Lie n-Algebra Cohomology

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Abstract

Ordinary Lie algebra cohomology of a Lie algebra \mathfrak{g} has a nice reformulation in terms of the Koszul dual differential algebra of the Lie 2-algebra of inner derivations of \mathfrak{g} . For every transgressive degree n element in \mathfrak{g} cohomology there is a short exact sequence of Lie n-algebras. These are characterized by the fact that n-connections taking values in them come from the corresponding Chern-Simons forms and characteristic classes.

A straightforward generalization of this construction yields a notion of cohomology, invariant polynomials and transgression elements for arbitrary Lie *n*-algebras. And in turn, each such element of degree d induces a new Lie max(n, d)-algebra.

From the invariant polynomials of a Lie n-algebra one obtains characteristic classes of the corresponding n-bundles.

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1 Introduction

URS: This is part of the stuff in collaboration with Jim Stashaeff which I am currently polishing. All mistakes and imperfections are mine. In particular, the last section about Lie *n*-algebra cohomology is a more recent contribution of mine, which still needs to be scrutinized. This addresses an issue I knew should have a nice answer based on the other stuff, but didn't seriously start looking into before John Baez highlighted the related issue of characteristic classes of String 2-bundles a while ago.

2 Lie algebra cohomology and $inn(\mathfrak{g})^*$

Lie algebra cohomology, invariant polynomials and Chern-Simons elements can all be conveniently conceived in terms of the quasi-free differential graded algebra corresponding to the Lie 2-algebra

$\operatorname{inn}(\mathfrak{g})$

of inner derivations of the Lie algebra \mathfrak{g} . This is nothing but the well-known Weil algebra. But by regarding it as a Lie *n*-algebra we can use it to build other Lie *n*-algebras.

The relation to the more common formulation of these phenomena in terms of the cohomology of the universal G-bundle comes from the fact that this universal bundle is the realization of the nerve of INN(G) [4].

2.1 Formulation in terms of the cohomlogy of EG

Let G be a compact, simply connected simple Lie group.

The classical formulation of

- Lie algebra cocycles
- invariant polynomials
- transgression induced by Chern-Simons elements

is the following.

Consider the fibration corresponding to the universal principal G-bundle:

$$G \longrightarrow EG \xrightarrow{p} BG$$

• A Lie algebra (2n + 1)-cocycle μ (with values in a trivial module) is an element

$$\mu \in H^{2n+1}(\mathfrak{g}, \mathbb{R})$$

By compactness of G, this is the same as an element in de Rham cohomology of G:

$$\mu \in H^{2n+1}(G,\mathbb{R})$$
.

• An invariant polynomial k of degree n + 1 represents an element in

$$k \in H^{2n+2}(BG, \mathbb{R})$$
.

• A transgression form mediating between μ and k is a cochain $cs\in \Omega^{2n+1}(EG)$ such that

$$cs|_G = \mu$$

and

$$d\,cs = p^*k$$

cocycle Chern-Simons inv. polynomial

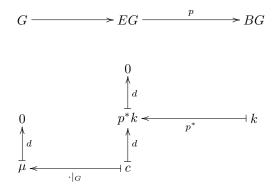


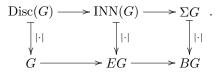
Figure 1: Lie algebra cocycles, invariant polynomials and transgression forms in terms of cohomology of the universal *G*-bundle.

2.2 Formulation in terms of the cohomology of $inn(g)^*$

The universal G-bundle may be obtained from the sequence of groupoids

$$\operatorname{Disc}(G) \to \operatorname{INN}(G) \to \Sigma G$$

by taking geometric realizations of nerves:



Disc(G) and INN(G) are strict 2-groups, coming from the crossed modules

 $\operatorname{Disc}(G) = (1 \to G)$

and

$$\text{INN}(G) = (\text{Id} : G \to G).$$

On the other hand, ΣG is a 2-group only if G is abelian.

2.2.1 Cocycles, invariant polynomials and Chern-Simons elements

Differentially, this corresponds to the sequence

$$\begin{array}{ccc} \operatorname{Disc}(G) & \longrightarrow & \operatorname{INN}(G) & \xrightarrow{p} & \Sigma G & . \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & &$$

In terms of this, we have

• A Lie algebra (2n + 1)-cocycle μ (with values in a trivial module) is an element

$$\mu \in \bigwedge^{(2n+1)} (s\mathfrak{g}^*)$$
$$d_{\mathfrak{g}}\mu = 0.$$

• An invariant polynomial k of degree n + 1 is an element

$$k \in \bigwedge^{n+1} (ss\mathfrak{g}^*)$$
$$d_{\mathrm{inn}(\mathfrak{g})}k = 0.$$

• A transgression form cs inducing a transgession between a (2n+1)-cocycle μ and a degree (n+1)-invariant polynomial is a degree (2n+1)-element

$$cs \in \bigwedge (s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)$$

such that

$$cs|_{\bigwedge^{\bullet}(s\mathfrak{g}^*)} = \mu$$

and

$$d_{\mathrm{inn}(\mathfrak{g})}cs = p^*k$$
.

2.2.2 Formulation in terms of components

In parts of the literature it is standard to express all these phenomena in components. Compare for instance [1] Then they read as follows.

For the given Lie algebra \mathfrak{g} choose a basis $\{t_a\}$. Let $\{t^a\}$ be the corresponding basis of $s\mathfrak{g}^*$ and $\{r^a\}$ that of $ss\mathfrak{g}^*$.

• A Lie (n)-cocylce is a completely antisymmetric tensor

$$\mu = \mu(t) = \mu_{a_1 \cdots a_n} t^{a_1} \wedge \cdots \wedge t^{a_n}$$

such that

$$\sum_{i=1}^{n} (-1)^{i} \mu_{[a_1 \cdots a_i \cdots a_n} C^{a_i}{}_{bc]} = 0.$$

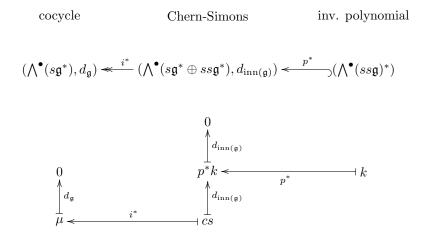


Figure 2: Lie algebra cocycles, invariant polynomials and transgression elements in terms of cohomology of $inn(\mathfrak{g})$.

• A degree n+1 symmetric invariant polynomial is a completely symmetric tensor

$$k = k(r) = k_{a_1 \cdots a_{n+1}} r^{a_1} \wedge \cdots \wedge r^{a_{n+1}}$$

such that

$$\sum_{i=1}^{n+1} k_{(a_1 \cdots a_{i-1}, a_i, a_{i+1} \cdots a_{n+1}} C^{a_i}{}_{|b|c)} = 0.$$

One finds, either by following Chern and Simons [3] or by using our homotopy operator for $inn(\mathfrak{g})$ as described in 2.2.4, that the restriction of a transgression element corresponding to the invariant polynomial k to g has components proportional to

$$k_{[a_1|b_2\cdots b_n|}C^{b_2}{}_{a_2a_3}C^{b_3}{}_{a_4a_5}\cdots C^{b_n}{}_{a_{2n}a_{2n+1}]}.$$

2.2.3Transgression and the trivializability of $inn(\mathfrak{g})$

It is important that

EG is contractible

- INN(G) is trivializable \Leftrightarrow
- $\begin{array}{ll} \Leftrightarrow & \text{the cohomology of } \operatorname{inn}(\mathfrak{g})^* = (\bigwedge^{\bullet}(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\operatorname{inn}(\mathfrak{g})}) \text{ is trivial} \\ \Leftrightarrow & \text{there is a homotopy } \tau : 0 \to \operatorname{Id}_{\operatorname{inn}(\mathfrak{g})}, \text{ i.e. } [d_{\operatorname{inn}(\mathfrak{g})}, \tau] = \operatorname{Id}_{\operatorname{inn}(\mathfrak{g})}. \end{array}$

This implies that if

cs

is to be a transgression element mediating between μ and k, then we have

$$cs = \tau(p^*k) + d_{\operatorname{inn}(\mathfrak{g})}q.$$

So for every invariant polynomial k

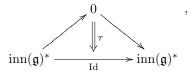
$$d_{\mathrm{inn}(\mathfrak{g})}k = 0$$

a "potential" cs does exist. The nontrivial condition is then that cs restricted to \mathfrak{g} is a cocyle.

Figure 3: The homotopy operator τ exists due to the trivializability of $inn(\mathfrak{g})$.

2.2.4 Computation of transgression elements

Using the fact that $inn(\mathfrak{g})$ is equivalent to the trivial Lie *n*-algebra, in that there is a 2-morphism



hence a chain homotopy τ satisfying

$$\mathrm{Id} = \left[d_{\mathrm{inn}(\mathfrak{g})}, \tau \right],$$

and using the fact that we have an explicit formula for these chain homotopies, we obtain an explicit algorithm for computing

$$cs := \tau(p^*k) \,.$$

To indicate how this works, we spell out the computation for the standard case where \mathfrak{g} is a Lie algebra with an invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$.

Choose a basis $\{t^a\}$ of $s\mathfrak{g}^*$ and let $\{r^a\}$ be the corresponding basis of $ss\mathfrak{g}^*$. Let the bilinear form in question have the components k_{ab} with respect to this basis. Then the $d_{inn(\mathfrak{g})}$ -closed element in $\wedge^2(ss\mathfrak{g}^*)$ whose transgression element we want to compute is

$$k = k_{ab} r^a \wedge r^b \,.$$

In order to compute

$$\tau(k_{ab}r^a \wedge r^b)$$

we need to first express k in terms of dt^a and $C^a{}_{bc}t^b\wedge t^c$ as

$$r^a = dt^a + \frac{1}{2}C^a{}_{bc}t^b \wedge t^c \,.$$

This yields

$$k = k_{ab}dt^a \wedge dt^b + k_{ab}dt^a \wedge C^a{}_{bc}t^b \wedge t^c + h_{abcd}t^a \wedge t^b \wedge t^c \wedge t^d ,$$

where the precise form of the last term does not matter for the following since it will be annihilated by τ .

The chain homotopy τ is fixed on these generators as

$$\tau:t^a\mapsto 0$$

and

$$\tau: dt^a \mapsto t^a$$

and then extended to a chain homotopy by the formula for 2-morphisms of Lie n-algebras.

Using this formula, we obtain

$$\tau: k \mapsto k_{ab}t^a \wedge dt^b + \frac{1}{3}C_{abc}t^a \wedge t^b \wedge t^c \,.$$

And indeed, this is the familiar transgression element associated to the Chern-Simons form. This is maybe more familar as we push it forward along any morphism

$$\operatorname{inn}(\mathfrak{g})^* \to \Omega^{\bullet}(X)$$

coming from a connection 1-form $A \in \Omega^1(X, \mathfrak{g})$ on some space X, in which case the above transgression element becomes

$$\operatorname{CS}_{\langle\cdot,\cdot\rangle}(A) := \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle.$$

2.2.5 Cocycles with values in arbitrary modules

The above discussion applies to Lie algebra cocycles and invariant polynomials with values in the trivial \mathfrak{g} -module. More generally, one considers Lie algebra cohomology with values in arbitrary modules.

Remarkably, discussing this more general cohomology in terms of $inn(\mathfrak{g})$ is already essentially tantamount to considering the families of Lie *n*-algebras to be discussed below.

Notice, however, that by *Whitehead's lemma* (compare [1]) the only nontrivial Lie algebra cohomology for finite-dimensional semisimple Lie algebras on finite dimensional modules does occur for the trivial module.

3 Lie *n*-algebras from cocycles and from invariant polynomials

Using the relation between Lie algebra cohomology and $inn(\mathfrak{g})$, we shall describe Lie *n*-algebras whose existence reflects the existence of Lie algebra cocycles, invariant polynomials and transgression elements.

- For each Lie algebra (n+1)-cocycle μ there is a Lie *n*-algebras \mathfrak{g}_{μ} , known from [2].
- For each degree n + 1 invariant polynomial k on \mathfrak{g} there is a Lie (2n + 1)algebra $\operatorname{ch}_k(\mathfrak{g})$ which we call a Chern Lie (2n + 1)-algebra.
- For each transgression element relating a degree (n + 1) invariant polynomial k and a (2n+1) Lie algebra cocycle μ_k there is a Lie (2n+1)-algebras $cs_k(\mathfrak{g})$ which we call a Chern-Simons Lie (2n + 1)-algebra.

3.1 Baez-Crans Lie *n*-algebras \mathfrak{g}_{μ} from Lie algebra cocycles μ

Definition 1 Let μ be an (n+1)-cocycle on \mathfrak{g} as in 2.2. Then the Lie n-algebra

 \mathfrak{g}_{μ}

is that dual to the qfDGCA whose underlying algebra is

 $\bigwedge^{\bullet}((s\mathfrak{g})^* \oplus (s^n\mathbb{R})^*)$

and whose differential d is defined by the fact that

$$d|_{\bigwedge^{\bullet}_{s\mathfrak{g}^*}} = d_{\mathfrak{g}}$$

and

$$db = -\mu$$

for b the canonical basis of $s^n \mathbb{R}^*$.

The property $d^2 = 0$ is equivalent to the Jacobi identity on \mathfrak{g} and the cocycle porperty of μ .

This has an obvious generalization to the case that the cocycle takes values in a Lie algebra module V.

Definition 2 Let μ be an (n + 1)-cocycle on \mathfrak{g} taking values in V. Let

$$\rho:\mathfrak{g}\otimes V\to V$$

be the Lie action on that module. Then the Lie n-algebra

 \mathfrak{g}_{μ}

is that dual to the qfDGCA whose underlying algebra is

$$\bigwedge^{\bullet}((s\mathfrak{g})^* \oplus (s^n V)^*).$$

and whose differential d is defined by the fact that

$$d|_{\bigwedge^{\bullet}{}_{s\mathfrak{g}^*}} = d_{\mathfrak{g}}$$

and

$$db^i = -\mu^i - \rho^i{}_{aj} t^a \wedge b^j ,$$

for $\{b^i\}$ any basis of $s^n V^*$ and $\{t^a\}$ a basis for sg^* .

Now the property $d^2 = 0$ is equivalent to the Jacobi identity on \mathfrak{g} , the cocycle porperty of μ and the action property of ρ .

We may reformulate this equivalently in L_{∞} -language:

Baez-Crans showed [2] that Lie n-algebras which are concentrated in top and bottom degree are all equivalent to Lie n-algebras of the following form.

Definition 3 For \mathfrak{g} any Lie algebra, V any \mathfrak{g} -module and

 $\mu \in H^{n+1}(\mathfrak{g}, V)$

a Lie algebra (n+1)-cocycle for \mathfrak{g} with values in m, the semistrict Lie n-algebra

 \mathfrak{g}_{μ}

is defined to be the L_{∞} -algebra on

$$S^c(s\mathfrak{g}\oplus s^nV)$$

with codifferential

$$D = d_1 + d_2 + d_{n+1}$$

defined by

$$d_2(sX \lor sY) = s[X,Y]$$
$$d_1(sX \lor s^n B) = s^n X(B)$$

$$d_{n+1}(sX_1 \lor \cdots \lor sX_{n+1}) = s^n \mu(X_1, \cdots, X_{n+1}),$$

for all $X, Y, X_i \in \mathfrak{g}$ and all $B \in m$.

We find that $D^2(sX \vee sY) = 0$ is the Jacobi identity on \mathfrak{g} , as before, and $D^2(sX \vee sY \vee B) = 0$ is the Lie module property of V. Finally

$$D^{2}(sX_{1} \vee \dots \vee sX_{n+2}) = D\left(\sum_{\sigma \in Sh(1,n+1)} \epsilon(\sigma)sX_{\sigma(1)} \vee s^{n}\mu(X_{\sigma(2)}, \dots, X_{\sigma(n+2)}) + \sum_{\sigma \in Sh(2,n)} \epsilon(\sigma)s[X_{\sigma(1)}, X_{\sigma(2)}] \vee sX_{\sigma(3)} \vee \dots \vee sX_{\sigma(n+2)}\right)$$
$$= s^{n}\sum_{\sigma \in Sh(1,n+1)} \epsilon(\sigma)X_{\sigma(1)} \left(\mu(X_{\sigma(2)}, \dots, X_{\sigma(n+2)})\right) + s^{n}\sum_{\sigma \in Sh(2,n)} \epsilon(\sigma)\mu([X_{\sigma(1)}, X_{\sigma(2)}], X_{\sigma(3)}, \dots X_{\sigma(n+2)})$$
$$= 0$$

is precisely the Lie cocycle property of μ .

URS: I think I have the signs right here, but should be checked again.

Remark. For \mathfrak{g} simple and $k = \langle \cdot, \cdot \rangle$ (a multiple of) the Killing form and

$$\mu = \langle \cdot, [\cdot, \cdot] \rangle$$

(a multiple of) the canonical 3-cocycle, the Baez-Crans Lie 2-algebra \mathfrak{g}_{μ} is the skeletal semistrict version of the String Lie 2-algebra.

3.2 Chern Lie (2n + 1)-algebra $cs_k(\mathfrak{g})$ from invariant polynomials k.

Definition 4 Let k be an invariant polynomial of degree (n + 1) on \mathfrak{g} . Then the Lie (2n + 1)-algebra

 $\mathrm{ch}_k(\mathfrak{g})$

is defined dually by a qfDGCA on

$$\bigwedge^{\bullet}((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n+1}\mathbb{R})^*)$$

whose differential d is fixed by demanding

$$d|_{\bigwedge^{\bullet}(s\mathfrak{g}^*\oplus ss\mathfrak{g}^*)} = d_{\operatorname{inn}(\mathfrak{g})}$$

and

$$dc = k$$

for $\{c\}$ the canonical basis of $s^{2n+1}\mathbb{R}^*$.

The property $d^2 = 0$ is nothing but the invariance property of k.

Again, we can generalize this to the the case where k takes values in an arbitrary g-module V.

Definition 5 Let k be an invariant polynomial of degree (n + 1) on \mathfrak{g} taking values in the Lie algebra module V. Then the Lie (2n + 1)-algebra

 $\mathrm{ch}_k(\mathfrak{g})$

is defined dually by a qfDGCA on

$$\bigwedge^{\bullet}((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n+1}V)^*)$$

whose differential d is fixed by demanding

$$d|_{\bigwedge^{\bullet}(s\mathfrak{g}^*\oplus ss\mathfrak{g}^*)} = d_{\operatorname{inn}(\mathfrak{g})}$$

and

$$dc^i = k^i - \rho^i{}_{aj} t^a \wedge c^j$$

for $\{c^i\}$ a basis of $s^{2n+1}V^*$.

Now $d^2 = 0$ is equivalent to the invariance of k together with the action property of ρ .

3.3 Chern-Simons Lie (2n + 1)-algebras $cs_k(\mathfrak{g})$ from invariant polynomials k

Definition 6 Let k be an invariant polynomial of degree n + 1 which is related by a transgression element cs to a degree (2n + 1) cocycle μ_k . Then the Lie (2n + 1)-algebra

 $\operatorname{cs}_k(\mathfrak{g})$

is defined dually on the free graded commutative algbra

$$\bigwedge^{\bullet} (s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n}\mathbb{R}^* \oplus s^{2n+1}\mathbb{R}^*)$$

equipped with a differential d defined by

$$d|_{\bigwedge^{\bullet}(s\mathfrak{g}^*\oplus ss\mathfrak{g}^*)}=d_{\mathrm{inn}(\mathfrak{g})}$$

and

$$db = c - cs$$
$$dc = k \,,$$

for $\{b\}$ the canonical basis of $s^{2n}\mathbb{R}^*$ and $\{c\}$ the canonical basis of $s^{2n+1}\mathbb{R}^*$.

Here $d^2 = 0$ is the invariance of k together with the fact that $d_{inn(g)}cs = k$. The generalization to arbitrary g-modules is again obvious: **Definition 7** Let k be an invariant polynomial of degree n+1 with values in the \mathfrak{g} -module V, which is related by a transgression element cs to a degree (2n+1) cocycle μ_k with values in V. Then the Lie (2n+1)-algebra

 $\operatorname{cs}_k(\mathfrak{g})$

is defined dually on the free graded commutative algbra

$$\bigwedge^{\bullet} (s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n}V^* \oplus s^{2n+1}V^*)$$

equipped with a differential d defined by

$$d|_{\bigwedge^{\bullet}(s\mathfrak{g}^*\oplus ss\mathfrak{g}^*)}=d_{\mathrm{inn}(\mathfrak{g})}$$

and

$$db^{i} = c^{i} - cs^{i} - \rho^{i}{}_{aj}t^{a} \wedge b^{j}$$
$$dc^{i} = k^{i} - \rho^{i}{}_{aj}t^{a} \wedge c^{j}.$$

Nilpotency $d^2 = 0$ is due to the invariance of k, the defining property of the transgression element cs and the action property of ρ .

3.4 Lie *n*-algebras of invariant polynomials

Definition 8 For g any Lie algebra, let

$$\operatorname{inv}(\mathfrak{g}) := \operatorname{ker}(d|_{\operatorname{inn}(\mathfrak{g})})|_{\bigwedge^{\bullet} ss\mathfrak{g}^{*}}$$

be the graded-commutative algebra of invariant polynomials of \mathfrak{g} . We may think of this as still equpped with the differential $d_{\operatorname{inn}(\mathfrak{g})}$, hence with a trivial differential. The abelian Lie n-algebra obtained this way, where n is the degree of the highest rank generator of $\operatorname{inv}(\mathfrak{g})$ we also denote

bg .

3.5 Morphisms

We exhibit some important Lie n-algebra morphisms involving Baez-Crans, Chern and Chern-Simons Lie n-algebras.

3.5.1 Higher connections and Chern-Simons forms

The Chern-Simons Lie (2n + 1)-algebras are characterized by the property that (2n+1)-connections taking values in them are given by degree (2n+1) differential forms which are constrained to be the respective Chern-Simons form coming from some Lie-algebra valued 1-form.

Proposition 1 (Chern and Chern-Simons forms) Connections with values in Chern and Chern-Simons Lie (2n + 1)-algebras encode the corresponding degree (2n + 1) differential forms.

• (2n+1)-Connections with values in the Chern Lie (2n+1)-algebras $ch_k(\mathfrak{g})$, *i.e.* morhisms

$$\operatorname{ch}_k(\mathfrak{g})^* \to \Omega^{\bullet}(X)$$

are in bijective correspondence with tuples

$$(A, C) \in \Omega^1(X, \mathfrak{g}) \times \Omega^{2n+1}(X)$$

such that

$$dC = k(F_A \wedge \cdots \wedge F_A).$$

• (2n+1)-Connections with values in the Chern-Simons Lie (2n+1)-algebras $\operatorname{cs}_k(\mathfrak{g})$, i.e. morphisms

$$\operatorname{ch}_k(\mathfrak{g})^* \to \Omega^{\bullet}(X),$$

are in bijective correspondence with tuples

$$(A, B, C) \in \Omega^1(X, \mathfrak{g}) \times \Omega^{2n}(X) \times \Omega^{2n+1}(X)$$

such that

$$C = dB + k \mathrm{CS}_k(A) \,.$$

Here $CS_k(A)$ is the k-Chern-Simons form, such that

$$dC = k(F_A \wedge \cdots \wedge F_A).$$

3.5.2 The isomorphism $inn(\mathfrak{g}_{\mu_k}) \simeq cs_k(\mathfrak{g})$

Proposition 2 We have an equivalence (even an isomorphism)

$$\operatorname{inn}(\mathfrak{g}_{\mu_k})\simeq \operatorname{cs}_k(\mathfrak{g})$$

whenever the latter exists.

Proof. One checks that the assignments

$$t^{a} \mapsto t^{a}$$
$$r^{a} \mapsto r^{a}$$
$$b \mapsto b$$
$$c \mapsto c \pm (cs - \mu)$$

in our standard basis define morphisms between the two Lie (2n + 1)-algebras. These are clearly strict inverses of each other.

3.5.3 The exact sequence $0 \to \mathfrak{g}_{\mu_k} \to \operatorname{cs}_k(\mathfrak{g}) \to \operatorname{ch}_k(\mathfrak{g}) \to 0$

Suppose that the degree n + 1 invariant polynomial k admits a Chern-Simons potential, i.e. such that all three Lie (2n + 1)-algebras

- $\mathfrak{g}_{\mu_k} 3.1$
- $\operatorname{cs}_k(\mathfrak{g}) 3.3$
- $\operatorname{ch}_k(\mathfrak{g}) 3.2$.

Then we have the following morphisms between these.

Proposition 3 We have a canonical surjection

$$i: \operatorname{cs}_k(\mathfrak{g}) \longrightarrow \operatorname{ch}_k(\mathfrak{g})$$

Proof. One checks that the canonical inclusion of vector spaces

$$\bigwedge^{\bullet} (s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n+1}\mathbb{R}^*) \hookrightarrow \bigwedge^{\bullet} (s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n}\mathbb{R}^* \oplus s^{2n+1}\mathbb{R}^*)$$

gives a monomorphic qfDGCA-morphism

$$\operatorname{ch}_k(\mathfrak{g})^* \to \operatorname{cs}_k(\mathfrak{g})^*$$

hence defines an epimorphic dual morphism.

Proposition 4 We have a canonical injection

$$i: \mathfrak{g}_{\mu_k} \longrightarrow \operatorname{cs}_k(\mathfrak{g})$$
.

Proof. One checks that the canonical surjection of vector spaces

$$\bigwedge^{\bullet}(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n}\mathbb{R}^*) \to \bigwedge^{\bullet}(s\mathfrak{g}^* \oplus s^{2n+1}\mathbb{R}^*)$$

gives an epimorphic qfDGCA-morphism

$$\mathrm{cs}_k(\mathfrak{g})^* o \mathfrak{g}^*_{\mu_k}$$

hence defines a monomorphic dual morphism.

Remark. It is the existence of this morphism which corresponds to the fact that the transgression element cs has the property that it restricts to the cocycle μ on $\bigwedge^{\bullet}(s\mathfrak{g}^*)$.

Proposition 5 The composite morphism

$$\mathfrak{g}_{\mu_k} \longrightarrow \operatorname{cs}_k(\mathfrak{g}) \longrightarrow \operatorname{ch}_k(\mathfrak{g})$$

is homotopic to the zero-morphism.

Proof. By the above, the dual morphism is the identity on the generators of $(s\mathfrak{g})^*$

$$f^*: t^a \mapsto t^a$$

and sends everything else to zero. But we have

$$f^* = [d\tau]$$

with τ given on generators as

$$\tau: t^{a} \mapsto 0$$
$$\tau: r^{a} \mapsto t^{a}$$
$$\tau: c \mapsto 0$$

and then extended as a 2-morphism.

In summary this gives

Corollary 1 Whenever the (2n + 1)-cocycle μ_k on \mathfrak{g} and the invariant degree (n + 1)-polynomial k are related by transgression, we have an exact sequence of Lie (2n + 1)-algebras

$$0 \to \mathfrak{g}_{\mu_k} \to \operatorname{cs}_k(\mathfrak{g}) \to \operatorname{ch}_k(\mathfrak{g}) \to 0$$
.

3.5.4 Other useful morphisms

Definition 9 For any $ch_k(\mathfrak{g})$ we have a canonical morphism

$$\operatorname{ch}_k(\mathfrak{g})^* \to \operatorname{inn}(\mathfrak{g})^*$$

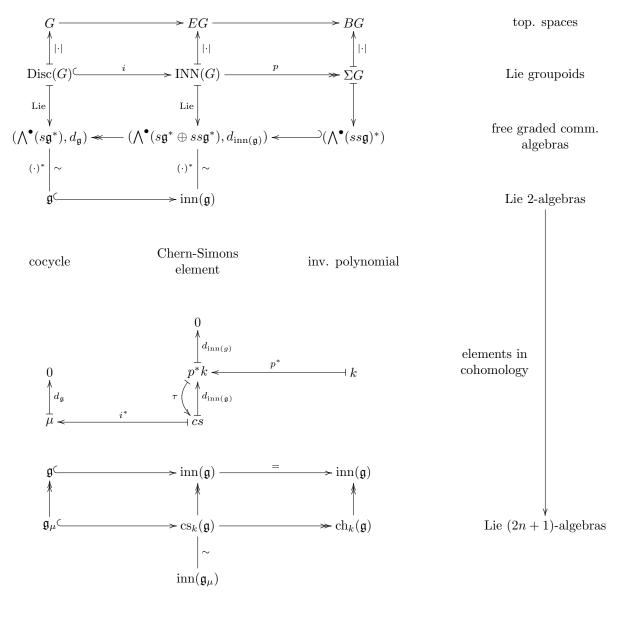
It sends the generator c of $s^{n-1}\mathbb{R}^*$ to $\tau(k)$, where τ is the trivializing homotopy of $\operatorname{inn}(\mathfrak{g})$.

$$c \mapsto \tau(k)$$
.

Definition 10 For any $ch_k(\mathfrak{g})$ we have a canonical morphism

$$\operatorname{Lie}(\Sigma^n U(1))^* \to \operatorname{ch}_k(\mathfrak{g})^*$$

which sends the single generator of $\operatorname{Lie}(\Sigma^n U(1))^*$ to k.





Chern-Simons

Chern

Figure 4: Chern Lie (2n + 1)-algebras: for each Lie algebra (2n + 1) cocycle μ which is related by transgression to an invariant polynomial k we obtain an exact sequence of Lie (2n + 1)-algebras.

4 Lie *n*-algebra cohomology

Above we described families of Lie *n*-algebras induced from the cohomology of ordinary Lie algebras. There is a rather obvious generalization of the entire discussion to what should be the cohomology of Lie *n*-algebras themselves.

4.1 Lie *n*-algebra cohomology

Let $\mathfrak{g}_{(n)}$ be any Lie *n*-algebra and let $\operatorname{inn}(\mathfrak{g}_{(n)})$ be the corresponding Lie (n+1)-algebra of inner derivations.

Write $(\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)}), d_{\mathfrak{g}_{(n)}})$ and $(\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}), d_{\operatorname{inn}(\mathfrak{g}_{(n)})}))$ for the corresponding dual qDGCAs, respectively.

Recalling that we have a canonical inclusion

$$i:\mathfrak{g}_{(n)}\to\operatorname{inn}(\mathfrak{g}_{(n)})$$

of Lie (n + 1)-algbras and a canonical inclusion

$$\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}) \leftarrow \bigwedge^{\bullet}(ss\mathfrak{g}^*_{(n)})$$

of graded-commutative algebras such that

$$(\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)}), d_{\mathfrak{g}_{(n)}}) \leftrightsquigarrow (\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}), d_{\mathrm{inn}(\mathfrak{g}_{(n)})}) \bigstar (ss\mathfrak{g}^*_{(n)})$$

can be thought of as differential forms on a universal $G_{(n)}$ -bundle [4]

$$G_{(n)} \to \mathrm{INN}_0(G_{(n)}) \to \Sigma G_{(n)}$$

we make the following definition, which is the obvious and straightforward generalization of 2.2.1.

Definition 11 For the above setup, we say

• A $\mathfrak{g}_{(n)}$ cocycle μ is a $d_{\mathfrak{g}_{(n)}}$ -closed element of $\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)})$

$$d_{\mathfrak{g}_{(n)}}\mu=0.$$

• A $\mathfrak{g}_{(n)}$ invariant polynomial k is a $d_{\operatorname{inn}(\mathfrak{g}_{(n)})}$ -closed element in $\bigwedge^{\bullet}(ss\mathfrak{g}_{(n)}^*)$

$$d_{\operatorname{inn}(\mathfrak{g}_{(n)})}k=0\,.$$

A g_(n) transgression element cs for a given g_(n) cocycle μ and a given g_(n) invariant polynomial k is an element cs in Λ[●](sg^{*}_(n) ⊕ ssg^{*}_(n)) such that

$$cs|_{\bigwedge \bullet_{s\mathfrak{g}^*_{(n)}}} = \mu$$

and

$$d_{\operatorname{inn}(\mathfrak{g}(n))}cs = k$$

We address these cocycles, invariant polynomials and transgression elements as being of degree d whenever the respective elements have degree r in $\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)} \oplus$ $ss\mathfrak{g}^*_{(n)})$. Notice that following this convention an ordinary degree n invariant polynomial is now addressed as having degree 2n. Which indeed is the more natural counting.

Definition 12 A Lie n-algebra cocycle is a Lie algebra coboundary simply if it is a $d_{\mathfrak{g}_{(n)}}$ coboundary. An invariant polynomial k is a coboundary if it is $d_{\operatorname{inn}(\mathfrak{g}_{(n)})}$ -coboundary

$$k = d_{\operatorname{inn}(\mathfrak{g}_{(n)})}\lambda$$

with the property that λ vanishes when restricted to $\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)})$.

Remark. This can be read as saying that an invariant polynomial is regarded as trivializable if it is in transgression with the trivial Lie algebra cocycle.

4.2 Lie $\max(n, d)$ -algebras from Lie *n*-algebra cohomology

Definition 13 For $\mathfrak{g}_{(n)}$ any Lie n-algebra and μ a degree (d+1) cocycle on it, we obtain a Lie $\max(n, d)$ -algebra $(\mathfrak{g}_{(n)})_{\mu}$ on

$$\bigwedge^{\bullet}(s\mathfrak{g}^*_{(n)}\oplus s^d\mathbb{R}^*)$$

dy defining a differential d by

$$d_{(\mathfrak{g}_{(n)})_{\mu}}|_{\bigwedge} \bullet_{s\mathfrak{g}_{(n)}^{*}} = d_{\mathfrak{g}_{(n)}}$$

and

$$d_{(\mathfrak{g}_{(n)})_{\mu}}b = -\mu$$

for $\{b\}$ the canonical basis of $s^d \mathbb{R}^*$.

Definition 14 For $\mathfrak{g}_{(n)}$ any Lie n-algebra and k a degree (d + 1) invariant polynomial on it, we obtain a Lie $\max(n, d)$ -algebra $\operatorname{ch}_k(\mathfrak{g}_{(n)})$ on

$$\bigwedge^{ullet}(s\mathfrak{g}^*_{(n)}\oplus ss\mathfrak{g}^*_{(n)}\oplus s^d\mathbb{R}^*)$$

dy defining a differential d by

$$d_{(\mathfrak{g}_{(n)})_{\mu}}|_{\bigwedge} \bullet_{s\mathfrak{g}_{(n)^*}\oplus ss\mathfrak{g}_{(n)}^*} = d_{\operatorname{inn}(\mathfrak{g}_{(n)})}$$

and

$$d_{\mathrm{ch}_k(\mathfrak{g}_{(n)})}c = k$$

for $\{c\}$ the canonical basis of $s^d \mathbb{R}^*$.

Definition 15 For $\mathfrak{g}_{(n)}$ any Lie n-algebra and cs a degree d transgression element interpolating between a degree d + 1 invariant polynomial k and a degree d cocycle μ on $\mathfrak{g}_{(n)}$ we obtain a Lie $\max(n, d)$ -algebra $\operatorname{cs}_k(\mathfrak{g}_{(n)})$ on

$$\bigwedge^{\bullet} (s\mathfrak{g}_{(n)}^* \oplus s^{d-1}\mathbb{R}^* \oplus s^d\mathbb{R}^*)$$

dy defining a differential d by

$$d_{(\mathfrak{g}_{(n)})_{\mu}}\Big|_{\bigwedge^{\bullet}s\mathfrak{g}_{(n)^{*}}\oplus ss\mathfrak{g}_{(n)}^{*}}=d_{\operatorname{inn}(\mathfrak{g}_{(n)})}$$

and

$$d_{\mathrm{cs}_{k}(\mathfrak{g}_{(n)})}b = cs - c$$
$$d_{\mathrm{cs}_{k}(\mathfrak{g}_{(n)})}c = k$$

for $\{b\}$ the canonical basis of $s^{d-1}\mathbb{R}^*$ and $\{c\}$ the canonical basis of $s^d\mathbb{R}^*$.

All these differentials square to 0, $d^2 = 0$ by exactly the same (simple) reasoning as in 3. And in fact, all the remaining discussion goes through just as before. So we get

Corollary 2 Whenever for $\mathfrak{g}_{(n)}$ any Lie n-algebra the d-cocycle μ_k on $\mathfrak{g}_{(n)}$ and the invariant degree (d+1) polynomial k are related by transgression, we have an exact sequence of Lie max(n, d)-algebras

$$0 \to (\mathfrak{g}_{(n)})_{\mu_k} \to \operatorname{cs}_k(\mathfrak{g}_{(n)}) \to \operatorname{ch}_k(\mathfrak{g}_{(n)}) \to 0.$$

4.3 Invariant polynomials of Baez-Crans type Lie *n*-algebras

Proposition 6 The cohomology classes of invariant polynomials of a Lie nalgebra \mathfrak{g}_{μ} of Baez-Crans type arising from an ordinary Lie algebra \mathfrak{g} and a degree (n+1)-cocycle μ_k on it which is in transgression with an invariant polynomial k are those of \mathfrak{g} itself modulo k:

$$\operatorname{inv}(\mathfrak{g}_{\mu_k}) \simeq \operatorname{inv}(\mathfrak{g})/k$$
.

Proof. By inspection one finds that \mathfrak{g}_{μ} has no invariant polynomials above those coming from \mathfrak{g} . But precisely k becomes a coboundary of invariant polynomials now, since

$$k = d\tau(k) = d((cs - \mu) + \mu) = d((cs - \mu) + c),$$

where c is the top degree generator of $\operatorname{inn}(\mathfrak{g}_{\mu})$. But $(cs - \mu) + c$ manifestly vanishes when restricted to $\bigwedge^{\bullet}(s\mathfrak{g}_{\mu}^{*})$. Hence, by definition 12, k is a coboundary of invariant polynomials.

5 Characteristic classes of *n*-bundles

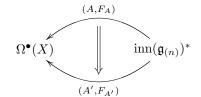
5.1 Lie *n*-algebra valued connections

Apart from being natural in itself, our definition of Lie n-algebra cohomology and of invariant polynomials on Lie n-algebras justifies itself in that it does give the right framework for the description of representatives of Chern classes of n-bundles.

Definition 16 For $\mathfrak{g}_{(n)}$ any Lie n-algebra and X a space, a connection on the trivial $\mathfrak{g}_{(n)}$ -n-bundle over X is a morphism

$$(A, F_A): \Omega^{\bullet}(X) \leftarrow \min(\mathfrak{g}_{(n)})^*$$

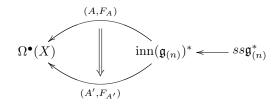
of differential graded commutative algebras. Following our general discussion of higher morphisms of Lie n-algebras, we define higher morphisms of such connections to be those homotopies



which vanish when pulled back along the canonical morphisms of graded algebras

$$\operatorname{inn}(\mathfrak{g}_{(n)})^* \longleftarrow ss\mathfrak{g}_{(n)}^* ,$$

i.e. such that



is the vanishing homotopy.

Here $\Omega^{\bullet}(X)$ is the deRham complex of X and $\operatorname{inn}(\mathfrak{g}_{(n)})^*$ is shorthand for the Koszul dual corresponding to $\operatorname{inn}(\mathfrak{g}_{(n)})$.

Remark. The condition on the (k > 1)-morphisms simply means that the component maps of the higher chain homotopies, which are maps

$$\bigwedge^{\bullet} (s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}) \to \Omega^{\bullet - (k-1)}(X)$$

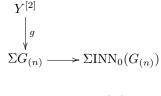
vanish on the generators in $ss\mathfrak{g}_{(n)}^*$. This makes the *n*-category of connections on the trivial $\mathfrak{g}_{(n)}$ -bundle contain interesting information, even though $\operatorname{inn}(\mathfrak{g}_{(n)})$ is equivalent, when all higher morphisms are admitted, to the trivial Lie *n*-algebra.

There are various ways to understand this condition. While these will be discussed in detail elsewhere, the following consideration indicates what is going on:

Suppose $G_{(n)}$ is some Lie *n*-group, to be thought of as integrating our Lie *n*-algebra $\mathfrak{g}_{(n)}$. Let $Y \to X$ be a good cover of base space X and $Y^{[2]}$ the corresponding groupoid. Then $G_{(n)}$ -bundles on X are classified by pseudo functors



These encode precisely a nonabelian $G_{(n)}$ -cocycle, hence the transition data of a locally trivialized $G_{(n)}$ -bundle. Equipping this $G_{(n)}$ -bundle with a connections amounts to extending this morphism along the canonical inclusion



to a square

$$\begin{array}{c} Y^{[2]} \longrightarrow \mathcal{C}_{n}(Y) \\ & \downarrow^{g} & \downarrow^{(g, \mathrm{tra, curv})} \\ \Sigma G_{(n)} \longrightarrow \Sigma \mathrm{INN}_{0}(G_{(n)}) \end{array}$$

where (tra, curv) is the integrated version of the morphism (A, F_A) appearing above, describing an integrated connection on the trivial $G_{(n)}$ bundle over Y obtained by locally trivializing a general $G_{(n)}$ -bundle on X. The point is that the horizontal arrows imply that while the connection takes values in $\text{INN}_0(G_{(n)})$, its transition morphisms (the descent gluing data), take values only in $G_{(n)}$. This is the integrated analog of our condition that higher morphisms of $\mathfrak{g}_{(n)}$ connections are restricted higher morphisms between 1-morphisms on $\text{inn}(\mathfrak{g})_{(n)}$.

We now first define Chern classes for trivial $\mathfrak{g}_{(n)}$ -bundles and then discuss their descent to Chern classes of possibly nontrivial $\mathfrak{g}_{(n)}$ -bundles.

Definition 17 (Chern classes for Lie *n*-algebras) Given a $\mathfrak{g}_{(n)}$ -connection

$$(A, F_A): \Omega^{\bullet}(X) \leftarrow \min(\mathfrak{g}_{(n)})^*$$

on a trivial $\mathfrak{g}_{(n)}$ -n-bundle over X, for any choice of degree r Lie n-algebra invariant polynomial k of $\mathfrak{g}_{(n)}$ we obtain an r-form

$$k(F_A): \ \Omega^{\bullet}(X) \xleftarrow{(A,F_A)} \operatorname{inn}(\mathfrak{g}_{(n)})^* \xleftarrow{} \operatorname{ch}_k(\mathfrak{g}_{(n)})^* \xleftarrow{} \operatorname{Lie}(\Sigma^{(r-1)}U(1))^*$$

where the two morphisms on the right are the canonical ones described in 3.5.4. This is the k-Chern-form of the connection (A, F_A) .

Remark. The r form $k(F_A)$ is nothing but the image of k under (A, F_A) . This, in turn, is nothing but the invariant polynomial k with the concrete curvature F_A substituted for the respective generators of $ssg^*_{(n)}$. But it is useful to restate this – simple but component-dependent – statement more intrinsically in terms of the above morphisms.

Example. For \mathfrak{g} an ordinary simple Lie 1-algebra and $k = \langle \cdot, \cdot \rangle$ the Killing form, and for (A, F_A) a \mathfrak{g} -connection, we have

$$k(F_A) = \langle F_A \wedge F_A \rangle$$

as one would expect.

5.2 *n*-Bundles with structure Lie *n*-algebra

For $\mathfrak{g}_{(n)}$ any Lie *n*-algebra, the sequence

$$\mathfrak{g}_{(n)}^{*}$$

$$\uparrow$$

$$\mathfrak{inn}(\mathfrak{g}_{(n)})^{*}$$

$$h$$

$$b\mathfrak{g}_{(n)}^{*}$$

with $b\mathfrak{g}$ as in definition 8 plays the role of the universal $G_{(n)}$ -n-bundle

$$\begin{array}{c}
G_{(n)} \\
\downarrow \\
EG_{(n)} \\
\downarrow \\
BG_{(n)}
\end{array}$$

in that it comes from its Lie n-groupoid realization

$$\begin{array}{c}
G_{(n)} \\
\downarrow^{i} \\
\text{INN}_{0}(G_{(n)}) \\
\downarrow^{p} \\
\Sigma G_{(n)}
\end{array}$$

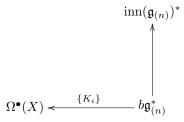
as described in [4].

Accordingly, we say that, for X a space, a qDGCA morphism

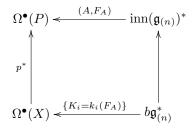
$$\Omega^{\bullet}(X) \xleftarrow{\{K_i\}} b\mathfrak{g}^*_{(n)}$$

is a classifying map for a $\mathfrak{g}_{(n)}$ -*n*-bundle: this morphism is nothing but a choice of a closed *r*-form K_i on X for each $\mathfrak{g}_{(n)}$ invariant polynomial $k_i \in inv(\mathfrak{g})_{(n)}$ of degree r.

Then, completing the cone

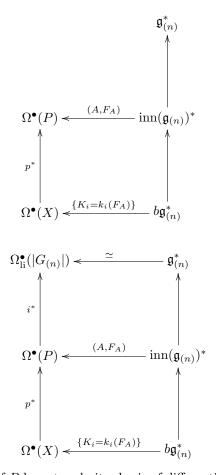


to a square



amounts to choosing a total space $p: P \to X$ over X with a $\mathfrak{g}_{(n)}$ -connection chosen on it that does induce the previoulsy chosen characteristic classes. Further

requiring that the pushout of



 \mathbf{exists}

says that the fibers of P have to admit a basis of differential forms that mimics the qDGCA of $\mathfrak{g}_{(n)}^*$. For n = 1 this just says that the fibers have to look like the group G and that the connection A restricts to the canonical 1-form θ on G, $i^*A = \theta$. Hence the top square is the first condition on a Cartan connection A.

(URS: I suspect that requiring the lower square to be a pushout is the second Cartan condition (equivariance of A). But I am not sure yet how to see this.)

For n > 1 our notation $\Omega_{\text{li}}^{\bullet}(|G_{(n)}|)$ indicates what one expects this statement to generalize to, though realizing an *n*-group $G_{(n)}$ integrating the Lie *n*-algebra $\mathfrak{g}_{(n)}$ as well as its nerve $|G_{(n)}|$ internal to smooth spaces is a currently unsolved problem.

Hence constructing smooth spaces P with the above properties is an issue beyond our present scope here. Nevertheless, we can proceed and study the properties P would have.

5.3 Characteristic classes of $\mathfrak{g}_{(n)}$ -*n*-Bundles

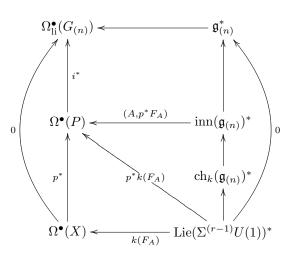


Figure 5: Characteristic classes on $\mathfrak{g}_{(n)}$ -*n*-bundles. Given the $\mathfrak{g}_{(n)}$ connection $(A, F_A) : \Omega^{\bullet}(P) \leftarrow \operatorname{inn}(\mathfrak{g}_{(n)})^*$ on the total space P the assumption that the top square exists as a pushout amounts to the assumption that $p : P \to X$ has fibers that look like the *n*-group integrating $\mathfrak{g}_{(n)}$. Each characteristic class of degree r, manifested in the existence of the Chern Lie (r+1) algebra $\operatorname{ch}_k(\mathfrak{g}_{(n)})$, leads to a differential *r*-form on P as indicated. By construction/definition this descends to the *r*-form representative $k(F_A)$ of the characteristic class as indicated.

Proposition 7 The images of invariant polynomials $k \in \bigwedge^{\bullet}(ss\mathfrak{g}^*_{(n)})$ of a Lie *n*-algebra $\mathfrak{g}_{(n)}$ under a choice of $\mathfrak{g}_{(n)}$ -connection

 $(A, F_A): \Omega^{\bullet}(X) \longleftarrow \operatorname{inn}(\mathfrak{g}_{(n)})$

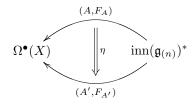
are invariant under morphisms of $g_{(n)}$ -connections: if

 $k(F_A), k(F_{A'}) \in \Omega^{\bullet}(X)$

are the images of k under (A, F_A) and $(A', F_{A'})$, respectively and if there exists a morphism $(A, F_A) \rightarrow (A', F_{A'})$ then in fact

$$k(F_A) = k(F_{A'}).$$

Proof. The existence of the morphism



implies that

$$k(F_{A'}) - k(F_A) = [d, \eta](k).$$

But

$$d(\eta(k)) = 0$$

since, by definition of invariant polynomials of Lie *n*-algebras, $k \in \bigwedge^{\bullet}(ss\mathfrak{g}_{(n)})$, which, by definition of morphisms of $\mathfrak{g}_{(n)}$ -connection, implies that $\eta(k) = 0$.

And

$$\eta(dk) = 0$$

since, by definition of invariant polynomials of Lie *n*-algebras,

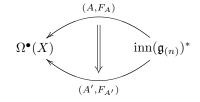
$$dk = d_{\operatorname{inn}(\mathfrak{g}_{(n)})}k = 0$$

Hence

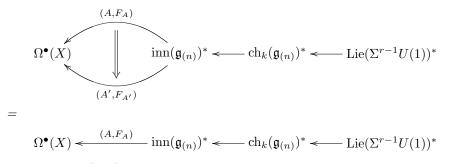
$$k(F_{A'}) - k(F_A) = d\eta(k) + \eta(dk) = 0.$$

Using the more intrinsic formulation of characteristic classes from definition 17 we may restate the above proposition concisely as

Corollary 3 Morphisms of $\mathfrak{g}_{(n)}$ connections



act trivially on the corresponding characteristic classes in that



for all $k \in inv(\mathfrak{g}_{(n)})$, or equivalently

$$\Omega^{\bullet}(X) \xrightarrow{(A,F_A)} \operatorname{inn}(\mathfrak{g}_{(n)})^* \longleftarrow b\mathfrak{g}_{(n)}^* = \Omega^{\bullet}(X) \xleftarrow{(A,F_A)} \operatorname{inn}(\mathfrak{g}_{(n)})^* \xleftarrow{b\mathfrak{g}_{(n)}^*} .$$

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