On String- and Chern-Simons n-Transport

Urs Schreiber

based in parts on work with

John Baez
Alissa Crans
David Roberts
Jim Stasheff
Danny Stevenson
Todd Trimble
Konrad Waldorf

December 11, 2007

Local Motivation

We want to get a handle on the theory and classification of n-bundles with n-functorial connections, in particular

- String 2-bundles
- Chern-Simons 3-bundles.

Global Motivation

We want to understand how the FRS description of 2-dimensional rational CFT generalizes to non-rational CFT and to SCFT. We have a bunch of hints that FRS is the

- local trivialization data
- of a certain push-forward ("quantization")
- of a transformation of parallel transport 3-functors
- describing a connection on a 3-bundle.

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There is much more to say about motivation.
A couple of more details are given in the following.
To skip further motivation

- continue with the plan of the further discussion
- or go directly to the detailed discussion at Lie n-algebra cohomology
- or jump to the Conclusion .

A Quantum Field Theory is a Functor

- Atiyah and Segal have famously axiomatized d-dimensional QFTs
- as functors

$$\mathrm{Z} : n\mathrm{Cob}_S \to \mathrm{Vect}$$

-

$$Z \;:\; \left(\partial_{\mathrm{in}}\Sigma \xrightarrow{(\Sigma,g)} \partial_{\mathrm{out}}\Sigma \right) \mapsto \left(H_{\mathrm{in}} \xrightarrow{U(\Sigma,g)} H_{\mathrm{out}} \right) .$$

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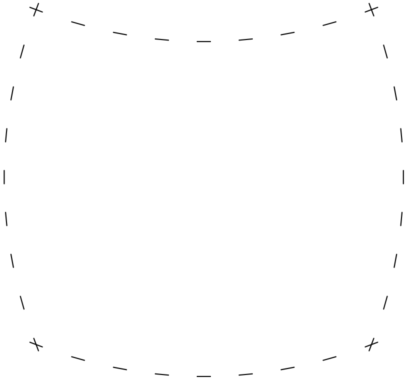
Cartoon of a 1-functorial QFT

$\langle \phi | U(\Sigma) | \psi \rangle$



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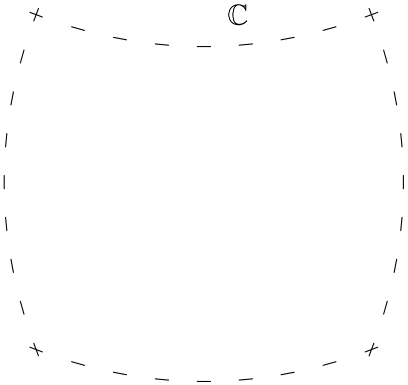

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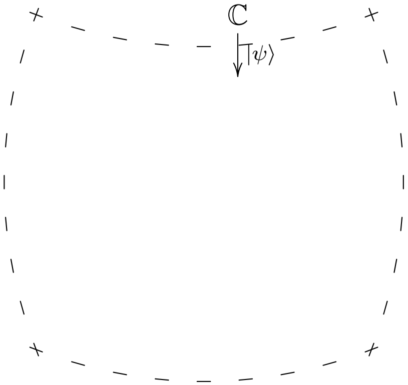

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$$\begin{tikzcd} \mathbb{C} \arrow[d, "|\psi\rangle"] \\ H \end{tikzcd}$$

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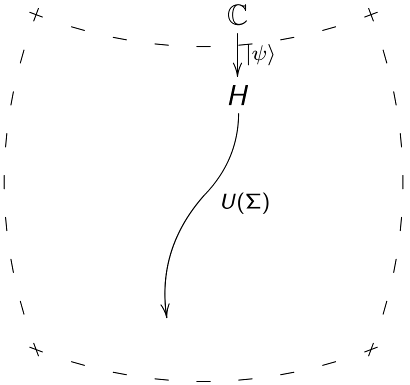

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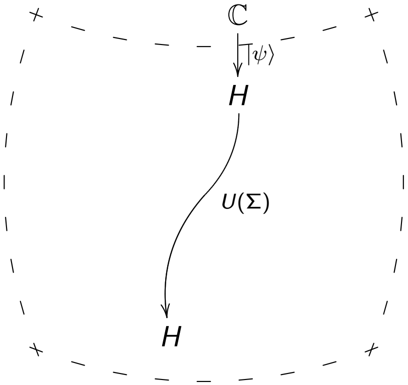

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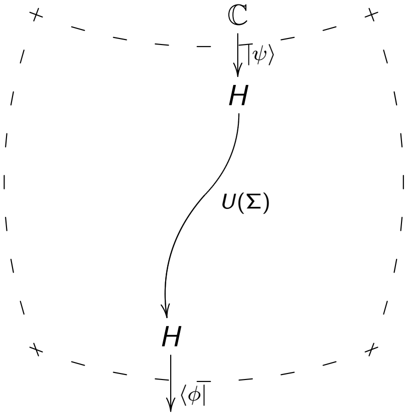

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H \arrow[d, "\langle \phi |"] \\
\mathbb{C}
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A Quantum Field Theory is an n-Functor

But later it was noticed that this is too imprecise if we want to be able to talk about

crucial requirements on QFT description

- *locality*
- *boundary conditions.*

Instead:

refined picture

An n-dimensional QFT should be an n-functor.
[Freed, Hopkins, Stolz, Teichner]

(remark on n-categories)

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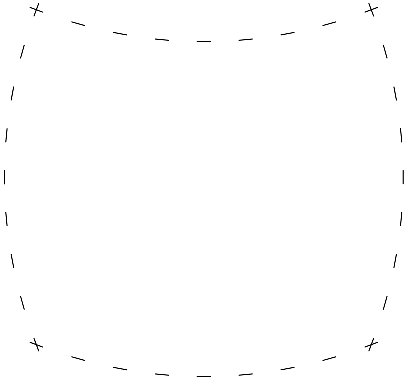

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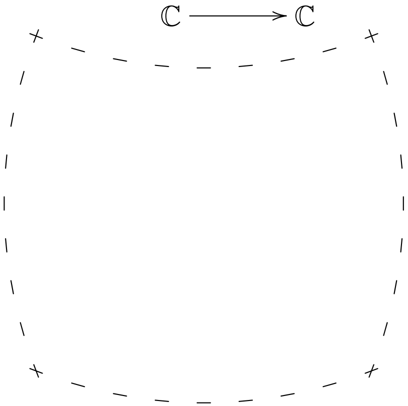

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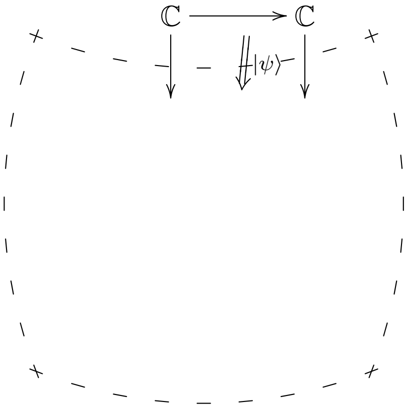

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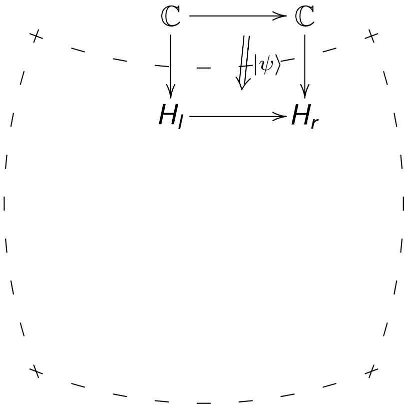

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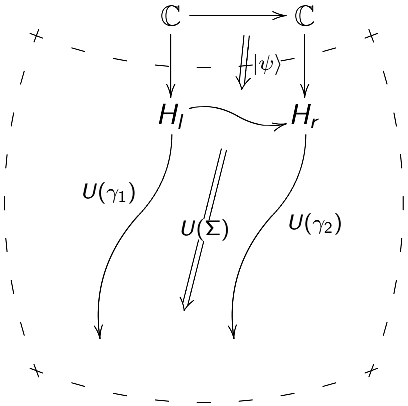

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$$
\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow |\psi\rangle"] & \mathbb{C} \arrow[d] \\
H_l \arrow[r, squiggly] \arrow[d, "U(\gamma_1)"'] \arrow[dr, phantom, "\Downarrow U(\Sigma)"] & H_r \arrow[d, "U(\gamma_2)"] \\
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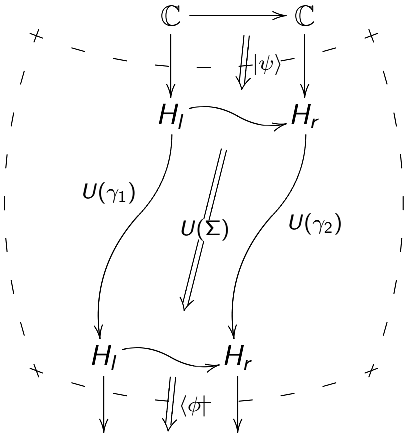

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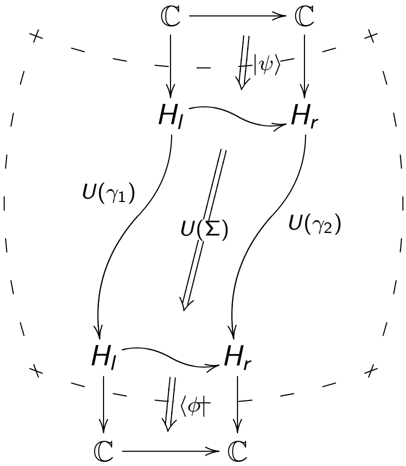

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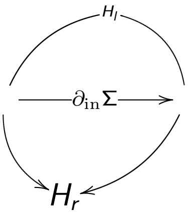

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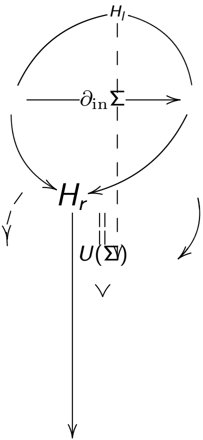

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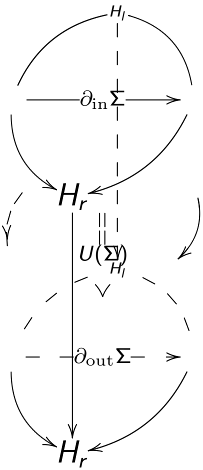

n-Particles and $(n-1)$-Branes

It follows that the action of the *n-particle*. . .

n-Particle

- $n = 1$: the point particle
- $n = 2$: the string
- $n = 3$: the membrane
- n-particle $\simeq$ $(n-1)$-brane

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n-Bundles and $(n-1)$-Gerbes

It follows that the action of the *n-particle* charged under an *n-bundle with connection*...

n-background fields

- $n = 1$: the electromagnetic field
- $n = 2$: the Kalb-Ramond field
- $n = 3$: the supergravity 3-form field
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Parallel n-Transport

It follows that the action of the *n-particle* charged under an *n-bundle with connection*

is itself an n-functor

- $$\mathrm{tra}_1 : \left(x \xrightarrow{\gamma} y \right) \mapsto \left(V_x \xrightarrow{P\exp\left(\int_\gamma A\right)} V_y \right)$$

- $$\mathrm{tra}_2 : \left(\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow \Sigma"] & y \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} V_x \arrow[rr, bend left, "P\exp\left(\int_{\gamma_1} A\right)"] \arrow[rr, bend right, "P\exp\left(\int_{\gamma_2} A\right)"'] \arrow[rr, phantom, "\Downarrow P_A \exp\left(\int_\Sigma B\right)"] & & V_y \end{tikzcd} \right)$$

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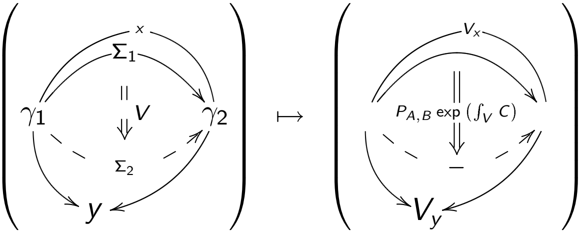
- tra_3: 

Parallel n-Transport

A parallel n-transport is (locally) an n-functor from the path n-groupoid to the structure n-group .

$$\mathrm{tra}_n : \mathcal{P}_n(X) \to \Sigma G_{(n)}$$

$(n+1)$-Curvature

Its $(n+1)$-curvature is (locally) an $(n+1)$-functor from the fundamental $(n+1)$-groupoid to the inner automorphism $(n+1)$-group of $G_{(n)}$.

$$d\mathrm{tra}_n := \mathrm{curv}_{(n+1)} : \Pi_{n+1}(X) \to \Sigma(\mathrm{INN}G_{(n)})$$

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$$d\mathrm{tra}_n := \mathrm{curv}_{(n+1)} : \Pi_{n+1}(X) \to \Sigma(\mathrm{INN}G_{(n)})$$

Strict 2-groups and crossed modules of groups

It is an old result that strict 2-groups are isomorphic to crossed modules of ordinary groups. The isomorphism is in fact almost canonical: only two minor choices are involved.

When differentiating 2-functors with values in strict Lie 2-groups, we make extensive use of this equivalence, the precise realization of which is spelled out below.

Definition

A **crossed module of groups** is a diagram

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

in Grp (meaning all objects are groups and all arrows are group homomorphisms) such that

$$\begin{tikzcd} H \arrow[rr, "\mathrm{Ad}"] \arrow[dr, "t"'] & & \mathrm{Aut}(H) \\ & G \arrow[ur, "\alpha"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} G \times H \arrow[r, "\mathrm{Id}\times t"] \arrow[d, "\alpha"'] & G \times G \arrow[d, "\mathrm{Ad}"] \\ H \arrow[r, "t"] & G \end{tikzcd} \quad .$$

Definition

A **strict 2-group** $G_{(2)}$ is any of the following equivalent entities

- a group object in Cat;
- a category object in Grp;
- a strict 2-groupoid with a single object.

As for groups, we shall write $G_{(2)}$ when we think of $G_{(2)}$ as a monoidal category, and $\Sigma G_{(2)}$ when we think of it as a 1-object 2-groupoid.

Proposition

Crossed modules of groups and strict 2-groups are isomorphic.

We now spell out this identification in detail. It is unique only up to a few conventional choices.

Our chosen isomorphism of 2-groups with crossed modules

The same is in principle already true for the identification of 1-groups with categories, which is unique only up to reversal of all arrows.

To start with, we take all principal actions to be from the *right*.

Our chosen isomorphism of 2-groups with crossed modules

So for G any group, $G\mathrm{Tor}$ denotes the category of right-principal G-spaces. This implies that if we want the canonical inclusion

$$i_G : \Sigma G \to G\mathrm{Tor}$$

to be covariant, we need to take composition in ΣG to work like

$$g_2 \circ g_1 = g_2 g_1 \,,$$

where on the left the composition is that of morphisms in ΣG, while on the right it is the product in G.

Our chosen isomorphism of 2-groups with crossed modules

Notice that this implies that diagrammatically we have

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \;=\; \bullet \xrightarrow{g_2 g_1} \bullet \,.$$

If G comes to us as a group of maps, we accordingly take the group product to be given by $g_2 g_1 := g_2 \circ g_1$.

Our chosen isomorphism of 2-groups with crossed modules

When we then pass to strict 2-groups $G_{(2)}$ coming from crossed modules $(t : H \to G)$ of groups, and want to label 2-morphisms in $\Sigma G_{(2)}$ with elements in H and G, we have one more convention to fix.

Our chosen isomorphism of 2-groups with crossed modules

Let $G_{(2)}$ be a (strict) 2-group which we may alternatively think of a crossed module $t\colon H \to G$. To recover $G_{(2)}$ from the crossed module $t\colon H \to G$ we set

$$\mathrm{Ob}(G_{(2)}) = G$$

$$\mathrm{Mor}(G_{(2)}) = G \ltimes H\,.$$

Here on the right we have the semidirect product group obtained from G and H using the action of G on H by way of α.

Our chosen isomorphism of 2-groups with crossed modules

A 2-morphism in $\Sigma G_{(2)}$ will be denoted by

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "g"{name=U}] \arrow[r, bend right=60, "g'"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "h"] \end{tikzcd}$$

for $g, g' \in G$ and $h \in H$, where g' will turn out to be fixed by $(g, h) \in G \ltimes H$. The semi-direct product structure on $G \ltimes H$, the source, target and composition homomorphisms are defined as follows.

Our chosen isomorphism of 2-groups with crossed modules

We shall agree that

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet
\end{tikzcd}
:=
\begin{tikzcd}
\bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "g"] & \bullet
\end{tikzcd}
.
$$

Our chosen isomorphism of 2-groups with crossed modules

From the requirement that $t : H \to G$ be a homomorphism, it follows that

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h')"'] \arrow[r, phantom, "\Downarrow h'"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h'h)"'] \arrow[r, phantom, "\Downarrow h'h"] & \bullet
\end{tikzcd}
.
$$

Our chosen isomorphism of 2-groups with crossed modules

Together with the convention above this means that the source-target matching condition then reads

$$g' = g\, t(h)\,.$$

Our chosen isomorphism of 2-groups with crossed modules

The exchange law then implies that

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=70, "\mathrm{Id}"] \arrow[r, "t(h)" description] \arrow[r, bend right=70] \arrow[r, phantom, bend left=35, "\Downarrow h"] \arrow[r, phantom, bend right=35, "\Downarrow h'"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(hh')"'] \arrow[r, phantom, "\Downarrow hh'"] & \bullet
\end{tikzcd}
.
$$

Our chosen isomorphism of 2-groups with crossed modules

Since in the crossed module we have $t(\alpha(g)(h)) = gt(h)g^{-1}$ we find that inner automorphisms in the 2-group have to be labeled like this:

$$
\begin{tikzcd}
\bullet \arrow[r, "g^{-1}"] & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "g"] & \bullet & = & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(\alpha(g)(h))"'] \arrow[r, phantom, "\Downarrow \alpha(g)(h)"] & \bullet
\end{tikzcd}
.
$$

Our chosen isomorphism of 2-groups with crossed modules

This then finally implies the rule for general horizontal compositions

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_1'"'] \arrow[r, phantom, "\Downarrow h_1"] & \bullet \arrow[r, bend left, "g_2"] \arrow[r, bend right, "g_2'"'] \arrow[r, phantom, "\Downarrow h_2"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[rr, bend left, "g_2 g_1"] \arrow[rr, bend right, "g_2' g_1'"'] \arrow[rr, phantom, "\Downarrow \alpha_{g_1^{-1}}(h_2) h_1"] & & \bullet \end{tikzcd} .$$

Tangent Categories

Inner automorphism $(n+1)$-Groups

- Every n-group $G_{(n)}$ has an $(n+1)$-group $\mathrm{AUT}(G_{(n)})$ of automorphisms.
- This sits inside an exact sequence
  $1 \to Z(G_{(n)}) \to \mathrm{INN}(G_{(n)}) \to \mathrm{AUT}(G_{(n)}) \to \mathrm{OUT}(G_{(n)}) \to 1$
- and INN_0 plays the role of the universal $G_{(n)}$-bundle
  $G_{(n)} \to \mathrm{INN}_0(G_{(n)}) \to \Sigma G_{(n)}$

We will re-encounter these crucial facts in their Lie n-algebra incarnation shortly.
[U.S., David Roberts]
(on tangent categories) (on inner automorphisms)

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Some structure n-Groups

Important structure (1-)Groups

electrically charged 1-particle: $G_{(1)} = U(1)$

spinning 1-particle: $G_{(1)} = \mathrm{Spin}(n)$

Important structure (2-)Groups

Kalb-Ramond charged 2-particle: $G_{(2)} = \Sigma U(1)$

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Important Structure 3-Groups

Chern-Simons charged 3-particle: $G_{(3)} = ?$

Tough question. Let's pass to the differential picture.

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[Bartels],[Baez,S],[S,Waldorf]

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Finding the Chern-Simons Lie 3-algebra

Problem

Identify that class of 3-transport – given by its structure 3-group – which evaluates to the Chern-Simons functional on 3-dimensional morphisms.

Strategy

- Differentiate. Pass from Lie n-groups to Lie n-algebras.
- Find that Lie 3-algebra $cs_k(g)$ with the property that connections taking values in it, $\mathrm{Vect}(-, cs_k(g))$, correspond to triples (A, B, C) of forms such that $C = CS_k(A) + dB$.

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Differentiation of parallel n-transport

Parallel n-transport is a morphism of Lie n-groupoids.
Differentiating it yields a morphism of Lie n-algebroids.

From parallel n-transport to Lie n-algebra valued connections

Lie n-groupoids	$\xrightarrow{\text{diff.}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	$\simeq$	**differential algebras** (qDGCAs)

$$
\begin{array}{ccc}
\Sigma(\mathrm{INN}(G_{(n)})) & \mathrm{inn}(\mathfrak{g}_{(n)}) & (\bigwedge^\bullet(s\mathfrak{g}_n^* \oplus ss\mathfrak{g}_{(n)}^*), d) \\
\uparrow F & \uparrow f & \downarrow f^* \\
\Pi_{n+1}(X) & \mathrm{Vect}(X) & (\Omega^\bullet(X), d)
\end{array}
$$

Parallel n-transport is a morphism of Lie $(n+1)$-groupoids.

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Lie n-groupoids	$\xrightarrow{\text{diff.}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	$\simeq$	**differential algebras** (qDGCAs)

$$\begin{tikzcd} \Sigma(\mathrm{INN}(G_{(n)})) \\ \Pi_{n+1}(X) \arrow[u, "F"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{inn}(\mathfrak{g}_{(n)}) \\ \mathrm{Vect}(X) \arrow[u, "f"] \end{tikzcd} \qquad \begin{tikzcd} (\bigwedge^\bullet(s\mathfrak{g}_n^* \oplus ss\mathfrak{g}_{(n)}^*), d) \arrow[d, "f^*"] \\ (\Omega^\bullet(X), d) \end{tikzcd}$$

Parallel n-transport is a morphism of Lie $(n+1)$-groupoids.

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Lie n-groupoids	$\xrightarrow{\text{diff.}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	$\simeq$	**differential algebras** (qDGCAs)

$$\begin{array}{ccccc} \Sigma(\mathrm{INN}(G_{(n)})) & & \mathrm{inn}(\mathfrak{g}_{(n)}) & & (\bigwedge^\bullet(s\mathfrak{g}_n^* \oplus ss\mathfrak{g}_{(n)}^*), d) \\ \uparrow F & & \uparrow f & & \downarrow f^* \\ \Pi_{n+1}(X) & & \mathrm{Vect}(X) & & (\Omega^\bullet(X), d) \end{array}$$

This morphism may be differentiated...

From parallel n-transport to Lie n-algebra valued connections

Lie n-groupoids	$\xrightarrow{\text{diff.}}$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	$\simeq$	**differential algebras** (qDGCAs)
$\Sigma(\mathrm{INN}(G_{(n)}))$		$\mathrm{inn}(\mathfrak{g}_{(n)})$		$(\bigwedge^\bullet(s\mathfrak{g}_n^* \oplus ss\mathfrak{g}_{(n)}^*), d)$
$\uparrow F$		$\uparrow f$		$\downarrow f^*$
$\Pi_{n+1}(X)$		$\mathrm{Vect}(X)$		$(\Omega^\bullet(X), d)$

This morphism may be differentiated...

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. . . to produce a morphism of Lie $(n+1)$-algebroids.

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\Pi_{n+1}(X) \arrow[u, "F"] & & \mathrm{Vect}(X) \arrow[u, "f"] & & (\Omega^\bullet(X), d)
\end{tikzcd}$$

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These are best handled in terms of their dual maps,

From parallel n-transport to Lie n-algebra valued connections

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which are morphisms of quasi-free differential-graded algebras.

Our

Main goal

is to understand n-bundles with connection for given structure Lie n-algebra $\mathfrak{g}_{(n)} = \mathrm{Lie}(G_{(n)})$ in terms of their differential *parallel transport*.

using the

Formulation

in terms of (co)differential (co)algebra to facilitate explicit computations

while following the

Structural Guidance

obtained by a theory of n-bundles with connection in terms of morphisms of n-groupoids and parallel transport n-functors.

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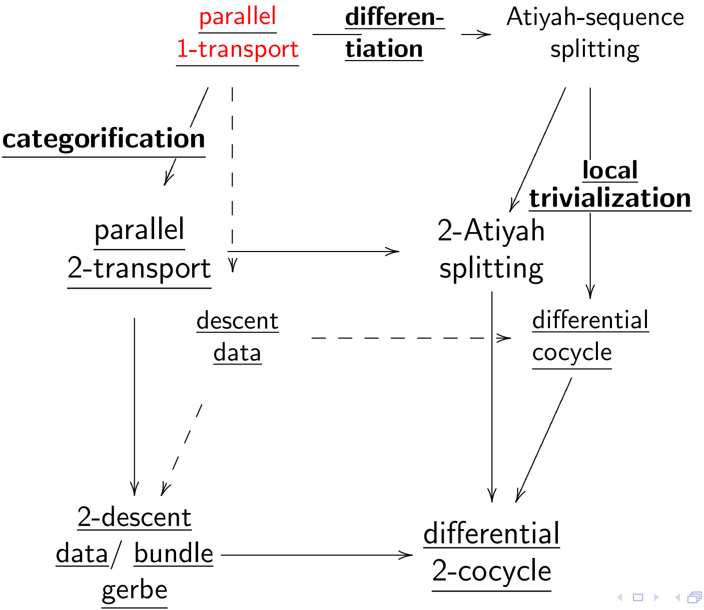
The classical Transport Cube:

The notions of classical parallel n-transport are conveniently thought of as arising from three orthogonal procedures from ordinary parallel transport in an ordinary bundle:

- categorification
- local trivialization
- differentiation

$$
\begin{tikzcd}
& \text{parallel 1-transport} \arrow[rr, "\textbf{differentiation}"] \arrow[dl, "\textbf{categorification}"'] \arrow[dd, dashed] & & \text{Atiyah-sequence splitting} \arrow[dl] \arrow[dd, "\textbf{local trivialization}"] \\
\text{parallel 2-transport} \arrow[rr] \arrow[dd] & & \text{2-Atiyah splitting} \arrow[dd] & \\
& \text{descent data} \arrow[rr, dashed] \arrow[dl, dashed] & & \text{differential cocycle} \arrow[dl] \\
\text{2-descent data/ bundle gerbe} \arrow[rr] & & \text{differential 2-cocycle} &
\end{tikzcd}
$$



Ordinary parallel transport in a bundle with connection is a functor from paths to fiber morphisms.

$$\mathcal{P}_1(X) \xrightarrow{\mathrm{tra}} G\mathrm{Tor}$$

parallel 1-transport — **differentiation** → Atiyah-sequence splitting

categorification

parallel 2-transport → 2-Atiyah splitting

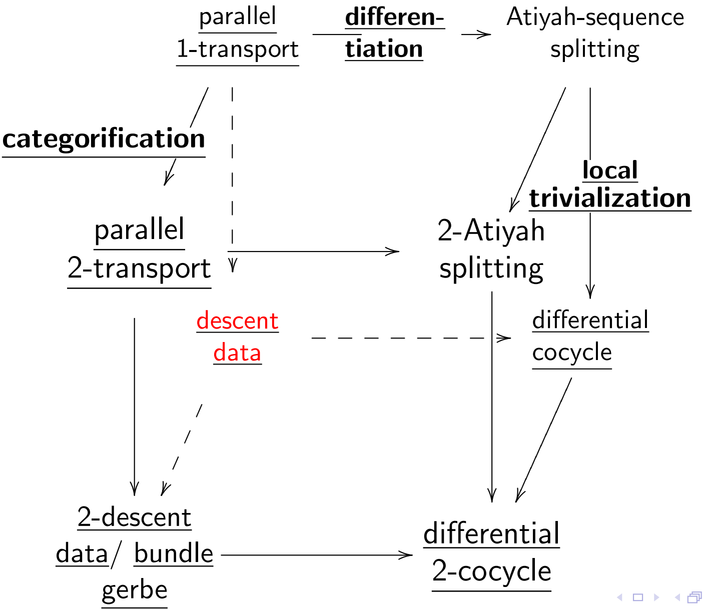
local trivialization

descent data - - → differential cocycle

2-descent data/ bundle gerbe → differential 2-cocycle

By locally trivializing the functor over a cover $Y \to X$, i.e. by choosing an isomorphism

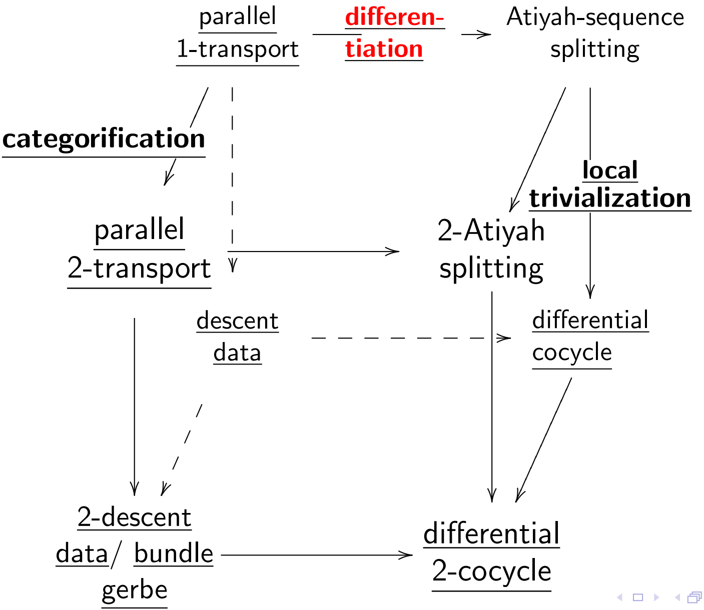
$$\begin{tikzcd} \mathcal{P}_1(Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{triv}"'] & \mathcal{P}_1(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\simeq"] \\ \Sigma G \arrow[r] & G\mathrm{Tor} \end{tikzcd}$$



we obtain descent data for the transport functor

$$\begin{tikzcd}
\pi_1^*\mathrm{triv} \arrow[dd, "\pi_{13}^* g"'] \arrow[dr, "\pi_{12}^* g"] & \\
& \pi_2^*\mathrm{triv} \arrow[dl, "\pi_{23}^* g"] \\
\pi_3^*\mathrm{triv} &
\end{tikzcd}$$

which may be thought of as an anafunctor



by differentiating the objects and morphisms in this descent data we pass from Lie groupoids and their morphisms to Lie algebroids and their morphisms

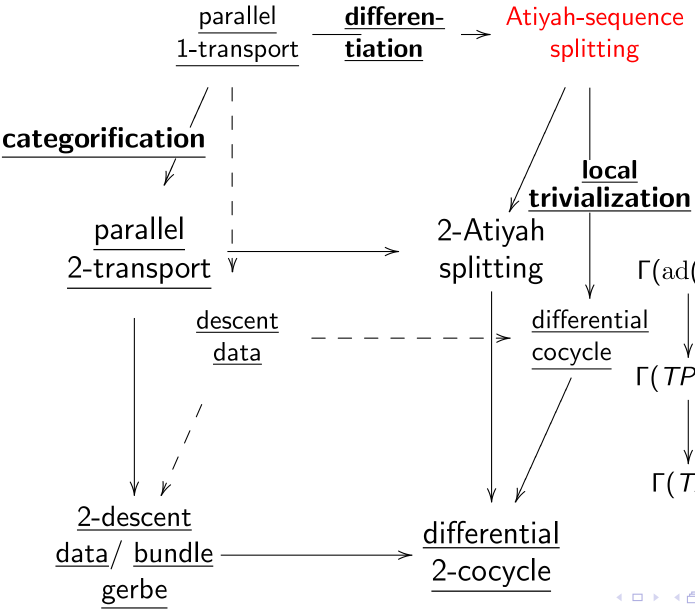
$$\begin{array}{ccc} \Pi_1(Y) & & TY \\ \downarrow{\scriptstyle \mathrm{tra}} & \stackrel{\mathrm{diff}}{\longmapsto} & \downarrow{\scriptstyle A} \\ \Sigma G & & \mathfrak{g} \end{array}$$

$$
\begin{tikzcd}
 & \underline{\text{parallel 1-transport}} \arrow[rr, "\textbf{differentiation}"] \arrow[dl, "\textbf{categorification}"'] \arrow[dr, dashed] & & \underline{\text{Atiyah-sequence splitting}} \arrow[dl] \arrow[dd, "\textbf{local trivialization}"] \\
\underline{\text{parallel 2-transport}} \arrow[dd] & & \text{2-Atiyah splitting} \arrow[dd] & \\
 & \underline{\text{descent data}} \arrow[rr, dashed] \arrow[dl, dashed] & & \underline{\text{differential cocycle}} \arrow[dl] \\
\underline{\text{2-descent data/ bundle gerbe}} \arrow[rr] & & \underline{\text{differential 2-cocycle}} &
\end{tikzcd}
$$

locally we thus obtain what is known as

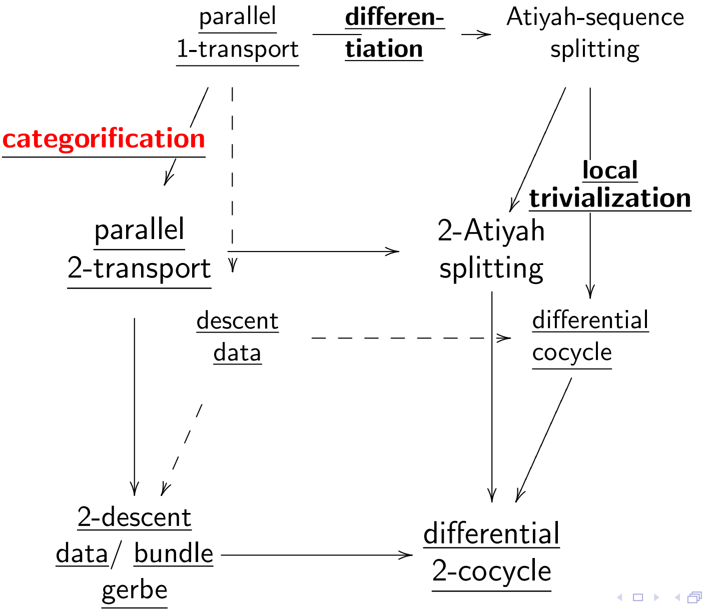
- transition data for a connection
- Ehresmann connection
- differential cocycle

$$g_{ij}g_{jk} = g_{ik}$$
$$A_j = g_{ij}(d + A_i)g_{ij}^{-1}$$

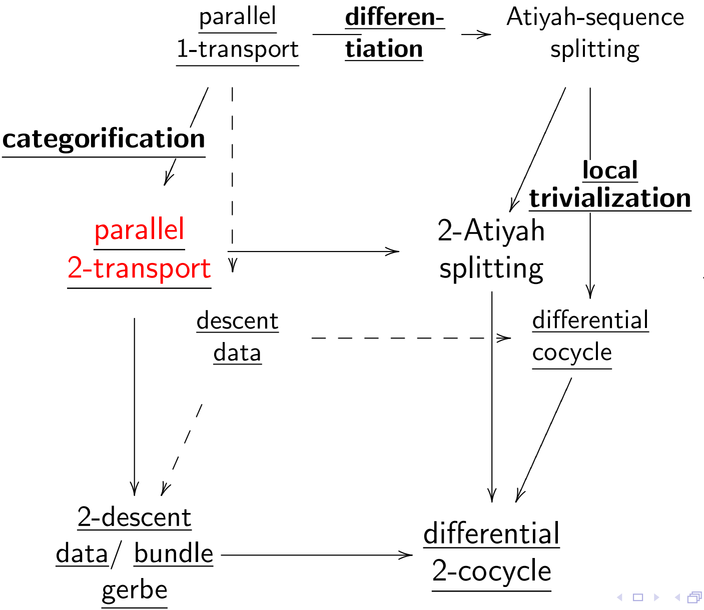


while globally this corresponds to a (weak) splitting of the Atiyah algebroid sequence

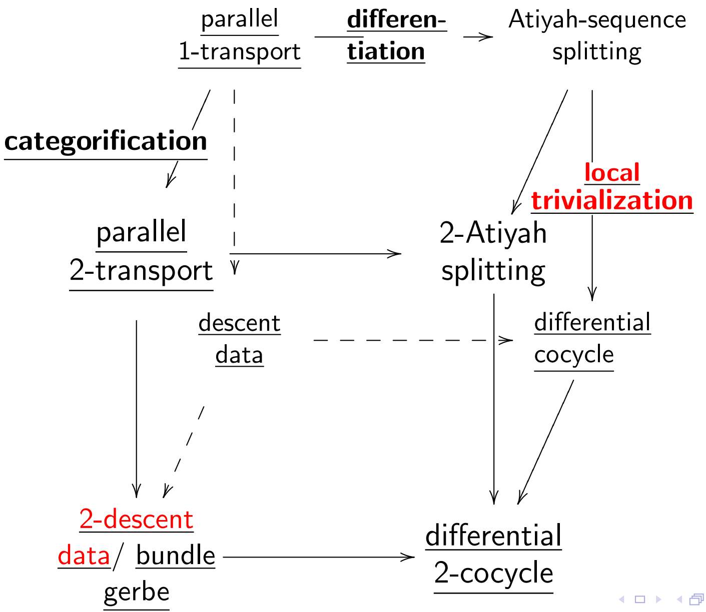
$$\begin{tikzcd} \Gamma(\mathrm{ad}(P)) \arrow[d] & \mathrm{der}(\Gamma(\mathrm{ad}(P))) \\ \Gamma(TP/G) \arrow[d] & \\ \Gamma(TX) \arrow[uur, bend right, "\nabla, F_\nabla"'] & \end{tikzcd}$$



This entire setup admits a straightforward categorification by replacing transport (1-)functors by n-functors.



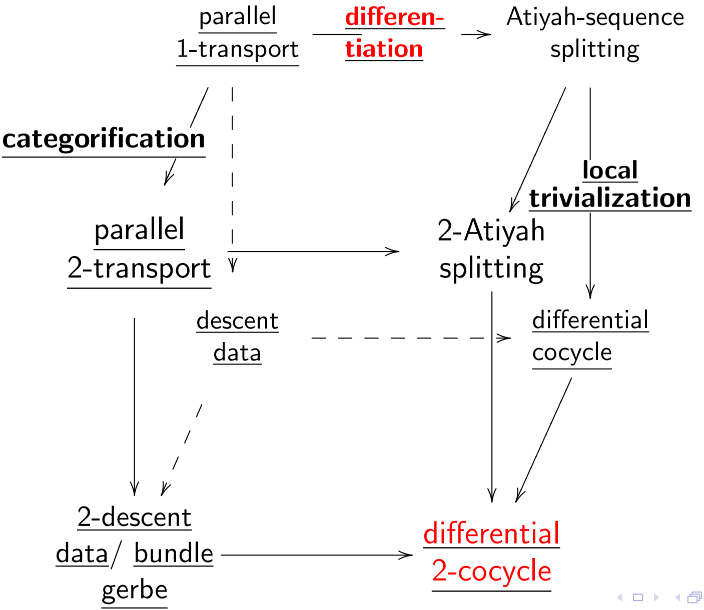
This way we obtain **parallel 2-transport** 2-functors that encode parallel transport along surfaces in 2-bundles with connection.



Their 2-descent data

$$\begin{tikzcd}
\pi_1^*\mathrm{triv} \arrow[dd, "\pi_{13}^* g"'] \arrow[dr, "\pi_{12}^* g"] & \\
 & \pi_2^*\mathrm{triv} \arrow[dl, "\pi_{23}^* g"] \arrow[l, phantom, "\stackrel{\Leftarrow}{\simeq}"] \\
\pi_3^*\mathrm{triv} &
\end{tikzcd}$$

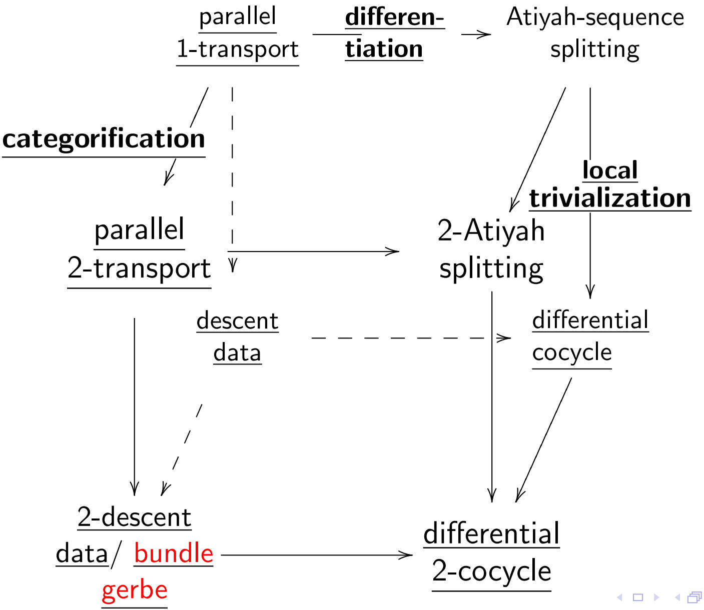
obtained from local trivialization now comes in two different stages:



the “full‘” local trivialization whose differentiation leads to a differential 2-cocycle

$$f_{ikl} f_{ijk} = f_{ijl} g_{ij}(f_{jkl})$$

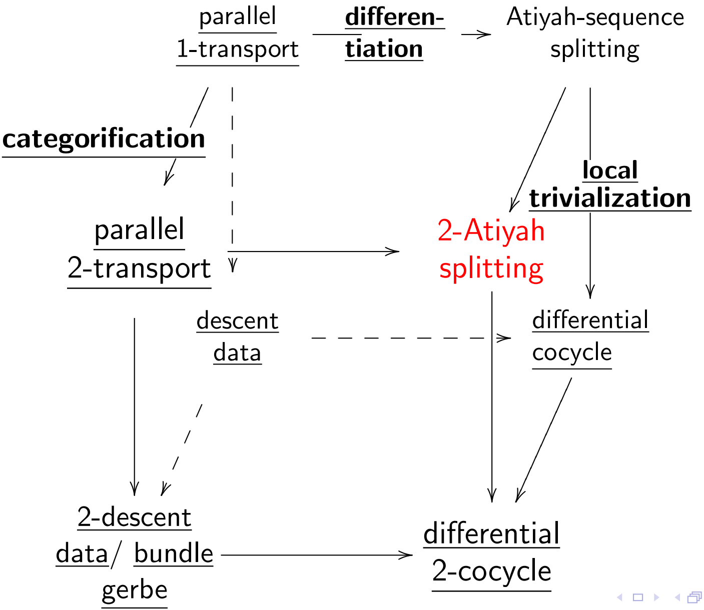
$$B_j = g_{ij}(B_i) + d_A a_{ij}$$



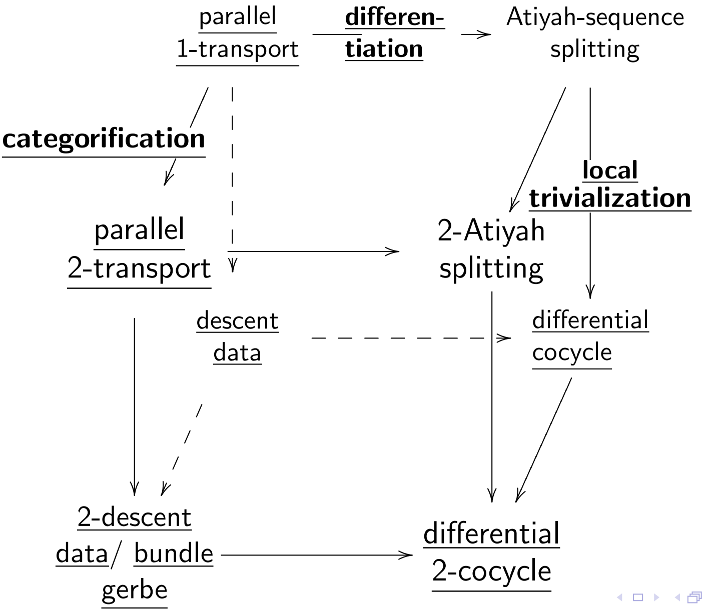
and an "intermediate" one: a bundle gerbe.

$$\begin{array}{ccc} (L, \nabla, \mu) & & \\ \downarrow & & \\ Y^{[2]} & \rightrightarrows & Y \\ & & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

which is a pseudofunctor $g : Y^{[2]} \to \Sigma 1d\mathrm{Vect}$



Finally we could categorify the Atiyah splitting to obtain a splitting of the Atiyah Lie 2-algebroid sequence. ([Stevenson, in preparation])

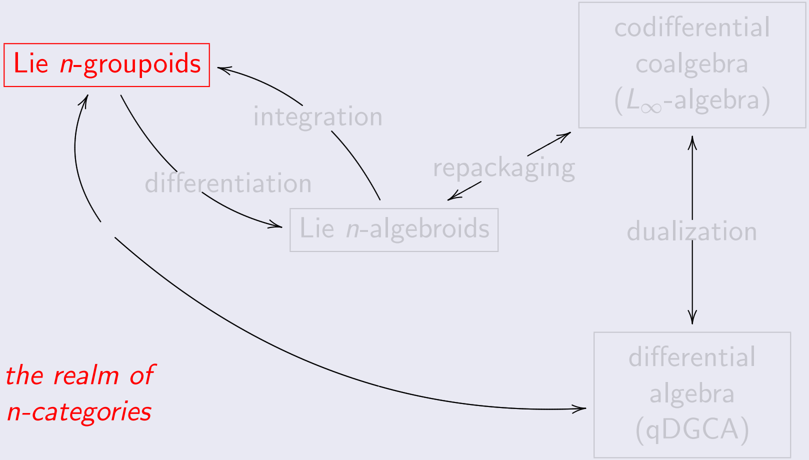

parallel 1-transport
differen- tiation
Atiyah-sequence splitting
categorification
parallel 2-transport
2-Atiyah splitting
local trivialization
descent data
differential cocycle
2-descent data/ bundle gerbe
differential 2-cocycle

The bridge between Lie n-algebra and differential graded algebra

By Koszul duality, semistrict Lie n-algebras are "the same" as differential graded-commutative algebras freely generated in positive degree smaller than n.

In principle this relation has been known for a long time to experts, going back to Quillen's 1968 paper on rational homotopy theory.

A bridge of concepts



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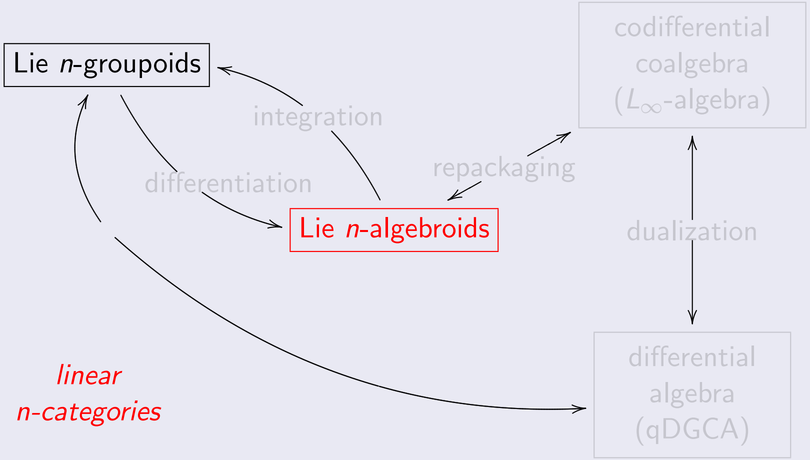

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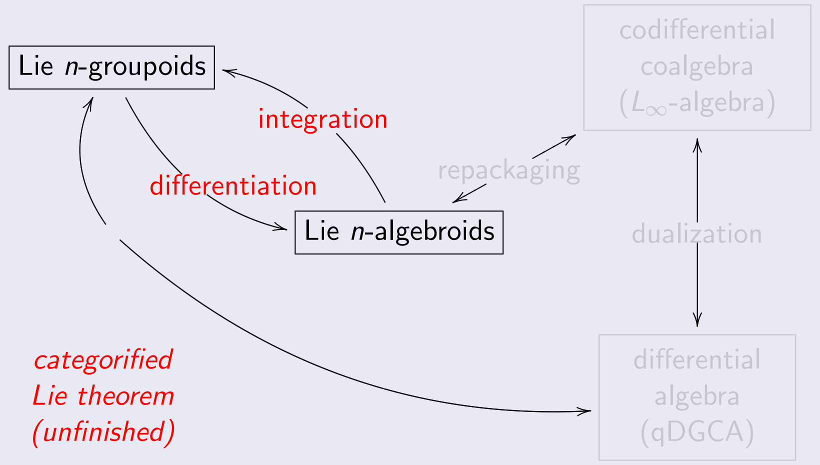

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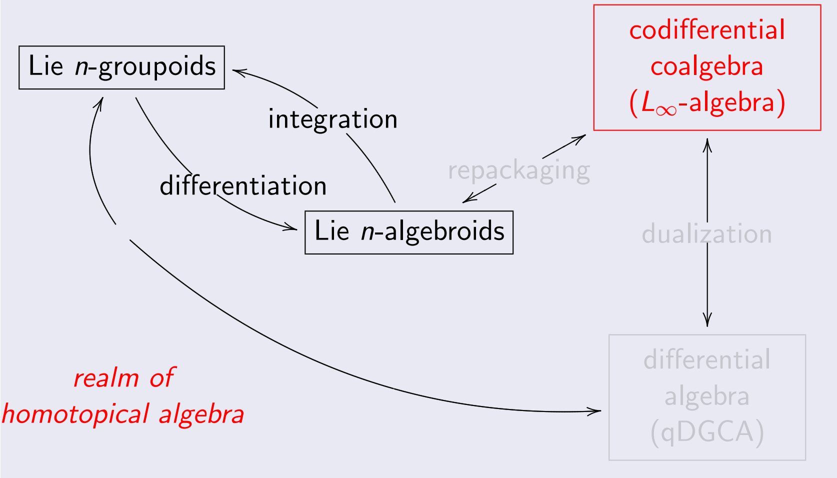

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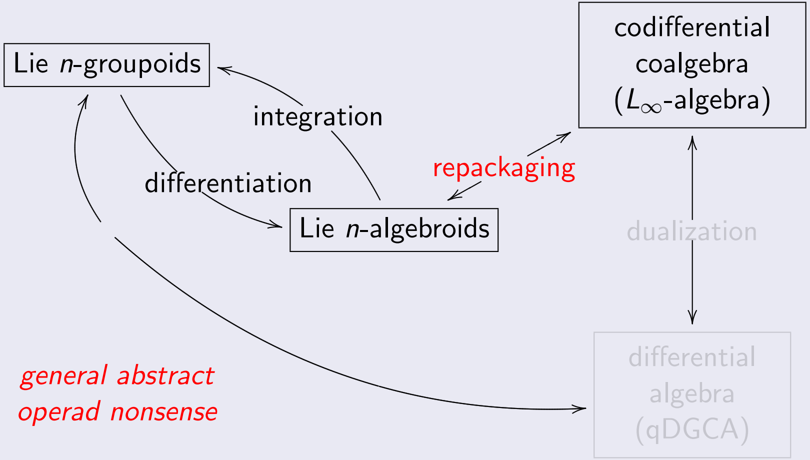

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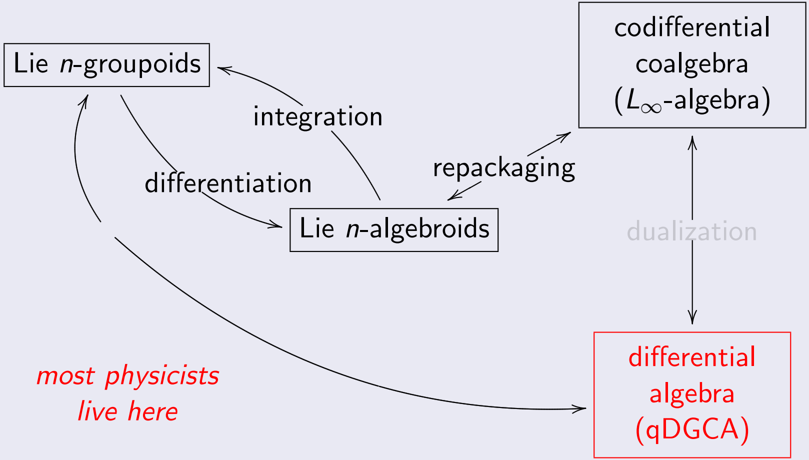

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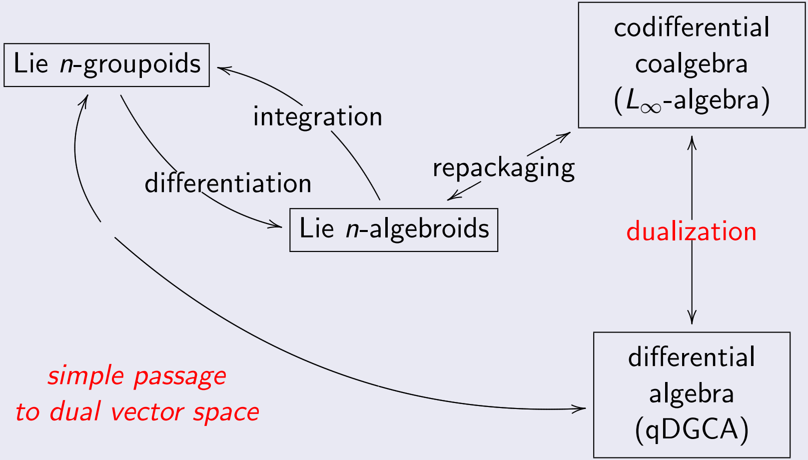

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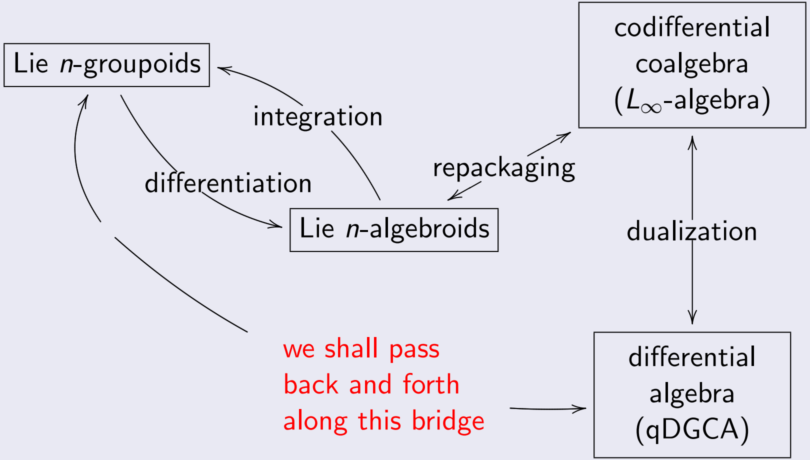

Bridging schools of thought

How to use the bridge

Lie n-groupoids	$\overset{\text{the bridge}}{\longleftrightarrow}$	differential algebra
conceptual understanding		computational accessibility
What is going on?		*How does it work?*
diagrammatics arrow theory		implementation

String n-transport

n-Groups from central extensions

Interesting examples of structure n-groups for parallel n-transport come from central extensions

$$1 \to \Sigma^{n-1} U(1) \to \hat{G} \to G \to 1$$

of an ardinary group G by a copy of $(n-1)$-fold shifted $U(1)$.

n-Groups from central extensions

The best-known example for this is the String 2-group

$$\mathrm{String}_k(G)$$

assignable to a compact, simple, simply connected Lie group G for every level $k \in H^3(G, \mathbb{Z})$:

$$1 \to \Sigma U(1) \to \mathrm{String}_k(G) \to G \to 1$$

Integrating the String Lie 2-algebra

$$\begin{tikzcd}
{|\mathrm{String}_k(G)|} \arrow[r, dash, "\simeq"] & {|G_\mu|} \\
\mathrm{String}_\mu(G) = (\hat{\Omega}_k G \to PG) \arrow[u, mapsto] & \color{red}{G_\mu} \arrow[u, mapsto] \\
\mathfrak{string}_\mu(\mathfrak{g}) = (\hat{\Omega}_k \mathfrak{g} \to P\mathfrak{g}) \arrow[u, mapsto, "\text{int.}"] & \color{red}{\mathfrak{g}_\mu} \arrow[l, "\simeq"'] \arrow[u, mapsto, "\text{integrate [Henriques]}"'] \arrow[ul, bend right, "\text{integrate [BCSS]}"]
\end{tikzcd}$$

Using a general procedure for integrating semistrict Lie n-algebras, the Baez-Crans type Lie 2-algebra $\mathfrak{g}_\mu$ for $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ may be integrated directly to a weak Lie 2-group G_μ.

Integrating the String Lie 2-algebra

$$
\begin{tikzcd}
{|\mathrm{String}_{\mathrm{k}}(\mathrm{G})|} \arrow[rr, no head, "\simeq"] & & {|G_\mu|} \\
\mathrm{String}_\mu(G) = (\hat{\Omega}_k G \to PG) \arrow[u, mapsto] & & G_\mu \arrow[u, mapsto] \\
\mathfrak{string}_\mu(\mathfrak{g}) = (\hat{\Omega}_k \mathfrak{g} \to P\mathfrak{g}) \arrow[u, mapsto, "\text{int.}"] & & \mathfrak{g}_\mu \arrow[ll, "\simeq"'] \arrow[u, mapsto, "\text{integrate [Henriques]}"] \arrow[ull, bend left, "\text{integrate [BCSS]}"]
\end{tikzcd}
$$

This was described by Henriques, following [Getzler].

Integrating the String Lie 2-algebra

$$
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|\mathrm{String}_k(G)| \arrow[r, dash, "\simeq"] & |G_\mu| \\
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\end{tikzcd}
$$

Alternatively, one can notice that the small but semistrict Lie 2-algebra $\mathfrak{g}_\mu$ is *equivalent* to a large but strict Lie 2-algebra $(\hat{\Omega}_k \mathfrak{g} \to P\mathfrak{g})$.

Integrating the String Lie 2-algebra

$$
\begin{tikzcd}
{|\mathrm{String}_k(G)|} \arrow[rr, dash, "\simeq"] & & {|G_\mu|} \\
\mathrm{String}_\mu(G) = (\hat{\Omega}_k G \to PG) \arrow[u, mapsto] & & G_\mu \arrow[u, mapsto] \\
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\end{tikzcd}
$$

This accordingly integrates to a strict Lie 2-group $(\hat{\Omega}_k G \to PG)$.

Integrating the String Lie 2-algebra

$$
\begin{tikzcd}
\left|\mathrm{String_k(G)}\right| \arrow[r, dash, "\simeq"] & \left|G_\mu\right| \\
\mathrm{String}_\mu(G) = (\hat{\Omega}_k G \to PG) \arrow[u, mapsto] & G_\mu \arrow[u, mapsto] \\
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\end{tikzcd}
$$

This was described in [BaezCransSchreiberStevenson].

Integrating the String Lie 2-algebra

$$\begin{tikzcd}
{|\mathrm{String}_k(G)|} \arrow[rr, dash, "\simeq"] & & {|G_\mu|} \\
\mathrm{String}_\mu(G) = (\hat{\Omega}_k G \to PG) \arrow[u, mapsto] & & G_\mu \arrow[u, mapsto] \\
\mathfrak{string}_\mu(\mathfrak{g}) = (\hat{\Omega}_k \mathfrak{g} \to P\mathfrak{g}) \arrow[u, mapsto, "\text{int.}"] & & \mathfrak{g}_\mu \arrow[ll, "\simeq"'] \arrow[u, mapsto, "\text{integrate [Henriques]}"'] \arrow[ull, bend right, mapsto, "\text{integrate [BCSS]}"]
\end{tikzcd}$$

In either case, the geometric realization is a model for the topological String (1-)group.

2-transport with local structure the strict String 2-group

Therefore

String-2-transport is parallel 2-transport with local structure given by $\Sigma\mathrm{String}_k(G)$.

The strict version of the String Lie 2-groups is useful for working out what this means in detail:

- Local cocycle data for **princial** $\mathrm{String}_k(G)$-2-transport is just a
  $$\mathrm{String}_k(G)$$
  nonabelian 2-cocycle (Breen-Messing data for $(\hat{\Omega}_k G \to PG)$).
- Associated string 2-transport is induced from the canonical 2-representation essentially like for any other strict Lie 2-group.

n-group from higher central extensions

But this is just the first in an infinite series of of higher central extensions, built from elements in Lie algebra cohomology

Definition and Proposition

From elements of $\mathrm{inn}(\mathfrak{g})^*$-cohomology we obtain Lie n-algebras:

Lie algebra cocycle	μ	Baez-Crans Lie n-algebra	$\mathfrak{g}_\mu$
invariant polynomial	k	Chern Lie n-algebra	$\mathrm{ch}_k(\mathfrak{g})$
transgression element	cs	Chern-Simons Lie n-algebra	$\mathrm{cs}_k(\mathfrak{g})$

For every transgression element cs these fit into a weakly exact sequences

$$\begin{tikzcd} \Sigma^{n-1}\mathfrak{u}(1) \arrow[r] & \mathfrak{g}_{\mu_k} \arrow[r] \arrow[d] & \mathfrak{g} \; . \\ & \mathrm{cs}_k(\mathfrak{g}) \simeq \mathrm{inn}(\mathfrak{g}_\mu) \arrow[d] & \\ & \mathrm{ch}_k(\mathfrak{g}) & \end{tikzcd}$$

Chern-Simons n-transport

Chern-Simons $(n+1)$-transport is the obstruction to lifting a G-transport through a higher central extension

$$1 \to \Sigma^{n-1}U(1) \to \hat{G} \to G \to 1\,.$$

The basic idea of parallel n-transport

Parallel transport is the consistent assignment of transformations of fibers to paths.

The basic idea of parallel n-transport

The principle of least resistance under categorification

There are many different definitions of the concept *connection on a possibly nontrivial bundle*.
Each definition behaves differently under categorification (= generalization to higher order structures).
We regard a definition as the more "fundamental" the more straightforwardly it categorifies.

Slogan

We understand the true nature of a concept the deeper, the more straightforwardly the definition we use to conceive it lends itself to categorification.

The basic idea of parallel n-transport

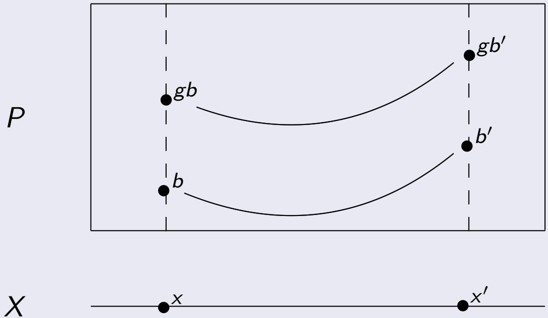
There is one definition of bundles with connection that stands out among all others with respect to the ease with which it lends itself to categorification:

general Fact

A bundle with connection is a parallel transport functor.

Parallel transport is a functor

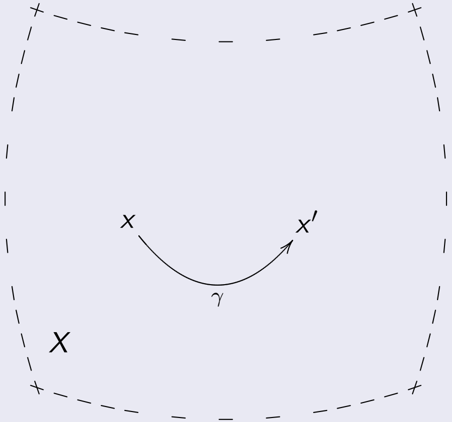
The parallel transport induced by a connection on a principal bundle $P \to X$



is a functorial map from paths to fiber morphisms.

Definition

The smooth path 1-groupoid $\mathcal{P}_1(X)$ is that whose morphisms $\gamma : x \to y$ are thin homotopy classes of paths in X.



Definition

A thin homotopy is a smooth homotopy whose differential has nonmaximal rank everywhere.

A thin homotopy between paths is a surface that degenerates to an at most 1-dimensional structure.
Invariance of parallel transport under thin homotopy means

- invariance under orientartion-preserving reparameterizations
- inversion under orientation-reversing reparameterizations.

In the physics literature this is sometimes addressed as *zig-zag-symmetry*.

The basic idea of parallel n-transport

Hence a connection ∇ on a trivial G-bundle gives rise to a smooth parallel transport functor

$$\mathrm{tra}_\nabla : \mathcal{P}_1(X) \to \Sigma G\,,$$

where

$$\Sigma G := \left\{ \bullet \xrightarrow{g} \bullet \,|g \in G \right\}$$

is the one-object Lie groupoid corresponding to the Lie group G.

The basic idea of parallel n-transport

A connection ∇ on a possibly *non*-trivial G-bundle gives rise to a parallel transport functor

$$\mathrm{tra}_\nabla : \mathcal{P}_1(X) \to G\mathrm{Tor}\,,$$

which *locally* looks like a functor $\mathcal{P}_1(X) \to \Sigma G$.
Here $G\mathrm{Tor}$ is the category whose objects are principal G-spaces isomorphic to G, and whose morphisms are maps between them preserving the G-action.

The basic idea of parallel n-transport

Proposition [S.-Waldorf]

The category of those functors

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$$

that admit a smooth local i-trivialization in that they fit into a diagram

$$\begin{tikzcd} P_1(Y) \arrow[r, two heads, "\pi"] \arrow[d, "\mathrm{triv}"'] & P_1(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "t"', "\simeq"] \\ \Sigma G \arrow[r, "i"'] & G\mathrm{Tor} \end{tikzcd}$$

is equivalent to that of principal G-bundles with connection on G.

This way we have described bundles with connection purely "arrow-theoretically" in terms of their parallel transport functors.
This, then, doesn't resist categorification anymore.

- Proceed with the discussion of parallel n-transport.
- First have a look at some of the history and the relation to Cheeger-Simons differential characters.

History

The point of view that characterizes bundles with connection in terms of their parallel transport has a long history, but was only recently [Baez-S.,S.-Waldorf] made fully explicit.
Building on older ideas, [Barrett:1991] and [CaetanoPicken:1994] noticed that (possibly nontrivial) G-bundles with connection on *connected* base spaces can be reconstructed, up to isomorphism, from their *holonomy* map from based *loops* to the structure group.
Inspired by John Baez, in [S.-Waldorf,2007] the generalization of this statement to parallel transport functors from paths to fiber morphisms was given.

History

The description of bundles with connection in terms of their holonomy maps around loops was generalized in [Mackaay-Picken,2002] to homomorphisms that label surfaces by an abelian Lie group. This describes abelian gerbes (abelian 2-bundles) with connection on simply connected spaces.

And this point of view is evidently closely related to that underlying Cheeger-Simons differential characters.

History

In [Baez,2002] the idea appears of refining the assignment of elements of $U(1)$ to surfaces to a 2-functor from surfaces to a 2-group.

The full description of of this idea in terms of descent/gluing data for 2-group-valued parallel transport 2-functors, and the observation that such data describes fake-flat nonabelian gerbes with connection, is given in [Baez.-S.,2004].

The development of this idea to a full theory of n-transport, as indicated in the following, is, as yet, largely unpublished, alas. But see [S.-Waldorf, in preparation].

Cheeger-Simons differential characters

To some extent, parallel n-transport can be regarded as a generalization of degree n Cheeger-Simons differential characters from the n-group $\Sigma^{n-1}U(1)$ to an arbitrary structure n-group.

Definition

Let $Z_n X$ be the group of smooth n-cycles in the manifold X. A degree n differential character on X is a group homomorphism

$$t : Z_n X \to U(1)$$

such that there is a closed $(n+1)$-form F_{n+1} satisfying

$$t(\partial V) = \exp(i \int_V F_{n+1})$$

for all smooth $(n+1)$-chains V.

Cheeger-Simons differential characters

Evidently, a degree n Cheegers-Simons differential character is a rule for assigning n-dimensional holonomy in $U(1)$ to closed n-dimensional volumes.
It is known that such degree n differential characters (the conventions for counting their degrees may vary) are equivalent to Deligne cohomology, which in turn is equivalent to abelian $(n+1)$-gerbes with connection.

Cheeger-Simons differential characters

There have been attempts to phrase differential characters in more functorial language. For instance [Turner, 2004].
We find that generalizing these holonomy assignments from abelian to nonabelian (n-)groups requires to generalize

- from closed volumes to volumes with boundary (from holonomy to parallel transport);
- from assigning data to n-dimensional volumes to assigning data to $(0 \leq d \leq n)$-dimensional volumes.

This means that maps from n-cycles to $U(1)$ need to be replaced by n-functors from n-paths to some n-group.

Associated transport

Notice, though, that our parallel n-transport, is, a priori, defined only on n-paths that have the topology of n-dimensional balls.

We can understand this requirement already for $n = 1$: in order for a nonabelian bundle with connection to yield a holonomy assignment defined on closed paths, we need the additional information of a linear representation and a notion of trace.

Similarly, parallel n-transport together with a linear n-representation yields associated n-vector transport. Categorified notions of traces then allow to obtain n-dimensional holonomy over arbitrary n-dimensional volumes.

Locally trivializable n-transport

We call an $n$functor locally i-trivializablle, if when pulled back to a cover of its domain, it becomes equivalent to a n-functor that factors through i.

Let's first introduce some useful

Terminology

- Write $P_n(X)$ for a Lie n-groupoid that plays the role of n-paths in X.
- Write T for a given Lie n-groupoid that a parallel n- transport might take values in.
- Write $G_{(n)}$ for a given Lie n-group which plays the role of the structure Lie n-group of the n-transport.

Definition

Let $\pi : P_n(Y) \twoheadrightarrow P_n(X)$ be an epimorphism and $i : \Sigma G_{(n)} \hookrightarrow T$ a monomorphism. Then an n-functor $\mathrm{tra} : P_n(X) \to T$ is called a π-locally i-trivial n-transport functor if there exists a square

$$\begin{tikzcd}
P_n(Y) \arrow[r, twoheadrightarrow, "\pi"] \arrow[d, "\mathrm{triv}"'] & P_n(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "t"', "\simeq"] \\
\Sigma G_{(n)} \arrow[r, "i"'] & T
\end{tikzcd}$$

such that the induced transition $g := \pi_2^* t^{-1} \circ \pi_1^* t$ is in components itself a locally trivializabel $(n-1)$-transport.

Descent data

Descent data

A descent datum for a locally i-trivializable transport n-functor is an i-trivial transport n-functor on a cover, together with an n-simplex of "gluing data".

The definition of locally trivializable n-transport connects two important points of view:

- **The global perspective.** The functor $\mathrm{tra} : \mathcal{P}_n(X) \to T$ is the global object corresponding to an n-bundle with connection. We will discuss theorems that assert that if tra has a smooth local trivialization, then this is *unique* up to equivalence. This means that tra contains all the relevant information.
- **The local perspective.** From any local trivialization $t : \pi^*\mathrm{tra} \to \mathrm{triv}$ one obtains straightforwardly the **descent data** (also: transition data or gluing data) which describes tra in terms of the descent of a "trivial" n-functor on Y down to X.

Descent data

Given a local trivialization t , we obtain a transition on $Y^{[2]}$ as

$$g := \pi_2^* t \circ \pi_1^* t^{-1} : \pi_1^* \mathrm{triv} \to \pi_2^* \mathrm{triv} \,,$$

i.e.

$$\begin{tikzcd}
 & \mathcal{P}_n(Y) \arrow[rr, "\mathrm{triv}"] \arrow[dr, "\pi"] & & \Sigma G_{(n)} \arrow[dr, "i"] & \\
\mathcal{P}_{Y^{[2]}} \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & \mathcal{P}_n(X) \arrow[rr, "\mathrm{tra}"] \arrow[ur, phantom, "\Downarrow t^{-1}"] \arrow[dr, phantom, "\Downarrow t"] & & T \\
 & \mathcal{P}_n(Y) \arrow[rr, "\mathrm{triv}"'] \arrow[ur, "\pi"] & & \Sigma G_{(n)} \arrow[ur, "i"'] &
\end{tikzcd}$$

Descent data

Each such transition gives rise to a descent object [Street], which is an $(n+1)$-simplex labeled by transitions pulled back to $Y^{[n+1]}$.

Descent data

Here is the filled triangle describing the descent of 2-transport:

$$
\begin{tikzcd}
 & \pi_2^*\mathrm{triv} \arrow[dr, "p_{23}^* g"] & \\
\pi_1^*\mathrm{triv} \arrow[ur, "p_{12}^* g"] \arrow[rr, "p_{13}^* g"'] \arrow[rr, phantom, shift left=4ex, "\Downarrow f"] & & \pi_3^*\mathrm{triv}
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
 & \pi_2^*\mathrm{triv} \arrow[d, bend left, "\pi_2^* t^{-1}"] \arrow[ddr, "\pi_{23}^* g"] & \\
 & \pi^*\mathrm{tra} \arrow[u, bend left, "\pi_2^* t"] \arrow[u, phantom, "\Downarrow"] \arrow[dr, "\pi_3^* t"] \arrow[d, phantom, "\Vert"] & \\
\pi_1^*\mathrm{triv} \arrow[uur, "\pi_{12}^* g"] \arrow[ur, "\pi_1^* t^{-1}"] \arrow[rr, "\pi_{13}^* g"'] & & \pi_3^*\mathrm{triv}
\end{tikzcd}
$$

Descent data

Definition

There is a more or less obvious n-category $\mathrm{Desc}^i_n(\pi)$ of π-local i-descent data.

We shall come to statements which identitfy these descent objects for transport functors as (nonabelian) differential cocycles of various kinds:
Deligne cocycles, line bundle gerbes with connection, nonabelian bundle gerbes with connection, Breen-Messing data for nonabelian gerbes with connection, and the like.
The crucial ingredient for these statements is the characterization of n-transport with values in an n-group in terms of differential form data.

Smooth n-functors from n-paths to Lie n-groups

A smooth n-functor is a morphism in n-categories internal to a suitable category of smooth spaces.
Manifolds are an insufficient model for smooth spaces, since maps between manifolds don't usually form a manifold themselves.
There are several options to generalize away from manifolds.
Sheaves on manifolds is one of them. A slightly smaller category is the most convenient for our purposes: Chen-smooth spaces.

Definition

A Chen-smooth structure S_X on a set X is a sheaf on manifolds quasi-representable by X.

This means that it is a sheaf on manifolds such that

$$S_X(U) \subset \mathrm{Hom}_{\mathrm{Set}}(U, X)$$

.

The elements of $S_X(U)$ are called *plots* from U to X.

- n-path spaces of Chen-smooth spaces are naturally Chen-smooth spaces themselves.
- Quotient spaces of Chen-smooth spaces are naturally Chen-smooth spaces themselves.

Definition

For X a Chen-smooth space,the smooth structure on its n-path space $P^nX = [I^n, X]$ is such that $\phi : U \to PX$ is a plot of P^nX if and only if the composite

$$U \times I^n \xrightarrow{\phi \times \mathrm{Id}} P^nX \times I^n \xrightarrow{\mathrm{ev}} X$$

is a plot of X.

n-Functors and differential forms

Smooth n-functors and differential forms

A smooth n-functor is entirely determined by its differentials at identity morphisms. Hence it encodes differential form data on the space of objects.

Since an n-transport locally looks like a smooth functor

$$\mathrm{tra} : \mathcal{P}_n(X) \to \Sigma G_{(n)}$$

$$\begin{tikzcd}[column sep=huge] x \arrow[r, bend left=60, "\gamma_1"] \arrow[r, bend right=60, "\gamma_2"'] \arrow[r, phantom, "\Downarrow \Sigma"] & y \end{tikzcd} \;\mapsto\; \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=60, "\mathrm{tra}(\gamma_1)"] \arrow[r, bend right=60, "\mathrm{tra}(\gamma_2)"'] \arrow[r, phantom, "\Downarrow \mathrm{tra}(\Sigma)"] & \bullet \end{tikzcd}$$

with values in a Lie n-group $G_{(n)}$, it is useful to first characterize such n-functors in terms of differential form data and generalized "path-ordered exponentials".

1-Functors and differential 1-forms

Proposition

For G a Lie group with Lie algebra $\mathfrak{g}$, smooth 1-functors

$$\mathcal{P}_1(X) \to \Sigma G$$

are in bijection with 1-forms $A \in \Omega^1(X, \mathfrak{g})$. The bijection is induced by the "path ordered exponential"

$$(x \xrightarrow{\gamma} y) \;\mapsto\; (\bullet \xrightarrow{P\exp(\int_\gamma A)} \bullet)$$

1-Functors and differential 1-forms

While the technically cleanest way to conceive this is in terms of solutions of differential equations, the best conceptual way to think of this is by conceiving the path ordered exponential as the limit obtained by applying the functor to ever smaller subdivisions of the given path:

1-Functors and differential 1-forms

$$x \xrightarrow{\quad\gamma\quad} y$$

$$\stackrel{\mathrm{tra}}{\longmapsto}$$

$$\bullet \xrightarrow{\quad P\exp(\int_\gamma A)\quad} \bullet$$

1-Functors and differential 1-forms

$$x \xrightarrow{\gamma_1} z_1 \xrightarrow{\gamma_2} y$$

$$\overset{\mathrm{tra}}{\longmapsto}$$

$$\bullet \xrightarrow{P\exp(\int_{\gamma_1} A)} z_1 \xrightarrow{P\exp(\int_{\gamma_2} A)} \bullet$$

1-Functors and differential 1-forms

$$\begin{tikzcd} x \arrow[r, "\gamma_1"] & z_1 \arrow[r, "\gamma_2"] & z_2 \arrow[r, "\gamma_3"] & y \end{tikzcd}$$

$$\overset{\mathrm{tra}}{\longmapsto}$$

$$\begin{tikzcd} \bullet \arrow[r, "P\exp(\int_{\gamma_1} A)"] & z_1 \arrow[r, "P\exp(\int_{\gamma_2} A)"] & z_2 \arrow[r, "P\exp(\int_{\gamma_3} A)"] & \bullet \end{tikzcd}$$

1-Functors and differential 1-forms

$$x \xrightarrow{\gamma_1} z_1 \xrightarrow{\gamma_2} z_2 \xrightarrow{\gamma_3} z_2 \xrightarrow{\gamma_4} y$$

$$\overset{\mathrm{tra}}{\longmapsto} \simeq$$

$$\bullet \xrightarrow{1+A(\gamma_1)} \bullet \xrightarrow{1+A(\gamma_2)} \bullet \xrightarrow{1+A(\gamma_3)} \bullet \xrightarrow{1+A(\gamma_4)} \bullet$$

2-Functors and differential 2-forms

Now let $G_{(2)}$ be a strict 2-group coming from the crossed module $H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$.

Propositon

Strict 2-functors

$$\mathcal{P}_2(X) \to \Sigma G_{(2)}$$

are in bijection with differential forms

$$(A, B) \in \Omega^1(X, \mathfrak{g}) \times \Omega^2(X, \mathfrak{h})$$

satisfying the *fake flatness condition*

$$F_A + t_* \circ B = 0\,.$$

2-Functors and differential 2-forms

The bijection is induced by a generalization of the concept of "path ordered exponential" to "surface ordered exponential".

$$\begin{tikzcd} x \arrow[r, bend left=50, "\gamma_1"] \arrow[r, bend right=50, "\gamma_2"'] \arrow[r, phantom, "\Downarrow \Sigma"] & y \end{tikzcd} \mapsto \begin{tikzcd} \bullet \arrow[r, bend left=50, "P\exp(\int_{\gamma_1} A)"] \arrow[r, bend right=50, "P\exp(\int_{\gamma_2} A)"'] \arrow[r, phantom, "\Downarrow P_A\exp(\int_\Sigma B)"] & \bullet \end{tikzcd}$$

This is again best understood as the result of applying the 2-functor to ever finer subdivisions of a surface and using the composition in the 2-group to compile the little contributions to the full surface transport.

2-Functors and differential 2-forms

$$
\begin{tikzcd}
x \arrow[r, "\gamma_1"] \arrow[d, "\gamma_3"'] & y_1 \arrow[d, "\gamma_2"] \arrow[dl, Rightarrow, "\Sigma"] \\
y_2 \arrow[r, "\gamma_4"'] & z
\end{tikzcd}
\quad \overset{\mathrm{tra}}{\longmapsto} \quad
\begin{tikzcd}
\bullet \arrow[r, "P\exp(\int_{\gamma_1} A)"] \arrow[d, "P\exp(\int_{\gamma_3} A)"'] & \bullet \arrow[d, "P\exp(\int_{\gamma_2} A)"] \arrow[dl, Rightarrow, "P_A\exp(\int_{\Sigma} B)"] \\
\bullet \arrow[r, "P\exp(\int_{\gamma_4} A)"'] & \bullet
\end{tikzcd}
$$

2-Functors and differential 2-forms

$$
\begin{tikzcd}
x \arrow[r, "\gamma_1"] \arrow[d, "\gamma_5"'] & y_1 \arrow[r, "\gamma_2"] \arrow[d] \arrow[dl, Rightarrow] \arrow[dl, phantom, "\Sigma_1", near start, shift left=3ex] & y_2 \arrow[d, "\gamma_3"] \arrow[dl, Rightarrow] \arrow[dl, phantom, "\Sigma_2", near start, shift left=3ex] \\
y_4 \arrow[r] \arrow[d, "\gamma_6"'] & y \arrow[r] \arrow[d] \arrow[dl, Rightarrow] \arrow[dl, phantom, "\Sigma_3", near start, shift left=3ex] & y_3 \arrow[d, "\gamma_4"] \arrow[dl, Rightarrow] \arrow[dl, phantom, "\Sigma_4", near start, shift left=3ex] \\
y_5 \arrow[r, "\gamma_7"] & y_6 \arrow[r, "\gamma_8"] & z
\end{tikzcd}
\quad \overset{\mathrm{tra}}{\longmapsto} \quad
\begin{tikzcd}
\bullet \arrow[r, "1+A(\gamma_1)"] \arrow[d, "P\exp(\int_{\gamma_5} A)"'] & \bullet \arrow[r, "1+A(\gamma_2)"] \arrow[d] \arrow[dl, Rightarrow, "1+B(\Sigma_1)"] & \bullet \arrow[d, "P\exp(\int_{\gamma_3} A)"] \arrow[dl, Rightarrow, "1+B(\Sigma_2)"] \\
\bullet \arrow[r] \arrow[d, "1+A(\gamma_6)"'] & \bullet \arrow[r] \arrow[d] \arrow[dl, Rightarrow, "1+B(\Sigma_3)"] & \bullet \arrow[d, "1+A(\gamma_4)"] \arrow[dl, Rightarrow, "1+B(\Sigma_4)"] \\
\bullet \arrow[r, "1+A(\gamma_7)"] & \bullet \arrow[r, "1+A(\gamma_8)"] & \bullet
\end{tikzcd}
$$

2-Functors and differential 2-forms

For the special case that $G_{(2)} = \mathrm{INN}(G) = G//G$ is the strict 2-group coming from the crossed module $G \stackrel{\mathrm{Id}}{\to} G \stackrel{\mathrm{Ad}}{\to} \mathrm{Aut}(G)$, the fake flatness condition implies that

$$B = -F_A$$

and the existence of the 2-morphism

$$\begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=60, "P\exp(\int_{\gamma_1} A)"] \arrow[r, bend right=60, "P\exp(\int_{\gamma_2} A)"'] \arrow[r, phantom, "\Downarrow P_A \exp(-\int_\Sigma F_A)"] & \bullet \end{tikzcd}$$

exhibits what is known as the nonabelian Stokes theorem.

Principal 1-transport

Principal 1-bundles with connection

Recall our claim concerning globally defined principal 1-transport:

Proposition

Let G be a Lie group and $i : \Sigma G \hookrightarrow T$ a monomorphic equivalence. Then locally i-trivializabel transport functors

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$$

are equivalent to G-bundles with connection.

This follows from using the relation between 1-forms and smooth 1-functors and inserting it into the corresponding descent object...

local connection 1-form $A \in \Omega^1(U, \mathfrak{g})$	smooth transport functor $\mathrm{triv} : \mathcal{P}_1(U) \to \Sigma(G)$
transition function $g \in \Omega^0(U^{[2]}, G)$	natural isomorphism $g : p_1^*\mathrm{triv} \to p_2^*\mathrm{triv}$
$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} d g_{ij}^{-1}$	$\begin{tikzcd} \bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "\mathrm{tra}_i(\gamma)"'] & \bullet \arrow[d, "\mathrm{tra}_j(\gamma)"] \\ \bullet \arrow[r, "g_{ij}(y)"'] & \bullet \end{tikzcd}$
$g_{ij} g_{jk} = g_{ik}$	$\begin{tikzcd} & p_2^*\mathrm{triv} \arrow[dr, "p_{23}^* g"] & \\ p_1^*\mathrm{triv} \arrow[ur, "p_{12}^* g"] \arrow[rr, "p_{13}^* g"'] & & p_3^*\mathrm{triv} \end{tikzcd}$

Vector bundles

The notion of functorial parallel transport in vector bundles was historically an important guiding light and motivation for the functorial conception of quantum field theory.

A complex (say) vector bundle with connection is simply a transport functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}_{\mathbb{C}}$$

with local i-structure, for

$$i : \cup_n \Sigma U(n) \to \mathrm{Vect}_{\mathbb{C}}$$

the canonical representation.

Sections of vector bundles

Let $1 : \mathcal{P}_1(X) \to \mathrm{Vect}_{\mathbb{C}}$ be the trivial such transport. Then

$$\Gamma := \mathrm{Hom}(1, \mathrm{tra})$$

is the space of *flat sections* of the given vector bundle.

Covariant derivative

Let $\mathrm{curv} : \Pi_2(X) \to \mathrm{Grpd}$ be the curvature 2-functor of the given vector bundle connection. Then

$$\Gamma := \mathrm{Hom}(1, \mathrm{curv})$$

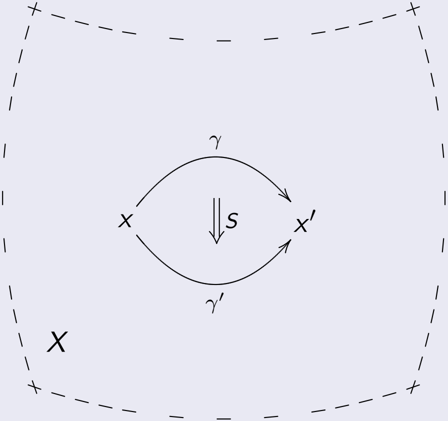
is the space of (not-necessarily flat) *sections* of the given vector bundle. The morphism part of the component map of these transformations encode the *covariant derivative* of these sections.

This procedure has a straightforward generalization to $n = 2$.

Parallel 2-Transport

Definition

$\mathcal{P}_2(X)$ is the (strict) 2-groupoid whose 2-morphisms $S : \gamma \to \gamma'$ are thin homotopy classes of cobounding surfaces.



Principal 2-bundles with connection

Descent data is now a 3-simplex

$$
\begin{tikzcd}
p_2^*\mathrm{tra}_U \arrow[r, "p_{23}^* g"] & p_3^*\mathrm{tra}_U \arrow[d, "p_{34}^* g"] \\
p_1^*\mathrm{tra}_U \arrow[u, "p_{12}^* g"] \arrow[ur, "p_{13}^* g" description] \arrow[r, "p_{14}^* g"'] \arrow[ur, phantom, "\Downarrow p_{123}^* f"', pos=0.75, shift left=6ex] \arrow[ur, phantom, "\Downarrow p_{134}^* f", pos=0.35, shift right=6ex] & p_4^*\mathrm{tra}_U
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
p_2^*\mathrm{tra}_U \arrow[r, "p_{23}^* g"] \arrow[dr, "p_{24}^* g" description] \arrow[dr, phantom, "\Downarrow p_{234}^* f", pos=0.3, shift left=6ex] \arrow[dr, phantom, "p_{124}^* f \Downarrow"', pos=0.7, shift right=6ex] & p_3^*\mathrm{tra}_U \arrow[d, "p_{34}^* g"] \\
p_1^*\mathrm{tra}_U \arrow[u, "p_{12}^* g"] \arrow[r, "p_{14}^* g"'] & p_4^*\mathrm{tra}_U
\end{tikzcd}
$$

$U(1)$ bundle gerbes with connection

Proposition

The descent catgeory for $\Sigma U(1)$ 2-transport is canonically isomorphic to $U(1)$-bundle gerbes with connection ("and curving").

Remark

Notice that this is asserting more than a mere equivalence. Having a canonical isomorphism here means that by starting with the concept of $\Sigma U(1)$-2-transport and turning the descent data crank, the very definition of a bundle gerbe with connection drops out, item by item.

In order to proceed, and to understand the meaning and relevance of fake flatness, we first need a better understanding of higher curvature.

Nonabelian differential cocycles

Nonabelian differential cocycles are descent data for locally $(i = \mathrm{Id}_{G_{(n)}})$-trivializable n-transport.

Definition

Given a strict 2-functor

$$\mathrm{triv} : \mathcal{P}_2(X) \to \Sigma G_{(2)}$$

we obtain a 1-form and a 2-form

$$(A, B)_{\mathrm{triv}} \in \Omega^1(X, \mathrm{Lie}(G)) \times \Omega^2(X, \mathrm{Lie}(H))$$

as follows.

The 1-form is that obtained by restricting triv to 1-morphisms, where it becomes a smooth functor

$$\mathrm{triv}_1 : \mathcal{P}_1(X) \to \Sigma G \,.$$

The value of the 2-form on a pair of vectors $v_1, v_2 \in T_x X$ is defined by choosing any smooth map

$$\Sigma : \mathbb{R}^2 \to X$$

with the property that

$$v_i = \Sigma_* \left(\frac{\partial}{\partial x_i}|_{(0,0)} \right)$$

and then setting

$$B(x)(v_1, v_2) := \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2}\Big|_{(0,0)} \Sigma^* \mathrm{triv}_2 \left(\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (x_1, 0) \arrow[d] \arrow[dl, Rightarrow] \\ (0, x_2) \arrow[r] & (x_1, x_2) \end{tikzcd} \right) .$$

Proposition

The 2-form B defined this way is well defined and smooth.

Proposition (vanishing of ("fake") 2-form curvature)

The forms $(A, B)_{\mathrm{triv}}$ obtained from a smooth functor triv this way satisfy the relation

$$F_A + t_* \circ B = 0 \,.$$

Proof. Differentiate the source and target matching condition

$$t(h)g = g'$$

for

$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr, phantom, "\swarrow h"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} = \Sigma^*\mathrm{triv}\left(\begin{tikzcd} (0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \cdot \arrow[d] \\ \cdot \arrow[r] & (x_1, x_2) \end{tikzcd} \right)$$

$\square$

Definition (integrating differential forms to a 2-functor)

For every pair of forms (A, B) with $F_A + t_* \circ B = 0$ as above, we define a strict 2-functor

$$\mathrm{triv}_{(A,B)} : \mathcal{P}_2(X) \to \Sigma G_{(2)}$$

as follows:

First, on 1-morphisms $\mathrm{triv}_{(A,B)}$ restricts to the 1-functor

$$\mathrm{triv}_A : \mathcal{P}_1(X) \to \Sigma G \,.$$

On 2-morphisms

$$\begin{tikzcd} x \arrow[rr, bend left, "\gamma"] \arrow[rr, bend right, "\gamma'"'] \arrow[rr, phantom, "\Downarrow\Sigma"] & & y \end{tikzcd}$$

coming from a smooth map

$$\Sigma : [0,1]^2 \to X$$

the element $h \in H$ assigned to the surface

$$\begin{tikzcd} \bullet \arrow[rr, bend left, "g"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\Downarrow h"] & & \bullet \end{tikzcd} \quad := \quad \mathrm{triv}\left(\begin{tikzcd} x \arrow[rr, bend left, "\gamma"] \arrow[rr, bend right, "\gamma'"'] \arrow[rr, phantom, "\Downarrow\Sigma"] & & y \end{tikzcd} \right)$$

is defined to be the path-ordered integral

$$P\exp\left(\int_0^1\left(\int_0^1 \alpha^{-1}_{\mathrm{triv}(\,(0,t)\longrightarrow(s,t)\,)}\Sigma^* B(\frac{\partial}{\partial s},\frac{\partial}{\partial t})\,ds\right)\,dt\right)$$

Proposition

Extracting the differential forms $(A, B)_{\mathrm{triv}}$ from a smooth 2-functor triv and then reconstructing a smooth 2-functor $\mathrm{triv}_{(A,B)_{\mathrm{triv}}}$ as above

$$\begin{tikzcd}
\mathrm{triv} \arrow[rr, equal] \arrow[dr, mapsto] & & \mathrm{triv}_{(A,B)_{\mathrm{triv}}} \\
& (A, B)_{\mathrm{triv}} \arrow[ur, mapsto] &
\end{tikzcd}$$

is the identity operation.

Next we obtain the data for gauge transformations by differentiating the component map of a pseudonatural transformation

$$g : \mathrm{triv}_{(A,B)} \to \mathrm{triv}_{(A',B')} .$$

This is a functor with values in squares

$$g : \left(x \xrightarrow{\gamma} y \right) \mapsto \begin{tikzcd} \bullet \arrow[r, "\mathrm{triv}(\gamma)"] \arrow[d, "g(x)"'] & \bullet \arrow[d, "g(y)"] \arrow[dl, Rightarrow, "g(\gamma)"] \\ \bullet \arrow[r, "\mathrm{tra}(\gamma')"'] & \bullet \end{tikzcd} .$$

Notice the following important fact, which pervades all of higher transport theory.

Fact

A morphism between two n-functors is itself an $(n-1)$-functor.

More precisely: its component map is. This yields the general

Fact

Transitions of n-transport is itself an $(n-1)$-transport

This may be familiar from bundle gerbes: these have transition bundles.

More on higher functors here.

The component map of our transformation has to satisfy a (pseudo)naturality condition: for every surface

$$\begin{tikzcd} y \arrow[r, bend left=50, "\gamma"{name=U}] \arrow[r, bend right=50, "\gamma'"'{name=D}] & y' \arrow[from=U, to=D, Rightarrow, "\Sigma"] \end{tikzcd}$$

we have

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "\mathrm{triv}(\gamma)"{name=U}] \arrow[r, "\mathrm{triv}\gamma'"'{name=D}] \arrow[d, "g(y)"'] & \bullet \arrow[d, "g(y')"] \arrow[from=U, to=D, Rightarrow, "\mathrm{triv}(\Sigma)"] \arrow[dl, Rightarrow, "g(\gamma')"] \\ \bullet \arrow[r, "\mathrm{triv}'(\gamma')"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, "\mathrm{triv}(\gamma)"] \arrow[d, "g(y)"'] & \bullet \arrow[d, "g(y')"] \arrow[dl, Rightarrow, "g(\gamma)"] \\ \bullet \arrow[r, "\mathrm{triv}'(\gamma)"{name=U}] \arrow[r, bend right=60, "\mathrm{triv}'(\gamma')"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "\mathrm{triv}'(\Sigma)"] \end{tikzcd}$$

Proposition

Smooth isomorphism

$$g : \mathrm{triv}_{(A,B)} \to \mathrm{triv}_{(A',B')}$$

of 2-functors are in bijective correspondence with pairs

$$(g, a) \in \Omega^0(Y, G) \times \Omega^1(Y, \mathrm{Lie}(H))$$

satisfying

$$A' + t_* \circ a = \mathrm{Ad}_g A + g^*\theta$$

and

$$B' = \alpha_g(B + F_a)\,,$$

where

$$F_a = da + a \wedge a + A(a)\,.$$

Proposition

Smooth 2-isomorphisms

$$\begin{tikzcd} & \mathrm{triv}_{(A_2,B_2)} \arrow[dr, "g_{23}"] & \\ \mathrm{triv}_{(A_1,B_1)} \arrow[ur, "g_{12}"] \arrow[rr, "g_{13}"'] & \arrow[u, phantom, "\Downarrow f"] & \mathrm{triv}_{(A_3,B_3)} \end{tikzcd}$$

are in bijection with

$$f \in \Omega^0(Y, H)$$

satisfying

$$g_{23}\, g_{12}\, t(f) = g_{13}$$

and

$$a_{12} + g_{12}^{-1}(a_{23}) + fA_1(f^{-1}) = +\mathrm{Ad}_f g_{13} + f^*\bar{\theta}$$

An undertanding of n-curvature generalizes this as follows:

Proposition

When we generalize from local 2-functors with values in $\Sigma G_{(2)}$ to local 3-functors with values in $\Sigma \mathrm{INN}_0(G_{(2)})$ the fake-flatness condition is lifted and the differential 2-coycle we obtain is the one given by Breen and Messing.

Deligne cohomology

nth Deligne cohomology is (equivalence classes) of descent data for locally ($i = \mathrm{Id}_{\Sigma^{n-1}U(1)}$)-trivializable n-transport.

Deligne cohomology is obtained as a special case from nonabelian differential cocycles by restricting the structure *n*-group to

$$G_{(n)} = \Sigma^{n-1} U(1) .$$

Bundle gerbes

Lie bundle gerbes with connection are precisely descent data for $(\Sigma^2 U(1) \stackrel{i}{\hookrightarrow} \Sigma 1d\mathrm{Vect})$-trivializable 2-transport: the curving 2-form is the local 2-functor, while the transition bundle is the pseudonatural gluing transformation.

We now explain

Proposition

Line bundle gerbes (Hitchin, Chatterjee, Murray) with connection are canonically isomorphic to descent data objects for $(\Sigma^2 U(1)) \hookrightarrow \Sigma 1d\mathrm{Vect})$-trivializable 2-transport.
Principal bundle gerbes are similarly obtained from $(\Sigma^2 U(1)) \hookrightarrow \Sigma U(1)\mathrm{Tor})$ 2-transport.

Definition [Murray, generalizing Hitchin]

A **line bundle gerbe** over X is

- a surjective submersion $\pi : Y \to X$
- a line bundle $L \to Y^{[2]}$
- a line bundle isomorphism $\mu : \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L$ which is associative in the obvious sense.

Remark

When we assume $Y = \sqcup_i U_i$ to be a good cover by open contractible sets the above definition restricts to that originally given by Nigel Hitchin.

Definition

A **connective structure** on a bundle gerbe (also known as **connection and curving** on a bundle gerbe) is

- a connection ∇ on L
- a 2-form $\omega \in \Omega^2(Y)$ on Y

such that on

$$\pi_2^*\omega - \pi_1^*\omega = F_\nabla \,.$$

Idea of the canonical isomorphism with transport descent

- The regular epimorphism $\pi : Y \to X$ is the same in both cases.
- The 2-form ω is the local trival 2-functor $\mathrm{triv} : \Pi_2(Y) \to \Sigma^2 U(1)$.
- The line bundle $L \to Y^{[2]}$ with connection ∇ is the component transport 1-functor of the transition $g : \pi_1^* \mathrm{triv} \to \pi_2^* \mathrm{triv}$.
- The condition $\pi_2^*\omega - \pi_1^*\omega = F_\nabla$ is pseudo-naturality of g.

The pseudonatural transformation $g : \pi_1^*\mathrm{triv} \to \pi_2^*\mathrm{triv}$ reads, in components, for:

$$\begin{tikzcd} y \arrow[rr, bend left, "\gamma"] \arrow[rr, bend right, "\gamma'"'] \arrow[rr, phantom, "\Downarrow \Sigma"] & & y' \end{tikzcd}$$

any surface in $Y^{[2]}$:

$$\begin{tikzcd} \bullet \arrow[r, "\mathbb{C}"] \arrow[r, bend left=60, "\mathbb{C}"] \arrow[r, phantom, bend left=30, "\mathrm{triv}(\pi_1(\Sigma))"] \arrow[d, "L_y"'] & \bullet \arrow[d, "L_{y'}"] \arrow[dl, phantom, "\swarrow g_\nabla(\gamma')"] \\ \bullet \arrow[r, "\mathbb{C}"'] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, "\mathbb{C}"] \arrow[d, "L_y"'] & \bullet \arrow[d, "L_{y'}"] \arrow[dl, phantom, "\swarrow g_\nabla(\gamma)"] \\ \bullet \arrow[r, "\mathbb{C}"] \arrow[r, bend right=60, "\mathbb{C}"'] \arrow[r, phantom, bend right=30, "\mathrm{triv}(\pi_2(\Sigma))"] & \bullet \end{tikzcd} \quad \Leftrightarrow \quad \pi_2^*\omega - \pi_1^*\omega = F_\nabla$$

Line bundle gerbes as categorified transition functions

The ana-2-functor obtained from this descent data is, when we forget the connection

$$\begin{tikzcd} Y^{[2]} \arrow[r, "g"] \arrow[d] & \Sigma 1d\mathrm{Vect} \\ X \end{tikzcd} .$$

Recall that an ordinary transition function for a G-bundle is an ana-1-functor

$$\begin{tikzcd} Y^{[2]} \arrow[r, "g"] \arrow[d] & \Sigma G \\ X \end{tikzcd} .$$

The canonical isomorphism between descent data for $(\Sigma^2 U(1)) \hookrightarrow)$-2-transport and bundle gerbes with connection extends to the entire 2-category of bundle gerbes:

Proposition

"Stable isomorphisms" of bundle gerbes (with connection) are canonically identitfied with morphisms of $(\Sigma^2 U(1)) \hookrightarrow)$-2-transport.
"Gerbe modules" for bundle gerbes are canonically identified with sections of the corresponding 2-transport after embedding $1d\mathrm{Vect} \hookrightarrow \Sigma\mathrm{Vect}$.

Definition [Murray]

A "stable morphism" $E : (Y, L, \mu) \to (Y, L', \mu')$ of bundle gerbes is a line bundle $E \to Y$ and a morphism

$$e : L \otimes \pi_1^* E \to \pi_2^* E \otimes L'$$

which is compatible with the connection and with μ in the obvious way.

A "stable morphism" $f : (Y, L, \mu) \to (Y, L', \mu')$ of bundle gerbes is precisely a transformation of the ana-2-functors encoding the descent data of the corresponding 2-transport:

$$\begin{tikzcd} Y^{[2]} \arrow[rr, bend left, "g"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\Downarrow (E,e)"] & & \Sigma 1d\mathrm{Vect} \end{tikzcd} .$$

Definition [Murray]

A "gerbe module" E is a vector bundle $E \to Y$ and a morphism

$$e : L \otimes \pi_1^* E \to \pi_2^* E$$

which is compatible with the connection and with μ in the obvious way.

A "gerbe module" is precisely a transformation from the trivial ana-2-functor to the ana-2-functors encoding the descent data of the corresponding 2-transport, after pushing forward along the inclusion $\Sigma 1d\mathrm{Vect} \hookrightarrow \Sigma\mathrm{Vect}$

$$\begin{tikzcd} Y^{[2]} \arrow[r, bend left, "g"] \arrow[rr, bend right, "0"'] & \Sigma 1d\mathrm{Vect} \arrow[r, hook] \arrow[d, phantom, "\Downarrow (E,e)"] & \Sigma\mathrm{Vect} \end{tikzcd} \,.$$

Interpretation in terms of line 2-bundles

We can understand this entire discussion as being about line 2-bundles, which are associated to $\Sigma U(1)$-2-transport by the canonical 2-representation

$$\rho : \Sigma^2 U(1) \to \Sigma \mathrm{Bim} \to 2\mathrm{Vect}\,.$$

Line bundle gerbes are transition data for line 2-bundles whose fibers are 2-vector spaces equivalent to

$$\mathrm{Mod}_{\mathbb{C}} \simeq \mathrm{Mod}_{K(H)}\,,$$

where $K(H)$ denotes the algebra of finite-rank operators on a Hilbert space.
The corresponding total spaces are a well known equivalent description for bundle gerbes: bundles of compact operators associated canonically to $PU(H)$-bundles.

Line bundle on loop space from 2-transport on base space

It is well known that a line bundle gerbe on X gives rise to a line bundle on loops in X. The curving of the gerbe translates into the connection of the bundle.
This transgression corresponds in terms of 2-transport functors simply to the application of

$$\mathrm{Hom}(\Sigma\mathbb{Z}, \cdot)$$

to the 2-transport and its local trivialization
More on transgression is here.

Paths on loop space

Definition

For $\mathcal{P}_2(X)$ a 2-path 2-groupoid on X, the corresponding 1-path groupoid on loops in X is

$$\mathcal{P}_1(LX) := \pi_1(\mathrm{Hom}(\Sigma\mathbb{Z}, \mathcal{P}_2(X)))\,.$$

Here the fact that we take π_1 (i.e. that we divide out 2-isomorphisms) means that we ignore the basepoint trajectories of the loops.

Observation

If the surjective submersion $\pi : Y \to X$ has connected fibers (i.e. is a smooth connected bundle on each connected component of X) then

$$\mathrm{Hom}(\Sigma\mathbb{Z}, \mathcal{P}_2(Y)) \xrightarrow{\mathrm{Hom}(\Sigma\mathbb{Z}),\pi_*} \mathrm{Hom}(\Sigma\mathbb{Z}, \mathcal{P}_2(X))$$

is epi.

In other words: every loop in X comes from a loop in Y by projection.
Notice that this is not saying that every loop in X has a *lift* to Y: the projection down to X is in general not injective on the points of the loop.

It follows that we can apply $\mathrm{Hom}(\Sigma\mathbb{Z}, \cdot)$ to everything in sight. This sends any 2-transport

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma 1d\mathrm{Vect}$$

to a loop 2-transport

$$\mathrm{Hom}(\Sigma\mathbb{Z}, \mathrm{tra}) : \mathrm{Hom}(\Sigma\mathbb{Z}, \mathcal{P}_2(X)) \to \mathrm{Hom}(\Sigma\mathbb{Z}, \Sigma 1d\mathrm{Vect})\,.$$

This naturally descends down to $\mathcal{P}_1(LX)$ by means of the pushout

$$\begin{tikzcd} \mathrm{Hom}(\Sigma\mathbb{Z}, \mathcal{P}_2(X)) \arrow[r, "\mathrm{Hom}(\Sigma\mathbb{Z},\mathrm{tra})"] \arrow[d] & \mathrm{Hom}(\Sigma\mathbb{Z}, \Sigma 1d\mathrm{Vect}) \\ \mathcal{P}_1(LX) & \end{tikzcd}$$

Noticing that

$$\pi_1(\mathrm{Hom}(\Sigma\mathbb{Z}, \Sigma 1d\mathrm{Vect})) \simeq 1d\mathrm{Vect}$$

the result of this pushout is a line bundle with connection on loop space.

$$\mathrm{tra}_{\Sigma Z} : \mathcal{P}_1(X) \to 1d\mathrm{Vect}\,.$$

Its local trivializability follows by applying $\mathrm{Hom}(\Sigma\mathbb{Z}, \cdot)$ to the entire local trivialization diagram of tra.

The local trivialization on loop space is the image of the local trivialization on base space under $\mathrm{Hom}(\Sigma\mathbb{Z}, \cdot)$.

$$
\begin{tikzcd}
\mathrm{Hom}(\Sigma\mathbb{Z}, P_n(Y)) \arrow[r, two heads, "\mathrm{Hom}(\Sigma\mathbb{Z},\pi)"] \arrow[d, "\mathrm{Hom}(\Sigma\mathbb{Z},\mathrm{triv})"'] & \mathrm{Hom}(\Sigma\mathbb{Z}, P_n(X)) \arrow[d, "\mathrm{Hom}(\Sigma\mathbb{Z},\mathrm{tra})"] \arrow[dl, Rightarrow, "\mathrm{Hom}(\Sigma\mathbb{Z},t)"', "\simeq"] \\
\mathrm{Hom}(\Sigma\mathbb{Z}, \Sigma\Sigma U(1)) \arrow[r, "\mathrm{Hom}(\Sigma\mathbb{Z},i)"'] & \mathrm{Hom}(\Sigma\mathbb{Z}, T)
\end{tikzcd}
$$

Nonabelian bundle gerbes

The 2-category $\Sigma U(1)\mathrm{Tor}$ underlying abelian bundle gerbes may be thought of as $\Sigma U(1)\mathrm{BiTor}$. As such it generalizes to any group H. Descent data for 2-transport with local

$$\Sigma H\mathrm{BiTor}$$

structure is canonically isomorphic to nonabelian bundle gerbes.

Nonabelian bundle gerbes with connection

Proposition

The descent catgeory for $(\Sigma\mathrm{AUT}(H) \stackrel{i}{\hookrightarrow} \Sigma H\mathrm{Bitor})$ 2-transport is canonically isomorphic to fake flat nonabelian bundle gerbes with connection.

These nonabelian bundle gerbes have been defined and studied by [AschieriJurco].

The delicate nature of the connection 1-form on the transition bi-bundle becomes transparent when we realize that this is the differential of the transition pseudonatural transformation

$$g : \pi_1^*\mathrm{triv} \to \pi_2^*\mathrm{triv}\,,$$

whose component map looks like

$$g : \left(x \xrightarrow{\gamma} y \right) \mapsto \begin{tikzcd} \bullet \arrow[r, "H_{\mathrm{tra}(\gamma)}"] \arrow[d, "L_x"'] & \bullet \arrow[d, "L_y"] \arrow[dl, Rightarrow, "g(\gamma)"] \\ \bullet \arrow[r, "H_{\mathrm{tra}(\gamma')}"'] & \bullet \end{tikzcd} .$$

$$g : \left(x \xrightarrow{\gamma} y \right) \mapsto \begin{tikzcd} \bullet \arrow[r, "H_{\pi_1^*\mathrm{triv}(\gamma)}"] \arrow[d, "L_x"'] & \bullet \arrow[d, "L_y"] \arrow[dl, Rightarrow, "g(\gamma)"] \\ \bullet \arrow[r, "H_{\pi_2^*\mathrm{triv}(\gamma)}"'] & \bullet \end{tikzcd} .$$

Here

- L_x is the H-bibundle fiber over x;
- H_g is the H-bitorsor which is, as an object, H itself, with the obvious left H-action and with the right H-action twisted by $\alpha(g)$;
- $g(\gamma)$ acts like the twisted parallel transport on the bibundle: instead of being a morphism $L_x \to L_y$ it is a twisted morphism $H_{\pi_1^*\mathrm{triv}(\gamma)} \otimes_H L_x \to L_Y \otimes_H H_{\pi_2^*\mathrm{triv}(\gamma)}$.

Rank one 2-vector bundles

A rank one 2-vector bundle is a 2-vector bundle associated by the canonical 2-representation of $\Sigma U(1)$ on bimodules.

$$\rho : \Sigma(\Sigma U(1)) \to \mathrm{Bim} \subset 2\mathrm{Vect}$$

$$\rho : \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow \cdot c"] & \mathbb{C} \end{tikzcd}$$

For simplicity, restrict attention to *finite*-dimensional module for the time being. Then a ρ-trivializable 2-transport

$$\mathcal{P}_2(X) \to \mathrm{Bim}$$

necessarily has fibers equivalent in Bim to the ground field $\mathbb{C}$. But equivalence in Bim is Morita equivalence.
The ground field is Morita equivalent to the full endomorphisms algebras on a (finite dimensional, by assumption) vector space.

The Morita equivalence $\mathbb{C} \simeq \mathrm{End}(V)$ is induced by the weakly invertible bimodules

$$\mathbb{C} \xrightarrow{V} \mathrm{End}(V)$$

and

$$\mathrm{End}(V) \xrightarrow{V^*} \mathbb{C} \,.$$

Which come with isomorphisms

$$\begin{tikzcd} \mathbb{C} \arrow[r, "V"] \arrow[rr, bend right, ""{name=B}] & \mathrm{End}(V) \arrow[r, "V^*"] \arrow[d, phantom, "\Downarrow\simeq"] & \mathbb{C} \\ & \mathbb{C} & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{End}(V) \arrow[r, "V^*"] \arrow[rr, bend right] & \mathbb{C} \arrow[r, "V"] \arrow[d, phantom, "\Downarrow\simeq"] & \mathrm{End}(V) \\ & \mathrm{End}(V) & \end{tikzcd}$$

Proposition

The descent data for rank one 2-vector transport is canonically ismorphic to line bundle gerbes with connection. The trivialization itself is the corresponding gerbe module.

In order to proceed, and to understand the meaning and relevance of fake flatness, we first need a better understanding of higher curvature.

Curvature

$$
\begin{tikzcd}
\Pi_0(X) \arrow[r, hook] \arrow[d, "g"'] & \Pi_n(X) \arrow[d, "(\mathrm{tra},\mathrm{curv})"] \arrow[dl, dotted, "\mathrm{tra}_{\mathrm{flat}}"'] \\
\Sigma G_{(n)} \arrow[r, hook] & \Sigma \mathrm{INN}(G_{(n)})
\end{tikzcd}
$$

Idea

n-Curvature is the obstruction to flat n-transport.

n-Curvature is controlled by a special case of Obstruction theory, namely the obstruction to lifting through

$$G_{(n)} \xrightarrow{\mathrm{Id}} G_{(n)} \longrightarrow 1$$

under certain constraints.

n-Curvature as $(n+1)$-Transport

Fact

n-Curvature is itself an $(n+1)$-Transport.

The generalized Bianchi identity

n	$(n+1)$	$(n+2)$
n-transport	n-curvature of n-transport	$(n+1)$-curvature of n-curvature
tra	$\mathrm{curv}_{\mathrm{tra}}$	$\mathrm{curv}_{\mathrm{curv}_{\mathrm{tra}}}$
arbitrary	flat	trivial

Curvature is the obstruction to lifting a trivial transport to a flat transport.

A $G_{(n)}$-n-bundle *without* connection on X is a transport n-functor

$$P : \Pi_0(X) \to \Sigma G_{(n)}$$

equipped with a smooth local $G_{(n)}$-trivialization.

A $G_{(n)}$-n-bundle on X with *flat* connection is a transport n-functor

$$\mathrm{tra} : \Pi_n(X) \to \Sigma G_{(n)} \,.$$

Given a $G_{(n)}$-bundle with connection, we may ask if we can extend it to a $G_{(n)}$-bundle with flat connection

$$\begin{tikzcd} \Pi_0(X) \arrow[r, hook] \arrow[d] & \Pi_n(X) \arrow[dl, "\mathrm{tra}_{\mathrm{flat}}"] \\ \Sigma G_{(n)} \end{tikzcd} .$$

In general we cannot. The obstruction is given by a $\mathrm{wcoker}(\mathrm{Id}_{G_{(n)}})$-transport.
To see this more clearly, we need a little bit of local data:

A possibly nontrivial $G_{(n)}$-bundle *without* connection on X is a surjective submersion $F \to Y \to X$ with connected fibers, together with a flat $\Sigma G_{(n)}$-transport on the fibers

$$P : \Pi_n(F) \to \Sigma G_{(n)} \,.$$

A flat $G_{(n)}$-connection, on this, is an extension of this to a functor on all of Y:

$$\mathrm{tra}_{\mathrm{flat}} : \Pi_n(Y) \to \Sigma G_{(n)} \,.$$

In general, this does not exist. What always exists, though, is the completely trivial bundle with connection

$$\mathrm{tra}_0 : \Pi_n(Y) \to \{\bullet\}\,,$$

i.e. the principal bundle for the trivial structure group.

Hence the question that we are asking when asking for curvature is:

Can we lift the connection for the trivial group through the exact sequence

$$G_{(n)} \xrightarrow{\mathrm{Id}} G_{(n)} \to \{\bullet\}$$

?

Curvature is a very degenerate case of general obstruction theory: we are asking for obstructions to extending the *trivial* structure group.

More precisely, we want to find a lift of tra which does restrict to the fixed functor $P : \Pi_n(F) \to \Sigma G_{(n)}$ on the fibers of the surjective submersion, meaning we want to lift to

$$\begin{tikzcd}
\Pi_n(F) \arrow[r, hook] \arrow[d] & \Pi_n(Y) \arrow[d, "\mathrm{tra}"] \\
\Sigma G_{(n)} \arrow[r, "\mathrm{Id}"] & \Sigma G_{(n)}
\end{tikzcd} .$$

In general this will not work. But we have obstruction theory as above to figure out what the obstructing $(n+1)$-bundle with connection will be: it will be an $(n+1)$-transport with values in

$$\mathrm{wcoker}(i)$$

obtained by first lifting the $\{\bullet\}$-transport tra_0 to an equivalent $\mathrm{wcoker}(\mathrm{Id}_{G_{(n)}})$-transport and then checking which mistake in $\mathrm{wcoker}(i)$ we make thereby:

$$
\begin{tikzcd}
\Pi_{n+1}(F) \arrow[r, hook, "i"] \arrow[d, "g"'] & \Pi_{n+1}(Y) \arrow[r, two heads, "\pi"] \arrow[d, "(\mathrm{tra},\mathrm{curv})"] \arrow[dl, Rightarrow, "\simeq"] & \Pi_{n+1}(X) \arrow[d, "K"] \arrow[dl, Rightarrow, "\simeq"] \\
\Sigma G_{(n)} \arrow[r, hook] & \Sigma \mathrm{INN}(G_{(n)}) \arrow[r, two heads] & T
\end{tikzcd}
$$

nonabelian cocycle / transition function/ descent data	differential cocycle / integrated Ehresmann n-connection	classifying map

Definition

To each parallel n-transport

$$\mathrm{tra} : \mathcal{P}_n(X) \to \Sigma G_{(n)}$$

we may canonically associate a curvature $(n+1)$-transport

$$\mathrm{curv} : \Pi_{n+1}(X) \to \Sigma \mathrm{INN}(G_{(n)})\,.$$

Flatness and Bianchi identity

Definition

We call an n-transport $\mathcal{P}_n(X) \to T$ flat when it factors through homotopy classes of n-paths

$$\begin{tikzcd}
\mathcal{P}_n(X) \arrow[rr, "\mathrm{tra}"] \arrow[dr] & & T \\
& \Pi_n(X) \arrow[ur] &
\end{tikzcd} .$$

Equivalently, an n-transport is flat if its curvature $(n+1)$-transport is trivial at top level.

Flatness and Bianchi identity

The following fact now is a tautology. But it is a useful tautology when translated to differential form data in concrete examples.

Fact (generalized Bianchi identity)

The curvature $(n+1)$-transport of any n-transport is itself always a flat $(n+1)$-transport.

There is a refinement of the definition of parallel n-transport which does incorporate non-vanishing fake-curvature.
Instead of regarding the n-transport

$$\mathrm{tra} : \mathcal{P}_n(X) \to \Sigma G_{(n)}$$

itself, we consider its $(n+1)$-curvature transport

$$\mathrm{curv} : \Pi_{n+1}(X) \to \Sigma \mathrm{INN}(G_{(n)}) \,.$$

This is itself a parallel $(n+1)$-transport, with the special property that its transition data factors through

$$\Sigma G_{(n)} \hookrightarrow \Sigma \mathrm{INN}(G_{(n)})$$

This property can be encoded by refining the diagram for local trivialization

Definition of non-fake-flat n-transport

Definition

We say that two composable strict n-functors

$$K \hookrightarrow G \twoheadrightarrow B$$

of strict n-groupoids form a short sequence, if the image of the first is in the preimage under the second of all identity morphisms in B.

Definition of non-fake-flat n-transport

For $G_{(n)}$ a strict Lie n-group and

$$\Sigma G_{(n)} \hookrightarrow \Sigma \mathrm{INN}(G_{(n)}) \twoheadrightarrow T$$

a short sequence of strict Lie $(n+1)$-groupoids, an $(n+1)$-curvature on a space X is an $(n+1)$-functor

$$K : \Pi_{(n+1)}(X) \to T$$

which fits into a diagram...

Definition of non-fake-flat n-transport

$$
\begin{tikzcd}
\Pi_n(F) \arrow[r, hook, "i"] \arrow[d, "g"'] & \Pi_n(Y) \arrow[r, two heads, "\pi"] \arrow[d, "(\mathrm{tra},\mathrm{curv})"] \arrow[dl, Rightarrow, "\simeq"] & \Pi_n(X) \arrow[d, "K"] \arrow[dl, Rightarrow, "\simeq"] \\
\Sigma G_{(n)} \arrow[r, hook] & \Sigma \mathrm{INN}(G_{(n)}) \arrow[r, two heads] & T
\end{tikzcd}
$$

nonabelian cocycle / transition function/ descent data	differential cocycle / integrated Ehresmann n-connection	classifying map

Definition of non-fake-flat n-transport

. . . where

$$\Pi_{(n+1)}(F) \hookrightarrow \Pi_{(n+1)}(Y) \twoheadrightarrow \Pi_{(n+1)}(X)$$

is a short sequence and where. . .

Definition of non-fake-flat n-transport

...the transformations respect the sequence property in that

$$
\begin{tikzcd}
\Pi_n(F) \arrow[r, hook, "i"] & \Pi_n(Y) \arrow[r, two heads, "\pi"] \arrow[d, "(\mathrm{tra},\mathrm{curv})"] & \Pi_n(X) \arrow[d, "K"] \arrow[dl, Rightarrow, "\simeq"] \\
 & \Sigma\mathrm{INN}(G_{(n)}) \arrow[r, two heads] & T
\end{tikzcd}
$$

is the identity transformation and ...

Definition of non-fake-flat n-transport

$$\begin{tikzcd}
\Pi_n(F) \arrow[r, hook, "i"] \arrow[d, "g"'] & \Pi_n(Y) \arrow[d, "(\mathrm{tra},\mathrm{curv})"] \arrow[dl, Rightarrow, "\simeq"] & \\
\Sigma G_{(n)} \arrow[r, hook] & \Sigma \mathrm{INN}(G_{(n)}) \arrow[r, two heads] & T
\end{tikzcd}$$

is the identity transformation.

Definition of non-fake-flat n-transport

Example

Let $\pi : P \to X$ be a principal $U(1)$-bundle with Ehresmann connection 1-form $A \in \Omega^1(P)$. Then

$$
\begin{tikzcd}
\Pi_2(U(1)) \arrow[r, hook, "i"] \arrow[d, "g"'] & \Pi_2(P) \arrow[r, two heads, "\pi"] \arrow[d, "(\mathrm{tra}_A, \mathrm{curv}_{F_A})"] \arrow[dl, Rightarrow, "="] & \Pi_2(X) \arrow[d, "\exp(\int F_A)"] \arrow[dl, Rightarrow, "="] \\
\Sigma U(1) \arrow[r, hook] & \Sigma \mathrm{INN}(U(1)) \arrow[r, two heads] & \Sigma\Sigma U(1)
\end{tikzcd}
$$

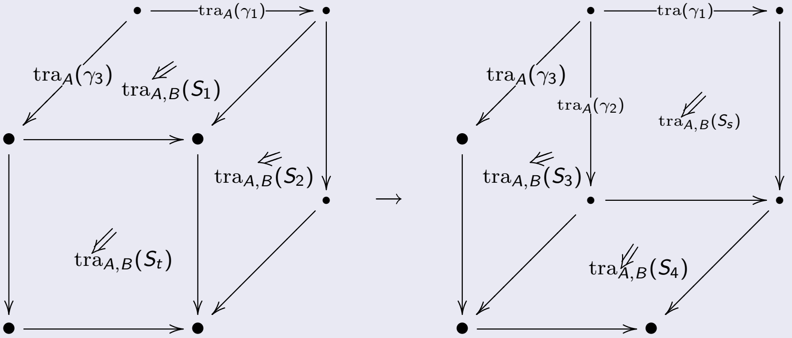
is the corresponding 2-curvature 2-functor.

There is something deeper going on here. For more hints see the discussion in
$G_{(n)}$-bundles with connection.

Proposition

Descent data for smoothly locally trivializable $\mathrm{INN}_0(\mathrm{AUT}(H))$ 3-transport whose transitions factor through $\mathrm{AUT}(H) \hookrightarrow \mathrm{INN}_0(\mathrm{AUT}(H))$ is equivalent to the Breen-Messing data.

The 2- and 3-curvature is now unrestricted



Associated transport

Requiring that the local structure

$$\Sigma G_{(n)} \to T$$

of a transport functor is an n-representation of $G_{(n)}$

$$\rho : \Sigma G_{(n)} \to n\mathrm{Vect}$$

leads to the notion of associated n-transport.

Associated fake-flat n-transport

Passing from principal to associated n-transport is merely a matter of replacing the principal local structure

$$i : \Sigma G_{(n)} \hookrightarrow G_{(n)}\mathrm{Tor}$$

by the desired n-representation

$$\rho : \Sigma G_{(n)} \to n\mathrm{Vect}\,,$$

i.e. by demanding local trivialization of n-transport of the form

$$\begin{tikzcd}
\mathcal{P}_n(Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{triv}"'] & \mathcal{P}_n(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\simeq"] \\
\Sigma G_{(n)} \arrow[r, "\rho"] & n\mathrm{Vect}
\end{tikzcd}$$

n-Vector spaces

Here $n\mathrm{Vect}$ indicates an n-category of n-vector spaces. Usually one considers

Definition: n-vector space

We address the monoid of complex numbers

$$\mathbb{C} := 0\mathrm{Vect}$$

as the 0-category of 0-vector spaces. Then the n-category of n-vector spaces is recursively defined as

$$n\mathrm{Vect} := (n-1)\mathrm{Vect} - \mathrm{Mod}\,.$$

As always, unwrapping this definition beyond low n requires first choosing a notion of n-categories and interpreting the notion of module over an $(n+1)$-monoid suitably.

Examples for n-Vector spaces

Example: Kapranov-Voevodsky

The category
$$\mathrm{Vect}^n \in 2\mathrm{Vect}$$
is the n-dimensional Kaparanov-Voevodsky 2-vector space. These KV 2-vector spaces form a 2-category
$$\mathrm{KV2Vect} \hookrightarrow 2\mathrm{Vect}\,.$$

Remark

This inclusion factors through the larger 2-category of algebras and bimodules
$$\mathrm{KV2Vect} \hookrightarrow \mathrm{Bimod} \to 2\mathrm{Vect}\,.$$

n-representations of n-groups

Definition

Given an n-group $G_{(n)}$ and notion of n-vector space, a linear n-representation of $G_{(n)}$ is an n-functor

$$\rho : \Sigma G_{(n)} \to n\mathrm{Vect}$$

Example

An ordinary linear representation of a (1-)group G is indeed a functor

$$\rho : \Sigma G \to \mathrm{Vect}$$

n-representations of n-groups

Example (canonical 2-representation)

For $G_{(2)}$ a strict 2-group coming from the crossed module $(H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H))$ and for

$$\rho_H : \Sigma G \to \mathrm{Vect}$$

an ordinary representation of H- we obtain an induced 2-representation

$$\rho : \Sigma G_{(2)} \to \mathrm{Bimod} \to 2\mathrm{Vect}$$

by sending

$$\rho : \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} \langle \rho_H \rangle \arrow[r, bend left, "\langle \rho_H \rangle_{\alpha(g)}"] \arrow[r, bend right, "\langle \rho_H \rangle_{\alpha(g)}"'] \arrow[r, phantom, "\Downarrow \cdot \rho(h)"] & \langle \rho_H \rangle \end{tikzcd}$$

n-representations of n-groups

Example (canonical 2-representation)

The canonical 2-representation for

$$G_{(2)} = \Sigma U(1) = (U(1) \to 1)$$

is the obvious one

$$\rho : \Sigma G_{(2)} \to \Sigma \mathrm{Vect} \to 2\mathrm{Vect}$$

which acts as

$$\rho : \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow \cdot c"] & \mathbb{C} \end{tikzcd}$$

n-representations of n-groups

Example (canonical 2-representation)

The canonical 2-representation for the strict version of the String 2-group [Baez-Crans-S.-Stevenson]

$$G_{(2)} = \mathrm{String}_k(G) = (\hat{\Omega}_k G \to PG)$$

would lead to a representation of the von Neumann algebra A generated by a positive energy rep of the Kac-Moody group $\hat{\Omega}_k(G)$.

$$\rho : \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[r, bend left, "A_{\alpha(g)}"] \arrow[r, bend right, "A_{\alpha(g>')}"'] \arrow[r, phantom, "\Downarrow \cdot\rho(h)"] & A \end{tikzcd} .$$

Associated String 2-transport

This is technically subtle due to issues with von Neumann bimodules ("Connes fusion" etc.). But seems to go through. The associated String 2-transport induced this way has an appearance very similar to the definitions proposed by [StolzTeichner].

Examples of n-vector transport

Proposition [S.-Waldorf]

The category of vector bundles with connection on X is equivalent to that of 1-transport functors on X with local structure being the canonical representation

$$\rho : \sqcup_n U(n) \to \mathrm{Vect}\,.$$

Examples of *n*-vector transport

Proposition [S.-Waldorf]

The 2-category of descent data for line 2-bundles with connection on X, coming from the standard 2-representation

$$\rho : \Sigma\Sigma U(1) \to \Sigma\mathrm{Vect} \hookrightarrow 2\mathrm{Vect}$$

is *canonically isomorphic* to that of line bundle gerbes with connection.

Remark

Here "canonically isomorphic" means: we don't just have an equivalence. Instead, feeding ρ into the machinery of n-transport and turning the crank, the very definition of line bundle gerbes with connection drops out.

Non-fake-flat associated n-transport

As we pass to non fake flat n-transport, we need to shift the n-representation of $G_{(n)}$ to a corresponding $(n+1)$-representation of $\mathrm{INN}(G_{(n)})$

Recall that the local structure is now encoded by a diagram

$$\begin{tikzcd}
\Pi_n(F) \arrow[r, hook, "i"] \arrow[d, "g"'] & \Pi_n(Y) \arrow[r, two heads, "\pi"] \arrow[dl, Rightarrow, "\simeq"] \arrow[d, "(\mathrm{tra},\mathrm{curv})"] & \Pi_n(X) \arrow[dl, Rightarrow, "\simeq"] \arrow[d, "K"] \\
\Sigma G_{(n)} \arrow[r, hook] & \Sigma\mathrm{INN}(G_{(n)}) \arrow[r, two heads] & T
\end{tikzcd}$$

Non-fake-flat associated n-transport

The n-representation $\rho : \Sigma G_{(n)} \to S$ can be attached at the bottom left corner of this diagram

$$\begin{tikzcd}
\Pi_n(F) \arrow[r, hook, "i"] \arrow[d, "g"'] & \Pi_n(Y) \arrow[r, two heads, "\pi"] \arrow[d, "(\mathrm{tra},\mathrm{curv})"] \arrow[dl, Rightarrow, "\simeq"] & \Pi_n(X) \arrow[d, "K"] \arrow[dl, Rightarrow, "\simeq"] \\
\Sigma G_{(n)} \arrow[r, hook] \arrow[d, "\rho"] & \Sigma \mathrm{INN}(G_{(n)}) \arrow[r, two heads] & T \\
S & &
\end{tikzcd}$$

Non-fake-flat associated n-transport

And then pushed-forward along the left bottom edge

$$\begin{tikzcd}
\Pi_n(F) \arrow[r, hook, "i"] \arrow[d, "g"'] & \Pi_n(Y) \arrow[r, two heads, "\pi"] \arrow[d, "(\mathrm{tra},\mathrm{curv})"] \arrow[dl, Rightarrow, "\simeq"] & \Pi_n(X) \arrow[d, "K"] \arrow[dl, Rightarrow, "\simeq"] \\
\Sigma G_{(n)} \arrow[r, hook] \arrow[d, "\rho"] & \Sigma \mathrm{INN}(G_{(n)}) \arrow[r, two heads] \arrow[d] & T \\
S \arrow[r] & \tilde{S} &
\end{tikzcd}$$

The induced $\mathrm{INN}(G)$-representation on the action groupouid

For $n = 1$, let V be the space that $\rho : \Sigma G \to S$ represents on. Then ρ factors as

$$\rho : \Sigma G \to \Sigma \mathrm{Aut}(V) \to S$$

and we may, for simplicity, consider the strict pushout of

$$\begin{tikzcd} \Sigma G \arrow[r, hook] \arrow[d, "\rho"'] & \Sigma \mathrm{INN}(G) \\ \Sigma \mathrm{Aut}(V) & \end{tikzcd}$$

in $2\mathrm{Cat}$.

The induced $\mathrm{INN}(G)$-rep on the action groupoid

Proposition

The strict pushout in $2\mathrm{Cat}$ of

$$\begin{tikzcd} \Sigma G \arrow[r, hook] \arrow[d, "\rho"'] & \Sigma \mathrm{INN}(G) \\ \Sigma \mathrm{Aut}(V) & \end{tikzcd}$$

is

$$\begin{tikzcd} \Sigma G \arrow[r, hook] \arrow[d, "\rho"'] & \Sigma \mathrm{INN}(G) \arrow[d] \\ \Sigma \mathrm{Aut}(V) \arrow[r] & \Sigma \mathrm{Aut}(V//G) \end{tikzcd},$$

where $V//G$ is the action groupoid of the action of G on V.

The action groupoid

Definition

The action groupoid $V//G$ has as objects the elements of V, and has a morphism for each pair $(v, \rho(g))$ for $v \in V$ and $g \in G$:

$$V//G := \left\{ v \stackrel{\rho(g)}{\to} \rho(g)(v) \mid v \in V, g \in G \right\}$$

.

The action groupoid

The action groupoid is still equpped with a canonical G-action

$$V//G \stackrel{\rho(g)}{\to} V//G$$

by endofunctors. This is such that any two such endofunctors are connected by a *unique* natural isomorphism

$$\begin{tikzcd} V//G \arrow[rr, bend left, "\rho(g)"] \arrow[rr, bend right, "\rho(g')"'] \arrow[rr, phantom, "\Downarrow \simeq"] & & V//G \end{tikzcd}.$$

Its component map is

$$v \mapsto (\ \rho(g)(v) \xrightarrow{\rho(g'g^{-1})} \rho(g')(v)\ ).$$

The induced $\mathrm{INN}(G)$-rep on the action groupouid

This makes manifest how with G represented on V,

$$\bullet \xrightarrow{g} \bullet \quad \mapsto \quad V \xrightarrow{\rho(g)} V$$

we have $\mathrm{INN}(G) = G//G$ represented on $V//G$:

$$\begin{tikzcd} \bullet \arrow[rr, bend left, "g"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\Downarrow\simeq"] & & \bullet \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} V//G \arrow[rr, bend left, "\rho(g)"] \arrow[rr, bend right, "\rho(g')"'] \arrow[rr, phantom, "\Downarrow\simeq"] & & V//G \end{tikzcd}$$

The induced $\mathrm{INN}(G)$-rep on the action groupoid

Notice that we can interpret the functor

$$\begin{tikzcd} V//G \arrow[rr, bend left, "\rho(g)"] & & V//G \end{tikzcd}$$

as a morphism in spans of groupoids over ΣG

$$\begin{tikzcd} & V//G \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\rho(g)"] & \\ V//G \arrow[dr] & \arrow[l, phantom, "\Swarrow \simeq"] & V//G \arrow[dl] \\ & \Sigma G & \end{tikzcd} .$$

Non-fake-flat associated 1-transport

So for $n = 1$ the picture of non-fake-flat associated n-transport is

$$\begin{tikzcd}
\Pi_n(F) \arrow[r, hook, "i"] \arrow[d, "g"'] & \Pi_n(Y) \arrow[r, two heads, "\pi"] \arrow[dl, Rightarrow, "\simeq"] \arrow[d, "(\mathrm{tra},\mathrm{curv})"] & \Pi_n(X) \arrow[dl, Rightarrow, "\simeq"] \arrow[d, "K"] \\
\Sigma G_{(n)} \arrow[r, hook] \arrow[d, "\rho"] & \Sigma \mathrm{INN}(G_{(n)}) \arrow[r, two heads] \arrow[d] & T \\
\Sigma \mathrm{Aut}(V) \arrow[r] & \Sigma \mathrm{Aut}(V//G) &
\end{tikzcd}$$

Associated n-Transport

For every representation of a structure Lie n-group we obtain a corresponding associated n-vector transport.

Associated n-transport is necessary and desireable for various reasons

- it admits tensor products (needed for K-theoretical applications)
- it admits global sections (needed for quantization).

Slogan

$$\begin{array}{ccccc} \text{associated} & & \text{principal} & & \text{representation} \\ n\text{-transport} & = & G_{(n)}\text{-transport} & + & \text{of } G_{(n)} \end{array}$$

What is a representation of an n-group?
Once we fix a notion of n-vector spaces, a representation of $G_{(n)}$ is

$$\rho : \Sigma G_{(n)} \to n\mathrm{Vect}\,.$$

The single object of $\Sigma G_{(n)}$ is sent to the representation n-space.

Example

Let $G_{(n)} = G_{(1)} = G$ be an ordinary group. Then an ordinary linear representation is a functor

$$\rho : \Sigma G \to \mathrm{Vect}$$

$$\bullet \xrightarrow{g} \bullet \quad \mapsto \quad V \xrightarrow{\rho(g)} V\,.$$

Proposition

Ordinary vector bundles with connection on X are equivalent to ρ-locally trivializabel 1-transport on X, for

$$\rho : \bigsqcup_n \Sigma U(n) \to \mathrm{Vect}$$

the defining representation.

$$\begin{tikzcd}
P_n(Y) \arrow[r, two heads, "\pi"] \arrow[d, "\mathrm{triv}"'] & P_n(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "t"', "\simeq"] \\
\Sigma G_{(n)} \arrow[r, "\color{red}\rho"'] & \mathrm{Vect}
\end{tikzcd}$$

In order to raise the categorical dimension now, we need to figure out what $2\mathrm{Vect}$ is.
Recall the the principle of least resistance under categorification: we want to start from a good definition of ordinary vector bundles.

Simple but useful observation in this context

$$\mathrm{Vect}_{\mathbb{C}} \simeq \mathrm{Mod}_{\mathbb{C}} \,.$$

Hence define, for $\mathcal{C}$ any (abelian) monoidal category

Definition

$$2\mathrm{Vect}_{\mathcal{C}} := \mathrm{Mod}_{\mathcal{C}}$$

where $\mathrm{Mod}_{\mathcal{C}}$ is the 2-category of categories equipped with a coherently associative and unital $\mathcal{C}$-action, and of functors respecting that action.

Just like there are different kinds of 1-vector space (real, complex, etc.) there are even more different kinds of 2-vector spaces:

Example

For $\mathcal{C} = \mathrm{Disc}(\mathbb{C})$ we get $2\mathrm{Vect}_{\mathrm{Disc}(\mathbb{C})} = \mathrm{BC2Vect}$, the 2-category of Baez-Crans 2-vector spaces, which satisfy

$$\mathrm{BC2Vect} \simeq 2\mathrm{Term}\,.$$

These 2-vector spaces are the right home for Lie 2-algebras. But as fibers for 2-vector bundles they apparently don't give rise to many interesting examples.

Here we shall be mostly interested in the following

Example

$$\mathrm{Mod}_{\mathrm{Vect}} := 2\mathrm{Vect}_{\mathrm{Vect}} \,.$$

For handling these, it is useful to make two observations:

- Such 2-vector spaces *with basis* corespond to ordinary algebras.
- Kapranov-Voevodsky's 2-vector spaces are contained, corresponding to the algebras of the form $\mathbb{C}^{\oplus n}$.

Observation

There is a canonical inclusion

$$\mathrm{Bim} \hookrightarrow 2\mathrm{Vect}_{\mathrm{Vect}}\,.$$

$$\begin{tikzcd} A \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow \rho"] & B \end{tikzcd} \mapsto \begin{tikzcd} \mathrm{Mod}_A \arrow[r, bend left, "-\otimes_A N"] \arrow[r, bend right, "-\otimes_A N'"'] \arrow[r, phantom, "\Downarrow -\otimes_A \rho"] & \mathrm{Mod}_B \end{tikzcd}$$

An algebra A .

Observation

There is a canonical inclusion

$$\mathrm{Bim} \hookrightarrow 2\mathrm{Vect}_{\mathrm{Vect}}\,.$$

$$\begin{tikzcd} A \arrow[r, bend left=50, "N"] \arrow[r, bend right=50, "N'"'] \arrow[r, phantom, "\Downarrow\rho"] & B \end{tikzcd} \mapsto \begin{tikzcd} {\color{red}\mathrm{Mod}_A} \arrow[r, bend left=50, "-\otimes_A N"] \arrow[r, bend right=50, "-\otimes_A N'"'] \arrow[r, phantom, "\Downarrow -\otimes_A \rho"] & \mathrm{Mod}_B \end{tikzcd}$$

An algebra A is sent to the category of its right modules.

Observation

There is a canonical inclusion

$$\mathrm{Bim} \hookrightarrow 2\mathrm{Vect}_{\mathrm{Vect}}\,.$$

$$\begin{tikzcd} A \arrow[rr, bend left, "N"] \arrow[rr, bend right, "N'"'] \arrow[rr, phantom, "\Downarrow\rho"] & & B \end{tikzcd} \mapsto \begin{tikzcd} \mathrm{Mod}_A \arrow[rr, bend left, "-\otimes_A N"] \arrow[rr, bend right, "-\otimes_A N'"'] \arrow[rr, phantom, "\Downarrow -\otimes_A \rho"] & & \mathrm{Mod}_B \end{tikzcd}$$

.

Notice that Mod_A is itself a module over Vect:

$$\mathrm{Vect} \times \mathrm{Mod}_A \to \mathrm{Mod}_A$$

$$V \times R \mapsto V \otimes R\,.$$

Observation

There is a canonical inclusion

$$\mathrm{Bim} \hookrightarrow 2\mathrm{Vect}_{\mathrm{Vect}}\,.$$

$$\begin{tikzcd} A \arrow[rr, bend left=50, "N"{name=U}] \arrow[rr, bend right=50, "\color{red}{N'}"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\rho"] & & B \end{tikzcd} \mapsto \begin{tikzcd} \mathrm{Mod}_A \arrow[rr, bend left=50, "-\otimes_A N"{name=U}] \arrow[rr, bend right=50, "-\otimes_A N'"'{name=D}] \arrow[from=U, to=D, Rightarrow, "-\otimes_A \rho"'] & & \mathrm{Mod}_B \end{tikzcd}$$

.

An A-B bimodule N'

Observation

There is a canonical inclusion

$$\mathrm{Bim} \hookrightarrow 2\mathrm{Vect}_{\mathrm{Vect}}\,.$$

$$\begin{tikzcd} A \arrow[rr, bend left, "N"] \arrow[rr, bend right, "N'"'] \arrow[rr, phantom, "\Downarrow\rho"] & & B \end{tikzcd} \mapsto \begin{tikzcd} \mathrm{Mod}_A \arrow[rr, bend left, "-\otimes_A N"] \arrow[rr, bend right, "\color{red}{-\otimes_A N'}"'] \arrow[rr, phantom, "\Downarrow -\otimes_A \rho"] & & \mathrm{Mod}_B \end{tikzcd}$$

.

An A-B bimodule N' is sent to the functor obtained by tensoring with N' on the right.

Observation

There is a canonical inclusion

$$\mathrm{Bim} \hookrightarrow 2\mathrm{Vect}_{\mathrm{Vect}}\,.$$

$$\begin{tikzcd} A \arrow[r, bend left=50, "N"] \arrow[r, bend right=50, "N'"'] \arrow[r, phantom, "\Downarrow \rho"] & B \end{tikzcd} \mapsto \begin{tikzcd} \mathrm{Mod}_A \arrow[r, bend left=50, "-\otimes_A N"] \arrow[r, bend right=50, "-\otimes_A N'"'] \arrow[r, phantom, "\Downarrow -\otimes_A \rho"] & \mathrm{Mod}_B \end{tikzcd}$$

.

A bimodule homomorphism ρ

Observation

There is a canonical inclusion

$$\mathrm{Bim} \hookrightarrow 2\mathrm{Vect}_{\mathrm{Vect}} \,.$$

$$\begin{tikzcd} A \arrow[rr, bend left, "N"] \arrow[rr, bend right, "N'"'] \arrow[rr, phantom, "\Downarrow\rho"] & & B \end{tikzcd} \mapsto \begin{tikzcd} \mathrm{Mod}_A \arrow[rr, bend left, "-\otimes_A N"] \arrow[rr, bend right, "-\otimes_A N'"'] \arrow[rr, phantom, "\Downarrow {\color{red}-\otimes_A \rho}"] & & \mathrm{Mod}_B \end{tikzcd}$$

.

A bimodule homomorphism ρ then induces a natural transformation.

Observation

We can think of Bim as being the 2-category of 2-vector space *with basis*.

Since

$$\mathrm{Mod}_A \simeq \mathrm{Hom}_{\mathrm{VectCat}}(\Sigma A, \mathrm{Vect})$$

just like

$$\mathbb{C}^n \simeq \mathrm{Hom}_{\mathrm{Set}}(\mathcal{B}, \mathbb{C})\,.$$

Here $\mathcal{B} = (v_1, v_2, \cdots)$ is a basis. Hence ΣA is like a 2-basis.

Hence we shall write

Definition

$$2\mathrm{Vect}_b := \mathrm{Bim}$$

to emphasize in which sense we are using the 2-category Bim.

Kapranov-Voevodsky 2-vector spaces

Remark

Inside $2\mathrm{Vect}_b$ sits the 2-category of Kaparanov-Voevodsky 2-vector spaces.

Definition

A KV 2-vector space of dimension n is the category Vect^n. A morphism of KV 2-vector spaces is a matrix in $M_{m,n}(\mathrm{Vect})$. Morphisms act like ordinary matrices, with product replaced by tensor product and sum replaced by direct sum.

Proposition

KV 2-vector spaces form the full sub 2-category of $2\mathrm{Vect}_b$ on the algebras of the form

$$A = \mathbb{C}^{\oplus n}$$

for $n \in \mathbb{N}$.

Hence we have a chain of inclusions

$$\mathrm{KV2Vect} \hookrightarrow 2\mathrm{Vect}_b := \mathrm{Bim} \hookrightarrow 2\mathrm{Vect}\,.$$

Now we can discuss 2-representations.

Observation and proposition

For every strict 2-group $G_{(2)} = (H \to G)$ and an ordinary representation of H, we canonically obtain a representation

$$\rho : \Sigma G_{(2)} \to 2\mathrm{Vect}_b$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} {\color{red}A_H} \arrow[rr, bend left, "{(A_H,\alpha(g))}"] \arrow[rr, bend right, "{(A_H,\alpha(g'))}"'] \arrow[rr, phantom, "\Downarrow \cdot h"] & & A_H \end{tikzcd}$$

The algebra A_H generated from the representation of H

Now we can discuss 2-representations.

Observation and proposition

For every strict 2-group $G_{(2)} = (H \to G)$ and an ordinary representation of H, we canonically obtain a representation

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serves as a bimodule over itself

Now we can discuss 2-representations.

Observation and proposition

For every strict 2-group $G_{(2)} = (H \to G)$ and an ordinary representation of H, we canonically obtain a representation

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with the right action twisted by $g \in G$.

Now we can discuss 2-representations.

Observation and proposition

For every strict 2-group $G_{(2)} = (H \to G)$ and an ordinary representation of H, we canonically obtain a representation

$$\rho : \Sigma G_{(2)} \to 2\mathrm{Vect}_b$$

$$\begin{tikzcd}
\bullet \arrow[rr, bend left=50, "g"] \arrow[rr, bend right=50, "g'"'] \arrow[rr, phantom, "\Downarrow h"] & & \bullet
\end{tikzcd}
\mapsto
\begin{tikzcd}
A_H \arrow[rr, bend left=50, "{(A_H,\alpha(g))}"] \arrow[rr, bend right=50, "{(A_H,\alpha(g'))}"'] \arrow[rr, phantom, "\Downarrow \cdot h"] & & A_H
\end{tikzcd}$$

One checks that multiplying with $h \in H$ provides a bimodule homomorphism.

Example

To be very canonical, take A_H to be the group algebra of H.

Example

One of the simplest examples is the canonical 2-representation of $\Sigma U(1)$ for the definiing representation of $U(1)$

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow c"] & \bullet
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=50, "\mathbb{C}"] \arrow[r, bend right=50, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow \cdot c"] & \mathbb{C}
\end{tikzcd}
$$

Form this we obtain an example of associated 2-transport: line 2-bundles.

Associated String 2-transport

In a very similar manner to line 2-bundles, we obtain associated 2-transport for the String 2-group by making use of the fact that it is represented by the *strict* 2-group

$$\mathrm{String}(G) = (\hat{\Omega}_1(G) \to PG)\,.$$

This yields a notion of String connections very close to the definition as conceived by Stolz-Teichner.

Transgression

The transgression of an n-transport to a space of maps from C into base space is the operation of applying

$$\mathrm{Hom}(C, \cdot)$$

to everything in sight.

Equivariance

An equivariant structure on an n-transport is the same as descent data for the case that the projection is given by the n-group action

$$\left(\, Y \xrightarrow{\pi} X \, \right) = \left(\, X \times G_{(n)} \xrightarrow{\rho} X \, \right).$$

Infinitesimal higher dimensional algebra

The concept of Lie n-algebras

- A Lie algebra $\mathfrak{g}$ is the infinitesimal version of a Lie group G:
$$\mathfrak{g} = \mathrm{Lie}(G)$$
- A group G is a one-object groupoid ΣG.
- An n-group $G_{(n)}$ is a one-object n-groupoid $\Sigma G_{(n)}$.
- A Lie n-algebra $\mathfrak{g}_{(n)}$ is the infinitesimal version of a one-object Lie n-groupoid:
$$\mathfrak{g}_{(n)} = \mathrm{Lie}(G_{(n)})\,.$$

Definition

A semistric Lie n-algebra is an n-category $\mathfrak{g}_{(n)}$ internal to Vect equipped with a skew symmetric functor $[\cdot,\cdot] : \mathfrak{g}_{(n)} \times \mathfrak{g}_{(n)} \to \mathfrak{g}_{(n)}$ which satisfies the Jacobi identity up to coherent isomorphism.

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$$\mathfrak{g}_{(n)} = \mathrm{Lie}(G_{(n)}) .$$

Definition

A semistric Lie n-algebra is an n-category $\mathfrak{g}_{(n)}$ internal to Vect equipped with a skew symmetric functor $[\cdot,\cdot] : \mathfrak{g}_{(n)} \times \mathfrak{g}_{(n)} \to \mathfrak{g}_{(n)}$ which satisfies the Jacobi identity up to coherent isomorphism.

Infinitesimal higher dimensional algebra

The concept of Lie n-algebras

- A Lie algebra $\mathfrak{g}$ is the infinitesimal version of a Lie group G:
$$\mathfrak{g} = \mathrm{Lie}(G)$$
- A group G is a one-object groupoid ΣG.
- An n-group $G_{(n)}$ is a one-object n-groupoid $\Sigma G_{(n)}$.
- A Lie n-algebra $\mathfrak{g}_{(n)}$ is the infinitesimal version of a one-object Lie n-groupoid:
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The relation between Lie n-groupoids and Lie n-algebras

Caveat: To what extent, and under which conditions, it is true that

Expected Statement: n-Lie's third theorem

- Every Lie n-algebra integrates to a Lie n-groupoid.
- Every Lie n-groupoid gives rise to a semistrict Lie n-algebra.

is still being investigated. Special cases are understood.
The statement hinges on

Issues still being discussed

- The precise definition of Lie n-groupoids.
- The question whether and when one may assume strict skew symmetry.

The Bridge: categorical Lie algebra to differential algebra

Principle

$$\left.\begin{array}{c}\textit{Lie } n\textit{-algebra } \mathfrak{g}_{(n)} - \\ \text{higher categorical} \\ \text{Lie algebra}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{graded-commutative} \\ \text{(co)differential} \\ \text{(co)algebra } (\bigwedge^\bullet sV^*, d_{\mathfrak{g}_{(n)}})\end{array}\right.$$

Dictionary

$\mathrm{Mor}_k(L)$	$\simeq$	$(sV)_1 \oplus \cdots \oplus (sV)_k$
structure morphisms	$\leftrightarrow$	$d_{\mathfrak{g}_{(n)}}$
coherence	$\leftrightarrow$	$(d_{\mathfrak{g}_{(n)}})^2 = 0$

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L_∞ and qDGCA

More precisely

Definition and Proposition

An n-term L_∞-algebra is a free graded commutative co-algebra $S^c(sV)$ on graded vector space $V = V_0 \oplus \cdots V_{n-1}$, which is equipped with a degree -1 codifferential $D : S^c(sV) \to S^c(sV)$ that squares to 0: $D^2 = 0$.

Dual statement

Dually, this is the exterior algebra $\wedge^\bullet(sV^*)$ equipped with the differential $d\omega := -\omega(D(\cdot))$. This we call an n-term quasi-free graded-commutative differential algebra, or n-term qDGCAs, for short.

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The standard example

Example: ordinary Lie algebra as L_∞-algebra

For $\mathfrak{g}$ an ordinary Lie (1-)algebra, the codifferential on the free graded-commutative coalgebra $S^c(s\mathfrak{g})$ acts as

$$D(sx_1 \vee sx_2) = s[x_1, x_2]$$

on all products of two generators $x_1, x_2 \in \mathfrak{g}$ and is freely extended as a codifferential to higher products of generators. The statement

$$D^2(sx_1 \vee sx_2 \vee sx_3) = 0$$

for a triples of generators is the Jacobi identity.

The standard example

Example: ordinary Lie algebra as qDGCA

For $\mathfrak{g}$ an ordinary Lie (1-)algebra, the differential on the exterior algebra $\wedge^\bullet(s\mathfrak{g}^*)$ acts as

$$d_{\mathfrak{g}} t^a = -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c$$

for $\{t^a\}$ any basis of $s\mathfrak{g}^*$ and $C^a{}_{bc}$ the structure constants of $\mathfrak{g}$ in the corresponding dual basis.

Of course this is nothing but the qDGCA of left-invariant forms on the group G

$$\mathfrak{g}^* := (\wedge^\bullet(s\mathfrak{g}^*), d_{\mathfrak{g}}) \simeq \Omega^\bullet_{\mathrm{li}}(G) .$$

The Bridge in more detail

The Bridge again, more precisely

Semistrict Lie n-algebras are "the same" as n-term L_∞-algebras, which in turn are dual (for finite dimensions) to n-term qDGCAs.

Caveat: "Semistrictness"

Here "semistrict" refers to the fact that the Jacobi identity is coherently weakened, while the skew symmetry is taken to hold strictly.

Caveat: higher morphisms

The general statement follows from general abstract operad nonsense. But explicit details on how *higher morphisms* pass over the bridge are hard to come by.

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The oidified Bridge: many objects

Lie algebroid version

On the qDGCA side the rather obvious generalization yields what should be addressed as Lie n-algebroids: in the literature the many-object qDGCAs are also known as *NQ-manifolds*.

The tangent Lie algebroid

The only Lie algebroid which we need here is the tangent algebroid $\mathrm{Vect}(X)$ of a manifold X. This is the differential of the *fundamental groupoid*

$$\mathrm{Vect}(X) := \mathrm{Lie}(\Pi_1(X)) .$$

This is very conveniently handled in its dual incarnation – there it is simply the deRham complex

$$\mathrm{Vect}(X)^* = (\Omega^\bullet(X), d) .$$

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The $\mathrm{inn}(\cdot)$-construction

Definition. (Inner derivation Lie $(n+1)$-algebra)

$\mathrm{inn}(\mathfrak{g}_{(n)})^* \simeq (\bigwedge(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}), \begin{pmatrix} d & 0 \\ \mathrm{Id} & d \end{pmatrix})$ corresponds to the mapping cone of the identity on $\mathfrak{g}_{(n)}$

Proposition

- There is a canonical injection $\mathfrak{g}_{(n)} \hookrightarrow \mathrm{inn}(\mathfrak{g})$.
- $\mathrm{inn}(\mathfrak{g}_{(n)})$ is *contractible*
- $(\bigwedge(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})})$ is the *Weil algebra* of $\mathfrak{g}_{(n)} = \mathfrak{g}$

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Hence $\mathrm{inn}(\mathfrak{g}_{(1)})^*$ plays the role of differential forms on the universal G-bundle.

The $\mathrm{inn}(\cdot)$-construction

The qDGCA of $\mathrm{inn}(\mathfrak{g})$: the Weil algebra

$\mathrm{inn}(\mathfrak{g})^* \simeq (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d)$ is spanned by generators $\{t^a\}$ in degree 1 and $\{r^a\}$ in degree 2, with differential

$$dt^a = -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c - r^a$$
$$dr^a = -C^a{}_{bc} t^b \wedge r^c \,.$$

We will now

- express the Lie algebra cohomology of $\mathfrak{g}$ in terms of the cohomology of the qDGCA underlying $\mathrm{inn}(\mathfrak{g})$.
- use the insight gained thereby to describe three families of Lie n-algebras: one for each cocycle, one for each invariant polynomial and one for each transgression element.
- then show that for the canonical 3-cocycle on a semisimple Lie algebra, connections with values in the Lie 3-algebra obtained this way describe the Chern-Simons parallel transport which we are after.

Lie algebra cohomology in terms of $\mathrm{inn}(\mathfrak{g})$

- A Lie algebra n-cocycle μ is

$$d|_{\bigwedge^\bullet(s\mathfrak{g}^*)}\mu = 0\,.$$

- An invariant degree n-polynomial k is

$$d|_{\bigwedge^\bullet(ss\mathfrak{g}^*)}k = 0\,.$$

- A transgression element cs is

$$\begin{aligned} \mathrm{cs}|_{\bigwedge^\bullet s\mathfrak{g}^*} &= \mu \\ d\mathrm{cs} &= k\,. \end{aligned}$$

The homotopy operator

- Recall that we said that $\mathrm{inn}(\mathfrak{g}_{(n)})$ is trivializable.
- This means there is a homotopy

$$\begin{tikzcd} \mathrm{inn}(\mathfrak{g}_{(n)}) \arrow[rr, bend left, "0"{name=U}] \arrow[rr, bend right, "\mathrm{Id}=[d,\tau]"'{name=D}] & & \mathrm{inn}(\mathfrak{g}_{(n)}) \arrow[from=U, to=D, Rightarrow, "\tau"] \end{tikzcd}$$

- We have τ *explicitly* (see Higher morphisms of Lie n-algebras) and hence an effective algorithm to always solve $k = d\mathrm{cs}$ as

$$\mathrm{cs} := \tau(k) + dq\,.$$

- The only nontrivial condition is hence $\mathrm{cs}|_{\wedge^\bullet s\mathfrak{g}^*} = \mu$.

A map of the cocycle situation

cocycle | Chern-Simons | inv. polynomial

$$\begin{tikzcd}
(\bigwedge^\bullet(s\mathfrak{g}^*), d_{\mathfrak{g}}) & (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}) \arrow[l, two heads, "i^*"'] & (\bigwedge^\bullet(ss\mathfrak{g})^*) \arrow[l, hook', "p^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 & \\
0 & p^* k \arrow[u, mapsto, "d_{\mathrm{inn}(\mathfrak{g})}"] \arrow[d, bend right, "\tau"'] & k \arrow[l, mapsto, "p^*"] \\
\mu \arrow[u, mapsto, "d_{\mathfrak{g}}"] & cs \arrow[l, mapsto, "i^*"'] \arrow[u, mapsto, "d_{\mathrm{inn}(\mathfrak{g})}"'] &
\end{tikzcd}$$

Lie n-algebras from cocycles

In the following we discuss

Definition and Proposition

From elements of $\mathrm{inn}(\mathfrak{g})^*$-cohomology we obtain Lie n-algebras:

Lie algebra cocycle	μ	Baez-Crans Lie n-algebra	$\mathfrak{g}_\mu$
invariant polynomial	k	Chern Lie n-algebra	$\mathrm{ch}_k(\mathfrak{g})$
transgression element	cs	Chern-Simons Lie n-algebra	$\mathrm{cs}_k(\mathfrak{g})$

For every transgression element cs these fit into a weakly exact sequence

$$\mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) .$$

Baez-Crans Lie n-algebras from cocycles

Definition and proposition [Baez,Crans]

For every Lie algebra $(n+1)$-cocycle μ of the Lie algebra $\mathfrak{g}$ there is a skeletal Lie n-algebra

$$\mathfrak{g}_\mu \,.$$

Construction.

Set $\mathfrak{g}_\mu \simeq (\bigwedge^\bullet(s\mathfrak{g}^* \oplus s^n\mathbb{R}^*), d)$ such that the differential is given by

$$\begin{aligned} dt^a &= -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c \\ db &= -\mu \end{aligned}$$

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Chern Lie n-algebras from invariant polynomials

Definition and proposition

For every degree $(n+1)$ Lie algebra invariant polynomial k of the Lie algebra $\mathfrak{g}$ there is a Lie $(2n+1)$-algebra

$$\mathrm{ch}_k(\mathfrak{g}) \,.$$

Construction.

Set $\mathrm{ch}_k(\mathfrak{g}) \simeq (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{(2n+1)}\mathbb{R}^*), d)$ such that we have

$$\begin{aligned} dt^a &= -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c - r^a \\ dr^a &= -C^a{}_{bc}t^b \wedge t^c \\ dc &= k \end{aligned}$$

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Chern-Simons Lie n-algebras from transgression elements

Definition and proposition

For every transgression element q of degree $(2n+1)$ there is a Lie $(2n+1)$-algebra

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Construction.

Set $\mathrm{cs}_k(\mathfrak{g}) \simeq (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus \oplus s^{2n}\mathbb{R}^* \oplus s^{(2n+1)}\mathbb{R}^*), d)$ such that

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Theorem

Whenever they exist, these Lie $(2n+1)$-algebras form a (weakly) short exact sequence:

$$0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0\,.$$

Theorem

Moreover, we have an isomorphism

$$\mathrm{cs}_k(\mathfrak{g}) \simeq \mathrm{inn}(\mathfrak{g}_{\mu_k})\,.$$

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Theorem

Moreover, we have an isomorphism

$$\mathrm{cs}_k(\mathfrak{g}) \simeq \mathrm{inn}(\mathfrak{g}_{\mu_k})\,.$$

The way we obtained Lie algebra cohomology from $\mathrm{inn}(\mathfrak{g})^*$ has a straightforward generalization with $\mathrm{inn}(\mathfrak{g})^*$ replaced by $\mathrm{inn}(\mathfrak{g}_{(n)})^*$, for $\mathfrak{g}_{(n)}$ any Lie *n*-algebra.

Lie n-algebra cohomology from $\mathrm{inn}(\mathfrak{g}_{(n)})^*$

- A Lie $\mathfrak{g}_{(n)}$-cocycle μ is

$$d_{\mathfrak{g}_\mu}\mu = 0\,.$$

- A $\mathfrak{g}_{(n)}$ invariant polynomial k is

$$d_{\mathrm{inn}(\mathfrak{g}_{(n)})}|_{\bigwedge^\bullet(ss\mathfrak{g}^*_{(n)})}k = 0\,.$$

- A transgression element cs is

$$\begin{aligned} \mathrm{cs}|_{\bigwedge^\bullet s\mathfrak{g}^*_{(n)}} &= \mu \\ d_{\mathrm{inn}(\mathfrak{g}_{(n)})}\mathrm{cs} &= k\,. \end{aligned}$$

Generalized String, Chern-Simons and Chern Lie n-algebras

Remark

The entire construction of String, Chern-Simons and Chern Lie n-algebras from ordinary Lie algebra cohomology accordingly has a straightforward analog for Lie n-algebra cohomology.

- $(\mathfrak{g}_{(n)})_\mu$, $\mathrm{cs}_k(\mathfrak{g}_{(n)})$, $\mathrm{ch}_k(\mathfrak{g}_{(n)})$

For the present discussion, however, we only need $\mathfrak{g}_{(n)}$ invariant polynomials. And we need to make manifest the qDGCA which they span.

Cohomology of the String Lie 2-algebra

Recall that the differential graded commutative algebra ("of left invariant differential forms") corresponding by Koszul duality to the String Lie 2-algebra $\mathrm{string}_k(\mathfrak{g}) = \mathfrak{g}_\mu$ is

$$(\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathbb{R}^*), d)$$

where the differential d is the ordinary Chevalley-Eilenberg differential on $s\mathfrak{g}^*$ and acts on the canonical degree 2 generator b as

$$db = \mu\,.$$

Cohomology of the String Lie 2-algebra

We may think of the 3-cocycle μ as the curvature 3-form of the canonical gerbe on G. It is hence suggestive to simply rename

$$\mu := H$$

such that

$$db = H\,.$$

Cohomology of the String Lie 2-algebra

It follows that the general degree n cochain on $\mathrm{string}_k(\mathfrak{g})$ is

$$\left(\sum_k \omega_k\right) \exp(b)|_n \,,$$

where $\omega_k \in \wedge^k(s\mathfrak{g}^*)$ and where $(\cdot)|_n$ denotes restricting an inhomogeneous cochain to its homogeneous part in degree n. This means that any n-cochain may be regarded as a $(n+2)$-cochain.

Cohomology of the String Lie 2-algebra

Moreover, the differential acts on such a cochain as

$$d\left(\sum_k \omega_k\right)\exp(b)|_n = \left((d + H\wedge)\sum_k \omega_k\right)\exp(b)|_{n+1}\,.$$

Cohomology of the String Lie 2-algebra

Therefore we find a $\mathbb{Z}_2$-graded complex

$$\left\{ \left(\sum_k \omega_k \right) \exp(b)|_{\dim(\mathfrak{g})-1} \right\} \xrightarrow{d} \left\{ \left(\sum_k \omega_k \right) \exp(b)|_{\dim(\mathfrak{g})} \right\}$$

which we may canonically identify with the complex of the twisted differential

$$d_H := d + H\wedge$$

acting on inhomogenous elements in $\wedge^\bullet(s\mathfrak{g}^*)$.

Almost a proposition

It seems that the cohomology of this complex is the ordinary Lie algebra cohomology of $\mathfrak{g}$ with the generator μ "killed".

Cohomology of the String-like Lie n-algebras

Analogous considerations apply to all string-like Lie n-algebras $\mathfrak{g}_\mu$ coming from odd-degree cocycles μ.
Denote by f the degree n generator of the Koszul dual

$$(\bigwedge^\bullet(s\mathfrak{g}^* \oplus s^n\mathbb{R}^*), d)$$

which satisfies

$$df = \mu\,.$$

We may suggestively rename μ as

$$df = H_{n+1}\,.$$

Cohomology of the String-like Lie n-algebras

We now get a $\mathbb{Z}_n$-graded complex

$$\left\{ \left(\sum_k \omega_k \right) \exp(f)|_{\dim(\mathfrak{g})-n} \right\} \xrightarrow{d} \cdots \xrightarrow{d} \left\{ \left(\sum_k \omega_k \right) \exp(f)|_{\dim(\mathfrak{g})} \right\}$$

which is canonically isomorphic to the complex of inhomogenous differential forms on G with the twisted differential

$$D_{H_{n+1}} := d + H_{n+1} \wedge \ .$$

Coboundaries for invariant polynomials

The qDGCA of $\mathfrak{g}_{(n)}$ invariant polynomials will turn out to play the role of differential forms on the classifying space of $\mathfrak{g}_{(n)}$-bundles. Therefore we will denote it $b\mathfrak{g}_{(n)}^*$.
Before defining this, we need to define coboundaries of $\mathfrak{g}_{(n)}$ invariant polynomials.

Coboundaries for invariant polynomials

Definition

An $\mathfrak{g}_{(n)}$ invariant polynomial $k \in \bigwedge^\bullet(ss\mathfrak{g}^*_{(n)})$ is a coboundary of invariant polynomials if it has a potential L such that

$$k = d_{\mathrm{inn}(\mathfrak{g}_{(n)})} L \,,$$

which vanishes "on the fibers" in that

$$L|_{\bigwedge^\bullet s\mathfrak{g}^*_{(n)}} = 0 \,.$$

Coboundaries for invariant polynomials

Remark

Recall that, due to the existence of the trivializing homotopy $\tau : 0 \to \mathrm{Id}_{\mathrm{inn}(\mathfrak{g})_{(n)}}$, every $d_{\mathrm{inn}(\mathfrak{g}_{(n)})}$ closed element k is $d_{\mathrm{inn}(\mathfrak{g}_{(n)})}$-exact

$$k = d(\tau k)\,.$$

- When $\mu \simeq (\tau k)|_{\bigwedge^\bullet(s\mathfrak{g}^*_{(n)})}$ is closed, then $\mathrm{cs} \simeq \tau k$ is a transgression element.
- When $L \simeq (\tau k)$ vanishes on $\bigwedge^\bullet(s\mathfrak{g}^*_{(n)})$ it is a coboundary of invariant polynomials.

Hence "coboundaries of invariant polynomials" are invariant polynomials that *suspend to zero*.

The algebra of $\mathfrak{g}_{(n)}$ invariant polynomials

Almost a proposition

- The strict kernel

$$\mathfrak{g}^*_{(n)} \xleftarrow{i^*} \mathrm{inn}(\mathfrak{g}_{(n)})^* \longleftarrow \ker(i^*)$$

is

$$b\mathfrak{g}^*_{(n)} := [\mathrm{inv}(\mathfrak{g}_{(n)})]\,,$$

which is the qDGCA freely generated from the nontrivial generators of the invariant polynomials of $\mathfrak{g}_{(n)}$, equipped with the trivial differential.

- The degree of $b\mathfrak{g}_{(n)}$ is that of the highest degree invariant polynomial.

The algebra of $\mathfrak{g}_{(n)}$ invariant polynomials

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The algebra of $\mathfrak{g}_{(n)}$ invariant polynomials

Example and Remark

The notation is derived from the important special abelian case where $\mathfrak{g}_{(n)} := \mathrm{Lie}(\Sigma^{n-1}U(1))$. In that case

$$b\mathrm{Lie}(\Sigma^{n-1}U(1)) = \mathrm{Lie}(\Sigma^{n}U(1))\,,$$

mimicking the fact that the classifying "space" of the n-group $\Sigma^{(n-1)U(1)}$ is the $(n+1)$-group $\Sigma^n U(1)$.

The algebra of $\mathfrak{g}_{(n)}$ invariant polynomials

Remark

A morphism

$$\Omega^\bullet(X) \xleftarrow{\{K_i\}} b\mathfrak{g}^*_{(n)}$$

is precisely the choice of closed r-forms K_i on X, one for each degree r generator k_i of $b\mathfrak{g}^*_{(n)}$.
There is a canonical morphism

$$\mathrm{ch}_{k_i}(\mathfrak{g}_{(n)})^* \longleftarrow b\mathfrak{g}^*_{(n)}$$

for each k_i, and composing this with a connection

$$\begin{tikzcd} \Omega^\bullet(X) & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "{(A,F_A)}"'] & \mathrm{ch}_k(\mathfrak{g}_{(n)})^* \arrow[l] & b\mathfrak{g}^*_{(n)}) \arrow[l] \arrow[lll, bend left, "k_i(F_A)"] \end{tikzcd}$$

picks out the Chern form of A with respect to k_i.

The algebra of $\mathfrak{g}_{(n)}$ invariant polynomials

Remark

For Lie 1-algebras $\mathfrak{g}_{(n)} = \mathfrak{g}$, the morphism

$$\Omega^\bullet(X) \xleftarrow{\{K_i\}} b\mathfrak{g}^*$$

is essentially the Chern-Weil homomorphism, once we impose the conditions described in Definition of $\mathfrak{g}_{(n)}$-bundles .

With these definitions in hand, we can now set out and try to explicitly compute $b\mathfrak{g}^*_{(n)}$ for concrete examples.
This will allow us then to make statements about the characteristic classes of $\mathfrak{g}_{(n)}$-bundles.

Invariant polynomials of the String Lie 2-algebra

Proposition

Let $\mathfrak{g}$ be a Lie algebra with transgressive invariant polynomial k. Then the algebra of invariant polynomials of the corresponding String (Baez-Crans type) Lie 2-algebra $\mathfrak{g}_{\mu_k}$ is that of $\mathfrak{g}$ modulo k:

$$b\mathfrak{g}^*_{\mu_k} \simeq b\mathfrak{g}^*/[k] \,.$$

Sketch of proof

In $\mathrm{inn}(\mathfrak{g}_{\mu_k})$, k becomes a coboundary of invariant polynomials:

$$\begin{aligned} k &= d_{\mathrm{inn}(\mathfrak{g}_{\mu_k})}\mathrm{cs} \\ &= d_{\mathrm{inn}(\mathfrak{g}_{\mu_k})}((\mathrm{cs}-\mu)+\mu) \\ &= d_{\mathrm{inn}(\mathfrak{g}_{\mu_k})}((\mathrm{cs}-\mu)+c) \end{aligned}$$

Invariant polynomials of the String Lie 2-algebra

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Invariant polynomials of the String Lie 2-algebra

Interpretation

In Bundles with Lie n-algebra connection we find that morphisms

$$\Omega^\bullet(X) \xleftarrow{\{K_i\}} b\mathfrak{g}^*_{(n)}$$

yield the characteristic classes of $\mathfrak{g}_{(n)}$-bundles. The above statement then amounts to saying that the characteristic classes of String bundles ($\mathfrak{g}_{\mu_k}$-bundles) are those of $\mathfrak{g}$-bundles modulo the element k.
Conversely, a $\mathfrak{g}$-bundle cannot be lifted to a $\mathfrak{g}_{\mu_k}$-bundle unless its characteristic class corresponding to k vanishes.

Connection and Curvature

Definition

For X some manifold and $\mathfrak{g}_{(n)}$ a Lie n-algebra, a $\mathfrak{g}_{(n)}$-connection on the trivial $\mathfrak{g}_{(n)}$-bundle over X is a morphism

$$\Omega^\bullet(X) \xleftarrow{(A,F_A)} \mathrm{inn}(\mathfrak{g}_{(n)})^* \ .$$

Morphisms of connections are higher qDGCA morphisms

$$\begin{tikzcd} \Omega^\bullet(X) & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, bend right, "{(A,F_A)}"'] \arrow[ll, bend left, "{(A',F_{A'})}"] \arrow[ll, phantom, "\Downarrow"] \end{tikzcd}$$

which vanish when pulled back along $\mathrm{inn}(\mathfrak{g}_{(n)})^* \leftarrow b\mathfrak{g}^*_{(n)}$.

Connection and Curvature

Example

For $\mathfrak{g}_{(n)} = \mathfrak{g}_{(1)} = \mathfrak{g}$ an ordinary Lie algebra, connections

$$\Omega^\bullet(X) \xleftarrow{(A,F_A)} \mathrm{inn}(\mathfrak{g}_{(n)})^*$$

are in bijection with $\mathfrak{g}$-valued 1-forms on X, and morphisms of them are linearized gauge transformations of these.

We have the following situation

$$\begin{tikzcd}
\mathfrak{g}_{(n)}^* \arrow[d, "{(A,F_A=0)}"'] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l] \arrow[d, "{(A,F_A)}"'] \\
\Omega^\bullet(X) & \Omega^\bullet(X)
\end{tikzcd}$$

Connection and Curvature

Remark

Recall that $\mathrm{inn}(\mathfrak{g}_{(n)})$ is trivializable. This makes the full $\mathrm{Hom}(\mathrm{inn}(\mathfrak{g}_{(n)})^*, \Omega^\bullet(X))$ also trivializable. But by restricting higher morphisms to those whose pullback along $\mathrm{inn}(\mathfrak{g}_{(n)})^* \leftarrow b\mathfrak{g}^*_{(n)}$ vanishes the crucial information is retained.

Definition

A morphism $0 \xrightarrow{(e,\nabla)} (A, F_A)$ is a section e (of the trivial $\mathfrak{g}_{(n)}$-bundle) together with its covariant derivative ∇e with respect to the connection A.

$$\begin{tikzcd}
\Omega^\bullet(X) & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, bend right=40, "0"', ""{name=U, below}] \arrow[ll, bend left=40, "{(A,F_A)}", ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "{(e,\nabla e)}"]
\end{tikzcd}$$

Connection and Curvature

Definition

The r-form

$$\begin{tikzcd} \Omega^\bullet(X) & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "{(A,F_A)}"'] & \mathrm{ch}_k(\mathfrak{g}_{(n)}) \arrow[l] & b\mathfrak{g}^*_{(n)} \arrow[l] \arrow[lll, bend left, "k(F_A)"] \end{tikzcd}$$

for k a degree r invariant polynomial on $\mathfrak{g}_{(n)}$ is the Chern-form of the connection A with respect to k.

Observation

A connection

$$\Omega^\bullet(X) \xleftarrow{(A,F_A)} \mathrm{inn}(\mathfrak{g}_{(n)})$$

on a trivial $\mathfrak{g}_{(n)}$-bundle is determined by an n-tuple of differential forms

$$A \in \Omega^1(X, V_1) \times \Omega^2(X, V_2) \times \cdots \times \Omega^n(X, V_n)\,,$$

where V_k is the degree k part of the graded vector space underlying $\mathfrak{g}_{(n)}$.
The corresponding curvatures forms

$$F_A \in \Omega^2(X, V_1) \times \Omega^3(X, V_2) \times \cdots \times \Omega^{n+1}(X, V_n)$$

are uniquely fixed.

The following lists some examples of $\mathfrak{g}_{(n)}$-connections and the nature of the differential form data corresponding to it.

Ordinary connection 1-forms

Ordinary connection 1-forms

n=1

$$\begin{tikzcd} \mathfrak{g} \\ \mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] \end{tikzcd}$$

for $A \in \Omega^1(X, \mathfrak{g})$.

Morphisms into $\mathfrak{g}_{(1)}$ come from *flat* connection 1-forms.

Ordinary connection 1-forms

Ordinary connection 1-forms

n=1 n=2

$$\begin{tikzcd} \mathfrak{g} \arrow[r, hook] & \mathrm{inn}(\mathfrak{g}) \\ \mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] & \mathrm{Vect}(X) \arrow[u, "(A)"] \end{tikzcd}$$

for $A \in \Omega^1(X, \mathfrak{g})$.

Morphisms into $\mathrm{inn}(\mathfrak{g}_{(1)})$ come from *arbitrary* connection 1-forms.

General Chern-Simons-like connections

Theorem

For every degree $(2n+1)$ Lie algebra transgressive element, $(2n+1)$-connections with values in $\mathrm{cs}_k(\mathfrak{g})$ are in bijection with $\mathfrak{g}$-Chern-Simons forms.

This means...

General Chern-Simons-like connections

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This means...

General Chern-Simons-like connections

$$\begin{tikzcd} 1 \\ \mathfrak{g} \\ \mathrm{Vect}(X) \arrow[u, "(A)", "F_A = 0"'] \end{tikzcd}$$

General Chern-Simons-like connections

Baez-Crans

$$\begin{tikzcd}
1 & 2n \\
\mathfrak{g} & \mathfrak{g}_{\mu_k} \arrow[l, two heads] \\
\mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] & \mathrm{Vect}(X) \arrow[u, "(A{,}B)", "{F_A=0,\ dB+\mathrm{CS}_k(A)=0}"']
\end{tikzcd}$$

General Chern-Simons-like connections

$$
\begin{tikzcd}
 & \text{Baez-Crans} & \text{Chern-Simons} \\
1 & 2n & 2n+1 \\
\mathfrak{g} & \mathfrak{g}_{\mu_k} \arrow[l, two heads] \arrow[r, hook] & \mathrm{cs}_k(\mathfrak{g}) \\
\mathrm{Vect}(X) \arrow[u, "(A)"', "F_A=0" description] & \mathrm{Vect}(X) \arrow[u, "(A{,}B)"', "\substack{F_A=0 \\ dB+\mathrm{CS}_k(A)=0}" description] & \mathrm{Vect}(X) \arrow[u, "(A{,}B{,}C)"', "C=dB+\mathrm{CS}_k(A)" description]
\end{tikzcd}
$$

General Chern-Simons-like connections

$$
\begin{tikzcd}
 & \text{Baez-Crans} & \text{Chern-Simons} & \text{Chern} \\
1 & 2n & 2n+1 & 2n+1 \\
\mathfrak{g} & \mathfrak{g}_{\mu_k} \arrow[l, two heads] \arrow[r, hook] & \mathrm{cs}_k(\mathfrak{g}) \arrow[r, two heads] & \mathrm{ch}_k(\mathfrak{g}) \\
\mathrm{Vect}(X) \arrow[u, "{(A)}" left, "{F_A=0}" description] & \mathrm{Vect}(X) \arrow[u, "{(A,B)}" left, "{\substack{F_A=0 \\ dB+\mathrm{CS}_k(A)=0}}" description] & \mathrm{Vect}(X) \arrow[u, "{(A,B,C)}" left, "{C=dB+\mathrm{CS}_k(A)}" description] & \mathrm{Vect}(X) \arrow[u, "{(A,C)}" left, "{dC=k((F_A)^{n+1})}" description]
\end{tikzcd}
$$

The standard Chern-Simons 3-connection

Finally: the case we wanted to understand

Let now $\mathfrak{g}$ be semisimple and let

$$\mu = \langle \cdot, [\cdot, \cdot] \rangle$$

be the canonical 3-cocycle.

Theorem (Baez, Crans, S, Stevenson)

The corresponding Baez-Crans Lie 2-algebra $\mathfrak{g}_\mu$ is equivalent to that of the corresponding String 2-group

$$\mathfrak{g}_\mu \simeq \mathrm{Lie}(\mathrm{String}_k(G)) .$$

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$$\mathfrak{g}_\mu \simeq \mathrm{Lie}(\mathrm{String}_k(G)) \,.$$

The standard Chern-Simons 3-connection

$$\begin{array}{c} \mathfrak{g} \\ \Big\| \\ \mathfrak{g} \\ {\scriptstyle (A)}\Big\uparrow {\scriptstyle F_A = 0} \\ \mathrm{Vect}(X) \end{array}$$

The standard Chern-Simons 3-connection

$$
\begin{tikzcd}
\mathfrak{g} \arrow[d, equal] & & \mathrm{string}_k(\mathfrak{g}) \arrow[ll, two heads] \arrow[d, equal, "\sim"] \\
\mathfrak{g} & & \mathfrak{g}_k \arrow[ll, two heads] \\
\mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] & & \mathrm{Vect}(X) \arrow[u, "(A{,}B)", "{F_A=0,\ dB+k\mathrm{CS}(A)=0}"']
\end{tikzcd}
$$

The standard Chern-Simons 3-connection

$$
\begin{tikzcd}
\mathfrak{g} & \mathrm{string}_k(\mathfrak{g}) \arrow[l, two heads] \arrow[r, hook] & \mathrm{inn}(\mathrm{string}_k(\mathfrak{g})) \\
\mathfrak{g} \arrow[u, equal] & \mathfrak{g}_k \arrow[l, two heads] \arrow[r, hook] \arrow[u, equal, "\sim"'] & \mathrm{cs}_k(\mathfrak{g}) \arrow[u, equal, "\sim"'] \\
\mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] & \mathrm{Vect}(X) \arrow[u, "{(A,B)}", "{\substack{F_A=0 \\ dB+k\mathrm{CS}(A)=0}}"'] & \mathrm{Vect}(X) \arrow[u, "{(A,B,C)}", "C=dB+k\mathrm{CS}(A)"']
\end{tikzcd}
$$

The standard Chern-Simons 3-connection

$$
\begin{tikzcd}
\mathfrak{g} \arrow[d, equal] & \mathrm{string}_k(\mathfrak{g}) \arrow[l, two heads] \arrow[r, hook] \arrow[d, equal, "\sim"] & \mathrm{inn}(\mathrm{string}_k(\mathfrak{g})) \arrow[d, equal, "\sim"] & \\
\mathfrak{g} & \mathfrak{g}_\mu \arrow[l, two heads] \arrow[r, hook] & \mathrm{cs}_k(\mathfrak{g}) \arrow[r, two heads] & \mathrm{ch}_k(\mathfrak{g}) \\
\mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] & \mathrm{Vect}(X) \arrow[u, "(A,B)", "{F_A=0,\ dB+\mathrm{CS}_k(A)=0}"'] & \mathrm{Vect}(X) \arrow[u, "(A,B,C)", "C=dB+\mathrm{CS}_\mu(A)"'] & \mathrm{Vect}(X) \arrow[u, "(A,C)", "dC=\langle F_A \wedge F_A \rangle"']
\end{tikzcd}
$$

General Chern-Simons-like connections

Remark.

The relevance of this statement is that this means that under the integration morphism

$$\boxed{\text{Lie } n\text{-algebroids}} \xrightarrow{\text{integration}} \boxed{\text{Lie } n\text{-groupoids}}$$

a morphism

$$\Omega^\bullet(X) \xleftarrow{(C=\mathrm{CS}_k(A)+dB)} \mathrm{ch}_k(\mathfrak{g})$$

should turn into a 4-functor

$$\Pi_4(X) \xrightarrow{\mathrm{tra}_C} G_{(4)}$$

which on 3-dimensional volumes V acts as the Chern-Simons *functional*

$$V \mapsto \exp(i \int_V \mathrm{CS}(A))\,.$$

Bundles with $\mathfrak{g}_{(n)}$-connection

We shall now give the central definition of a *global* $\mathfrak{g}_{(n)}$-connection. This is the differential version of the definition of non fake-flat parallel $G_{(n)}$-transport.

$\mathfrak{g}_{(n)}$-Connections on nontrivial bundles

Recall:

Remark

For $\mathfrak{g}_{(n)}$ any Lie n-algebra, the sequence

$$\mathfrak{g}_{(n)}^* \xleftarrow{i_u^*} \mathrm{inn}(\mathfrak{g}_{(n)})^* \xleftarrow{p_u^*} b\mathfrak{g}_{(n)}^*$$

plays the role of differential forms on the universal $\mathfrak{g}_{(n)}$-n-bundle.

For more background on this, see

- Universal $G_{(n)}$-bundles in terms of n-groupoids
- $G_{(n)}$-bundles with connection

in n-C ategorical background.

Bundles with $\mathfrak{g}_{(n)}$-connection

Definition

A bundle $p : P \to X$ with $\mathfrak{g}_{(n)}$-connection is a morphism (A, F_A) and a morphism i^* such that $i^*A \to i_u^*$; and a morphism p^* and a choice of r-forms $\{K_i\}$ such that $p^*K_i \simeq k_i(F_A)$.

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{li}}(|G_{(n)}|) & \mathfrak{g}^*_{(n)} \arrow[l] \arrow[dl, Rightarrow, "="'] \\
\Omega^\bullet(P) \arrow[u, "i^*"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "(A,F_A)"'] \arrow[u, "i_u^*"'] \arrow[dl, Rightarrow, "\simeq"'] \\
\Omega^\bullet(X) \arrow[u, "p^*"] & b\mathfrak{g}^*_{(n)} \arrow[l, "\{K_i\}"'] \arrow[u, "p_u^*"']
\end{tikzcd}
$$

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Bundles with $\mathfrak{g}_{(n)}$-connection

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and a morphism i^*
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and a morphism p^*
and a choice
of r-forms $\{K_i\}$
such that $p^* K_i \simeq k_i(F_A)$.

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$$
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\Omega^\bullet_{\mathrm{li}}(|G_{(n)}|) & & \mathfrak{g}^*_{(n)} \arrow[ll] \arrow[ddll, Rightarrow, "="'] \\
& & \\
\Omega^\bullet(P) \arrow[uu, "i^*"] & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, "{(A,F_A)}"'] \arrow[uu, "i_u^*"'] \arrow[ddll, Rightarrow, "\simeq"'] \\
& & \\
\Omega^\bullet(X) \arrow[uu, "p^*"] & & b\mathfrak{g}^*_{(n)} \arrow[ll, "\{K_i\}"'] \arrow[uu, "p_u^*"']
\end{tikzcd}
$$

Bundles with $\mathfrak{g}_{(n)}$-connection

More precisely. . .

Bundles with $\mathfrak{g}_{(n)}$-connection

Let $\mathfrak{g}_{(n)}$ be a Lie n-algebra. A $\mathfrak{g}_{(n)}$-n-bundle with connection over a manifold X is a diagram

$$\begin{tikzcd}
\Omega^\bullet(F) & & \mathfrak{g}^*_{(n)} \arrow[ll] \arrow[dll, Rightarrow, "\simeq"] \\
\Omega^\bullet(Y) \arrow[u, "i^*"] & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, "{(A,F_A)}"'] \arrow[u] \arrow[dll, Rightarrow, "\simeq"] \\
\Omega^\bullet(X) \arrow[u, "\pi^*"] \arrow[uu, bend left, "0"] & & b\mathfrak{g}^*_{(n)} \arrow[ll, "{\{K_i = k_i(F_A)\}}"] \arrow[u] \arrow[uu, bend right, "0"']
\end{tikzcd}$$

where we have...

Bundles with $\mathfrak{g}_{(n)}$-connection

- $Y \xrightarrow{\pi} X$ is a surjective submersion whose kernel $F = \ker(\pi)$ exists, $F \xrightarrow{i} Y \xrightarrow{\pi} X$;
- a characteristic map

$$\Omega^\bullet(X) \xleftarrow{\{K_i\}} b\mathfrak{g}^*_{(n)}$$

- a n-Cartan-Ehresmann connection

$$\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{inn}(\mathfrak{g}_{(n)})^*$$

and where...

Bundles with $\mathfrak{g}_{(n)}$-connection

...the homotopies are required to respect the sequence property in that

$$
\begin{tikzcd}
\Omega^\bullet(F) & \mathfrak{g}^*_{(n)} \arrow[l] \arrow[dl, Rightarrow, "\simeq"] \\
\Omega^\bullet(Y) \arrow[u, "i^*"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "{(A,F_A)}"'] \arrow[u] \\
& b\mathfrak{g}^*_{(n)} \arrow[u] \arrow[uu, bend right, "0"']
\end{tikzcd}
$$

and...

Bundles with $\mathfrak{g}_{(n)}$-connection

$$
\begin{tikzcd}
\Omega^\bullet(F) & \\
\Omega^\bullet(Y) \arrow[u, "i^*"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "(A,F_A)"'] \arrow[dl, Rightarrow, "\simeq"] \\
\Omega^\bullet(X) \arrow[u, "\pi^*"] \arrow[uu, bend left, "0"] & b\mathfrak{g}_{(n)}^* \arrow[l, "\{K_i = k_i(F_A)\}"] \arrow[u]
\end{tikzcd}
$$

are required to vanish.

Bundles with $\mathfrak{g}_{(n)}$-connection

Here

$$\begin{tikzcd} \Omega^\bullet(F) & & \mathfrak{g}_{(n)}^* \arrow[ll] \arrow[dll, Rightarrow, "\simeq"] \\ \Omega^\bullet(Y) \arrow[u, "i^*"] & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, "(A, F_A)"'] \arrow[u] \end{tikzcd}$$

is the *first Ehresmann condition*: this says that the connection n-forms pulled back to the fiber have to look like "the canonical left-invariant n-forms".

Bundles with $\mathfrak{g}_{(n)}$-connection

And

$$\begin{tikzcd}
\Omega^\bullet(Y) & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, "{(A,F_A)}"'] \arrow[dll, Rightarrow, "\simeq"] \\
\Omega^\bullet(X) \arrow[u, "\pi^*"] & & b\mathfrak{g}_{(n)}^* \arrow[ll, "{\{K_i = k_i(F_A)\}}"] \arrow[u]
\end{tikzcd}$$

is the *second Ehresmann condition*: this says that the connection data has to transform equivariantly (since it requests that $k(F_A)$ descends to the base and is hence invariant under vertical transformations along the fibers.)

Bundles with $\mathfrak{g}_{(n)}$-connection

Definition

The morphism

$$\Omega^\bullet(X) \xleftarrow[\{K_i = k_i(F_A)\}]{} b\mathfrak{g}^*_{(n)}$$

here is the Chern-Weil homomorphism of the given n-bundle: its image are the characteristic classes of the n-bundle with $\mathfrak{g}_{(n)}$-connection.

This leads us to have a closer look at the characteristic classes of n-bundles with $\mathfrak{g}_{(n)}$-connection.

We have seen that the morphism

$$\Omega^\bullet(X) \xleftarrow[\{K_i = k_i(F_A)\}]{} b\mathfrak{g}_{(n)}^*$$

in the diagram

$$\begin{tikzcd}
\Omega^\bullet(F) & & \mathfrak{g}_{(n)}^* \arrow[ll] \arrow[dll, Rightarrow, "\simeq"] \\
\Omega^\bullet(Y) \arrow[u, "i^*"] & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, "(A, F_A)"'] \arrow[u] \arrow[dll, Rightarrow, "\simeq"] \\
\Omega^\bullet(X) \arrow[u, "\pi^*"] \arrow[uu, bend left, "0"] & & b\mathfrak{g}_{(n)}^* \arrow[ll, "\{K_i = k_i(F_A)\}"] \arrow[u] \arrow[uu, bend right, "0"']
\end{tikzcd}$$

describes the characteristic classes of the n-bundle with $\mathfrak{g}_{(n)}$-connection.

Reminder on characteristic classes

$$
\begin{tikzcd}
\mathrm{inv}(\mathfrak{g}) \arrow[r] \arrow[rr, bend right, "\text{Chern-Weil homomorphism}"'] & \Omega^\bullet(X) \arrow[r] & H^\bullet(X,\mathbb{R}) \\
k \arrow[r, mapsto] & k(F_A) \arrow[r, mapsto] & [k(f_A)]
\end{tikzcd}
$$

invariant polynomial | characteristic form | characteristi class

The Chern-Weil homomorphism sends, for each G-bundle $P \to X$, any degree n invariant polynomial on $\mathfrak{g} = \mathrm{Lie}(G)$ to the deRham class of the differential form $k(F_A) = k(F_A \wedge \cdots \wedge F_A)$ obtained by inserting the curvature 2-form of any connection on P into k.

Remark.

Beware, though, that we are at the moment making statements only about deRham classes. As we can see from the $n = 1$-case, where the diagram encodes an ordinary Ehresmann connection, the diagram should also encode the integral classes. This needs to better understood.

We can understand from this point of view how the condition arises, that
<u>invariant polynomials which suspend to zero do not contribute</u>...

The existence of the transformation

$$\begin{tikzcd}
\Omega^\bullet(Y) & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "(A, F_A)"'] \arrow[dl, Rightarrow, "\simeq"] \\
\Omega^\bullet(X) \arrow[u, "\pi^*"] & b\mathfrak{g}_{(n)}^* \arrow[l, "\{K_i = k_i(F_A)\}"] \arrow[u]
\end{tikzcd}$$

says that $k(F_A)$ and $\pi^* K$ may differ by an exact term $d\omega$ on Y...

Bundles with $\mathfrak{g}_{(n)}$-connection

...where however ω has to vanish on the fibers, since

$$\begin{tikzcd}
\Omega^\bullet(F) & \\
\Omega^\bullet(Y) \arrow[u, "i^*"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "(A,F_A)"'] \arrow[dl, Rightarrow, "\simeq"] \\
\Omega^\bullet(X) \arrow[u, "\pi^*"] \arrow[uu, bend left, "0"] & b\mathfrak{g}^*_{(n)} \arrow[u] \arrow[l, "\{K_i = k_i(F_A)\}"]
\end{tikzcd}$$

has to vanish.

But this says that invariant polynomials that suspend to zero can always be absorbed by this transformation.

$\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$	$\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$	$\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$
ch_k	c_k	$\mathrm{p}_{k/2}$
$\mathrm{ch}(X) = \mathrm{tr}(\exp(itX)) = \sum_k t^k \mathrm{ch}_k(X, \cdots, X)$	$\mathrm{c}(X) = \det(t + iX) = \sum_k t^{n-k} \mathrm{c}_k(X, \cdots, X)$	$\det(t - X) = \sum_k t^{n-k} p_{k/2}(X, \cdots, X)$
Chern character	Chern class	Pontryagin class

Characteristic classes for matrix Lie algebras obtained from the trace and the determinant.

Ordinary bundles

Example

For an ordinary Lie algebra $\mathfrak{g}_{(n)} = \mathfrak{g}$ this reproduces the definition of a Cartan-Ehresmann connection:

Ordinary bundles

Example

For an ordinary Lie algebra $\mathfrak{g}_{(n)} = \mathfrak{g}$ this reproduces the definition of a Cartan-Ehresmann connection:

The morphism

$$\Omega^\bullet(P) \xleftarrow{(A, F_A)} \mathrm{inn}(\mathfrak{g})^*$$

is a $\mathfrak{g}$-valued 1-form A on the total space P of the bundle.

Ordinary bundles

Example

For an ordinary Lie algebra $\mathfrak{g}_{(n)} = \mathfrak{g}$ this reproduces the definition of a Cartan-Ehresmann connection:

The square

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{li}}(G) & \mathfrak{g}^* \arrow[l, "\simeq"'] \\ \Omega^\bullet(P) \arrow[u, "i^*"] & \mathrm{inn}(\mathfrak{g})^* \arrow[l, "{(A,F_A)}"'] \arrow[u] \end{tikzcd}$$

says that A restricted to the fiber is the canonical 1-form on G.

Ordinary bundles

Example

For an ordinary Lie algebra $\mathfrak{g}_{(n)} = \mathfrak{g}$ this reproduces the definition of a Cartan-Ehresmann connection:

The square

$$\begin{tikzcd} \Omega^\bullet(P) & \mathrm{inn}(\mathfrak{g})^* \arrow[l, "(A,F_A)"'] \arrow[dl, Rightarrow, "\simeq"'] \\ \Omega^\bullet(X) \arrow[u, "p^*"] & b\mathfrak{g}^* \arrow[l, "\{K_i = k_i(F_A)\}"'] \arrow[u] \end{tikzcd}$$

says that the Chern forms $k_i(F_A)$ on the total space have to descend to the characteristic classes on the base space. A sufficient condition for this is the $\mathfrak{g}$-equivariance of A.

Line 2-bundles (abelian gerbes)

Example

For $\mathfrak{g}_{(2)} = \mathrm{Lie}(\Sigma U(1))$ the morphism

$$\Omega^\bullet(X) \xleftarrow{K} b\mathfrak{g}^*_{(2)}$$

defines a closed 3-form on X.
The condition

$$\Omega^\bullet_{\mathrm{li}}(|G_{(2)}|) \xleftarrow{\simeq} \mathfrak{g}^*_{(2)}$$

says that the fibers have

$$H^\bullet(|G_{(2)}|) = H^2(|G_{(2)}|) \simeq \mathbb{R}\,.$$

They look like $PU(H)$.

String 2-bundles

Example

For $\mathfrak{g}$ simple and $\mathfrak{g}_{(2)} = \mathfrak{g}_{\langle\cdot,[\cdot,\cdot]\rangle}$ the String Lie 2-algebra, the morphism

$$\Omega^\bullet(X) \xleftarrow{\{K_i = k_i(F_A)\}} b\mathfrak{g}^*_{(2)}$$

assigns, due to the nature of the invariant polynomials of the String Lie 2-algebra, the characteristic classes of a $\mathfrak{g}$-bundle with $[\langle F_A \wedge F_A\rangle]$ vanishing.

The condition

$$\Omega^\bullet_{\mathrm{li}}(|G_{(2)}|) \xleftarrow{\simeq} \mathfrak{g}^*_{(2)}$$

says that the fibers are like G but with

$$H^3(|G_{(2)}|) \simeq 0\,.$$

This says they look like the String group.

String 2-bundles

Example

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$$H^3(|G_{(2)}|) \simeq 0\,.$$

This says they look like the String group.

Chern-Simons 3-bundles

Example

For $\mathfrak{g}$ simple and $\mathfrak{g}_{(2)} = \mathrm{ch}_{\langle\cdot,\cdot\rangle}(\mathfrak{g})$ the Chern Lie 3-algebra corresponding to the Killing form, the morphism

$$\Omega^\bullet(X) \xleftarrow{\{K_i = k_i(F_A)\}} b\mathfrak{g}^*_{(2)}$$

assigns the Pontryagin class of a $\mathfrak{g}$-bundle.

The basic idea of String- and Chern-Simons n-transport

A Chern-Simons $(n+1)$-transport is the obstruction to lifting a G-1-transport through a String-like extension

$$\Sigma^{n-1}U(1) \to \hat{G} \to G \, .$$

The String-2-group and its 2-transport

Killingback and Witten noticed that

1	super particles	couple to	$\mathrm{Spin}(n)$-bundles	with connection
	like			
2	super strings	couple to	$\mathrm{String}(n)$-bundles	with (?)

Using the Atiyah-Segal observation that

1	quantum (super) particles	are functors	$1\mathrm{Cob}_S \to \mathrm{Hilb}_S$

like

2	quantum (super) strings	are functors	$2\mathrm{Cob}_S \to \mathrm{Hilb}_S$

this should translate into a precise statement (about representations of cobordisms categories).

Back then few people thought of categorification. But Stolz and Teichner later made two remarks.

First Remark.

First, following Dan Freed, Segal's original viewpoint should be refined to

1	quantum (super) particles	are functors	$1\mathrm{Cob}_S \to \mathrm{Hilb}_S$
	like		
2	quantum (super) strings	are 2-functors	$\mathrm{Cob}_S^{\mathrm{ext}} \to 2\mathrm{Hilb}_S$

This is nowadays known as *extended quantum field theory*.

Second Remark

Moreover, it should be true that

1	$\mathrm{Spin}(n)$ bundles	with connection	are related to	K-cohomology

like

2	$\mathrm{String}(n)$ bundles	with connection	are related to	elliptic cohomology

All in all, this is supposed to be considerable reason to be interested in $\mathrm{String}(n)$-bundles with connection.

What is $\mathrm{String}(n)$, anyway?

There is the classical definition of $\mathrm{String}(n)$, and there is a "revisionist" one. The latter is maybe intuitively more accessible.

Revisionist definition: $\mathrm{String}(n)$ as stringy $\mathrm{Spin}(n)$.

In the old days, superstrings (in their RNS incarnation) were sometimes called *spinning strings*. Indeed, a superstring is much like a continuous line of spinors.
This suggests that the corresponding gauge group is the loop group

$$\Omega\mathrm{Spin}(n)$$

or maybe its Kac-Moody central extension

$$\hat{\Omega}_k\mathrm{Spin}(n)$$

or maybe the path group

$$P\mathrm{Spin}(n)\,.$$

Or maybe all of these.

In fact, there are canonical group homomorphisms

$$\hat{\Omega}_k \mathrm{Spin}(n) \xrightarrow{t} P\mathrm{Spin}(n) \xrightarrow{\alpha} \mathrm{Aut}(\hat{\Omega}_k \mathrm{Spin}(n)) \,.$$

These satisfy two compatibility conditions which say that the groups here conspire to form a (strict Fréchet-Lie) *2-group*

$$G_{(2)} \,.$$

A 2-group is a category which behaves like a group.

Every toplogical 2-group like this may be turned into a big ordinary topological group by taking its nerve. For $G_{(2)}$, this nerve is [Henriques,BCSS]

$$|G_{(2)}| \simeq \mathrm{String}(n)\,.$$

Classical definition

This is all that is needed about $\mathrm{String}(n)$ in the following. But for completeness, here is the classical definition.

Definition

The **string group** String_G of a simple, simply connected, compact topological group G is (a model for) the 3-connected topological group with the same homotopy groups as G, except

$$\pi_3(\mathrm{String}_G) = 0\,,$$

which, furthermore, fits into the exact sequence

$$1 \longrightarrow (BU(1) \simeq K(\mathbb{Z},2)) \longrightarrow \mathrm{String}_G \longrightarrow G \longrightarrow 1$$

of topological groups.

The string group proper is obtained by setting $G = \mathrm{Spin}(n)$.

$$\mathrm{String}(n) := \mathrm{String}_{\mathrm{Spin}(n)}\,.$$

The way to see that such a group is a plausible candidate for something generalizing the Spin-group, which, recall, fits into the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 1\,,$$

is to note that the first few homotopy groups π_k of $O(n)$ are

$k =$	0	1	2	3	4	5	6	7
$\pi_k(O(n)) =$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$

.

Starting with $O(n)$, we can successively "kill" the lowest nonvanishing homotopy groups, thus obtaining first $SO(n)$ (the connected component), then $\mathrm{Spin}(n)$ (the universal cover) and finally $\mathrm{String}(n)$ (the 3-connected cover). Notice that with π_3 vanishing, $\mathrm{String}(n)$ cannot be a compact Lie group – but it can be a Lie 2-group.

Usually (see [**?**]), the definition of String_G includes also a condition on the boundary map $\pi_3(G) \xrightarrow{\partial} \pi_2(K(\mathbb{Z}, 2))$. Our definition above is really geared towards the application where $G = \mathrm{Spin}(n)$, for which we find it more natural.

Namely, recall that every short exact sequence of topological groups

$$0 \to A \to B \to C \to 0\,,$$

which happens to be a fibration, gives rise to a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(B) \longrightarrow \pi_n(C) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots\,.$$

In our case this becomes

$$\cdots \longrightarrow \pi_n(K(\mathbb{Z},2)) \longrightarrow \pi_n(\mathrm{String}_G) \longrightarrow \pi_n(G) \xrightarrow{\partial} \pi_{n-1}(K(\mathbb{Z},2)$$

Demanding that $\pi_3(\mathrm{String}_G) = 0$ and assuming that also $\pi_2(\mathrm{String}_G) = 0$ (which we noticed above is the case for $G = \mathrm{Spin}(n)$) implies that we find inside this long exact sequence the short exact sequence

$$0 \longrightarrow (\pi_3(G) \simeq \mathbb{Z}) \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0 \; .$$

But this implies that the boundary map ∂ here is an isomorphism, hence that it acts on $\mathbb{Z}$ either by multiplication with $k = 1$ or $k = -1$. (This number is really the "level" governing this construction. If I find the time I will explain this later.)

In [StolzTeichner] this logic is applied the other way around. Instead of demanding that $\pi_3(\mathrm{String}_G) = 0$ it is demanded that the boundary map

$$\pi_3(G) \xrightarrow{\partial} \mathbb{Z}$$

is given by multiplication with the level, namely a specified element in $H^4(BG)$.

String 2-transport

Principal String 2-transport is principal 2-transport with structure 2-group $\mathrm{String}_k(G) : (\hat{\Omega}_k G \to PG)$.
2-Vector String 2-transport is 2-transport associated to that by the canonical 2-rep

$$\rho : \mathrm{String}_k(G) \to \mathrm{Bimod}_{\mathrm{vN}} \hookrightarrow 2\mathrm{Vect}\,.$$

When considering $\mathrm{String}(n)$-transport, there is a simple example to keep in mind: rank-1 2-vector bundles, line 2-bundles

- let $G_{(2)} = \Sigma U(1)$
- then $|G_{(2)}| \simeq PU(H)$
- local semi trivialization of ρ-associated $\Sigma U(1)$-2-bundles are line bundle gerbes [S.-Waldorf]
- indeed, these have same classification as $PU(H)$-bundles, namely class in $H^3(X, \mathbb{Z})$
- canonical 2-rep on algebras equivalent to $\mathbb{C}$: finite rank operators $K(H)$

Compare:

	line 2-bundle	**String 2-bundle**
structure 2-group	$(U(1) \to 1)$	$(\hat{\Omega}\mathrm{Spin}(n) \to P\mathrm{Spin}$
nerve of that	$PU(H)$	$\mathrm{String}(n)$
associated 2-vector bundle	finite-rank operators	von-Neumann algeb

Obstruction theory

In obstruction theory we study the failure of existences of lifts

$$\begin{tikzcd} & K \arrow[d] \\ & G \arrow[d] \\ P \arrow[ur, dashed] \arrow[r] & B \end{tikzcd}$$

The idea of obstruction theory

In some suitable categorical context, let

$$P \longrightarrow B$$

be a morphism (a parallel n-transport in our context) and

$$\begin{tikzcd} K \arrow[r, "t"] & G \arrow[d] \\ & B \end{tikzcd}$$

an exact sequence (of transport codomains, in our context).

Then obstruction theory studies ...

The idea of obstruction theory

Then obstruction theory studies the failure of being able to construct a lift

$$\begin{tikzcd} K \arrow[r] & G \arrow[d] \\ P \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

The idea of obstruction theory

The obstruction to this should be the composite denoted obst in

$$
\begin{tikzcd}
K \arrow[r, "t"] & G \arrow[r, "i"] \arrow[d] & \mathrm{wcoker}(t) \arrow[r] & \mathrm{coker}(i) \\
P \arrow[r] \arrow[ur, dotted] \arrow[urrr, bend right, "\mathrm{obst}"'] & B \arrow[ur, "f^{-1}"'] & &
\end{tikzcd}
$$

The idea of obstruction theory

where wcoker denotes a *weak* cokernel construction and where f^{-1} is some suitable "local inverse" to the universal f defined by

$$\begin{tikzcd} K \arrow[r, "t"] & G \arrow[r] \arrow[d] & \mathrm{wcoker}(t) \arrow[dl, "f"] \arrow[dl, phantom, shift right=3, "\Leftarrow \simeq"] \\ & B & \end{tikzcd} .$$

Weak cokernels of group homomorphisms

The weak cokernel $\mathrm{wcoker}(t)$ of a morphism of groups

$$H \xrightarrow{t} G$$

is defined to be the cokernel of 2-groups when H and G are regarded as discrete 2-groups.

Weak cokernels of 2-group homomorphisms

Similarly the weak cokernel $\mathrm{wcoker}(t)$ of a morphism of 2-groups

$$H_{(2)} \xrightarrow{t} G_{(2)}$$

is defined to be the cokernel of 3-groups when H and G are regarded as discrete 2-groups.
This has been studied in [CarrascoGarzónVitale:2006].

Weak cokernels of 2-group homomorphisms

Proposition

For $H_{(2)}$ and $G_{(2)}$ strict 2-groups, and t a morphism of strict 2-groups, the weak cokernel wcoker is isomorphic to the mapping cone

$$\mathrm{wcoker} = (H_{(2)} \xrightarrow{t} G_{(2)})$$

of 2-groups.

See mapping cones.

Weak cokernels of 2-group homomorphisms

Example

Let $H \xrightarrow{t} G$ be a crossed module of groups. Then $\mathrm{wcoker}(t)$ is the corresponding strict 2-group.

Example

Let $t_{\mathrm{Id}_{G_{(2)}}}$. Then

$$\mathrm{wcoker}(t) = \mathrm{INN}_0(G_{(2)})$$

is the inner automorphism 3-group studied in [RobertsSchreiber:2007].

Weak cokernels of 2-group homomorphisms

It follows that for any given short exact sequence of strict 2-groups

$$K_{(2)} \xrightarrow{t} G_{(2)} \longrightarrow B_{(2)}$$

one obtains the setup

$$\begin{tikzcd}
K_{(2)} \arrow[r, "t"] & G_{(2)} \arrow[r] \arrow[d] & (H_{(2)} \xrightarrow{t} G_{(2)}) \arrow[dl, "f"] \arrow[dl, phantom, bend right, "\swarrow \simeq"] \\
& B_{(2)} &
\end{tikzcd} .$$

Weak cokernels of morphisms of Lie n-algebras

Most of these constructions are computationally easier and easier to generalize to arbitrary n when we pass from Lie n-groups to Lie n-algebras.

Weak cokernels of morphisms of Lie n-algebras

We can reproduce the construction analogous to the above one for sequences of Lie n-algebras

$$\mathfrak{k}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)} \longleftarrow \mathfrak{b}^*_{(n)}$$

with t^* assumed to be particularly well behaved. (A condition always satisfied in the examples we shall study. A generalization away from this assumption is certainly expected to exists, but not studied here.)

Weak cokernels of morphisms of Lie n-algebras

Thinking of the weak cokernel of 2-groups as a mapping cone proves to be useful for the generalization to Lie n-algebras: we can define the mapping cone Lie $(n+1)$-algebra

$$(\mathfrak{k}_{(n)} \xrightarrow{t} \mathfrak{g}_{(n)})$$

and show that it does fit into

$$\begin{tikzcd}
\mathfrak{k}^*_{(n)} & \mathfrak{g}^*_{(n)} \arrow[l, "t^*"'] & (\mathfrak{k}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)}) \arrow[l] \\
& \mathfrak{b}^*_{(n)} \arrow[u] \arrow[ur, dashed, "f"'] &
\end{tikzcd} .$$

Weak cokernels of morphisms of Lie n-algebras

Moreover, in this context now the map f does have a weak inverse

$$f^{-1} : (\mathfrak{k}^*_{(n)} \xleftarrow{t^*} \mathfrak{g}^*_{(n)}) \to \mathfrak{b}^*_{(n)} .$$

This we can use to compute obstructions quite explicitly.
You may first look at the families of extensions of Lie n-algebras that we are going to consider: String-like central extensions.
Or see how the obstructions to lifting $\mathfrak{g}_{(n)}$-connections through these extensions are computed: Obstructions to n-bundle lifts

String-like central extensions

We now describe a class of central extensions of Lie n-algebras whose obstruction theory is relevant in the context of Chern-Simons theory and its generalizations.

Recall the main statement about the Baez-Crans type Lie n-algebras

Proposition

Let $\mathfrak{g}$ be a Lie algebra. Then for any Lie algebra $2n+1$ cocycle μ which is in transgression with an invariant polynomial k there is a (weakly exact) sequence

$$\mathfrak{g}_\mu \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g})$$

of Lie $2n+1$-algebras. Here $\mathfrak{g}_\mu$ is a $2n$-algebra which is a central extension

$$\mathrm{Lie}(\Sigma^{(n-1)}U(1)) \to \mathfrak{g}_\mu \to \mathfrak{g}$$

of $\mathfrak{g}$ by the shifted abelian Lie n-algebra and we have a canonical isomorphism

$$\mathrm{cs}_k(\mathfrak{g}) \simeq \mathrm{inn}(\mathfrak{g}_\mu)\,.$$

By combining these two sequences we obtain the Lie n-algebra description of the extension of the universal $\mathfrak{g}$-bundle by the universal $\Sigma^{(n-1)}U(1)$-bundle to the universal $\mathfrak{g}_\mu$-bundle:

universal $\Sigma^{n-1}U(1)$-bundle	**universal $\mathfrak{g}_\mu$-bundle**	**universal $\mathfrak{g}$-bundle**

$$\begin{tikzcd}
\mathrm{Lie}(\Sigma^{n-1}U(1)) \arrow[r] \arrow[d] & \mathfrak{g}_\mu \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\
\mathrm{inn}(\mathrm{Lie}(\Sigma^{n-1}U(1))) \arrow[r] \arrow[d] & \mathrm{inn}(\mathfrak{g}_\mu) \arrow[r] \arrow[d] & \mathrm{inn}(\mathfrak{g}) \arrow[d] \\
\mathrm{Lie}(\Sigma^{n}U(1)) \arrow[r] & b\mathfrak{g}_\mu \arrow[r] & b\mathfrak{g}
\end{tikzcd}$$

Obstructing n-bundles: integral picture

Lifting line 2-bundle (lifting gerbes)

Given an ordinary central extension

$$U(1) \to \hat{G} \to G$$

we find from

$$G = (1 \to G) \simeq (U(1) \to \hat{G})$$

and

$$\hat{G} \overset{i}{\hookrightarrow} (U(1) \to \hat{G}) \to \mathrm{coker}(i) = (U(1) \to 1)$$

that...

Lifting line 2-bundle (lifting gerbes)

...that the obstruction to lifting a G-cocycle

$$\begin{tikzcd} & {} \arrow[dr, "g_{23}"] & \\ {} \arrow[ur, "g_{12}"] \arrow[rr, "g_{13}"'] & & {} \end{tikzcd}$$

to a $\hat{G}$ cocycle is obtained by first lifting to $(U(1) \to \hat{G})$

$$\begin{tikzcd} & {} \arrow[dr, "\hat{g}_{23}"] \arrow[d, phantom, "c \Downarrow \simeq"] & \\ {} \arrow[ur, "\hat{g}_{12}"] \arrow[rr, "\hat{g}_{13}"'] & {} & {} \end{tikzcd}$$

Lifting line 2-bundles (lifting gerbes)

. . . which is always possible, and then extracting the resulting $(U(1) \to 1)$-cocycle

$$\begin{tikzcd} & {} \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "c \Downarrow \simeq"] & \\ {} \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}"'] & {} & {} \end{tikzcd}.$$

Its nontriviality measure the failure of the lift.

Lifting line 3-bundles

The same principle works for the String extension

$$\Sigma U(1) \to \mathrm{String}_k(G) \to G\,.$$

Lifting line 3-bundles

We use

$$G = (1 \to 1 \to G) \simeq (1 \to \Omega G \to PG) \simeq (U(1) \to \hat{\Omega}_k G \to PG)$$

and

$$\mathrm{String}_k(G) = (\hat{\Omega}_k G \to PG) \stackrel{i}{\hookrightarrow} (U(1) \to \hat{\Omega}_k G \to PG) \to \mathrm{coker}(i)$$

with

$$\mathrm{coker}(i) = (U(1) \to 1 \to 1)$$

and then proceed as before.

Lifting line 3-bundles

Definition

A Chern-Simons 3-bundle (Chern-Simons 2-gerbe) is a 3-bundle obstructing the lift of a G-bundle to a $\mathrm{String}_k(G)$-2-bundle.

Obstructing n-bundles: differential picture

When we have a $\mathfrak{g}$-transport given by

$$
\begin{tikzcd}
\Sigma^{n-1}\mathfrak{u}(1) \arrow[r] \arrow[dd] & \mathfrak{g}_\mu \arrow[rr] \arrow[dd] & & \mathfrak{g} \arrow[dd] \\
& & TF \arrow[ur] \arrow[dd] & \\
\mathrm{inn}(\Sigma^{n-1}\mathfrak{u}(1)) \arrow[r] \arrow[dd] & \mathrm{inn}(\mathfrak{g}_\mu) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{inn}(\mathfrak{g}) \arrow[dd] \\
& & TY \arrow[ur, "{(A,F_A)}"] \arrow[dd] & \\
\Sigma^{n}\mathfrak{u}(1) \arrow[r] & b\mathfrak{g}_\mu \arrow[rr, crossing over] & & b\mathfrak{g} \\
& & TX \arrow[ur, "{\{K_i\}}"] &
\end{tikzcd}
$$

we may want to try to factor it

$$
\begin{tikzcd}
\Sigma^{n-1}\mathfrak{u}(1) \arrow[r] \arrow[dd] & \mathfrak{g}_\mu \arrow[rr] \arrow[dd] & & \mathfrak{g} \arrow[dd] \\
& & TF \arrow[ul, dashed] \arrow[ur] \arrow[dd] & \\
\mathrm{inn}(\Sigma^{n-1}\mathfrak{u}(1)) \arrow[r] \arrow[dd] & \mathrm{inn}(\mathfrak{g}_\mu) \arrow[rr] \arrow[dd] & & \mathrm{inn}(\mathfrak{g}) \arrow[dd] \\
& & TY \arrow[ul, dashed, "{(A',F_{A'})}"] \arrow[ur, "{(A,F_A)}"'] \arrow[dd] & \\
\Sigma^{n}\mathfrak{u}(1) \arrow[r] & b\mathfrak{g}_\mu \arrow[rr] & & b\mathfrak{g} \\
& & TX \arrow[ul, dashed] \arrow[ur, "\{K_i\}"'] &
\end{tikzcd}
$$

...through a $\mathfrak{g}_\mu$-transport. That is: we may try to lift the $\mathfrak{g}$-bundle through the string-like extension $\mathrm{Lie}(\Sigma^{(n-1)}U(1)) \to \mathfrak{g}_\mu \to \mathfrak{g}$ to a $\mathfrak{g}_\mu$-bundle.

To measure the obstruction to being able to do this we postcompose with a suitably weak cokernel of $\mathfrak{g}_\mu \to \mathfrak{g}$.

The result...

$$
\begin{tikzcd}
 & & & & & \Sigma^n\mathfrak{u}(1) \arrow[dd] \\
 & & & & (\Sigma^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[ur, "\mathrm{coker}(i)"] \arrow[dd] & \\
\Sigma^{n-1}\mathfrak{u}(1) \arrow[r] \arrow[dd] & \mathfrak{g}_\mu \arrow[rr, hook] \arrow[urrr, hook, "i"] \arrow[dd] & & \mathfrak{g} \arrow[ur, "\simeq"] \arrow[dd] & & \mathrm{inn}(\Sigma^n\mathfrak{u}(1)) \arrow[dd] \\
 & & TF \arrow[ul, dashed] \arrow[ur] \arrow[dd] & & \mathrm{inn}(\Sigma^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[ur] \arrow[dd] & \\
\mathrm{inn}(\Sigma^{n-1}\mathfrak{u}(1)) \arrow[r] \arrow[dd] & \mathrm{inn}(\mathfrak{g}_\mu) \arrow[rr] \arrow[dd] & & \mathrm{inn}(\mathfrak{g}) \arrow[ur, "\simeq"] \arrow[dd] & & \Sigma^{n+1}\mathfrak{u}(1) \\
 & & TY \arrow[ul, "(A',F_{A'})"] \arrow[ur, "(A,F_A)"'] \arrow[dd] & & b(\Sigma^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[ur] & \\
\Sigma^n\mathfrak{u}(1) \arrow[r] & b\mathfrak{g}_\mu \arrow[rr] & & b\mathfrak{g} \arrow[ur, "\simeq"] & & \\
 & & TX \arrow[ul, dashed] \arrow[ur, "\{K_i\}"'] & & &
\end{tikzcd}
$$

...is a $\Sigma^n U(1)$-connection. This we call a Chern-Simons $(n+1)$-bundle with connection.

Proposition

The $(n+1)$-line bundle obstructing the lift of a $\mathfrak{g}$-bundle to a $\mathfrak{g}_\mu$-n-bundle for μ an $(n+1)$-cocycle in transgression with the invariant polynomial k has the characteristic class $k(F_A)$ with F_A the curvature of any connection on the original $\mathfrak{g}$-bundle.

n-Bundles with connection

$G_{(n)}$ n-bundles ($(n-1)$-gerbes) with connection are

- locally trivializable parallel transport n-functors
- or rather their curvature $(n+1)$-functors
- from the fundamental $(n+1)$-groupoid of the base space
- to (a representation of) the structure Lie n-group $G_{(n)}$
- or rather (locally) to its inner automorphism $(n+1)$-group $\mathrm{INN}_0(G_{(n)})$.

Examples of n-Bundles with connection

- Ordinary bundles with connection are parallel transport 1-functors
- $U(1)$ bundle gerbes with connection are descent data of $\Sigma U(1)$ 2-transport.
- Line bundle gerbes with connection are descent data of $1d\mathrm{Vect}$ 2-transport.
- Aschieri-Jurco nonabelian bundle gerbes with connection are descent data of $\mathrm{Bitor}(H)$ 2-transport.
- Breen-Messing nonabelian gerbe connection data is descent data for $\mathrm{INN}_0(\mathrm{AUT}(G))$ 3-curvature of 2-transport.
- Stolz-Teichner String connection is like associated $\mathrm{String}_k(G)$ 2-transport.

Universal n-bundles in terms of n-groupoids

- For every n-group $G_{(n)}$ there is an $(n+1)$-group $\mathrm{INN}_0(G_{()})$ of inner automorphisms.
- It sits in a sequence
$$Z(G_{(n)}) \to \mathrm{INN}_0(G_{(n)}) \to \mathrm{AUT}(G_{(n)}) \to \mathrm{OUT}(G_{(n)})$$
- Its underlying n-groupoid plays the role of the universal $G_{(n)}$-bundle $\quad G_{(n)} \to \mathrm{INN}_0(G_{(n)}) \to \Sigma G_{(n)}$
- For $n = 1$ shown by Segal in the 60s:
$$\xrightarrow{|\cdot|} (G \to EG \to BG)$$
- For $n = 2$ discussed in [RobertsSchreiber].

Passage between Lie n-groupoids and Lie n-algebroids

- Lie n-algebras and Lie n-algebroids are to Lie n-groups and Lie n-groupoids like Lie algebras are to Lie groups.
- A full n-Lie theorem – concerning differentiation of Lie n-groupoids and integration of Lie n-algebroids – is expected, even though only partially understood so far.
- Still, we can transfer structural understanding between the two realsm.
- Parallel n-transport is a morphism of Lie n-groupoids. Hence it corresponds differentially to a morphisms of Lie n-algebroids.

Passage between Lie n-algebras and differential algebra

- General abstract operad nonsense implies equivalence between Lie n-algebras and n-term L_∞-algebras, or their duals: free graded commutative algebras with a nilpotent degree $+1$ differential (qDGCAs).
- qDGCAs are useful for concrete computations.
- qDGCAs prevail in physics literature (compare in particular AKSZ-BV). Making the explicit n-categorical structure explicit is often useful.
- For instance pairing the qDGCA description with its understanding in terms of Lie n-algebra yields understanding of Lie n-algebra cohomology and n-characteristic classes.

Lie n-algebra cohomology

The notion of Lie-cocycle, invariant polynomial and transgression elements can be generalized to Lie n-algebras.

Lie algebra cocycle	μ	Baez-Crans Lie n-algebra	$\mathfrak{g}_\mu$
invariant polynomial	k	Chern Lie n-algebra	$\mathrm{ch}_k(\mathfrak{g})$
transgression element	cs	Chern-Simons Lie n-algebra	$\mathrm{cs}_k(\mathfrak{g})$

For every transgression element cs these fit into a weakly exact sequence

$$\mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g})\,.$$

Cokernels, mapping cones and inner derivations

- Crucial for considerations of $\mathfrak{g}_{(n)}$-connections is the strict cokernel
$$\mathfrak{h}_{(n)} \overset{t}{\hookrightarrow} \mathfrak{g}_{(n)} \twoheadrightarrow \mathrm{coker}(t)$$
of Lie n-algebra injections
- and its weak analog, the mapping cone Lie $(n+1)$-algebra $(\ \mathfrak{h}_{(n)} \overset{t}{\hookrightarrow} \mathfrak{g}_{(n)}\ )$.
-
    - $(\ \mathfrak{g}_{(n)} \xrightarrow{\mathrm{Id}} \mathfrak{g}_{(n)}\ ) = \mathrm{inn}(\mathfrak{g}_{(n)})$ is the inner derivation Lie $(n+1)$-algebra of $\mathfrak{g}_{(n)}$ – codomain for $\mathfrak{g}_{(n)}$-connections
    - $\mathrm{coker}(\ \mathfrak{g}_{(n)} \hookrightarrow \mathrm{inn}(\mathfrak{g}_{(n)})\ ) = b\mathfrak{g}_{(n)}$ is the Lie n'-algebra generated from the classes of invariant $\mathfrak{g}_{(n)}$ polynomials – it plays the role of the classifying space for $\mathfrak{g}_{(n)}$
    - $\mathrm{coker}(\mathfrak{g}_{(n)} \hookrightarrow (\ \mathfrak{h}_{(n)} \xrightarrow{t} \mathfrak{g}_{(n)}\ ))$ is the structure Lie n'-algebra for obstructions of extensions through $\mathfrak{g}_{(n)} \to \mathrm{coker}(t)$.

$\mathfrak{g}_{(n)}$-Bundles with connection

After passing from Lie n-groupoids to Lie n-algebroids

- The curvature $(n+1)$-functor
$$\mathrm{curv} : \Pi_{n+1}(P) \to \Sigma\mathrm{INN}_0(G_{(n)})$$
turns into a qDGCA morphism
$$\Omega^\bullet(P) \xleftarrow{\quad (A,F) \quad} \mathrm{inn}(\mathfrak{g}_{(n)})^* \ .$$

- The n-groupoid version of the universal $G_{(n)}$-bundle
$$G_{(n)} \to \mathrm{INN}(G_{(n)}) \to \Sigma\mathrm{INN}(G_{(n)})$$
turns into the sequence
$$\mathfrak{g}_{(n)}^* \longleftarrow \mathrm{inn}(\mathfrak{g}_{(n)})^* \longleftarrow b\mathfrak{g}_{(n)}^*$$

The n-Ehresmann conditon

And the descent condition on (A, F_A) says we have a pullback of the universal $G_{(n)}$-bundle in that

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{li}}(|G_{(n)}|) & & \mathfrak{g}^*_{(n)} \arrow[ll, "\simeq"'] \arrow[dll, Rightarrow, "="'] \\
\Omega^\bullet(P) \arrow[u, "i^*"] & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, "{(A,F_A)}"'] \arrow[u] \arrow[dll, Rightarrow, "\simeq"'] \\
\Omega^\bullet(X) \arrow[u, "p^*"] & & b\mathfrak{g}^*_{(n)} \arrow[ll, "{\{K_i = k_i(F_A)\}}"'] \arrow[u]
\end{tikzcd}
$$

n-Chern-Weil and characteristic classes

- Here $\Omega^\bullet(X) \xleftarrow{\{K_i = k_i(F_A)\}} b\mathfrak{g}^*_{(n)}$ is the n-Chern-Weil homomorphism, assigning the characteristic classes K_i to the given $(n+1)$-curvature F_A.
- For instance: the characteristic classes of $\mathfrak{g}_{\mu_k}$-bundles (String 2-bundles) are those of the underlying $\mathfrak{g}$-bundles, but modulo $K = k(F_A) = \langle F_A \wedge F_A \rangle$
$$b\mathfrak{g}^*_{\mu_k} \simeq b\mathfrak{g}/[k]\,.$$

Remark.

There is an obvious and straightforward generalization of all Lie n-algebra construction from the world of vector spaces to that of super vector spaces (i.e. to the category of $\mathbb{Z}_2$-graded vector spaces equipped with the unique nontrivial symmetric braiding).

The supergravity Lie 3-algebra

D'Auria and Fré noticed that (rephrased in our language) 11-dimensional supergravity is governed by the Baez-Crans type Lie 3-algebra

$$\mathrm{sugra}_{11} := \mathfrak{siso}(11)_\mu$$

coming from a 4-cocylce μ of the super-Poincaré Lie algebra $\mathfrak{siso}(11)$.

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coming from a 4-cocylce μ of the super-Poincaré Lie algebra $\mathfrak{siso}(11)$.

Sugra configurations are sugra_{11}-connections

A field configuration of supergravity is nothing but a $\mathrm{sugra}_1$1-connection

$$\Omega^\bullet(X) \xleftarrow{(A, F_A)} \mathrm{inn}(\mathrm{sugra}_{11})^* \ ,$$

where A encodes

- the graviton, in terms of
  - the vielbein
  - the spin connection
- the gravitino
- the 3-form field .

This suggests that 11-dimensional supergravity is a theory of $\mathfrak{siso}(11)_\mu$-bundles with connection.
The *n*-Ehresmann condition would give the global description.

Morphisms of 2-Functors

Strict morphisms between strict 2-functors simply preserve all compositions strictly. Still, the morphisms between these morphisms, called pseudonatural transformations, add a new crucial level of complexity.

Definition

Let $S \xrightarrow{F_1} T$ and $S \xrightarrow{F_2} T$ be two 2-functors. A **pseudonatural transformation**

$$\begin{tikzcd} S \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "F_2"'] \arrow[r, phantom, "\Downarrow \rho"] & T \end{tikzcd}$$

is . . .

a map

$$
\begin{tikzcd}
x \arrow[r, "\gamma"] & y
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swarrow \rho(\gamma)"] & F_1(y) \arrow[d, "\rho(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y)
\end{tikzcd}
$$

which is functorial in the sense that

$$
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma_1)"] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swArrow \rho(\gamma_1)"] & F_1(y) \arrow[r, "F_1(\gamma_2)"] \arrow[d, "\rho(y)"] \arrow[dr, phantom, "\swArrow \rho(\gamma_2)"] & F_1(z) \arrow[d, "\rho(z)"] \\
F_2(x) \arrow[r, "F_2(\gamma_1)"'] & F_2(y) \arrow[r, "F_2(\gamma_2)"'] & F_2(z)
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma_1 \circ \gamma_2)"] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swArrow \rho(\gamma_1 \circ \gamma_2)"] & F_1(z) \arrow[d, "\rho(z)"] \\
F_2(x) \arrow[r, "F_2(\gamma_1 \circ \gamma_2)"'] & F_2(z)
\end{tikzcd}
$$

and which makes the pseudonaturality tin can 2-commute

$$
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma_1)"] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swarrow \rho(\gamma_1)"] & F_1(y) \arrow[d, "\rho(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma_1)"] \arrow[r, bend right=60, "F_2(\gamma_2)"'] \arrow[r, phantom, shift right=4ex, "\Downarrow F_2(S)"] & F_2(y)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_1(x) \arrow[r, bend left=60, "F_1(\gamma_1)"] \arrow[r, phantom, shift left=4ex, "\Downarrow F_2(S)"] \arrow[r, "F_1(\gamma_2)"'] \arrow[d, "\rho(x)"'] \arrow[dr, phantom, "\swarrow \rho(\gamma_2)"] & F_1(y) \arrow[d, "\rho(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma_2)"] & F_2(y)
\end{tikzcd}
$$

for all
$$\begin{tikzcd} x \arrow[rr, bend left, "\gamma_1"] \arrow[rr, bend right, "\gamma_2"'] \arrow[rr, phantom, "\Downarrow s"] & & y \end{tikzcd} \in \mathrm{Mor}_2(S).$$

The vertical **composition of pseudonatural transformations**

$$
\begin{tikzcd}
S \arrow[rr, bend left=50, "F_1"] \arrow[rr, bend right=50, "F_3"'] \arrow[rr, phantom, "\Downarrow\rho"] && T
\end{tikzcd}
\;:=\;
\begin{tikzcd}
S \arrow[rr, bend left=60, "F_1", ""{name=U, below}] \arrow[rr, "F_2" description, ""{name=M}] \arrow[rr, bend right=60, "F_3"', ""{name=D, above}] && T
\arrow[Rightarrow, from=U, to=M, "\rho_1"]
\arrow[Rightarrow, from=M, to=D, "\rho_2"]
\end{tikzcd}
$$

is given by

$$
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho(x)"'] & F_1(y) \arrow[d, "\rho(y)"] \arrow[dl, phantom, "\swarrow \rho(\gamma)"] \\
F_3(x) \arrow[r, "F_3(\gamma)"] & F_3(y)
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho_1(x)"'] & F_1(y) \arrow[d, "\rho_1(y)"] \arrow[dl, phantom, "\swarrow \rho_1(\gamma)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"] \arrow[d, "\rho_2(x)"'] & F_2(y) \arrow[d, "\rho_2(y)"] \arrow[dl, phantom, "\swarrow \rho_2(\gamma)"] \\
F_3(x) \arrow[r, "F_3(\gamma)"] & F_3(y)
\end{tikzcd}
$$

Let $F_1 \xrightarrow{\rho_1} F_2$ $F_1 \xrightarrow{\rho_2} F_2$ be two pseudonatural transformations. A **modification** (of pseudonatural transformations)

$$\begin{tikzcd} F_1 \arrow[rr, bend left, "\rho_1"] \arrow[rr, bend right, "\rho_2"'] \arrow[rr, phantom, "\Downarrow\mathcal{A}"] & & F_2 \end{tikzcd}$$

is a map

$$\mathrm{Obj}(S) \ni x \mapsto \begin{tikzcd} F_1(x) \arrow[rr, bend left, "\rho_1(x)"] \arrow[rr, bend right, "\rho_2(x)"'] \arrow[rr, phantom, "\Downarrow\mathcal{A}(x)"] & & F_2(x) \end{tikzcd} \in \mathrm{Mor}_2(T)$$

such that

$$
\begin{tikzcd}[column sep=large, row sep=huge]
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho_1(x)"] \arrow[d, bend right=70, "\rho_2(x)"'] \arrow[d, phantom, shift right=6ex, "\overset{\Leftarrow}{\mathcal{A}(x)}"] \arrow[dr, phantom, "\swarrow \rho_1(\gamma)"] & F_1(y) \arrow[d, "\rho_1(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=large, row sep=huge]
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho_2(x)"] \arrow[dr, phantom, "\swarrow \rho_2(\gamma)"] & F_1(y) \arrow[d, "\rho_2(y)"'] \arrow[d, bend left=70, "\rho_1(y)"] \arrow[d, phantom, shift left=6ex, "\overset{\Leftarrow}{\mathcal{A}(y)}"] \\
F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y)
\end{tikzcd}
$$

for all $x \xrightarrow{\gamma} y \in \mathrm{Mor}_1(S)$.

Definition

The horizontal and vertical composite of modifications is, respectively, given by the horizontal and vertical composites of their component maps.

Definition

Let S and T be two 2-categories. The **2-functor 2-category** T^S is the 2-category

1. whose objects are functors $F : S \to T$
2. whose 1-morphisms are pseudonatural transformations
   $F_1 \xrightarrow{\rho} F_2$
3. whose 2-morphisms are modifications

$$\begin{tikzcd} F_1 \arrow[rr, bend left, "\rho_1"] \arrow[rr, bend right, "\rho_2"'] \arrow[rr, phantom, "\Downarrow \mathcal{A}"] & & F_2 \end{tikzcd} .$$

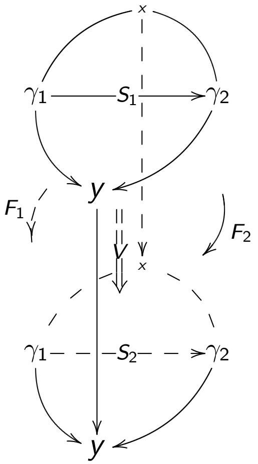
Morphisms of 3-functors

We shall regard 3-categories as special categories internal to $2\mathrm{Cat}$. From this point of view, a 3-category has a 2-category of objects S, each of which looks like

$$\begin{tikzcd} & x \arrow[dl, dash] \arrow[dr, dash] & \\ \gamma_1 \arrow[rr, "S"] \arrow[dr] & & \gamma_2 \arrow[dl] \\ & y & \end{tikzcd} .$$

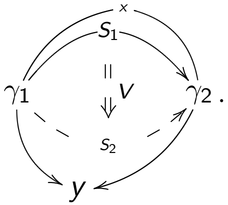
In a general category internal to $2\mathrm{Cat}$, we similarly have a

2-category of morphisms $S_1 \xrightarrow{V} S_2$, that look like

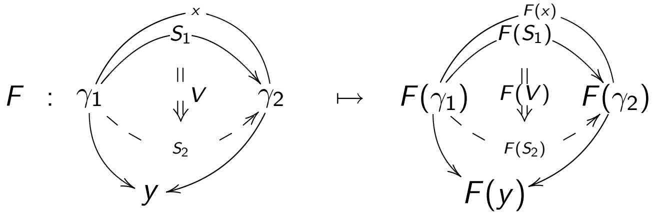


.

We shall restrict attention to the special case where the vertical faces here are identities. Then the above shape looks like



Instead of saying that V is a morphism of a category internal to $2\mathrm{Cat}$, we say V is a 3-morphism. Similarly, S_1, S_2 are 2-morphisms, γ_1, γ_2 are 1-morphisms and x and y are objects. We would have arrived at the same picture had we regarded categories enriched over $2\mathrm{Cat}$. However, we find that thinking of 3-morphisms as morphisms of a category internal to $2\mathrm{Cat}$ facilitates handling morphisms of 3-functors, to which we now turn. A 3-functor $F : S \to T$ between 3-categories S and T is a functor internal to $2\mathrm{Cat}$, hence a map



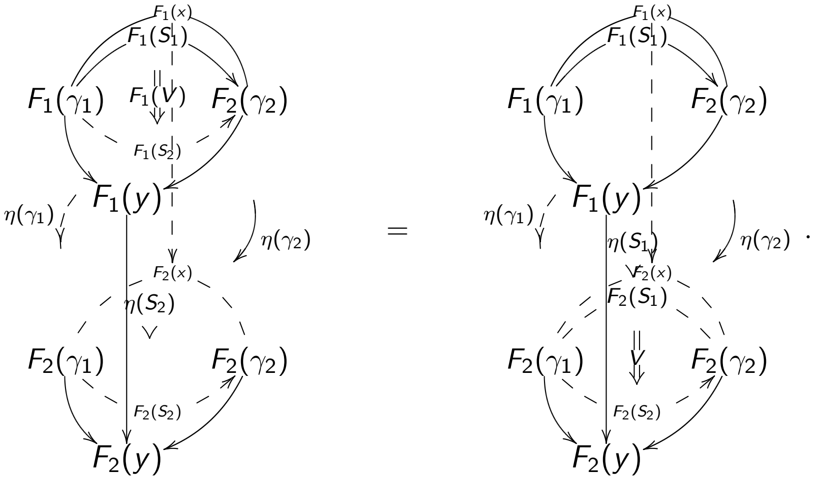
that respects vertical composition strictly and is 2-functorial up to coherent 3-isomorphisms with respect to the composition perpendicular to that.

A 1-morphism $F_1 \xrightarrow{\eta} F_2$ between two such 3-functors is a natural transformation internal to $2\mathrm{Cat}$, hence a 2-functor from the object 2-category to the morphism 2-category, hence a

2-functorial assignment

$$
\eta \quad : \quad
\begin{tikzcd}
 & x \arrow[dl, dash, bend right] \arrow[dr, dash, bend left] & \\
\gamma_1 \arrow[rr, "S"] \arrow[dr, bend right] & & \gamma_2 \arrow[dl, bend left] \\
 & y &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & F_1(x) \arrow[dl, dash, bend right] \arrow[dr, dash, bend left] \arrow[ddd, dashed] & \\
F_1(\gamma_1) \arrow[rr, "F_1(S)"] \arrow[dr, bend right] & & F_1(\gamma_2) \arrow[dl, bend left] \\
\arrow[r, phantom, "\eta(\gamma_1)"] & F_1(y) \arrow[ddd] \arrow[d, phantom, "\eta(S)"] & \arrow[l, phantom, "\eta(\gamma_2)"] \\
 & F_2(x) \arrow[dl, dash, dashed, bend right] \arrow[dr, dash, dashed, bend left] & \\
F_2(\gamma_1) \arrow[rr, "F_2(S)"] \arrow[dr, bend right] & & F_2(\gamma_2) \arrow[dl, bend left] \\
 & F_2(y) &
\end{tikzcd}
$$

that satisfies the naturality condition

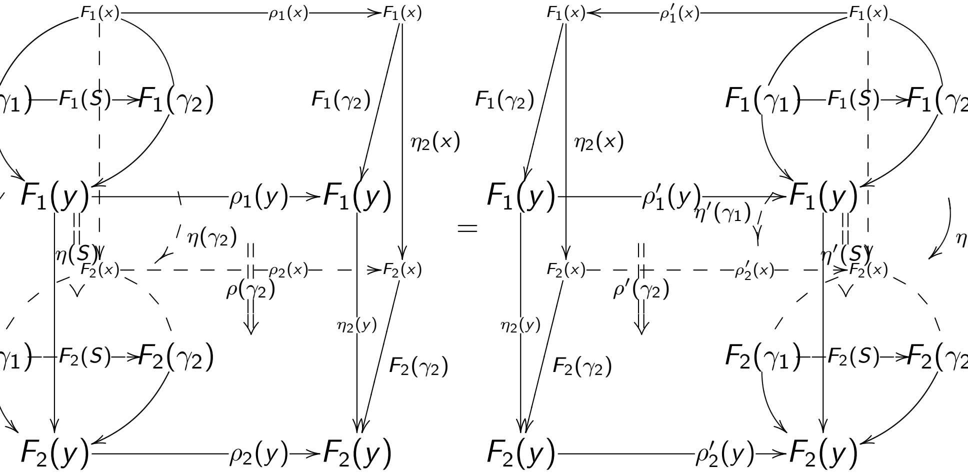


Accordingly, 2-morphisms and 3-morphisms of our 3-functors are 1-morphisms and 2-morphisms of these 2-functors η.

Hence a 2-morphism $\eta \xrightarrow{\rho} \eta'$ of our 3-functors is a 1-functorial assignment

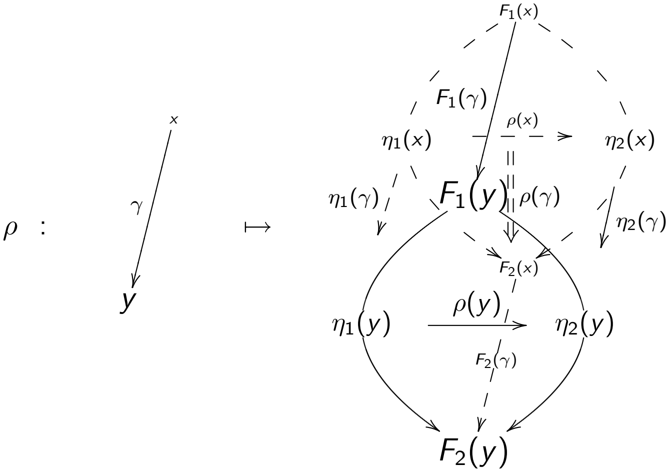
$$
\rho \ : \ \begin{tikzcd} x \arrow[d, "\gamma"'] \\ y \end{tikzcd} \quad \mapsto \quad
\begin{tikzcd}
 & F_1(x) \arrow[rr, "\rho_1(x)"] \arrow[dl, "F_1(\gamma)"'] \arrow[dd, dashed, "\eta_1(x)"] & & F_1(x) \arrow[dl, "F_1(\gamma)"'] \arrow[dd, "\eta_2(x)"] \\
F_1(y) \arrow[rr, "\rho_1(y)"] \arrow[dd, "\eta_1(y)"'] & & F_1(y) \arrow[dd, "\eta_2(y)"'] & \\
 & F_2(x) \arrow[rr, dashed, "\rho_2(x)"] \arrow[dl, dashed, "F_2(\gamma)"] & \arrow[u, phantom, "\Downarrow \rho(\gamma)"] & F_2(x) \arrow[dl, "F_2(\gamma)"] \\
F_2(y) \arrow[rr, "\rho_2(y)"'] & & F_2(y) &
\end{tikzcd}
$$

such that



We want to restrict attention to those ρ for which the horizontal

1-morphisms $\rho_1(x)$, $\rho_2(x)$, etc. are identities.

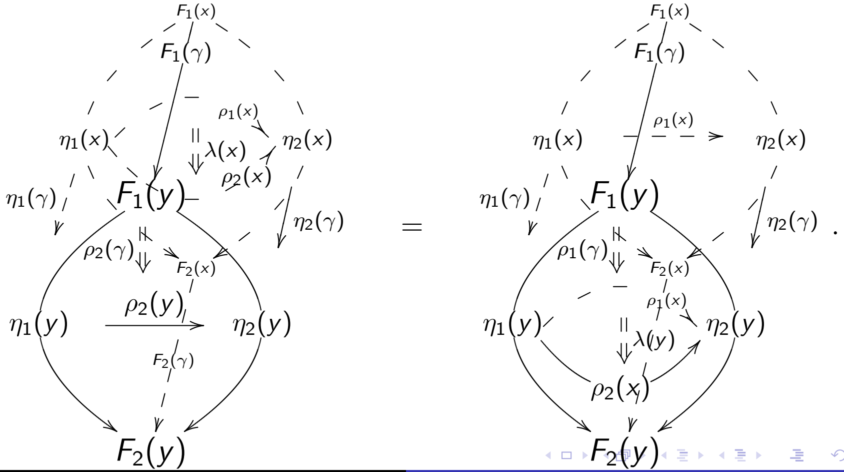


Proceeding this way, a modification $\lambda : \rho_1 \to \rho_2$ of transformations

ρ gives us a 3-morphisms of 3-functors. This now is a map

$$\lambda \;:\; x \;\mapsto\; \begin{tikzcd} & F_1(x) \arrow[dl, dash] \arrow[dr, dash] & \\ \eta_1(x) \arrow[dr] \arrow[rr, bend left, dashed, "\rho_1(x)"] \arrow[rr, bend right, "\rho_2(x)"'] \arrow[rr, phantom, "\Downarrow \lambda(x)"] & & \eta_2(x) \arrow[dl] \\ & F_2(x) & \end{tikzcd}$$

such that


$F_1(x)$
$F_1(\gamma)$
$\rho_1(x)$
$\eta_1(x)$
$\lambda(x)$
$\eta_2(x)$
$\rho_2(x)$
$\eta_1(\gamma)$
$F_1(y)$
$\eta_2(\gamma)$
$\rho_2(\gamma)$
$F_2(x)$
$\rho_2(y)$
$\eta_1(y)$
$\eta_2(y)$
$F_2(\gamma)$
$F_2(y)$
=
$F_1(x)$
$F_1(\gamma)$
$\rho_1(x)$
$\eta_1(x)$
$\eta_2(x)$
$\eta_1(\gamma)$
$F_1(y)$
$\eta_2(\gamma)$
$\rho_1(\gamma)$
$F_2(x)$
$\rho_1(x)$
$\eta_1(y)$
$\lambda(y)$
$\eta_2(y)$
$\rho_2(x)$
$F_2(y)$

Strict 2-groups and crossed modules of groups

It is an old result that strict 2-groups are isomorphic to crossed modules of ordinary groups. The isomorphism is in fact almost canonical: only two minor choices are involved.
When differentiating 2-functors with values in strict Lie 2-groups, we make extensive use of this equivalence, the precise realization of which is spelled out below.

Definition

A **crossed module of groups** is a diagram

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

in Grp (meaning all objects are groups and all arrows are group homomorphisms) such that

$$\begin{tikzcd} H \arrow[rr, "\mathrm{Ad}"] \arrow[dr, "t"'] & & \mathrm{Aut}(H) \\ & G \arrow[ur, "\alpha"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} G \times H \arrow[r, "\mathrm{Id}\times t"] \arrow[d, "\alpha"'] & G \times G \arrow[d, "\mathrm{Ad}"] \\ H \arrow[r, "t"] & G \end{tikzcd} .$$

Definition

A **strict 2-group** $G_{(2)}$ is any of the following equivalent entities

- a group object in Cat;
- a category object in Grp;
- a strict 2-groupoid with a single object.

As for groups, we shall write $G_{(2)}$ when we think of $G_{(2)}$ as a monoidal category, and $\Sigma G_{(2)}$ when we think of it as a 1-object 2-groupoid.

Proposition

Crossed modules of groups and strict 2-groups are isomorphic.

We now spell out this identification in detail. It is unique only up to a few conventional choices.

Our chosen isomorphism of 2-groups with crossed modules

The same is in principle already true for the identification of 1-groups with categories, which is unique only up to reversal of all arrows.
To start with, we take all principal actions to be from the *right*.

Our chosen isomorphism of 2-groups with crossed modules

So for G any group, $G\mathrm{Tor}$ denotes the category of right-principal G-spaces. This implies that if we want the canonical inclusion

$$i_G : \Sigma G \to G\mathrm{Tor}$$

to be covariant, we need to take composition in ΣG to work like

$$g_2 \circ g_1 = g_2 g_1 \,,$$

where on the left the composition is that of morphisms in ΣG, while on the right it is the product in G.

Our chosen isomorphism of 2-groups with crossed modules

Notice that this implies that diagrammatically we have

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \; = \; \bullet \xrightarrow{g_2 g_1} \bullet \, .$$

If G comes to us as a group of maps, we accordingly take the group product to be given by $g_2 g_1 := g_2 \circ g_1$.

Our chosen isomorphism of 2-groups with crossed modules

When we then pass to strict 2-groups $G_{(2)}$ coming from crossed modules $(t : H \to G)$ of groups, and want to label 2-morphisms in $\Sigma G_{(2)}$ with elements in H and G, we have one more convention to fix.

Our chosen isomorphism of 2-groups with crossed modules

Let $G_{(2)}$ be a (strict) 2-group which we may alternatively think of a crossed module $t\colon H \to G$. To recover $G_{(2)}$ from the crossed module $t\colon H \to G$ we set

$$\mathrm{Ob}(G_{(2)}) = G$$

$$\mathrm{Mor}(G_{(2)}) = G \ltimes H\,.$$

Here on the right we have the semidirect product group obtained from G and H using the action of G on H by way of α.

Our chosen isomorphism of 2-groups with crossed modules

A 2-morphism in $\Sigma G_{(2)}$ will be denoted by

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd}$$

for $g, g' \in G$ and $h \in H$, where g' will turn out to be fixed by $(g, h) \in G \ltimes H$. The semi-direct product structure on $G \ltimes H$, the source, target and composition homomorphisms are defined as follows.

Our chosen isomorphism of 2-groups with crossed modules

We shall agree that

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & \bullet
\end{tikzcd}
:=
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\mathrm{Id}"{name=U}] \arrow[r, bend right=50, "t(h)"'{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & \bullet \arrow[r, "g"] & \bullet
\end{tikzcd}
\;.
$$

Our chosen isomorphism of 2-groups with crossed modules

From the requirement that $t : H \to G$ be a homomorphism, it follows that

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "t(h')"'] \arrow[r, phantom, "\Downarrow h'"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "t(h'h)"'] \arrow[r, phantom, "\Downarrow h'h"] & \bullet
\end{tikzcd}
.
$$

Our chosen isomorphism of 2-groups with crossed modules

Together with the convention above this means that the source-target matching condition then reads

$$g' = g\, t(h)\,.$$

Our chosen isomorphism of 2-groups with crossed modules

The exchange law then implies that

$$
\begin{tikzcd}
\bullet \arrow[rr, bend left=70, "\mathrm{Id}"] \arrow[rr, "t(h)" description] \arrow[rr, bend right=70] \arrow[rr, phantom, shift left=4ex, "\Downarrow h"] \arrow[rr, phantom, shift right=4ex, "\Downarrow h'"] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[rr, bend left=50, "\mathrm{Id}"] \arrow[rr, bend right=50, "t(hh')"'] \arrow[rr, phantom, "\Downarrow hh'"] & & \bullet
\end{tikzcd}
\,.
$$

Our chosen isomorphism of 2-groups with crossed modules

Since in the crossed module we have $t(\alpha(g)(h)) = gt(h)g^{-1}$ we find that inner automorphisms in the 2-group have to be labeled like this:

$$
\begin{tikzcd}
\bullet \arrow[r, "g^{-1}"] & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "g"] & \bullet & = & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(\alpha(g)(h))"'] \arrow[r, phantom, "\Downarrow \alpha(g)(h)"] & \bullet
\end{tikzcd}
.
$$

Our chosen isomorphism of 2-groups with crossed modules

This then finally implies the rule for general horizontal compositions

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_1'"'] \arrow[r, phantom, "\Downarrow h_1"] & \bullet \arrow[r, bend left=50, "g_2"] \arrow[r, bend right=50, "g_2'"'] \arrow[r, phantom, "\Downarrow h_2"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=50, "g_2 g_1"] \arrow[r, bend right=50, "g_2' g_1'"'] \arrow[r, phantom, "\Downarrow \alpha_{g_1^{-1}}(h_2) h_1"] & \bullet \end{tikzcd} .$$

Tangent categories

Tangent categories are categories of images of the fat point $\{\bullet \stackrel{\simeq}{\to} \circ\}$ whose left end is fixed, while the right end is allowed to float.

Tangent categories are related to weak cokernels of identity morphisms, to inner automorphism $(n+1)$-groups, to vector fields on Lie n-groupoids and hence to Lie n-algebroids.

The sequences of Lie n-algebras which appeared in Bundles with Lie n-algebra connection and which were related to universal $\mathfrak{g}_{(n)}$-bundles have their origin in a very fundamental n-categorical construction which we address as the construction of *tangent n-categories*.

Definition

Let

$$2 := \{ \bullet \longrightarrow \circ \}$$

be the category wtih two objects and one nontrivial morphism, going between them.

Definition

For C any category, the tangent category TC is the strict pullback

$$\begin{tikzcd} TC \arrow[r] \arrow[d] & C^2 \arrow[d] \\ C_0 \arrow[r, hook] & C \end{tikzcd}$$

in Cat.

Proposition

- $\mathrm{Mor}(C) \to TC \to C$ is exact
- for C a (Lie) groupoid, $TC \simeq C_0$
- sections $\Gamma(TC)$ of $TC \to C_0$ inherit a 2-group structure through the inclusion $\Gamma(TC) \hookrightarrow T_{\mathrm{Id}}\mathrm{End}(C)$
- $\Gamma_{\mathbb{R}}(TC) := \mathrm{Hom}(\mathbb{R}, \Gamma(TC))$ is the Lie algebroid of C
- for $C = \Sigma G$, $TC := \mathrm{INN}(G)$ is the inner automorphism 2-group of G.

Remark.

These statements have more or less obvious generalizations to $n > 1$. For $n = 2$ this is done in [RobertsSchreiber]

Inner automorphism $(n+1)$-Groups

- Every n-group $G_{(n)}$ has an $(n+1)$-group $\mathrm{AUT}(G_{(n)})$ of automorphisms.
- This sits inside an exact sequence $1 \to Z(G_{(n)}) \to \mathrm{INN}(G_{(n)}) \to \mathrm{AUT}(G_{(n)}) \to \mathrm{OUT}(G_{(n)}) \to 1$
- and INN_0 plays the role of the universal $G_{(n)}$-bundle $G_{(n)} \to \mathrm{INN}_0(G_{(n)}) \to \Sigma G_{(n)}$

[David Roberts, U.S.]

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[David Roberts, U.S.]

Observation

Given a cover $Y \to X$ and a G-cycle $g : Y^{[2]} \to \Sigma G$ its pullback

$$\begin{tikzcd} Y^{[2]} \times_g \mathrm{INN}(G) \arrow[r] \arrow[d] & \mathrm{INN}(G) \arrow[d] \\ Y^{[2]} \arrow[r, "g"] & \Sigma G \end{tikzcd}$$

plays the role of the total space of the G-bundle clasified by g.

Analogous statements hold for $n > 1$.

Mapping Cones

The notion of tangent categories generalizes to a notion of mapping cones of n-categories.

Definition

The Gray groupoid which we denote either $T\Sigma G_{(2)}$ and address it as the tangent 2-groupoid of $\Sigma G_{(2)}$, or $\mathrm{INN}_0(G_{(2)})$ and address it as the inner automorphism 2-groupoid of $\Sigma G_{(2)}$ or simply

$\left(G_{(2)} \xrightarrow{\mathrm{Id}} G_{(2)} \right)$ and address it as the mapping cone of $\mathrm{Id}_{G_{(2)}}$ or as the 2-crossed module induced by $\mathrm{Id}_{G_{(2)}}$.

This 2-groupoid $T\Sigma G_{(2)}$ is defined to be the the strict pullback

$$\begin{tikzcd}
T\Sigma G_{(2)} \arrow[r] \arrow[d] & (\Sigma G_{(2)})^2 \arrow[d, "\mathrm{dom}"] \\
\{\bullet\} \arrow[r] & \Sigma G_{(2)}
\end{tikzcd}$$

An object of $T\Sigma G_{(2)}$ is a morphism

$$\bullet \xrightarrow{q} \bullet$$

in $\Sigma G_{(2)}$, hence an object of $G_{(2)}$.

A 1-morphism in $T\Sigma G_{(2)}$ is a filled triangle

$$\begin{tikzcd} & \bullet \arrow[dd, "f"] \\ \bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "\Downarrow F"] & \\ & \bullet \end{tikzcd}$$

in $\Sigma G_{(2)}$.

Finally, a 2-morphism in $T^t \Sigma G_{(2)}$ looks like

$$
\begin{tikzcd}
 & \bullet \arrow[dd, bend right=40, "f"'] \arrow[dd, bend left=40, "f'"] \arrow[dd, phantom, "\overset{L}{\Longrightarrow}"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "F \Downarrow \; \Downarrow F'"] & \\
 & \bullet
\end{tikzcd}
\,.
$$

The monoidal structure on $T\Sigma G_{(2)}$ is that induced from the embedding

$$T\Sigma G_{(2)} := \mathrm{INN}_0(\Sigma G_{(2)}) \hookrightarrow \mathrm{AUT}(G_{(2)})$$

discussion in [RobertsSchreiber:2007].
This canonically sits in the sequence

$$G_{(2)} \hookrightarrow T\Sigma G_{(2)} \twoheadrightarrow \Sigma G_{(2)} \ .$$

Mapping cone of a faithful morphism

This has an obvious generalization to non-identity but faithful morphisms:

Let $G_{(2)}$ and $H_{(2)}$ be strict 2-groups and write $\Sigma G_{(2)}$ and $\Sigma H_{(2)}$ be the corresponding strict one object 2-groupoids.

Let

$$t : H_{(2)} \hookrightarrow G_{(2)}$$

be a morphism of strict 2-groups, faithful as a functor of the underlying 1-groupoids. This means we have a strict 2-functor

$$\Sigma t : \Sigma H_{(2)} \hookrightarrow \Sigma G_{(2)} .$$

Definition

The morphism t defines a strict 2-groupoid with a weak monoidal structure that makes it a Gray groupoid, which we denote either $T^t\Sigma G_{(2)}$ and address it as the tangent 2-groupoid of $\Sigma G_{(2)}$ relative to t, or $\mathrm{INN}_0^t(G_{(2)})$ and address it as the inner automorphism 2-groupoid of $\Sigma G_{(2)}$ relative to t or simply $(\; H_{(2)} \xrightarrow{t} G_{(2)} \;)$ and address it as the mapping cone of t or as the 2-crossed module induced by t.

This 2-groupoid $T^t\Sigma G_{(2)}$ is defined to be the the strict pullback

$$\begin{tikzcd}
 & & T^t\Sigma G_{(2)} \arrow[dll] \arrow[d] \arrow[drr] & & \\
\{\bullet\} \arrow[dr] & & (\Sigma G_{(2)})^2 \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{codom}"] & & \Sigma H_{(2)} \arrow[dl, "\Sigma t"'] \\
 & \Sigma G_{(2)} & & \Sigma G_{(2)} &
\end{tikzcd}$$

Here

$$2 := \{ \bullet \xrightarrow{\simeq} \circ \}$$

is the fat point.

Equivalently this means that $T^t \Sigma G_{(2)}$ is the strict pullback

$$\begin{tikzcd} T^t \Sigma G_{(2)} \arrow[r] \arrow[d] & \Sigma G_{(2)} \arrow[d, "="] \\ \Sigma H_{(2)} \arrow[r, "\Sigma t"] & \Sigma G_{(2)} \end{tikzcd} .$$

An object of $T^t \Sigma G_{(2)}$ is a morphism

$$\bullet \xrightarrow{q} \bullet$$

in $\Sigma G_{(2)}$, hence an object of $G_{(2)}$.

A 1-morphism in $T^t\Sigma G_{(2)}$ is a filled triangle

$$
\begin{tikzcd}
 & \bullet \arrow[dd, "t(f)"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "\Downarrow F"] & \\
 & \bullet
\end{tikzcd}
$$

in $\Sigma G_{(2)}$, with f a morphism in $\Sigma H_{(2)}$, hence an object of $H_{(2)}$.

Finally, a 2-morphism in $T^t \Sigma G_{(2)}$ looks like

$$
\begin{tikzcd}
 & \bullet \arrow[dd, bend right, "t(f)"'] \arrow[dd, bend left, "t(f')"] \arrow[dd, phantom, "\overset{t(L)}{\Rightarrow}"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "F \Downarrow \; \Downarrow F'"] & \\
 & \bullet
\end{tikzcd}
$$

with

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow L"] & \bullet
\end{tikzcd}
$$

a 2-morphism in $\Sigma H_{(2)}$, hence a morphism in $H_{(2)}$.

The monoidal structure on $T^t \Sigma G_{(2)}$ is that induced from the embedding

$$T^t \Sigma G_{(2)} \hookrightarrow T \Sigma G_{(2)} \,.$$

Proposition

The 2-groupoid $T^t \Sigma G_{(2)}$ is codiscrete at top level. Therefore it is equivalent to its quotient by its 2-morphisms

$$T^t \Sigma G_{(2)} \simeq \pi_1(T^t \Sigma G_{(2)}) \,.$$

This quotient is isomorphic to what in [CarrascoGarzónVitale:2006] is called (p. 595) the *quotient pointed groupoid*: $G_{(2)}/\langle H_{(2)}, t\rangle$:

$$\pi_1(T^t \Sigma G_{(2)}) \simeq G_{(2)}/\langle H_{(2)}, t\rangle \,.$$

[CarrascoGarzónVitale:2006] prove that $G_{(2)}/\langle H_{(2)}, t\rangle$ is indeed the cokernel of t. See the last paragraph on p. 595 and item 2 on p. 596.

$G_{(n)}$-bundles with connection from universal $\mathrm{INN}_0(G_{(n)})$-bundles

The following presents the arrow-theory of universal n-bundles and their pullbacks and connections (explicitly only for $n = 1$) in a way that shows how the definition of bundles with $\mathfrak{g}_{(n)}$-connection arises.

$$
\begin{tikzcd}
G \arrow[d, hook] \\
\mathrm{INN}(G) \arrow[d, two heads] \\
\Sigma G
\end{tikzcd}
$$

The universal G 1-bundle.

$$\begin{tikzcd} G \arrow[d, hook] \\ \mathrm{INN}(G) \arrow[d, two heads] \\ \Sigma G \end{tikzcd}$$

The universal G 1-bundle. Now suppose that $G = U(1)$.

$$\begin{tikzcd} G \arrow[d, hook] \\ \mathrm{INN}(G) \arrow[d, two heads] \\ \Sigma G \end{tikzcd}$$

The universal G 1-bundle. Then ΣG is itself a 2-group.

$$
\begin{tikzcd}
G \arrow[d, hook] \\
\mathrm{INN}(G) \arrow[d, two heads] \\
\Sigma G
\end{tikzcd}
$$

The universal G 1-bundle. And what used to be the classifying space for G 1-bundles...

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[d, two heads] & & \\
\Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

…becomes the fiber of the universal ΣG 2-bundle.

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[d, two heads] & & \\
\Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Given a space X, let $\Pi_2(X)$ be its fundamental 2-groupoid.

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[d, two heads] & & \Pi_2(X) \\
\Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Given a space X, let $\Pi_2(X)$ be its fundamental 2-groupoid.

$$\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[d, two heads] & & \Pi_2(X) \\
\Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}$$

Given a space X, let $\Pi_2(X)$ be its fundamental 2-groupoid.

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
& & \Pi_2(X) \\
\Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Then a (smooth) morphism from $\Pi_2(X)$ to $\Sigma\Sigma G$

$$\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
& & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}$$

Then a (smooth) morphism from $\Pi_2(X)$ to $\Sigma\Sigma G$

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
 & & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Is a choice of 2-form $K \in \Omega^2(X)$ on X.

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
& & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

This we may regard as a trivial $\Sigma^2 G$ 2-bundle with connection on X.

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
& & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Hence we may ask if we can lift the structure 2-group

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
& & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Hence we may ask if we can lift the structure 2-group through $\Sigma \mathrm{INN}(G) \to \Sigma\Sigma G$.

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
& & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Hence we may ask if we can lift the structure 2-group through $\Sigma\mathrm{INN}(G) \to \Sigma\Sigma G$. We can, if we can form

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd] & & \\
& \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[d, "{(g,\mathrm{tra},\mathrm{curv})}"'] & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Hence we may ask if we can lift the structure 2-group through $\Sigma\mathrm{INN}(G) \to \Sigma\Sigma G$. We can, if we can form this.

$$
\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
& \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[d, "{(g,\mathrm{tra},\mathrm{curv})}"'] & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

Here $\pi : Y \to X$ is a choice of cover of X.

$$\begin{tikzcd}
G \arrow[d, hook] & & \\
\mathrm{INN}(G) \arrow[dd, two heads] & & \\
& \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[d, "{(g,\mathrm{tra},\mathrm{curv})}"'] & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}$$

And $\mathcal{C}_2(Y)$ is generated from $\Pi_2(Y)$ and from $Y^{[2]}$, modulo an obvious relation.

$$
\begin{tikzcd}
 & G \arrow[d, hook] & & & & \\
 & \mathrm{INN}(G) \arrow[dd, two heads] & & & & \\
Y^{[2]} \arrow[rr, hook] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

And $\mathcal{C}_2(Y)$ is generated from $\Pi_2(Y)$ and from $Y^{[2]}$, modulo an obvious relation.

$$\begin{tikzcd}
 & G \arrow[d, hook] & & & & \\
 & \mathrm{INN}(G) \arrow[dd, two heads] & & & & \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}$$

Hence $g : Y^{[2]} \to \Sigma G$ is the classifying map of a G 1-bundle.

$$
\begin{tikzcd}
 & G \arrow[d, hook] & & & & \\
 & \mathrm{INN}(G) \arrow[dd, two heads] & & & & \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "(g{,}\mathrm{tra}{,}\mathrm{curv})"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

While the smooth parallel transport 2-functor $(\mathrm{tra}, \mathrm{curv}) : \Pi_2(Y) \to \Sigma\mathrm{INN}(G)$ encodes a compatible connection 1-form A and its curvature 2-form F_A.

$$
\begin{tikzcd}
 & G \arrow[d, hook] & & & & \\
 & \mathrm{INN}(G) \arrow[dd, two heads] & & & & \\
Y^{[2]} \arrow[rrr, hook] \arrow[dr, "g"'] & & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "(g{,}\mathrm{tra}{,}\mathrm{curv})"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rrr, hook] & & & \Sigma \mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

Requiring the left square to commute is the gluing condition on a G-bundle with connection.

$$\begin{tikzcd}
 & G \arrow[d, hook] & & \\
 & \mathrm{INN}(G) \arrow[dd, two heads] & & \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[d, "{(g,\mathrm{tra},\mathrm{curv})}"] & \Pi_2(X) \arrow[d, "K"] \\
 & \Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}$$

Requiring the right square to commute says that the 2-form $K = F_A$ is the curvature 2-form of this connection.

$$
\begin{tikzcd}
 & G \arrow[d, hook] & & & & \\
 & \mathrm{INN}(G) \arrow[dd, two heads] & & & & \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "(g,\mathrm{tra},\mathrm{curv})"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

Requiring the right square to commute up to natural isomorphism says that K represents the Chern class of g.

$$
\begin{tikzcd}
 & G \arrow[d] & & & & \\
 & \mathrm{INN}(G) \arrow[dd] & & & & \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "(g,\mathrm{tra},\mathrm{curv})"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

Finally, we obtain the total "space" of the G-bundle thus classified by pulling back g along $\mathrm{INN}(G) \to \Sigma G$.

$$
\begin{tikzcd}
 & G \arrow[dd, hook] & & & & \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dd] \arrow[dr] & & & & & \\
 & \mathrm{INN}(G) \arrow[dd, two heads] & & & & \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma \mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

Finally, we obtain the total space of the G-bundle thus classified by pulling back g along $\mathrm{INN}(G) \to \Sigma G$.

$$
\begin{tikzcd}
 & G \arrow[d, hook] & & & & \\
 & Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[ddl] \arrow[d] \arrow[d, two heads] & & & & \\
 & \mathrm{INN}(G) \arrow[dd, two heads] & & & & \\
Y^{[2]} \arrow[rrr, hook] \arrow[dr, "g"'] & & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "(g{,}\mathrm{tra}{,}\mathrm{curv})"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rrr, hook] & & & \Sigma \mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

There is in fact an entire lattice of universal n-bundles in the background.

$$
\begin{tikzcd}
 & G \arrow[r, hook] \arrow[dd, hook] & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & & & \\
 & \mathrm{INN}(G) \arrow[r, hook] \arrow[dd, two heads] & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd, two heads] \arrow[d, equal] & & \Sigma\mathrm{INN}(G) \arrow[ddl, two heads] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[dl, Rightarrow, "\simeq"'] \arrow[d, "{(g,\mathrm{tra},\mathrm{curv})}"] & \Pi_2(X) \arrow[d, "K"] & \\
 & \Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G &
\end{tikzcd}
$$

There is in fact an entire lattice of universal n-bundles in the background.

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dl, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd, two heads] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dl, equal] \arrow[dd, two heads] & & \Sigma\mathrm{INN}(G) \arrow[dd, two heads] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dl, Rightarrow, "\simeq"'] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"'] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

Where the middle row and column give the universal $\mathrm{INN}(G)$ 2-bundle.

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[dll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dd] \arrow[dr] & & & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd, two heads] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd, two heads] & & \Sigma\mathrm{INN}(G) \arrow[dd, two heads] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[ur, equal] \arrow[dl, Rightarrow, "\simeq"'] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] & & \Sigma\Sigma G
\end{tikzcd}
$$

Notice that, since $\mathrm{INN}(G)$ is trivializable, that universal 2-bundle admits a canonical 2-section e.

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd, two heads] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd, two heads] & & \Sigma\mathrm{INN}(G) \arrow[dd, two heads] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[ur, equal] \arrow[dl, Rightarrow, "\simeq"'] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"'] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

Notice that, since $\mathrm{INN}(G)$ is trivializable, that universal 2-bundle admits a canonical 2-section e.

$$
\begin{tikzcd}
 & & G \arrow[rr, hook] \arrow[ddl, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[ddl, hook] & & \Sigma G \arrow[ddl, hook] \arrow[ddlll, Rightarrow, "\simeq"'] \\
 & Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[u] \arrow[d] \arrow[ddl] & & & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd, two heads] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd, two heads] & & \Sigma\mathrm{INN}(G) \arrow[dd, two heads] & \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[ur, equal] \arrow[dl, Rightarrow, "\simeq"'] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"'] & & \Pi_2(X) \arrow[dr, "K"] & & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G &
\end{tikzcd}
$$

We can further pull back our data along this lattice, for instance in the middle.

$$
\begin{tikzcd}
 & & G \arrow[rr, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
 & Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dr] \arrow[dd] & & & \\
 & & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd] & & \Sigma\mathrm{INN}(G) \arrow[dd] \\
 & Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[ur, equal] \arrow[dl, Rightarrow, "\simeq"'] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

We can further pull back our data along this lattice, for instance in the middle.

$$
\begin{tikzcd}
 & & G \arrow[rr, hook] \arrow[ddl, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dr] \arrow[dd] & & & \\
 & \mathrm{INN}(G) \arrow[rrr, hook] \arrow[dd, two heads] & & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd, two heads] \arrow[dl, equal] & & \Sigma\mathrm{INN}(G) \arrow[dd, two heads] \\
Y^{[2]} \arrow[rrr, hook] \arrow[dr, "g"'] & & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dll, Rightarrow, "\simeq"] \arrow[dr, "(g,\mathrm{tra},\mathrm{curv})"] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rrr, hook] & & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, bend right, hook, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

This yields essentially the Atiyah groupoid $\mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G))$.

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[ddll, Rightarrow, "\simeq"'] \arrow[dd, hook] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dr] \arrow[dd] & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd] & & \Sigma\mathrm{INN}(G) \arrow[dd] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "(g,\mathrm{tra},\mathrm{curv})"] \arrow[dl, Rightarrow, "\simeq"'] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

And we find that the choice $(g, \mathrm{tra}, \mathrm{curv})$ lifts the canonical section e to a splitting of the Atiyah groupoid projection.

$$\begin{tikzcd}
 & & G \arrow[rr, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
 & Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dd] \arrow[dr] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dd] \arrow[dr] & & & \\
 & & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd] & & \Sigma\mathrm{INN}(G) \arrow[dd, two heads] \\
 & Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] \arrow[dl, Rightarrow, "\simeq"'] \arrow[uu, Rightarrow, hook, bend left] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}$$

And we find that the choice $(g, \mathrm{tra}, \mathrm{curv})$ lifts the canonical section e to a splitting of the Atiyah groupoid projection.

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dd] \arrow[dr] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dd] \arrow[dr] & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd] & & \Sigma\mathrm{INN}(G) \arrow[dd] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dl, Rightarrow, "\simeq"] \arrow[dr, "(g,\mathrm{tra},\mathrm{curv})"] \arrow[uu, bend right, Rightarrow] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

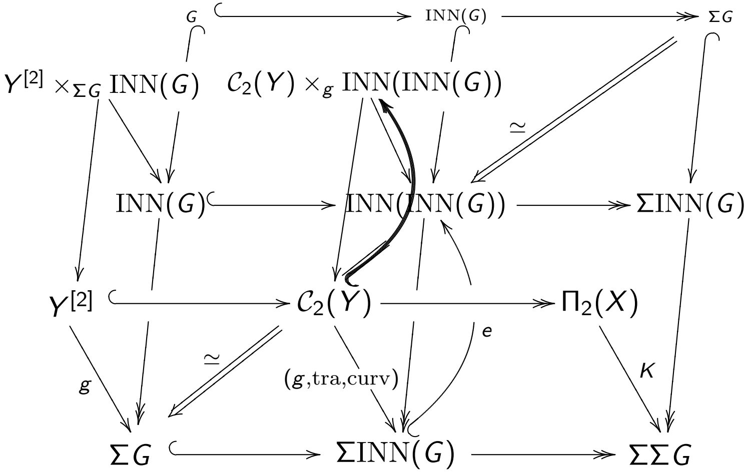
We should probably read this as follows:

$$
\begin{tikzcd}
 & G \arrow[rrr, hook] \arrow[dd, hook] & & & \mathrm{INN}(G) \arrow[rrr, two heads] \arrow[dd, hook] & & & \Sigma G \arrow[dd, hook] \arrow[dlll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dr] \arrow[dd] & & & & \\
 & \mathrm{INN}(G) \arrow[rrr, hook] \arrow[dd] & & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rrr, two heads] \arrow[dd] & & & \Sigma\mathrm{INN}(G) \arrow[dd] \\
Y^{[2]} \arrow[rrr, hook] \arrow[dr, "g"'] & & & \mathcal{C}_2(Y) \arrow[rrr, two heads] \arrow[dll, Rightarrow, "\simeq"] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] \arrow[uu, hook, bend left] & & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rrr, hook] & & & \Sigma\mathrm{INN}(G) \arrow[rrr, two heads] \arrow[uu, hook, bend right, "e"'] & & & \Sigma\Sigma G
\end{tikzcd}
$$

We should probably read this as follows:
$\Sigma\mathrm{INN}(G)$ plays the role of the fundamental 2-groupoid of BG.

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[dll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dr] \arrow[dd] & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd] & & \Sigma\mathrm{INN}(G) \arrow[dd] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dl, Rightarrow, "\simeq"] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] \arrow[uu, hook, bend right, Rightarrow] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

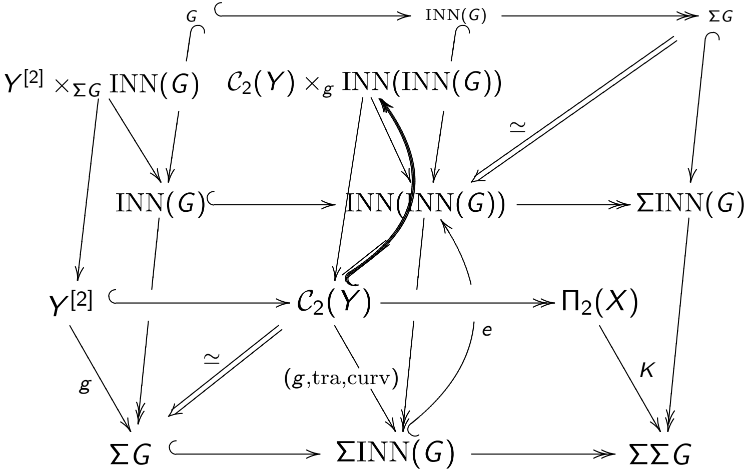
We should probably read this as follows:
The section e of the $\mathrm{INN}(G)$ 2-bundle plays the role of the universal connection on the universal G-bundle.



We should probably read this as follows:
The choice $(g, \mathrm{tra}, \mathrm{curv})$ pulls back the universal connection.

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dl, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dl, hook] \arrow[dd, hook] & & \Sigma G \arrow[dll, Rightarrow, "\simeq"'] \arrow[dd, hook] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dr] \arrow[dd] & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd, two heads] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd, two heads] & & \Sigma \mathrm{INN}(G) \arrow[dd, two heads] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dl, Rightarrow, "\simeq"] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] \arrow[uu, Rightarrow, bend right, hook] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma \mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

Finally, recall that we assumed G to be abelian.



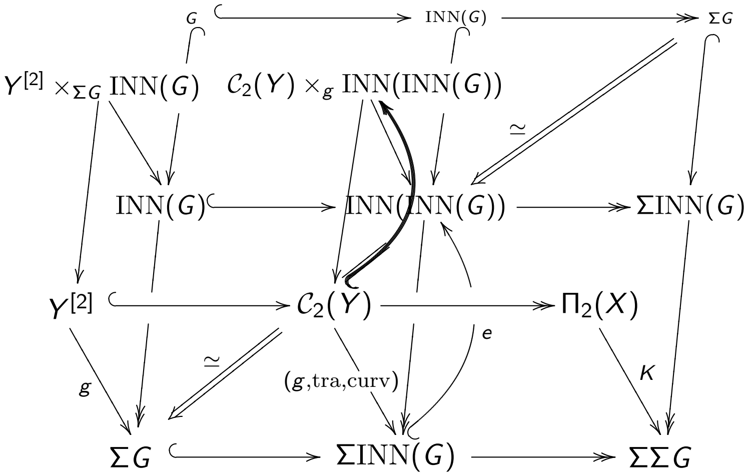
The reason is that otherwise the 2-groupoid $\Sigma\Sigma G$ does not exist.

$$
\begin{tikzcd}
 & & G \arrow[rr, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dd, hook] & & \Sigma G \arrow[ddll, Rightarrow, "\simeq"'] \arrow[dd, hook] \\
 & Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dr] \arrow[dd] & & & \\
 & & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd] & & \Sigma\mathrm{INN}(G) \arrow[dd] \\
 & Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr] \arrow[dl, Rightarrow, "\simeq"'] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"'] \arrow[uu, hook, Rightarrow, bend right] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

But we shall pass to the differential picture now,. . .

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dl, hook] \arrow[dd, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dl, hook] \arrow[dd, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dr] \arrow[dd] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dr] \arrow[dd] & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd] & & \Sigma\mathrm{INN}(G) \arrow[dd] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dl, Rightarrow, "\simeq"] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] \arrow[uu, hook, bend right] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

...and find that for nonabelian G, $\Sigma\Sigma G$ may be thought of as being replaced by an r-groupoid...



…for r the degree of the highest generator of the algebra of invariant polynomials of $\mathfrak{g} = \mathrm{Lie}(G)$.

$$
\begin{tikzcd}
 & G \arrow[rr, hook] \arrow[dl, hook] & & \mathrm{INN}(G) \arrow[rr, two heads] \arrow[dl, hook] & & \Sigma G \arrow[dd, hook] \arrow[ddll, Rightarrow, "\simeq"'] \\
Y^{[2]} \times_{\Sigma G} \mathrm{INN}(G) \arrow[dd] \arrow[dr] \arrow[dr, shift left] & & \mathcal{C}_2(Y) \times_g \mathrm{INN}(\mathrm{INN}(G)) \arrow[dd] \arrow[dr] \arrow[dr, shift left] & & & \\
 & \mathrm{INN}(G) \arrow[rr, hook] \arrow[dd] & & \mathrm{INN}(\mathrm{INN}(G)) \arrow[rr, two heads] \arrow[dd] & & \Sigma\mathrm{INN}(G) \arrow[dd] \\
Y^{[2]} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathcal{C}_2(Y) \arrow[rr, two heads] \arrow[dl, Rightarrow, "\simeq"'] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}"] \arrow[uu, Rightarrow, bend right] & & \Pi_2(X) \arrow[dr, "K"] & \\
 & \Sigma G \arrow[rr, hook] & & \Sigma\mathrm{INN}(G) \arrow[rr, two heads] \arrow[uu, hook, bend right, "e"'] & & \Sigma\Sigma G
\end{tikzcd}
$$

To get there, we first suppress everything except for the front face of our diagram...

$$
\begin{tikzcd}
Y^{[2]} \arrow[r, hook] \arrow[d, "g"'] & \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[d, "{(g,\mathrm{tra},\mathrm{curv})}"] & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma\mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

$$
\begin{tikzcd}
Y^{[2]} \arrow[r, hook] \arrow[dr, "g"'] & \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[dr, "(g,\mathrm{tra},\mathrm{curv})"] & \Pi_2(X) \arrow[dr, "K"] & \\
& \Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}
$$

...and then restrict attention to the special case where we take the cover Y to be the total space P of the G-bundle $P \to X$ itself, $Y = P$.

$$\begin{tikzcd}
Y^{[2]} \arrow[r, hook] \arrow[dr, "g"'] & \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[dr, "{(g,\mathrm{tra},\mathrm{curv})}" description] & \Pi_2(X) \arrow[dr, "K"] & \\
& \Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}$$

Then the cocycle data $g : Y^{[2]} \to \Sigma G$ is canonically given as $g : (p, p \cdot g_1) \mapsto g_1$.

$$\begin{tikzcd}
Y^{[2]} \arrow[r, hook] \arrow[d, "g"'] & \mathcal{C}_2(Y) \arrow[r, two heads] \arrow[d, "{(g,\mathrm{tra},\mathrm{curv})}"] & \Pi_2(X) \arrow[d, "K"] \\
\Sigma G \arrow[r, hook] & \Sigma \mathrm{INN}(G) \arrow[r, two heads] & \Sigma\Sigma G
\end{tikzcd}$$

This way we should arrive at the following differential formulation...

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{li}(G)} & \Omega^\bullet(P) \arrow[l, two heads, "i^*"'] & \Omega^\bullet(X) \arrow[l, hook', "p^*"'] \\
\mathfrak{g}^* \arrow[u, "\simeq"] & \mathrm{inn}(\mathfrak{g})^* \arrow[l, two heads] \arrow[u, "{(A,F_A)}"] & b\mathfrak{g}^* \arrow[l, hook'] \arrow[u, "K"']
\end{tikzcd}
$$