unique morphism

$$\begin{tikzcd}
K^{\bullet} \arrow[r, dashed] & Q
\end{tikzcd}$$

Proposition 1 *$K^{\bullet}$ is given in terms of generators and relations as follows. The generators are the morphism of K together with new morphisms – the* **gluing morphisms** *–*

$$\begin{tikzcd}
p_1(x) \arrow[r] & p_2(x)
\end{tikzcd}$$

and their inverses, for all $x \in \mathrm{Obj}(K^{[2]})$. The relations are

$$\begin{tikzcd}
p_1(x) \arrow[r, "p_1(\gamma)"] \arrow[d] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & p_1(y) \arrow[d] \\
p_2(x) \arrow[r, "p_2(\gamma)"'] & p_2(y)
\end{tikzcd}$$

for all $\gamma \in \mathrm{Mor}(K^{[2]})$ and

$$\begin{tikzcd}
& p_2(x) \arrow[dr] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \\
p_1(x) \arrow[ur] \arrow[rr] & {} & p_3(x)
\end{tikzcd}$$

for all $x \in \mathrm{Obj}(K^{[3]})$.

Example 3

In terms of the previous example, the gluing morphism would form precisely the groupoid

$$U^{[2]} \rightrightarrows U$$

of the ordinary cover of X. In other words, there is then a unique gluing morphism

$$(x,i) \longrightarrow (x,j)$$

for every $(x,i,j) \in U^{[2]}$. If we denote by

$$(\gamma, i): \ (x,i) \longrightarrow (y,i)$$

any path in U_i, then the first kind of relation says that

$$\begin{tikzcd} (x,i) \arrow[r, "(\gamma,i)"] \arrow[d] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & (y,i) \arrow[d] \\ (x,j) \arrow[r, "(\gamma,j)"'] & (y,j) \end{tikzcd}$$

for every path

$$(\gamma, i, j): \ (x,i,j) \longrightarrow (y,i,j)$$

in $U^{[2]}$.

Notice the following:

for $Q = \Sigma(G)$ a category with a single object and a Lie group G worth of morphisms, a smooth functor $\mathcal{P}_1(U) \to \Sigma(G)$, for $\mathcal{P}_1(U)$ the groupoid of thin homotopy classes of paths in U, is precisely a **trivial G-bundle with connection** on U.

Moreover, a smoth functor $U^{[2]} \to \Sigma(G)$ is precisely a **G-cocycle** relative to U. Or in other words: the transition function of a G-principal bundle locally trivialized with respect to U.

Finally, a smooth functor $K^\bullet \to \Sigma(G)$ is both of that, together with the compatibility condition, induced by the respect for the rectangular relation relation above, which makes the cocycle a **differential G-cocycle**, hence the transition data of a locally trivialized G-bundle with connection.

Proposition 2 *For any*

$$\begin{tikzcd} K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[d, "F"] \\ K \arrow[r, "F"'] & Q \end{tikzcd}$$

satisfying a triangle law, the morphism

$$K^\bullet \dashrightarrow Q$$

is the functor

$$K^\bullet \xrightarrow{(F,g)} Q$$

that acts as F on the generators from K and assigns $g(x)$ to the gluing morphism at x.

Proposition 3 *The obvious epimorphism*

$$p: \; K^\bullet \longrightarrow A$$

has the property that it has unique lifts with given source and target.

Proof. The relations mentioned above precisely ensure that any two lifts with given source and target are equal. □

It follows that

Proposition 4 *From a given cover and a functor F with transition g on that cover, we get an anafunctor*

$$(F,g): \; A \longrightarrow Q$$

given by the span

$$\begin{tikzcd} K^\bullet \arrow[r, "(F,g)"] \arrow[d, "p"'] & Q \\ A & \end{tikzcd} .$$

Also the converse is true:

Proposition 5 *For every anafunctor*

$$\mathbf{F} : A \to Q$$

there is a cover $K \to A$ and a functor $F : K \to Q$ with transition g such that $\mathbf{F}$ *is the corresponding anafunctor according to the above proposition.*

Proof. Identitfy all nontrivial morphisms in $|\mathbf{F}|$ that get sent to identity morphisms in A with gluing morphisms.

Then take K to be the minimal sub-category of $|\mathbf{F}|$ such that K together with the gluing morphisms generate all of $|\mathbf{F}|$. This implies that the image of $F : K \to A$ generates all of A.

The two relations to be satisfied by the generators follow directly from the fact that F_0 has unique lifts with given source and target.

Finally, identify F with the restriction of $\mathbf{F}$ to K and g with the restriction of $\mathbf{F}$ to the gluing morphisms. □

Equivalence of Anafunctors and Local id**-Trivializations.** Let $i = \mathrm{id}_{T'}$ be the identity on a smooth category T'.

We have already seen that every smooth π-local id-trivialization gives rise to an anafunctor

$$\begin{tikzcd} \mathcal{C}_1(\pi) \arrow[r, "\mathrm{R}(g,\mathrm{triv})"] \arrow[d, "\pi_{\mathcal{C}}"'] & T' \\ \mathcal{P}_1(X) & \end{tikzcd}$$

Conversely, we want to find a condition which guarantees that a smooth anafunctor

$$F : \mathcal{P}_1(X) \to T'$$

is of this form, for some surjective submersion $\pi : Y \to X$.

Proposition 6 *If*

$$p : |F| \to \mathcal{P}_1(X)$$

is a smooth surjective equivalence, whose component maps are surjective submersions, then there exists a surjective submersion

$$\pi : Y \to X$$

such that

$$|F| = \mathcal{C}_1(\pi)\,.$$

Proof. We simply define

$$Y := \mathrm{Obj}(|F|)\,.$$

Then we need to show that indeed $\mathcal{C}_1(\pi) = |F|$, for $\pi = p_0$.

In order to do so, we repeatedly make use of the fact that, since p is a surjective equivalence, there is, for every morphism in $\mathcal{P}_1(X)$ and every lift of its endpoints to $\mathrm{Obj}(|F|)$, a *unique* lift of the entire morphism.

This immediately implies that we have pullback squares of the form

$$\begin{tikzcd}
\mathrm{Mor}(\mathcal{P}_1(Y)) \arrow[r, hook] \arrow[d, equal] & \mathrm{Mor}(|F|) \arrow[d, "p_1"] \\
\mathrm{Mor}(\mathcal{P}_1(Y)) \arrow[r, hook] & \mathrm{Mor}(\mathcal{P}_1(X))
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y^{[2]} \arrow[r, hook] \arrow[d, equal] & \mathrm{Mor}(|F|) \arrow[d, "p_1"] \\
Y^{[2]} \arrow[r, "r"] & \mathrm{Mor}(\mathcal{P}_1(X))
\end{tikzcd} \quad ,$$

which define the inclusions

$$\mathcal{P}_1(Y) \hookrightarrow \mathrm{Mor}(|F|)$$

and

$$Y^{[2]} \hookrightarrow \mathrm{Mor}(|F|) \ .$$

Here r sends (x, y) to $\mathrm{Id}_{\pi(x)}$ $(= \mathrm{Id}_{\pi(y)})$.

The fact that these generators satisfy the relations that hold in $\mathcal{C}_1(\pi)$ again follows from uniqueness of lifts. Therefore we even have an inclusion

$$\mathcal{C}_1(\pi) \hookrightarrow |F| \ .$$

Finally, by lifting any path in X piecewise to morphisms in $\mathcal{P}_1(Y)$ and in $Y^{[2]}$ we obtain a lift for each choice of lift of the endpoints. By the uniqueness of lifts, this means that $\mathcal{C}_1(\pi)$ already coincides with $|F|$. $\square$

Proposition 7 *Let $F : \mathcal{P}_1(X) \to T'$ be a smooth anafunctor such that the component maps of*

$$p : |F| \to \mathcal{P}_1(X)$$

are surjective submersions. Then there is a smoothly locally id*-trivializable transport functor*

$$\mathrm{tra}_F : \mathcal{P}_1(X) \to T'$$

with transition data (triv, g) *such that*

$$\tilde{F} : |F| \to T'$$

equals

$$R_{(\mathrm{triv},g)} : \mathcal{C}_1(\pi) \to T' \ .$$

Proof. According to prop. 6 there is a surjective submersion $\pi : Y \to X$ such that $|F| = \mathcal{C}_1(\pi)$, so that

$$\tilde{F} : \mathcal{C}_1(\pi) \to T' \ .$$

But using the equivalence of such functors with transition data, it follows that there is $(\mathrm{triv}, g) \in \mathrm{TD}^\infty_\pi(i)$ such that $\tilde{F} = R_{(\mathrm{triv},g)}$. Finally, by applying Ex_π we get the corresponding transport functor $\mathrm{Ex}_\pi(\mathrm{triv}, g)$. $\square$

3.3 Ana-2-functors.

(** like much of this document, the following needs polishing and, in fact, more details **)

I would like to use the above equivalence between anafunctors and descent data in order to formulate higher anafunctors. The reason is that a good notion of higher versions of transitions is relatively obvious and has proven its value in applications.

I'll work with strict 2-categories, pseudonatural transformation between them and modifications between these.

There are obvious higher versions of the definitions in the previous paragraph:

Definition 10 *For A any 2-category, a* **cover** *of A is a morphism*

$$p : K \to A$$

such that the image of p generates *A.*

Definition 11 *Given a 2-category A and a cover $K \to A$, denote by $K^\bullet$ the object sitting in a diagram*

$$\begin{tikzcd} K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\swArrow v\ \sim"] & K \arrow[d, "i"] \\ K \arrow[r, "i"'] & K^\bullet \end{tikzcd}$$

together with a morphism

$$\begin{tikzcd} & p_2^* i \arrow[dr, "p_{23}^* v"] \arrow[d, phantom, "w\Downarrow"] & \\ p_1^* i \arrow[ur, "p_{12}^* v"] \arrow[rr, "p_{13}^* v"'] & {} & p_3^* i \end{tikzcd}$$

on $K^{[3]}$ that satisfies a tetrahedron law on $K^{[4]}$ and that is (strictly) universal *in the sense that for any other*

$$
\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[d] \\
K \arrow[r] & Q
\end{tikzcd}
$$

satisfying a tetrahedron law we have

$$
\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[ddr] & \\
K \arrow[drr] & & \\
& & Q
\end{tikzcd}
=
\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{v}{\swarrow}\sim"] & K^{[2]} \arrow[d, "i"] \arrow[ddr] & \\
K^{[2]} \arrow[r, "i"] \arrow[drr] & K^{\bullet} \arrow[dr, dashed] & \\
& & Q
\end{tikzcd}
$$

for a unique morphism

$$K^{\bullet} \dashrightarrow Q \ .$$

This morphism should be addressed as a **2-anafunctor**:

Definition 12 *A* **2-anafunctor**

$$\mathbf{F} : A \to Q$$

between 2-categories A and Q is a cover $K \to A$ of A together with a 2-functor $F : K \to Q$ and its transition data such that

$$\mathbf{F} : K^{\bullet} \to Q$$

is the universal morphism obtained from this transition data as above.

4 Concordance and transformations of ana-n-functors

Definition 13 (concordance) *Given two anafunctors*

$$
\begin{tikzcd}
Y \arrow[d, two heads, "\simeq"'] \arrow[dr, "f"] & \\
C & D
\end{tikzcd}
$$

and

$$\begin{tikzcd} Y' \arrow[d, "\simeq"'] \arrow[drr, "f'"] & & \\ C & & D \end{tikzcd}$$

from C to D, a **concordance** *between them is an anafunctor*

$$\begin{tikzcd} \hat{Y} \arrow[d, "\simeq"'] \arrow[drr] & & \\ C \otimes I & & D \end{tikzcd}$$

which restricts to two given anafunctors over the endpoints of $I = 2_1 = \{ \bullet \longrightarrow \circ \}$.

I want to show that concordance of "charted 2-bundles" (i.e. ana-2-functors) as defined in [8] is equivalent to the other notion of morphisms of ana-2-functors.

To do so, two steps need to be taken:

1. Show that we can always refine a cover, and in particular that for any two ana-2-functors given we can always find a common refinement for the covers that they are defined on.

2. Show that concordances between anafunctors defined for the same cover coincide with transformations of anafunctors.

The first point is obvious using the notion of anafunctor following Makkai. It seems also to be easily true when using the notion of concordance.

The second point should then essentially boil down to using the closed structure of ωCat.

Let f and f' be anafunctors as above, for $Y = Y'$ the same cover. Then we can take

$$\hat{Y} = Y \times I$$

and thus turn the concordance

$$Y \otimes I \to D$$

into a transformation

$$Y \to \mathrm{hom}(I, D)\,.$$

Example. Let us say that again for the special case that our anafunctors are n-bundle cocycles. Then it amounts to saying that for two n-bundles defined on the same cover of X, we can always choose on $X \times [0,1]$ the product cover obtained from that of X and from covering $[0,1]$ by two sets, one containing the left, the other containing the right endpoint.

$$
h : \begin{tikzcd}
(y_1,\bullet) \arrow[rr] \arrow[dr] \arrow[dd] \arrow[dr, phantom, "\Leftarrow", near start] & & (y_1,\circ) \arrow[dr] \arrow[dd] \arrow[ddll, Rightarrow] \arrow[dr, phantom, "\Leftarrow", near start] & \\
& (y_2,\bullet) \arrow[rr] \arrow[dl] & & (y_2,\circ) \arrow[dl] \\
(y_3,\bullet) \arrow[rr] & & (y_3,\circ) &
\end{tikzcd}
\mapsto
\begin{tikzcd}
\bullet \arrow[rr] \arrow[dr, "g(y_1,y_2)"] \arrow[dd, "g(y_1,y_3)"'] \arrow[dr, phantom, "\Leftarrow", near start] & & \bullet \arrow[dr, "g'(y_2,y_3)"] \arrow[dd, "g'(y_1,y_3)"'] \arrow[ddll, Rightarrow] \arrow[dr, phantom, "\Leftarrow", near start] & \\
& \bullet \arrow[rr] \arrow[dl, "g(y_2,y_3)"] & & \bullet \arrow[dl, "g'(y_2,y_3)"] \\
\bullet \arrow[rr] & & \bullet &
\end{tikzcd}
$$

A Morphisms of higher functors

We review the basic definitions for morphisms of strict n-functors between strict n-categories for low n.

(** some of the discussion here needs polishing **)

A.1 Morphisms of 2-functors

Definition 14 *Let* $S \xrightarrow{F_1} T$ *and* $S \xrightarrow{F_2} T$ *be two 2-functors. A* **pseudo-natural transformation**

$$
\begin{tikzcd}
S \arrow[r, bend left=50, "F_1"{name=U}] \arrow[r, bend right=50, "F_2"'{name=D}] & T
\arrow[from=U, to=D, Rightarrow, "\rho"]
\end{tikzcd}
$$

is a map

$$
\mathrm{Mor}_1(S) \ni \quad x \xrightarrow{\gamma} y \quad \mapsto \quad
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho(x)"'] & F_1(y) \arrow[d, "\rho(y)"] \arrow[dl, phantom, "\swarrow \rho(\gamma)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y)
\end{tikzcd}
\quad \in \mathrm{Mor}_2(T)
$$

which is functorial in the sense that

$$\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma_1)"] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swarrow\rho(\gamma_1)"] & F_1(y) \arrow[r, "F_1(\gamma_2)"] \arrow[d, "\rho(y)"] \arrow[dr, phantom, "\swarrow\rho(\gamma_2)"] & F_1(z) \arrow[d, "\rho(z)"] \\
F_2(x) \arrow[r, "F_2(\gamma_1)"'] & F_2(y) \arrow[r, "F_2(\gamma_2)"'] & F_2(z)
\end{tikzcd}
=
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma_2\circ\gamma_1)"] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swarrow\rho(\gamma_2\circ\gamma_1)"] & F_1(z) \arrow[d, "\rho(z)"] \\
F_2(x) \arrow[r, "F_2(\gamma_2\circ\gamma_1)"'] & F_2(z)\circ
\end{tikzcd}$$

and which makes the pseudonaturality tin can 2-commute

$$\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma_1)"] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swarrow\rho(\gamma_1)"] & F_1(y) \arrow[d, "\rho(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma_1)"] \arrow[r, bend right=60, "F_2(\gamma_2)"'] \arrow[r, phantom, bend right=30, "\Downarrow F_2(S)"] & F_2(y)
\end{tikzcd}
=
\begin{tikzcd}
F_1(x) \arrow[r, bend left=60, "F_1(\gamma_1)"] \arrow[r, phantom, bend left=30, "\Downarrow F_2(S)"] \arrow[r, "F_1(\gamma_2)"'] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swarrow\rho(\gamma_2)"] & F_1(y) \arrow[d, "\rho(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma_2)"'] & F_2(y)
\end{tikzcd}$$

for all $\begin{tikzcd} x \arrow[r, bend left=50, "\gamma_1"] \arrow[r, bend right=50, "\gamma_2"'] \arrow[r, phantom, "\Downarrow S"] & y \end{tikzcd} \in \mathrm{Mor}_2(S)$.

Definition 15 *The vertical* **composition of pseudonatural transformations**

$$\begin{tikzcd}
S \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "F_3"'] \arrow[r, phantom, "\Downarrow\rho"] & T
\end{tikzcd}
:=
\begin{tikzcd}
S \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, bend left=30, "\Downarrow\rho_1"] \arrow[r, phantom, bend right=30, "\Downarrow\rho_2"] & T
\end{tikzcd}$$

is given by

$$
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[dd, "\rho(x)"'] & F_1(y) \arrow[dd, "\rho(y)"] \\
\arrow[r, phantom, "\swArrow \rho(\gamma)"] & \\
F_3(x) \arrow[r, "F_3(\gamma)"] & F_3(y)
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho_1(x)"'] \arrow[dr, phantom, "\swArrow \rho_1(\gamma)"] & F_1(y) \arrow[d, "\rho_1(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"] \arrow[d, "\rho_2(x)"'] \arrow[dr, phantom, "\swArrow \rho_2(\gamma)"] & F_2(y) \arrow[d, "\rho_2(y)"] \\
F_3(x) \arrow[r, "F_3(\gamma)"] & F_3(y)
\end{tikzcd}
$$

Definition 16 *Let* $F_1 \xrightarrow{\rho_1} F_2$ $F_1 \xrightarrow{\rho_2} F_2$ *be two pseudonatural transformations. A* **modification** *(of pseudonatural transformations)*

$$
\begin{tikzcd}
F_1 \arrow[r, bend left, "\rho_1"] \arrow[r, bend right, "\rho_2"'] \arrow[r, phantom, "\Downarrow \mathcal{A}"] & F_2
\end{tikzcd}
$$

is a map

$$
\mathrm{Obj}(S) \ni x \mapsto
\begin{tikzcd}
F_1(x) \arrow[r, bend left, "\rho_1(x)"] \arrow[r, bend right, "\rho_2(x)"'] \arrow[r, phantom, "\Downarrow \mathcal{A}(x)"] & F_2(x)
\end{tikzcd}
\in \mathrm{Mor}_2(T)
$$

such that

$$
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho_1(x)"] \arrow[d, bend right=60, "\rho_2(x)"'] \arrow[d, phantom, "\Leftarrow \mathcal{A}(x)"] \arrow[dr, phantom, "\swArrow \rho_1(\gamma)"] & F_1(y) \arrow[d, "\rho_1(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y)
\end{tikzcd}
=
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho_2(x)"] \arrow[dr, phantom, "\swArrow \rho_2(\gamma)"] & F_1(y) \arrow[d, "\rho_2(y)"'] \arrow[d, bend left=60, "\rho_1(y)"] \arrow[d, phantom, "\Leftarrow \mathcal{A}(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y)
\end{tikzcd}
$$

for all $x \xrightarrow{\gamma} y \in \mathrm{Mor}_1(S)$.

Definition 17 *The horizontal and vertical composite of modifications is, respectively, given by the horizontal and vertical composites of the maps to 2-morphisms in* $\mathrm{Mor}_2(T)$.

Definition 18 *Let S and T be two 2-categories. The* **2-functor 2-category** *T^S is the 2-category*

1. *whose objects are functors $F : S \to T$*
2. *whose 1-morphisms are pseudonatural transformations $F_1 \xrightarrow{\rho} F_2$*
3. *whose 2-morphisms are modifications*

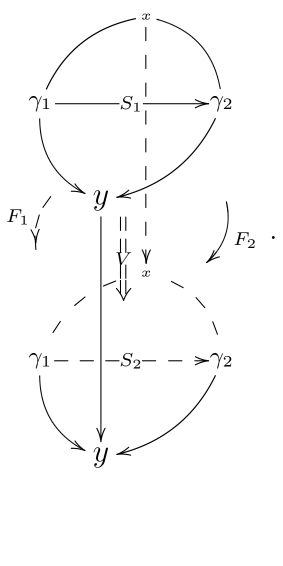
$$\begin{tikzcd} F_1 \arrow[rr, bend left, "\rho_1"] \arrow[rr, bend right, "\rho_2"'] \arrow[rr, phantom, "\Downarrow \mathcal{A}"] & & F_2 \end{tikzcd} .$$

A.2 Morphisms of 3-Functors

We shall regard 3-categories as special categories internal to 2Cat. From this point of view, a 3-category has a 2-category of objects S, each of which looks like

$$\begin{tikzcd} & x \arrow[dr, dash, bend left] & \\ \gamma_1 \arrow[ur, dash, bend left] \arrow[rr, "S" description] \arrow[dr, bend right] & & \gamma_2 \arrow[dl, bend left] \\ & y & \end{tikzcd} .$$

In a general category internal to 2Cat, we similarly have a 2-category of morphisms $S_1 \xrightarrow{V} S_2$, that look like



We shall restrict attention to the special case where the vertical faces here are identities. Then the above shape looks like

$$
\begin{tikzcd}[row sep=large]
\gamma_1 \arrow[rr, bend left=60, dash, "x"] \arrow[rr, bend left, "S_1"] \arrow[rr, dashed, bend right=20, "S_2"'] \arrow[dr, bend right] & \overset{\|}{\Downarrow} V & \gamma_2 \arrow[dl, bend left] \\
& y &
\end{tikzcd}
.
$$

Instead of saying that V is a morphism of a category internal to 2Cat, we say V is a 3-morphism. Similarly, S_1, S_2 are 2-morphisms, γ_1, γ_2 are 1-morphisms and x and y are objects.

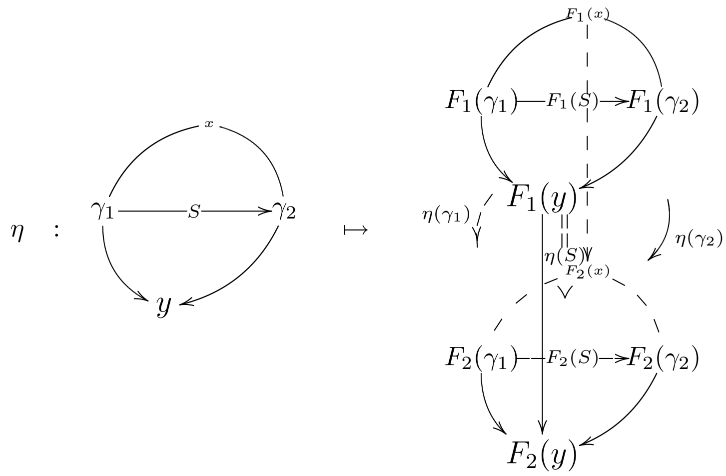
We would have arrived at the same picture had we regarded categories enriched over 2Cat. However, we find that thinking of 3-morphisms as morphisms of a category internal to 2Cat facilitates handling morphisms of 3-functors, to which we now turn.

A 3-functor $F : S \to T$ between 3-categories S and T is a functor internal to 2Cat, hence a map

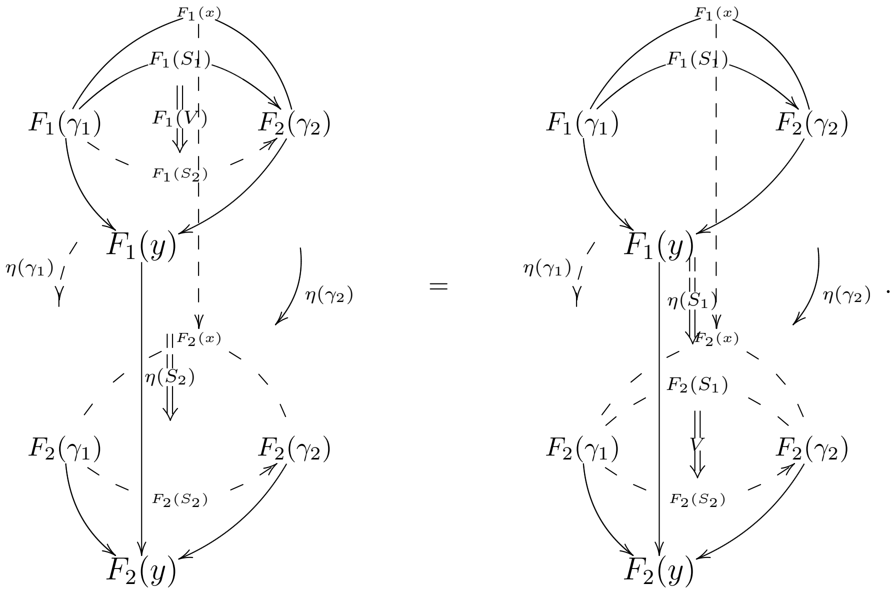
$$
F \;:\;
\begin{tikzcd}[row sep=large]
\gamma_1 \arrow[rr, bend left=60, dash, "x"] \arrow[rr, bend left, "S_1"] \arrow[rr, dashed, bend right=20, "S_2"'] \arrow[dr, bend right] & \overset{\|}{\Downarrow} V & \gamma_2 \arrow[dl, bend left] \\
& y &
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}[row sep=large]
F(\gamma_1) \arrow[rr, bend left=60, dash, "F(x)"] \arrow[rr, bend left, "F(S_1)"] \arrow[rr, dashed, bend right=20, "F(S_2)"'] \arrow[dr, bend right] & \overset{\|}{\Downarrow} F(V) & F(\gamma_2) \arrow[dl, bend left] \\
& F(y) &
\end{tikzcd}
$$

that respects vertical composition strictly and is 2-functorial up to coherent 3-isomorphisms with respect to the composition perpendicular to that.

A 1-morphism $F_1 \xrightarrow{\eta} F_2$ between two such 3-functors is a natural transformation internal to 2Cat, hence a 2-functor from the object 2-category to the morphism 2-category, hence a 2-functorial assignment

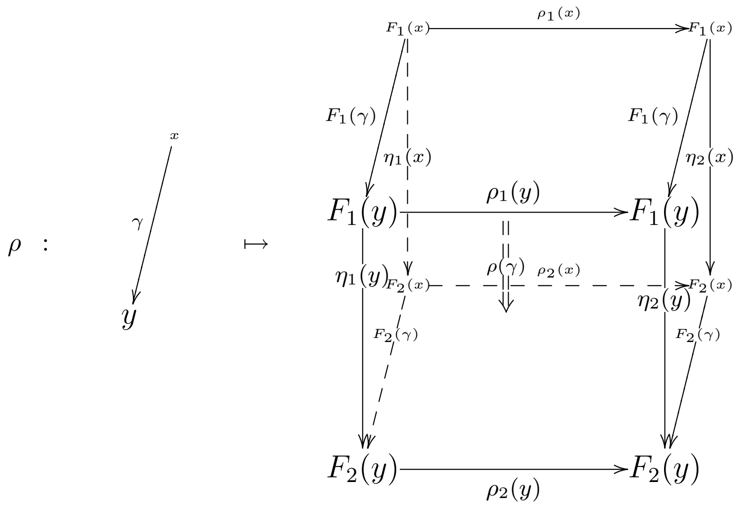


that satisfies the naturality condition

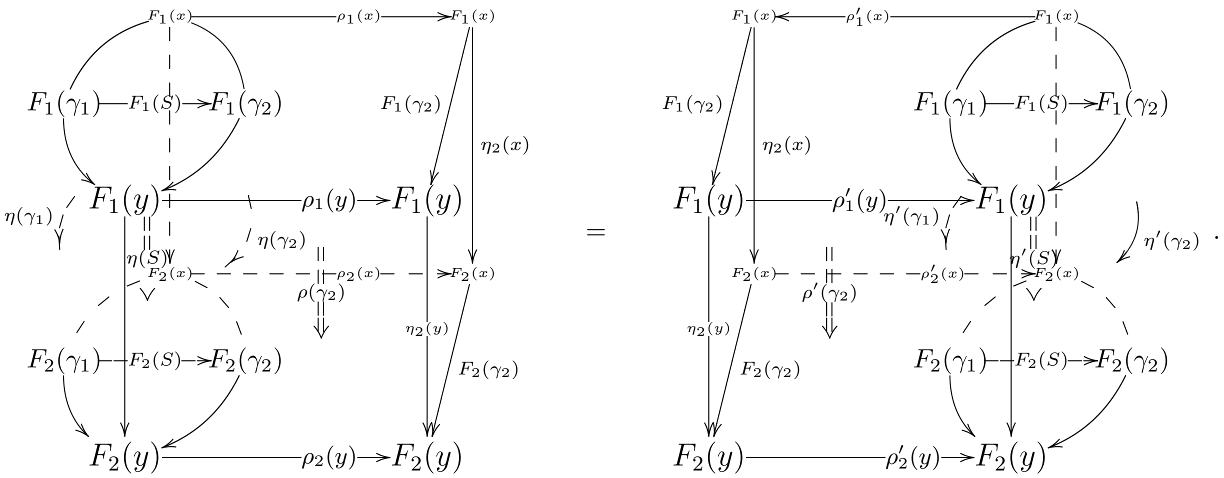


Accordingly, 2-morphisms and 3-morphisms of our 3-functors are 1-morphisms and 2-morphisms of these 2-functors η.

Hence a 2-morphism $\eta \xrightarrow{\rho} \eta'$ of our 3-functors is a 1-functorial assignment

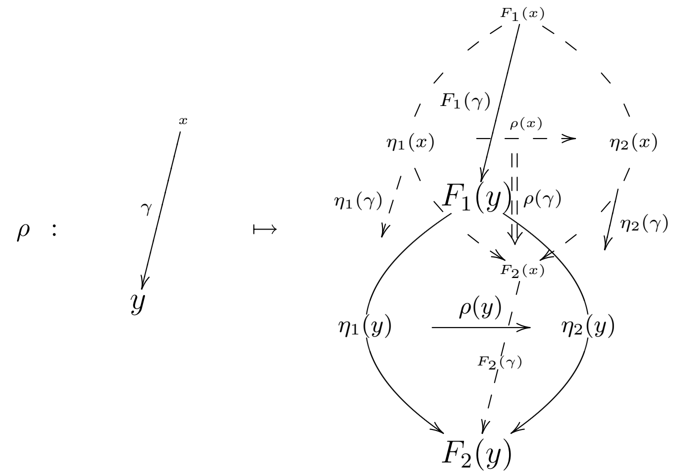


such that

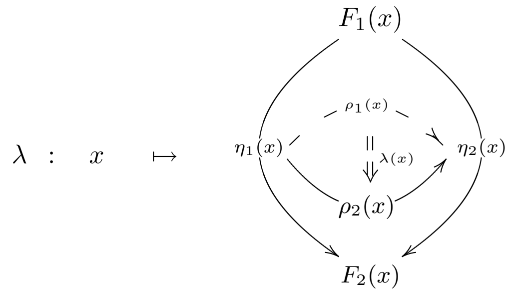


We want to restrict attention to those ρ for which the horizontal 1-morphisms

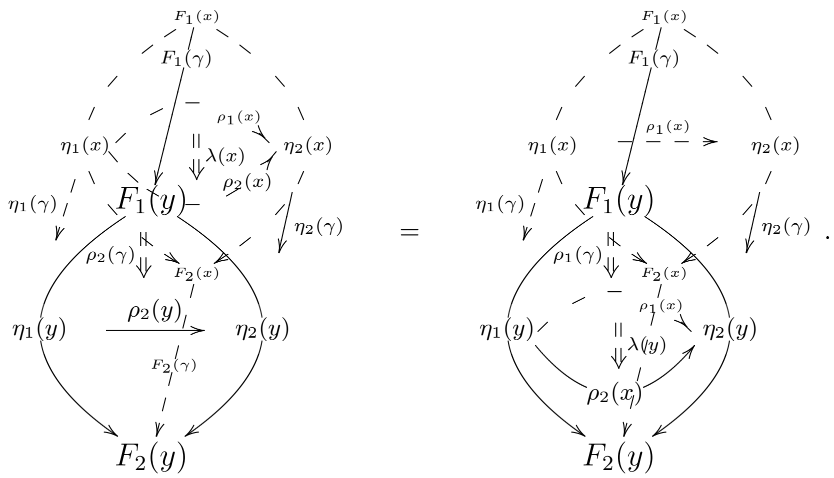
$\rho_1(x)$, $\rho_2(x)$, etc. are identities.



Proceeding this way, a modification $\lambda : \rho_1 \to \rho_2$ of transformations ρ gives us a 3-morphisms of 3-functors. This now is a map



such that



We thus get a 3-category of 3-morphisms of 3-functors.

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