# Lie $n$-Algebra Cohomology

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September 7, 2007

**Abstract**

Ordinary Lie algebra cohomology of a Lie algebra $\mathfrak{g}$ has a nice reformulation in terms of the Koszul dual differential algebra of the Lie 2-algebra of inner derivations of $\mathfrak{g}$. For every transgressive degree $n$ element in $\mathfrak{g}$-cohomology there is a short exact sequence of Lie $n$-algebras. These are characterized by the fact that $n$-connections taking values in them come from the corresponding Chern-Simons forms and characteristic classes.

A straightforward generalization of this construction yields a notion of cohomology, invariant polynomials and transgression elements for arbitrary Lie $n$-algebras. And in turn, each such element of degree $d$ induces a new Lie $\max(n,d)$-algebra.

From the invariant polynomials of a Lie $n$-algebra one obtains characteristic classes of the corresponding $n$-bundles.

# Contents

# 1 Introduction

URS: This is part of the stuff in collaboration with Jim Stashaeff which I am currently polishing. All mistakes and imperfections are mine. In particular, the last section about Lie $n$-algebra cohomology is a more recent contribution of mine, which still needs to be scrutinized. This addresses an issue I knew should have a nice answer based on the other stuff, but didn't seriously start looking into before John Baez highlighted the related issue of characteristic classes of String 2-bundles a while ago.

# 2 Lie algebra cohomology and $\mathrm{inn}(\mathfrak{g})^*$

Lie algebra cohomology, invariant polynomials and Chern-Simons elements can all be conveniently conceived in terms of the quasi-free differential graded algebra corresponding to the Lie 2-algebra

$$\mathrm{inn}(\mathfrak{g})$$

of inner derivations of the Lie algebra $\mathfrak{g}$. This is nothing but the well-known Weil algebra. But by regarding it as a Lie $n$-algebra we can use it to build other Lie $n$-algebras.

The relation to the more common formulation of these phenomena in terms of the cohomology of the universal $G$-bundle comes from the fact that this universal bundle is the realization of the nerve of $\mathrm{INN}(G)$ [4].

## 2.1 Formulation in terms of the cohomlogy of $EG$

Let $G$ be a compact, simply connected simple Lie group.

The classical formulation of

- Lie algebra cocycles
- invariant polynomials
- transgression induced by Chern-Simons elements

is the following.

Consider the fibration corresponding to the universal principal $G$-bundle:

$$G \longrightarrow EG \xrightarrow{p} BG \ .$$

- A Lie algebra $(2n+1)$-cocycle $\mu$ (with values in a trivial module) is an element
$$\mu \in H^{2n+1}(\mathfrak{g}, \mathbb{R})\,.$$
By compactness of $G$, this is the same as an element in de Rham cohomology of $G$:
$$\mu \in H^{2n+1}(G, \mathbb{R})\,.$$

- An invariant polynomial $k$ of degree $n+1$ represents an element in
$$k \in H^{2n+2}(BG, \mathbb{R}) .$$
- A transgression form mediating between $\mu$ and $k$ is a cochain $cs \in \Omega^{2n+1}(EG)$ such that
$$cs|_G = \mu$$
and
$$d\, cs = p^* k .$$

$$
\begin{tikzcd}
\text{cocycle} & \text{Chern-Simons} & \text{inv. polynomial} \\
G \arrow[r] & EG \arrow[r, "p"] & BG \\
 & 0 & \\
0 & p^*k \arrow[u, mapsto, "d"'] & k \arrow[l, mapsto, "p^*"] \\
\mu \arrow[u, mapsto, "d"'] & c \arrow[u, mapsto, "d"'] \arrow[l, mapsto, "\cdot|_G"] &
\end{tikzcd}
$$

Figure 1: **Lie algebra cocycles, invariant polynomials and transgression forms** in terms of cohomology of the universal $G$-bundle.

## 2.2 Formulation in terms of the cohomology of $\mathrm{inn}(\mathfrak{g})^*$

The universal $G$-bundle may be obtained from the sequence of groupoids
$$\mathrm{Disc}(G) \to \mathrm{INN}(G) \to \Sigma G$$
by taking geometric realizations of nerves:
$$
\begin{tikzcd}
\mathrm{Disc}(G) \arrow[r] \arrow[d, mapsto, "|\cdot|"] & \mathrm{INN}(G) \arrow[r] \arrow[d, mapsto, "|\cdot|"] & \Sigma G \arrow[d, mapsto, "|\cdot|"] \\
G \arrow[r] & EG \arrow[r] & BG
\end{tikzcd} .
$$

$\mathrm{Disc}(G)$ and $\mathrm{INN}(G)$ are strict 2-groups, coming from the crossed modules
$$\mathrm{Disc}(G) = (1 \to G)$$
and
$$\mathrm{INN}(G) = (\mathrm{Id} : G \to G) .$$

On the other hand, $\Sigma G$ is a 2-group only if $G$ is abelian.

### 2.2.1 Cocycles, invariant polynomials and Chern-Simons elements

Differentially, this corresponds to the sequence

$$\begin{tikzcd} \mathrm{Disc}(G) \arrow[r] \arrow[d, mapsto, "\mathrm{Lie}"] & \mathrm{INN}(G) \arrow[r, "p"] \arrow[d, mapsto, "\mathrm{Lie}"] & \Sigma G \arrow[d, mapsto] \\ \bigwedge^\bullet s\mathfrak{g}^* & \bigwedge^\bullet (s\mathfrak{g}^* \oplus ss\mathfrak{g}^*) \arrow[l] & \bigwedge^\bullet (ss\mathfrak{g}^*) \arrow[l, "p^*"'] \end{tikzcd} .$$

In terms of this, we have

- A Lie algebra $(2n+1)$-cocycle $\mu$ (with values in a trivial module) is an element
$$\mu \in \bigwedge^{(2n+1)}(s\mathfrak{g}^*)$$
$$d_{\mathfrak{g}}\mu = 0\,.$$

- An invariant polynomial $k$ of degree $n+1$ is an element
$$k \in \bigwedge^{n+1}(ss\mathfrak{g}^*)$$
$$d_{\mathrm{inn}(\mathfrak{g})}k = 0\,.$$

- A transgression form $cs$ inducing a transgession between a $(2n+1)$-cocycle $\mu$ and a degree $(n+1)$-invariant polynomial is a degree $(2n+1)$-element
$$cs \in \bigwedge(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)$$
such that
$$cs|_{\bigwedge^\bullet(s\mathfrak{g}^*)} = \mu$$
and
$$d_{\mathrm{inn}(\mathfrak{g})}cs = p^*k\,.$$

### 2.2.2 Formulation in terms of components

In parts of the literature it is standard to express all these phenomena in components. Compare for instance [1] Then they read as follows.

For the given Lie algebra $\mathfrak{g}$ choose a basis $\{t_a\}$. Let $\{t^a\}$ be the corresponding basis of $s\mathfrak{g}^*$ and $\{r^a\}$ that of $ss\mathfrak{g}^*$.

- A Lie $(n)$-cocylce is a completely antisymmetric tensor
$$\mu = \mu(t) = \mu_{a_1\cdots a_n} t^{a_1} \wedge \cdots \wedge t^{a_n}$$
such that
$$\sum_{i=1}^{n} (-1)^i \mu_{[a_1\cdots a_i \cdots a_n} C^{a_i}{}_{bc]} = 0\,.$$

$$
\begin{tikzcd}
\text{cocycle} & \text{Chern-Simons} & \text{inv. polynomial} \\
(\bigwedge^\bullet(s\mathfrak{g}^*), d_{\mathfrak{g}}) & (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}) \arrow[l, two heads, "i^*"'] & (\bigwedge^\bullet(ss\mathfrak{g})^*) \arrow[l, hook', "p^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 & \\
0 & p^*k \arrow[u, mapsto, "d_{\mathrm{inn}(\mathfrak{g})}"'] & k \arrow[l, mapsto, "p^*"] \\
\mu \arrow[u, mapsto, "d_{\mathfrak{g}}"'] & cs \arrow[l, mapsto, "i^*"'] \arrow[u, mapsto, "d_{\mathrm{inn}(\mathfrak{g})}"'] &
\end{tikzcd}
$$

Figure 2: **Lie algebra cocycles, invariant polynomials and transgression elements** in terms of cohomology of $\mathrm{inn}(\mathfrak{g})$.

- A degree $n+1$ symmetric invariant polynomial is a completely symmetric tensor
$$k = k(r) = k_{a_1 \cdots a_{n+1}} r^{a_1} \wedge \cdots \wedge r^{a_{n+1}}$$
such that
$$\sum_{i=1}^{n+1} k_{(a_1 \cdots a_{i-1}, a_i, a_{i+1} \cdots a_{n+1}} C^{a_i}{}_{|b|c)} = 0\,.$$

One finds, either by following Chern and Simons [3] or by using our homotopy operator for $\mathrm{inn}(\mathfrak{g})$ as described in 2.2.4, that the restriction of a transgression element corresponding to the invariant polynomial $k$ to $\mathfrak{g}$ has components proportional to

$$k_{[a_1|b_2 \cdots b_n|} C^{b_2}{}_{a_2 a_3} C^{b_3}{}_{a_4 a_5} \cdots C^{b_n}{}_{a_{2n} a_{2n+1}]}\,.$$

#### 2.2.3 Transgression and the trivializability of $\mathrm{inn}(\mathfrak{g})$

It is important that

$$
\begin{array}{rl}
 & EG \text{ is contractible} \\
\Leftrightarrow & \mathrm{INN}(G) \text{ is trivializable} \\
\Leftrightarrow & \text{the cohomology of } \mathrm{inn}(\mathfrak{g})^* = (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}) \text{ is trivial} \\
\Leftrightarrow & \text{there is a homotopy } \tau : 0 \to \mathrm{Id}_{\mathrm{inn}(\mathfrak{g})}, \text{ i.e. } [d_{\mathrm{inn}(\mathfrak{g})}, \tau] = \mathrm{Id}_{\mathrm{inn}(\mathfrak{g})}\,.
\end{array}
$$

This implies that if

$$cs$$

is to be a transgression element mediating between $\mu$ and $k$, then we have

$$cs = \tau(p^*k) + d_{\mathrm{inn}(\mathfrak{g})} q\,.$$

So for every invariant polynomial $k$

$$d_{\mathrm{inn}(\mathfrak{g})} k = 0$$

a "potential" $cs$ does exist. The nontrivial condition is then that $cs$ restricted to $\mathfrak{g}$ is a cocyle.

$$\begin{tikzcd}
\text{cocycle} & \text{Chern-Simons} & \text{inv. polynomial} \\
(\bigwedge^\bullet(s\mathfrak{g}^*), d_{\mathfrak{g}}) & (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}) \arrow[l, two heads, "i^*"'] & (\bigwedge^\bullet(ss\mathfrak{g})^*) \arrow[l, hook', "p^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 & \\
0 & p^*k \arrow[u, mapsto, "d_{\mathrm{inn}(\mathfrak{g})}"'] \arrow[d, bend right, "\tau"'] & k \arrow[l, mapsto, "p^*"] \\
\mu \arrow[u, mapsto, "d_{\mathfrak{g}}"'] & cs \arrow[l, mapsto, "i^*"'] \arrow[u, mapsto, "d_{\mathrm{inn}(\mathfrak{g})}"'] &
\end{tikzcd}$$

Figure 3: **The homotopy operator** $\tau$ exists due to the trivializability of $\mathrm{inn}(\mathfrak{g})$.

#### 2.2.4 Computation of transgression elements

Using the fact that $\mathrm{inn}(\mathfrak{g})$ is equivalent to the trivial Lie $n$-algebra, in that there is a 2-morphism

$$\begin{tikzcd}
 & 0 \arrow[dr] \arrow[d, phantom, "\Downarrow\tau"] & \\
\mathrm{inn}(\mathfrak{g})^* \arrow[ur] \arrow[rr, "\mathrm{Id}"'] & & \mathrm{inn}(\mathfrak{g})^*
\end{tikzcd},$$

hence a chain homotopy $\tau$ satisfying

$$\mathrm{Id} = [d_{\mathrm{inn}(\mathfrak{g})}, \tau]\,,$$

and using the fact that we have an explicit formula for these chain homotopies, we obtain an explicit algorithm for computing

$$cs := \tau(p^*k)\,.$$

To indicate how this works, we spell out the computation for the standard case where $\mathfrak{g}$ is a Lie algebra with an invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$.

Choose a basis $\{t^a\}$ of $s\mathfrak{g}^*$ and let $\{r^a\}$ be the corresponding basis of $ss\mathfrak{g}^*$. Let the bilinear form in question have the components $k_{ab}$ with respect to this basis. Then the $d_{\mathrm{inn}(\mathfrak{g})}$-closed element in $\wedge^2(ss\mathfrak{g}^*)$ whose transgression element we want to compute is

$$k = k_{ab} r^a \wedge r^b \,.$$

In order to compute

$$\tau(k_{ab} r^a \wedge r^b)$$

we need to first express $k$ in terms of $dt^a$ and $C^a{}_{bc} t^b \wedge t^c$ as

$$r^a = dt^a + \frac{1}{2} C^a{}_{bc} t^b \wedge t^c \,.$$

This yields

$$k = k_{ab} dt^a \wedge dt^b + k_{ab} dt^a \wedge C^a{}_{bc} t^b \wedge t^c + h_{abcd} t^a \wedge t^b \wedge t^c \wedge t^d \,,$$

where the precise form of the last term does not matter for the following since it will be annihilated by $\tau$.

The chain homotopy $\tau$ is fixed on these generators as

$$\tau : t^a \mapsto 0$$

and

$$\tau : dt^a \mapsto t^a$$

and then extended to a chain homotopy by the formula for 2-morphisms of Lie $n$-algebras.

Using this formula, we obtain

$$\tau : k \mapsto k_{ab} t^a \wedge dt^b + \frac{1}{3} C_{abc} t^a \wedge t^b \wedge t^c \,.$$

And indeed, this is the familiar transgression element associated to the Chern-Simons form. This is maybe more familar as we push it forward along any morphism

$$\mathrm{inn}(\mathfrak{g})^* \to \Omega^\bullet(X)$$

coming from a connection 1-form $A \in \Omega^1(X, \mathfrak{g})$ on some space $X$, in which case the above transgression element becomes

$$\mathrm{CS}_{\langle \cdot, \cdot \rangle}(A) := \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \,.$$

#### 2.2.5 Cocycles with values in arbitrary modules

The above discussion applies to Lie algebra cocycles and invariant polynomials with values in the trivial $\mathfrak{g}$-module. More generally, one considers Lie algebra cohomology with values in arbitrary modules.

Remarkably, discussing this more general cohomology in terms of $\mathrm{inn}(\mathfrak{g})$ is already essentially tantamount to considering the families of Lie $n$-algebras to be discussed below.

Notice, however, that by *Whitehead's lemma* (compare [1]) the only non-trivial Lie algebra cohomology for finite-dimensional semisimple Lie algebras on finite dimensional modules does occur for the trivial module.

# 3 Lie $n$-algebras from cocycles and from invariant polynomials

Using the relation between Lie algebra cohomology and $\mathrm{inn}(\mathfrak{g})$, we shall describe Lie $n$-algebras whose existence reflects the existence of Lie algebra cocycles, invariant polynomials and transgression elements.

- For each Lie algebra $(n+1)$-cocycle $\mu$ there is a Lie $n$-algebras $\mathfrak{g}_\mu$, known from [2].
- For each degree $n+1$ invariant polynomial $k$ on $\mathfrak{g}$ there is a Lie $(2n+1)$-algebra $\mathrm{ch}_k(\mathfrak{g})$ which we call a Chern Lie $(2n+1)$-algebra.
- For each transgression element relating a degree $(n+1)$ invariant polynomial $k$ and a $(2n+1)$ Lie algebra cocycle $\mu_k$ there is a Lie $(2n+1)$-algebras $\mathrm{cs}_k(\mathfrak{g})$ which we call a Chern-Simons Lie $(2n+1)$-algebra.

## 3.1 Baez-Crans Lie $n$-algebras $\mathfrak{g}_\mu$ from Lie algebra cocycles $\mu$

**Definition 1** *Let $\mu$ be an $(n+1)$-cocycle on $\mathfrak{g}$ as in 2.2. Then the Lie $n$-algebra*

$$\mathfrak{g}_\mu$$

*is that dual to the qfDGCA whose underlying algebra is*

$$\bigwedge^\bullet((s\mathfrak{g})^* \oplus (s^n\mathbb{R})^*)$$

*and whose differential $d$ is defined by the fact that*

$$d|_{\bigwedge^\bullet s\mathfrak{g}^*} = d_\mathfrak{g}$$

*and*

$$db = -\mu$$

*for $b$ the canonical basis of $s^n\mathbb{R}^*$.*

The property $d^2 = 0$ is equivalent to the Jacobi identity on $\mathfrak{g}$ and the cocycle porperty of $\mu$.

This has an obvious generalization to the case that the cocycle takes values in a Lie algebra module $V$.

**Definition 2** *Let $\mu$ be an $(n+1)$-cocycle on $\mathfrak{g}$ taking values in $V$. Let*

$$\rho : \mathfrak{g} \otimes V \to V$$

*be the Lie action on that module.*

*Then the Lie $n$-algebra*

$$\mathfrak{g}_\mu$$

*is that dual to the qfDGCA whose underlying algebra is*

$$\bigwedge^\bullet((s\mathfrak{g})^* \oplus (s^n V)^*)\,.$$

*and whose differential $d$ is defined by the fact that*

$$d|_{\bigwedge^\bullet s\mathfrak{g}^*} = d_\mathfrak{g}$$

*and*

$$db^i = -\mu^i - \rho^i{}_{aj} t^a \wedge b^j\,,$$

*for $\{b^i\}$ any basis of $s^n V^*$ and $\{t^a\}$ a basis for $s\mathfrak{g}^*$.*

Now the property $d^2 = 0$ is equivalent to the Jacobi identity on $\mathfrak{g}$, the cocycle porperty of $\mu$ and the action property of $\rho$.

We may reformulate this equivalently in $L_\infty$-language:

Baez-Crans showed [2] that Lie $n$-algebras which are concentrated in top and bottom degree are all equivalent to Lie $n$-algebras of the following form.

**Definition 3** *For $\mathfrak{g}$ any Lie algebra, $V$ any $\mathfrak{g}$-module and*

$$\mu \in H^{n+1}(\mathfrak{g}, V)$$

*a Lie algebra $(n+1)$-cocycle for $\mathfrak{g}$ with values in $m$, the semistrict Lie $n$-algebra*

$$\mathfrak{g}_\mu$$

*is defined to be the $L_\infty$-algebra on*

$$S^c(s\mathfrak{g} \oplus s^n V)$$

*with codifferential*

$$D = d_1 + d_2 + d_{n+1}$$

*defined by*

$$d_2(sX \vee sY) = s[X,Y]$$

$$d_1(sX \vee s^n B) = s^n X(B)$$

$$d_{n+1}(sX_1 \vee \cdots \vee sX_{n+1}) = s^n \mu(X_1, \cdots, X_{n+1})\,,$$

*for all $X, Y, X_i \in \mathfrak{g}$ and all $B \in m$.*

We find that $D^2(sX \vee sY) = 0$ is the Jacobi identity on $\mathfrak{g}$, as before, and $D^2(sX \vee sY \vee B) = 0$ is the Lie module property of $V$. Finally

$$\begin{aligned}
D^2(sX_1 \vee \cdots \vee sX_{n+2}) &= D\Bigg(\sum_{\sigma \in \mathrm{Sh}(1,n+1)} \epsilon(\sigma) sX_{\sigma(1)} \vee s^n \mu(X_{\sigma(2)}, \cdots, X_{\sigma(n+2)}) \\
&\quad + \sum_{\sigma \in \mathrm{Sh}(2,n)} \epsilon(\sigma) s[X_{\sigma(1)}, X_{\sigma(2)}] \vee sX_{\sigma(3)} \vee \cdots \vee sX_{\sigma(n+2)}\Bigg) \\
&= s^n \sum_{\sigma \in \mathrm{Sh}(1,n+1)} \epsilon(\sigma) X_{\sigma(1)} \left(\mu(X_{\sigma(2)}, \cdots, X_{\sigma(n+2)})\right) \\
&\quad + s^n \sum_{\sigma \in \mathrm{Sh}(2,n)} \epsilon(\sigma) \mu([X_{\sigma(1)}, X_{\sigma(2)}], X_{\sigma(3)}, \cdots X_{\sigma(n+2)}) \\
&= 0
\end{aligned}$$

is precisely the Lie cocycle property of $\mu$.

URS: I think I have the signs right here, but should be checked again.

**Remark.** For $\mathfrak{g}$ simple and $k = \langle \cdot, \cdot \rangle$ (a multiple of) the Killing form and

$$\mu = \langle \cdot, [\cdot, \cdot] \rangle$$

(a multiple of) the canonical 3-cocycle, the Baez-Crans Lie 2-algebra $\mathfrak{g}_\mu$ is the skeletal semistrict version of the String Lie 2-algebra.

## 3.2 Chern Lie $(2n+1)$-algebra $\mathrm{cs}_k(\mathfrak{g})$ from invariant polynomials $k$.

**Definition 4** *Let $k$ be an invariant polynomial of degree $(n+1)$ on $\mathfrak{g}$. Then the Lie $(2n+1)$-algebra*

$$\mathrm{ch}_k(\mathfrak{g})$$

*is defined dually by a qfDGCA on*

$$\bigwedge^\bullet((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n+1}\mathbb{R})^*)$$

*whose differential $d$ is fixed by demanding*

$$d|_{\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)} = d_{\mathrm{inn}(\mathfrak{g})}$$

*and*

$$dc = k$$

*for $\{c\}$ the canonical basis of $s^{2n+1}\mathbb{R}^*$.*

The property $d^2 = 0$ is nothing but the invariance property of $k$.

Again, we can generalize this to the the case where $k$ takes values in an arbitrary $\mathfrak{g}$-module $V$.

**Definition 5** *Let $k$ be an invariant polynomial of degree $(n+1)$ on $\mathfrak{g}$ taking values in the Lie algebra module $V$. Then the Lie $(2n+1)$-algebra*

$$\mathrm{ch}_k(\mathfrak{g})$$

*is defined dually by a qfDGCA on*

$$\bigwedge^{\bullet}((s\mathfrak{g})^* \oplus (ss\mathfrak{g})^* \oplus (s^{2n+1}V)^*)$$

*whose differential $d$ is fixed by demanding*

$$d|_{\bigwedge^{\bullet}(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)} = d_{\mathrm{inn}(\mathfrak{g})}$$

*and*

$$dc^i = k^i - \rho^i{}_{aj} t^a \wedge c^j$$

*for $\{c^i\}$ a basis of $s^{2n+1}V^*$.*

Now $d^2 = 0$ is equivalent to the invariance of $k$ together with the action property of $\rho$.

### 3.3 Chern-Simons Lie $(2n+1)$-algebras $\mathrm{cs}_k(\mathfrak{g})$ from invariant polynomials $k$

**Definition 6** *Let $k$ be an invariant polynomial of degree $n+1$ which is related by a transgression element $cs$ to a degree $(2n+1)$ cocycle $\mu_k$. Then the Lie $(2n+1)$-algebra*

$$\mathrm{cs}_k(\mathfrak{g})$$

*is defined dually on the free graded commutative algbra*

$$\bigwedge^{\bullet}(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n}\mathbb{R}^* \oplus s^{2n+1}\mathbb{R}^*)$$

*equipped with a differential $d$ defined by*

$$d|_{\bigwedge^{\bullet}(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)} = d_{\mathrm{inn}(\mathfrak{g})}$$

*and*

$$db = c - cs$$

$$dc = k\,,$$

*for $\{b\}$ the canonical basis of $s^{2n}\mathbb{R}^*$ and $\{c\}$ the canonical basis of $s^{2n+1}\mathbb{R}^*$.*

Here $d^2 = 0$ is the invariance of $k$ together with the fact that $d_{\mathrm{inn}(\mathfrak{g})} cs = k$.
The generalization to arbitrary $\mathfrak{g}$-modules is again obvious:

**Definition 7** *Let $k$ be an invariant polynomial of degree $n+1$ with values in the $\mathfrak{g}$-module $V$, which is related by a transgression element $cs$ to a degree $(2n+1)$ cocycle $\mu_k$ with values in $V$. Then the Lie $(2n+1)$-algebra*

$$\mathrm{cs}_k(\mathfrak{g})$$

*is defined dually on the free graded commutative algbra*

$$\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n}V^* \oplus s^{2n+1}V^*)$$

*equipped with a differential $d$ defined by*

$$d|_{\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*)} = d_{\mathrm{inn}(\mathfrak{g})}$$

*and*

$$db^i = c^i - cs^i - \rho^i{}_{aj} t^a \wedge b^j$$

$$dc^i = k^i - \rho^i{}_{aj} t^a \wedge c^j \,.$$

Nilpotency $d^2 = 0$ is due to the invariance of $k$, the defining property of the transgression element $cs$ and the action property of $\rho$.

## 3.4 Lie $n$-algebras of invariant polynomials

**Definition 8** *For $\mathfrak{g}$ any Lie algebra, let*

$$\mathrm{inv}(\mathfrak{g}) := \ker(d|_{\mathrm{inn}(\mathfrak{g})})|_{\bigwedge^\bullet ss\mathfrak{g}^*}$$

*be the graded-commutative algebra of invariant polynomials of $\mathfrak{g}$. We may think of this as still equpped with the differential $d_{\mathrm{inn}(\mathfrak{g})}$, hence with a trivial differential. The abelian Lie $n$-algebra obtained this way, where $n$ is the degree of the highest rank generator of $\mathrm{inv}(\mathfrak{g})$ we also denote*

$$bg \,.$$

## 3.5 Morphisms

We exhibit some important Lie $n$-algebra morphisms involving Baez-Crans, Chern and Chern-Simons Lie $n$-algebras.

### 3.5.1 Higher connections and Chern-Simons forms

The Chern-Simons Lie $(2n+1)$-algebras are characterized by the property that $(2n+1)$-connections taking values in them are given by degree $(2n+1)$ differential forms which are constrained to be the respective Chern-Simons form coming from some Lie-algebra valued 1-form.

**Proposition 1 (Chern and Chern-Simons forms)** *Connections with values in Chern and Chern-Simons Lie $(2n+1)$-algebras encode the corresponding degree $(2n+1)$ differential forms.*

- $(2n{+}1)$-*Connections with values in the Chern Lie* $(2n{+}1)$*-algebras* $\mathrm{ch}_k(\mathfrak{g})$*, i.e. morhisms*

$$\mathrm{ch}_k(\mathfrak{g})^* \to \Omega^\bullet(X)$$

*are in bijective correspondence with tuples*

$$(A, C) \in \Omega^1(X, \mathfrak{g}) \times \Omega^{2n+1}(X)$$

*such that*

$$dC = k(F_A \wedge \cdots \wedge F_A)\,.$$

- $(2n{+}1)$-*Connections with values in the Chern-Simons Lie* $(2n{+}1)$*-algebras* $\mathrm{cs}_k(\mathfrak{g})$*, i.e. morphisms*

$$\mathrm{ch}_k(\mathfrak{g})^* \to \Omega^\bullet(X)\,,$$

*are in bijective correspondence with tuples*

$$(A, B, C) \in \Omega^1(X, \mathfrak{g}) \times \Omega^{2n}(X) \times \Omega^{2n+1}(X)$$

*such that*

$$C = dB + k\mathrm{CS}_k(A)\,.$$

Here $\mathrm{CS}_k(A)$ is the $k$-Chern-Simons form, such that

$$dC = k(F_A \wedge \cdots \wedge F_A)\,.$$

### 3.5.2 The isomorphism $\mathrm{inn}(\mathfrak{g}_{\mu_k}) \simeq \mathrm{cs}_k(\mathfrak{g})$

**Proposition 2** *We have an equivalence (even an isomorphism)*

$$\mathrm{inn}(\mathfrak{g}_{\mu_k}) \simeq \mathrm{cs}_k(\mathfrak{g})$$

*whenever the latter exists.*

Proof. One checks that the assignments

$$t^a \mapsto t^a$$

$$r^a \mapsto r^a$$

$$b \mapsto b$$

$$c \mapsto c \pm (cs - \mu)$$

in our standard basis define morphisms between the two Lie $(2n+1)$-algebras. These are clearly strict inverses of each other. $\square$

#### 3.5.3 The exact sequence $0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0$

Suppose that the degree $n+1$ invariant polynomial $k$ admits a Chern-Simons potential, i.e. such that all three Lie $(2n+1)$-algebras

- $\mathfrak{g}_{\mu_k}$ – 3.1
- $\mathrm{cs}_k(\mathfrak{g})$ – 3.3
- $\mathrm{ch}_k(\mathfrak{g})$ – 3.2 .

Then we have the following morphisms between these.

**Proposition 3** *We have a canonical surjection*

$$i : \mathrm{cs}_k(\mathfrak{g}) \twoheadrightarrow \mathrm{ch}_k(\mathfrak{g}) \ .$$

Proof. One checks that the canonical inclusion of vector spaces

$$\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n+1}\mathbb{R}^*) \hookrightarrow \bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n}\mathbb{R}^* \oplus s^{2n+1}\mathbb{R}^*)$$

gives a monomorphic qfDGCA-morphism

$$\mathrm{ch}_k(\mathfrak{g})^* \to \mathrm{cs}_k(\mathfrak{g})^*$$

hence defines an epimorphic dual morphism. □

**Proposition 4** *We have a canonical injection*

$$i : \mathfrak{g}_{\mu_k} \hookrightarrow \mathrm{cs}_k(\mathfrak{g}) \ .$$

Proof. One checks that the canonical surjection of vector spaces

$$\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^* \oplus s^{2n}\mathbb{R}^*) \to \bigwedge^\bullet(s\mathfrak{g}^* \oplus s^{2n+1}\mathbb{R}^*)$$

gives an epimorphic qfDGCA-morphism

$$\mathrm{cs}_k(\mathfrak{g})^* \to \mathfrak{g}^*_{\mu_k}$$

hence defines a monomorphic dual morphism. □

**Remark.** It is the existence of this morphism which corresponds to the fact that the transgression element $cs$ has the property that it restricts to the cocycle $\mu$ on $\bigwedge^\bullet(s\mathfrak{g}^*)$.

**Proposition 5** *The composite morphism*

$$\mathfrak{g}_{\mu_k} \hookrightarrow \mathrm{cs}_k(\mathfrak{g}) \twoheadrightarrow \mathrm{ch}_k(\mathfrak{g})$$

*is homotopic to the zero-morphism.*

Proof. By the above, the dual morphism is the identity on the generators of $(s\mathfrak{g})^*$

$$f^* : t^a \mapsto t^a$$

and sends everything else to zero. But we have

$$f^* = [d\tau]$$

with $\tau$ given on generators as

$$\tau : t^a \mapsto 0$$

$$\tau : r^a \mapsto t^a$$

$$\tau : c \mapsto 0$$

and then extended as a 2-morphism. □

In summary this gives

**Corollary 1** *Whenever the $(2n+1)$-cocycle $\mu_k$ on $\mathfrak{g}$ and the invariant degree $(n+1)$-polynomial $k$ are related by transgression, we have an exact sequence of Lie $(2n+1)$-algebras*

$$0 \to \mathfrak{g}_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}) \to \mathrm{ch}_k(\mathfrak{g}) \to 0\,.$$

#### 3.5.4 Other useful morphisms

**Definition 9** *For any $\mathrm{ch}_k(\mathfrak{g})$ we have a canonical morphism*

$$\mathrm{ch}_k(\mathfrak{g})^* \to \mathrm{inn}(\mathfrak{g})^*\,.$$

*It sends the generator $c$ of $s^{n-1}\mathbb{R}^*$ to $\tau(k)$, where $\tau$ is the trivializing homotopy of $\mathrm{inn}(\mathfrak{g})$.*

$$c \mapsto \tau(k)\,.$$

**Definition 10** *For any $\mathrm{ch}_k(\mathfrak{g})$ we have a canonical morphism*

$$\mathrm{Lie}(\Sigma^n U(1))^* \to \mathrm{ch}_k(\mathfrak{g})^*$$

*which sends the single generator of $\mathrm{Lie}(\Sigma^n U(1))^*$ to $k$.*

$$
\begin{tikzcd}
G \arrow[r] & EG \arrow[r] & BG & \text{top. spaces} \\
\mathrm{Disc}(G) \arrow[u, mapsto, "|\cdot|"'] \arrow[r, hook, "i"] \arrow[d, mapsto, "\mathrm{Lie}"'] & \mathrm{INN}(G) \arrow[u, mapsto, "|\cdot|"'] \arrow[r, two heads, "p"] \arrow[d, mapsto, "\mathrm{Lie}"'] & \Sigma G \arrow[u, mapsto, "|\cdot|"'] \arrow[d, mapsto] & \text{Lie groupoids} \\
(\bigwedge^\bullet(s\mathfrak{g}^*), d_{\mathfrak{g}}) & (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}) \arrow[l, two heads] & (\bigwedge^\bullet(ss\mathfrak{g})^*) \arrow[l, hook'] & \text{free graded comm. algebras} \\
\mathfrak{g} \arrow[u, dash, "(\cdot)^*", "\sim"'] \arrow[r, hook] & \mathrm{inn}(\mathfrak{g}) \arrow[u, dash, "(\cdot)^*", "\sim"'] & & \text{Lie 2-algebras} \arrow[dddd]
\\
\text{cocycle} & \text{Chern-Simons element} & \text{inv. polynomial} & \\
& 0 & & \\
0 & p^*k \arrow[u, mapsto, "d_{\mathrm{inn}(g)}"'] \arrow[d, bend right, "\tau"'] & k \arrow[l, mapsto, "p^*"'] & \text{elements in cohomology} \\
\mu \arrow[u, mapsto, "d_{\mathfrak{g}}"'] & cs \arrow[l, mapsto, "i^*"'] \arrow[u, mapsto, "d_{\mathrm{inn}(\mathfrak{g})}"'] & & \\
\mathfrak{g} \arrow[r, hook] & \mathrm{inn}(\mathfrak{g}) \arrow[r, "="] & \mathrm{inn}(\mathfrak{g}) & \\
\mathfrak{g}_\mu \arrow[u, two heads] \arrow[r, hook] & \mathrm{cs}_k(\mathfrak{g}) \arrow[u, two heads] \arrow[r, two heads] & \mathrm{ch}_k(\mathfrak{g}) \arrow[u, two heads] & \text{Lie } (2n+1)\text{-algebras} \\
& \mathrm{inn}(\mathfrak{g}_\mu) \arrow[u, dash, "\sim"'] & & \\
\text{Baez-Crans} & \text{Chern-Simons} & \text{Chern} &
\end{tikzcd}
$$

Figure 4: **Chern Lie $(2n+1)$-algebras**: for each Lie algebra $(2n+1)$ cocycle $\mu$ which is related by transgression to an invariant polynomial $k$ we obtain an exact sequence of Lie $(2n+1)$-algebras.

# 4 Lie $n$-algebra cohomology

Above we described families of Lie $n$-algebras induced from the cohomology of ordinary Lie algebras. There is a rather obvious generalization of the entire discussion to what should be the cohomology of Lie $n$-algebras themselves.

## 4.1 Lie $n$-algebra cohomology

Let $\mathfrak{g}_{(n)}$ be any Lie $n$-algebra and let $\mathrm{inn}(\mathfrak{g}_{(n)})$ be the corresponding Lie $(n+1)$-algebra of inner derivations.

Write $(\bigwedge^\bullet(s\mathfrak{g}^*_{(n)}), d_{\mathfrak{g}_{(n)}})$ and $(\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}), d_{\mathrm{inn}(\mathfrak{g}_{(n)})})$ for the corresponding dual qDGCAs, respectively.

Recalling that we have a canonical inclusion

$$i : \mathfrak{g}_{(n)} \to \mathrm{inn}(\mathfrak{g}_{(n)})$$

of Lie $(n+1)$-algbras and a canonical inclusion

$$\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}) \leftarrow \bigwedge^\bullet(ss\mathfrak{g}^*_{(n)})$$

of graded-commutative algebras such that

$$(\bigwedge^\bullet(s\mathfrak{g}^*_{(n)}), d_{\mathfrak{g}_{(n)}}) \twoheadleftarrow (\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}), d_{\mathrm{inn}(\mathfrak{g}_{(n)})}) \hookleftarrow \bigwedge^\bullet(ss\mathfrak{g}^*_{(n)})$$

can be thought of as differential forms on a universal $G_{(n)}$-bundle [4]

$$G_{(n)} \to \mathrm{INN}_0(G_{(n)}) \to \Sigma G_{(n)}$$

we make the following definition, which is the obvious and straightforward generalization of 2.2.1.

**Definition 11** *For the above setup, we say*

- *A $\mathfrak{g}_{(n)}$ cocycle $\mu$ is a $d_{\mathfrak{g}_{(n)}}$-closed element of $\bigwedge^\bullet(s\mathfrak{g}^*_{(n)})$*

$$d_{\mathfrak{g}_{(n)}}\mu = 0\,.$$

- *A $\mathfrak{g}_{(n)}$ invariant polynomial $k$ is a $d_{\mathrm{inn}(\mathfrak{g}_{(n)})}$-closed element in $\bigwedge^\bullet(ss\mathfrak{g}^*_{(n)})$*

$$d_{\mathrm{inn}(\mathfrak{g}_{(n)})}k = 0\,.$$

- *A $\mathfrak{g}_{(n)}$ transgression element $cs$ for a given $\mathfrak{g}_{(n)}$ cocycle $\mu$ and a given $\mathfrak{g}_{(n)}$ invariant polynomial $k$ is an element $cs$ in $\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)})$ such that*

$$cs|_{\bigwedge^\bullet s\mathfrak{g}^*_{(n)}} = \mu$$

*and*

$$d_{\mathrm{inn}(\mathfrak{g}(n))}cs = k\,.$$

We address these cocycles, invariant polynomials and transgression elements as being of degree $d$ whenever the respective elements have degree $r$ in $\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)})$. Notice that following this convention an ordinary degree $n$ invariant polynomial is now addressed as having degree $2n$. Which indeed is the more natural counting.

**Definition 12** *A Lie $n$-algebra cocycle is a Lie algebra coboundary simply if it is a $d_{\mathfrak{g}_{(n)}}$ coboundary. An invariant polynomial $k$ is a coboundary if it is $d_{\mathrm{inn}(\mathfrak{g}_{(n)})}$-coboundary*

$$k = d_{\mathrm{inn}(\mathfrak{g}_{(n)})}\lambda$$

*with the property that $\lambda$ vanishes when restricted to $\bigwedge^\bullet(s\mathfrak{g}^*_{(n)})$.*

**Remark.** This can be read as saying that an invariant polynomial is regarded as trivializable if it is in transgression with the trivial Lie algebra cocycle.

## 4.2 Lie $\max(n,d)$-algebras from Lie $n$-algebra cohomology

**Definition 13** *For $\mathfrak{g}_{(n)}$ any Lie $n$-algebra and $\mu$ a degree $(d+1)$ cocycle on it, we obtain a Lie $\max(n,d)$-algebra $(\mathfrak{g}_{(n)})_\mu$ on*

$$\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus s^d\mathbb{R}^*)$$

*dy defining a differential $d$ by*

$$d_{(\mathfrak{g}_{(n)})_\mu}|_{\bigwedge^\bullet s\mathfrak{g}_{(n)^*}} = d_{\mathfrak{g}_{(n)}}$$

*and*

$$d_{(\mathfrak{g}_{(n)})_\mu} b = -\mu$$

*for $\{b\}$ the canonical basis of $s^d\mathbb{R}^*$.*

**Definition 14** *For $\mathfrak{g}_{(n)}$ any Lie $n$-algebra and $k$ a degree $(d+1)$ invariant polynomial on it, we obtain a Lie $\max(n,d)$-algebra $\mathrm{ch}_k(\mathfrak{g}_{(n)})$ on*

$$\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)} \oplus s^d\mathbb{R}^*)$$

*dy defining a differential $d$ by*

$$d_{(\mathfrak{g}_{(n)})_\mu}|_{\bigwedge^\bullet s\mathfrak{g}_{(n)^*}\oplus ss\mathfrak{g}^*_{(n)}} = d_{\mathrm{inn}(\mathfrak{g}_{(n)})}$$

*and*

$$d_{\mathrm{ch}_k(\mathfrak{g}_{(n)})} c = k$$

*for $\{c\}$ the canonical basis of $s^d\mathbb{R}^*$.*

**Definition 15** *For $\mathfrak{g}_{(n)}$ any Lie $n$-algebra and cs a degree $d$ transgression element interpolating between a degree $d+1$ invariant polynomial $k$ and a degree $d$ cocycle $\mu$ on $\mathfrak{g}_{(n)}$ we obtain a Lie* $\max(n,d)$*-algebra* $\mathrm{cs}_k(\mathfrak{g}_{(n)})$ *on*

$$\bigwedge^\bullet (s\mathfrak{g}_{(n)}^* \oplus s^{d-1}\mathbb{R}^* \oplus s^d\mathbb{R}^*)$$

*dy defining a differential $d$ by*

$$d_{(\mathfrak{g}_{(n)})_\mu}|_{\bigwedge^\bullet s\mathfrak{g}_{(n)^*} \oplus ss\mathfrak{g}_{(n)}^*} = d_{\mathrm{inn}(\mathfrak{g}_{(n)})}$$

*and*

$$d_{\mathrm{cs}_k(\mathfrak{g}_{(n)})} b = cs - c$$

$$d_{\mathrm{cs}_k(\mathfrak{g}_{(n)})} c = k$$

*for $\{b\}$ the canonical basis of $s^{d-1}\mathbb{R}^*$ and $\{c\}$ the canonical basis of $s^d\mathbb{R}^*$.*

All these differentials square to 0, $d^2 = 0$ by exactly the same (simple) reasoning as in 3. And in fact, all the remaining discussion goes through just as before. So we get

**Corollary 2** *Whenever for $\mathfrak{g}_{(n)}$ any Lie $n$-algebra the $d$-cocycle $\mu_k$ on $\mathfrak{g}_{(n)}$ and the invariant degree $(d+1)$ polynomial $k$ are related by transgression, we have an exact sequence of Lie* $\max(n,d)$*-algebras*

$$0 \to (\mathfrak{g}_{(n)})_{\mu_k} \to \mathrm{cs}_k(\mathfrak{g}_{(n)}) \to \mathrm{ch}_k(\mathfrak{g}_{(n)}) \to 0\,.$$

### 4.3 Invariant polynomials of Baez-Crans type Lie $n$-algebras

**Proposition 6** *The cohomology classes of invariant polynomials of a Lie $n$-algebra $\mathfrak{g}_\mu$ of Baez-Crans type arising from an ordinary Lie algebra $\mathfrak{g}$ and a degree $(n+1)$-cocycle $\mu_k$ on it which is in transgression with an invariant polynomial $k$ are those of $\mathfrak{g}$ itself modulo $k$:*

$$\mathrm{inv}(\mathfrak{g}_{\mu_k}) \simeq \mathrm{inv}(\mathfrak{g})/k\,.$$

Proof. By inspection one finds that $\mathfrak{g}_\mu$ has no invariant polynomials above those coming from $\mathfrak{g}$. But precisely $k$ becomes a coboundary of invariant polynomials now, since

$$k = d\tau(k) = d((\mathrm{cs} - \mu) + \mu) = d((\mathrm{cs} - \mu) + c)\,,$$

where $c$ is the top degree generator of $\mathrm{inn}(\mathfrak{g}_\mu)$. But $(\mathrm{cs} - \mu) + c$ manifestly vanishes when restricted to $\bigwedge^\bullet(s\mathfrak{g}_\mu^*)$. Hence, by definition 12, $k$ is a coboundary of invariant polynomials. $\square$

# 5 Characteristic classes of $n$-bundles

## 5.1 Lie $n$-algebra valued connections

Apart from being natural in itself, our definition of Lie $n$-algebra cohomology and of invariant polynomials on Lie $n$-algebras justifies itself in that it does give the right framework for the description of representatives of Chern classes of $n$-bundles.

**Definition 16** *For $\mathfrak{g}_{(n)}$ any Lie $n$-algebra and $X$ a space, a connection on the trivial $\mathfrak{g}_{(n)}$-$n$-bundle over $X$ is a morphism*

$$(A, F_A) : \ \Omega^\bullet(X) \longleftarrow \mathrm{inn}(\mathfrak{g}_{(n)})^*$$

*of differential graded commutative algebras. Following our general discussion of higher morphisms of Lie $n$-algebras, we define higher morphisms of such connections to be those homotopies*

$$\begin{tikzcd} \Omega^\bullet(X) \arrow[r, phantom, "\Downarrow"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, bend right, "{(A,F_A)}"'] \arrow[l, bend left, "{(A',F_{A'})}"] \end{tikzcd}$$

*which vanish when pulled back along the canonical morphisms of graded algebras*

$$\mathrm{inn}(\mathfrak{g}_{(n)})^* \longleftarrow ss\mathfrak{g}^*_{(n)} \ ,$$

*i.e. such that*

$$\begin{tikzcd} \Omega^\bullet(X) \arrow[r, phantom, "\Downarrow"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, bend right, "{(A,F_A)}"'] \arrow[l, bend left, "{(A',F_{A'})}"] & ss\mathfrak{g}^*_{(n)} \arrow[l] \end{tikzcd}$$

*is the vanishing homotopy.*

Here $\Omega^\bullet(X)$ is the deRham complex of $X$ and $\mathrm{inn}(\mathfrak{g}_{(n)})^*$ is shorthand for the Koszul dual corresponding to $\mathrm{inn}(\mathfrak{g}_{(n)})$.

**Remark.** The condition on the $(k > 1)$-morphisms simply means that the component maps of the higher chain homotopies, which are maps

$$\bigwedge^\bullet(s\mathfrak{g}^*_{(n)} \oplus ss\mathfrak{g}^*_{(n)}) \to \Omega^{\bullet-(k-1)}(X)$$

vanish on the generators in $ss\mathfrak{g}^*_{(n)}$. This makes the $n$-category of connections on the trivial $\mathfrak{g}_{(n)}$-bundle contain interesting information, even though $\mathrm{inn}(\mathfrak{g}_{(n)})$ is equivalent, when all higher morphisms are admitted, to the trivial Lie $n$-algebra.

There are various ways to understand this condition. While these will be discussed in detail elsewhere, the following consideration indicates what is going on:

Suppose $G_{(n)}$ is some Lie $n$-group, to be thought of as integrating our Lie $n$-algebra $\mathfrak{g}_{(n)}$. Let $Y \to X$ be a good cover of base space $X$ and $Y^{[2]}$ the corresponding groupoid. Then $G_{(n)}$-bundles on $X$ are classified by pseudo functors

$$\begin{tikzcd} Y^{[2]} \arrow[d] \\ \Sigma G_{(n)} \end{tikzcd} .$$

These encode precisely a nonabelian $G_{(n)}$-cocycle, hence the transition data of a locally trivialized $G_{(n)}$-bundle. Equipping this $G_{(n)}$-bundle with a connections amounts to extending this morphism along the canonical inclusion

$$\begin{tikzcd} Y^{[2]} \arrow[d, "g"] & \\ \Sigma G_{(n)} \arrow[r] & \Sigma \mathrm{INN}_0(G_{(n)}) \end{tikzcd}$$

to a square

$$\begin{tikzcd} Y^{[2]} \arrow[r] \arrow[d, "g"] & \mathcal{C}_n(Y) \arrow[d, "(g,\mathrm{tra},\mathrm{curv})"] \\ \Sigma G_{(n)} \arrow[r] & \Sigma \mathrm{INN}_0(G_{(n)}) \end{tikzcd} ,$$

where $(\mathrm{tra}, \mathrm{curv})$ is the integrated version of the morphism $(A, F_A)$ appearing above, describing an integrated connection on the trivial $G_{(n)}$ bundle over $Y$ obtained by locally trivializing a general $G_{(n)}$-bundle on $X$. The point is that the horizontal arrows imply that while the connection takes values in $\mathrm{INN}_0(G_{(n)})$, its *transition morphisms* (the descent gluing data), take values only in $G_{(n)}$. This is the integrated analog of our condition that higher morphisms of $\mathfrak{g}_{(n)}$-connections are restricted higher morphisms between 1-morphisms on $\mathrm{inn}(\mathfrak{g})_{(n)}$.

We now first define Chern classes for trivial $\mathfrak{g}_{(n)}$-bundles and then discuss their descent to Chern classes of possibly nontrivial $\mathfrak{g}_{(n)}$-bundles.

**Definition 17 (Chern classes for Lie $n$-algebras)** *Given a $\mathfrak{g}_{(n)}$-connection*

$$(A, F_A) : \ \Omega^\bullet(X) \longleftarrow \mathrm{inn}(\mathfrak{g}_{(n)})^*$$

*on a trivial $\mathfrak{g}_{(n)}$-$n$-bundle over $X$, for any choice of degree $r$ Lie $n$-algebra invariant polynomial $k$ of $\mathfrak{g}_{(n)}$ we obtain an $r$-form*

$$k(F_A) : \ \Omega^\bullet(X) \xleftarrow{(A,F_A)} \mathrm{inn}(\mathfrak{g}_{(n)})^* \longleftarrow \mathrm{ch}_k(\mathfrak{g}_{(n)})^* \longleftarrow \mathrm{Lie}(\Sigma^{(r-1)}U(1))^* \ ,$$

*where the two morphisms on the right are the canonical ones described in 3.5.4.*

*This is the* $k$*-Chern-form of the connection* $(A, F_A)$*.*

**Remark.** The $r$ form $k(F_A)$ is nothing but the image of of $k$ under $(A, F_A)$. This, in turn, is nothing but the invariant polynomial $k$ with the concrete curvature $F_A$ substituted for the respective generators of $ss\mathfrak{g}^*_{(n)}$. But it is useful to restate this – simple but component-dependent – statement more intrinsically in terms of the above morphisms.

**Example.** For $\mathfrak{g}$ an ordinary simple Lie 1-algebra and $k = \langle \cdot, \cdot \rangle$ the Killing form, and for $(A, F_A)$ a $\mathfrak{g}$-connection, we have

$$k(F_A) = \langle F_A \wedge F_A \rangle$$

as one would expect.

## 5.2 $n$-Bundles with structure Lie $n$-algebra

For $\mathfrak{g}_{(n)}$ any Lie $n$-algebra, the sequence

$$\begin{tikzcd} \mathfrak{g}^*_{(n)} \\ \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[u] \\ b\mathfrak{g}^*_{(n)} \arrow[u] \end{tikzcd}$$

with $b\mathfrak{g}$ as in definition 8 plays the role of the universal $G_{(n)}$-$n$-bundle

$$\begin{tikzcd} G_{(n)} \arrow[d] \\ EG_{(n)} \arrow[d] \\ BG_{(n)} \end{tikzcd}$$

in that it comes from its Lie $n$-groupoid realization

$$\begin{tikzcd} G_{(n)} \arrow[d, "i"] \\ \mathrm{INN}_0(G_{(n)}) \arrow[d, "p"] \\ \Sigma G_{(n)} \end{tikzcd}$$

as described in [4].

Accordingly, we say that, for $X$ a space, a qDGCA morphism

$$\begin{tikzcd} \Omega^\bullet(X) & b\mathfrak{g}^*_{(n)} \arrow[l, "\{K_i\}"'] \end{tikzcd}$$

is a classifying map for a $\mathfrak{g}_{(n)}$-$n$-bundle: this morphism is nothing but a choice of a closed $r$-form $K_i$ on $X$ for each $\mathfrak{g}_{(n)}$ invariant polynomial $k_i \in \mathrm{inv}(\mathfrak{g})_{(n)}$ of degree $r$.

Then, completing the cone

$$\begin{tikzcd} & \mathrm{inn}(\mathfrak{g}_{(n)})^* \\ \Omega^\bullet(X) & b\mathfrak{g}^*_{(n)} \arrow[l, "\{K_i\}"'] \arrow[u] \end{tikzcd}$$

to a square

$$\begin{tikzcd} \Omega^\bullet(P) & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "(A, F_A)"'] \\ \Omega^\bullet(X) \arrow[u, "p^*"] & b\mathfrak{g}^*_{(n)} \arrow[l, "\{K_i = k_i(F_A)\}"'] \arrow[u] \end{tikzcd}$$

amounts to choosing a total space $p : P \to X$ over $X$ with a $\mathfrak{g}_{(n)}$-connection chosen on it that does induce the previoulsy chosen characteristic classes. Further requiring that the pushout of

$$\begin{tikzcd}
 & \mathfrak{g}^*_{(n)} \\
\Omega^\bullet(P) & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "{(A,F_A)}"'] \arrow[u] \\
\Omega^\bullet(X) \arrow[u, "p^*"] & b\mathfrak{g}^*_{(n)} \arrow[l, "{\{K_i = k_i(F_A)\}}"'] \arrow[u]
\end{tikzcd}$$

exists

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{li}}(|G_{(n)}|) & \mathfrak{g}^*_{(n)} \arrow[l, "\simeq"'] \\
\Omega^\bullet(P) \arrow[u, "i^*"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "{(A,F_A)}"'] \arrow[u] \\
\Omega^\bullet(X) \arrow[u, "p^*"] & b\mathfrak{g}^*_{(n)} \arrow[l, "{\{K_i = k_i(F_A)\}}"'] \arrow[u]
\end{tikzcd}$$

says that the fibers of $P$ have to admit a basis of differential forms that mimics the qDGCA of $\mathfrak{g}^*_{(n)}$. For $n = 1$ this just says that the fibers have to look like the group $G$ and that the connection $A$ restricts to the canonical 1-form $\theta$ on $G$, $i^*A = \theta$. Hence the top square is the first condition on a Cartan connection $A$.

(URS: I suspect that requiring the lower square to be a pushout is the second Cartan condition (equivariance of $A$). But I am not sure yet how to see this.)

For $n > 1$ our notation $\Omega^\bullet_{\mathrm{li}}(|G_{(n)}|)$ indicates what one expects this statement to generalize to, though realizing an $n$-group $G_{(n)}$ integrating the Lie $n$-algebra $\mathfrak{g}_{(n)}$ as well as its nerve $|G_{(n)}|$ internal to smooth spaces is a currently unsolved problem.

Hence constructing smooth spaces $P$ with the above properties is an issue beyond our present scope here. Nevertheless, we can proceed and study the properties $P$ would have.

## 5.3 Characteristic classes of $\mathfrak{g}_{(n)}$-$n$-Bundles

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{li}}(G_{(n)}) & \mathfrak{g}^*_{(n)} \arrow[l] \\
\Omega^\bullet(P) \arrow[u, "i^*"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "{(A,p^*F_A)}"'] \arrow[u] \\
 & \mathrm{ch}_k(\mathfrak{g}_{(n)})^* \arrow[u] \\
\Omega^\bullet(X) \arrow[uu, "p^*"] \arrow[uuu, bend left, "0"] & \mathrm{Lie}(\Sigma^{(r-1)}U(1))^* \arrow[l, "k(F_A)"] \arrow[u] \arrow[uul, "p^*k(F_A)"'] \arrow[uuu, bend right, "0"']
\end{tikzcd}$$

Figure 5: **Characteristic classes** on $\mathfrak{g}_{(n)}$-$n$-bundles. Given the $\mathfrak{g}_{(n)}$ connection $(A, F_A) : \Omega^\bullet(P) \leftarrow \mathrm{inn}(\mathfrak{g}_{(n)})^*$ on the total space $P$ the assumption that the top square exists as a pushout amounts to the assumption that $p : P \to X$ has fibers that look like the $n$-group integrating $\mathfrak{g}_{(n)}$. Each characteristic class of degree $r$, manifested in the existence of the Chern Lie $(r+1)$ algebra $\mathrm{ch}_k(\mathfrak{g}_{(n)})$, leads to a differential $r$-form on $P$ as indicated. By construction/definition this descends to the $r$-form representative $k(F_A)$ of the characteristic class as indicated.

**Proposition 7** *The images of invariant polynomials $k \in \bigwedge^\bullet(ss\mathfrak{g}^*_{(n)})$ of a Lie $n$-algebra $\mathfrak{g}_{(n)}$ under a choice of $\mathfrak{g}_{(n)}$-connection*

$$(A, F_A) : \ \Omega^\bullet(X) \longleftarrow \mathrm{inn}(\mathfrak{g}_{(n)})$$

*are invariant under morphisms of $\mathfrak{g}_{(n)}$-connections: if*

$$k(F_A),\ k(F_{A'}) \in \Omega^\bullet(X)$$

*are the images of $k$ under $(A, F_A)$ and $(A', F_{A'})$, respectively and if there exists a morphism $(A, F_A) \to (A', F_{A'})$ then in fact*

$$k(F_A) = k(F_{A'})\,.$$

Proof. The existence of the morphism

$$\begin{tikzcd}
\Omega^\bullet(X) \arrow[r, phantom, "\Downarrow\eta"] & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, bend right, "{(A,F_A)}"'] \arrow[l, bend left, "{(A',F_{A'})}"]
\end{tikzcd}$$

implies that

$$k(F_{A'}) - k(F_A) = [d,\eta](k)\,.$$

But

$$d(\eta(k)) = 0$$

since, by definition of invariant polynomials of Lie $n$-algebras, $k \in \bigwedge^\bullet(s s\mathfrak{g}_{(n)})$, which, by definition of morphisms of $\mathfrak{g}_{(n)}$-connection, implies that $\eta(k) = 0$.

And

$$\eta(dk) = 0$$

since, by definition of invariant polynomials of Lie $n$-algebras,

$$dk = d_{\mathrm{inn}(\mathfrak{g}_{(n)})} k = 0\,.$$

Hence

$$k(F_{A'}) - k(F_A) = d\eta(k) + \eta(dk) = 0\,.$$

$\square$

Using the more intrinsic formulation of characteristic classes from definition 17 we may restate the above proposition concisely as

**Corollary 3** *Morphisms of $\mathfrak{g}_{(n)}$ connections*

$$\begin{tikzcd} \Omega^\bullet(X) & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, bend right, "(A,F_A)"'] \arrow[ll, bend left, "(A',F_{A'})"] \arrow[ll, phantom, "\Downarrow"] \end{tikzcd}$$

*act trivially on the corresponding characteristic classes in that*

$$\begin{tikzcd} \Omega^\bullet(X) & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, bend right, "(A,F_A)"'] \arrow[ll, bend left, "(A',F_{A'})"] \arrow[ll, phantom, "\Downarrow"] & \mathrm{ch}_k(\mathfrak{g}_{(n)})^* \arrow[l] & \mathrm{Lie}(\Sigma^{r-1}U(1))^* \arrow[l] \end{tikzcd}$$

$=$

$$\begin{tikzcd} \Omega^\bullet(X) & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "(A,F_A)"'] & \mathrm{ch}_k(\mathfrak{g}_{(n)})^* \arrow[l] & \mathrm{Lie}(\Sigma^{r-1}U(1))^* \arrow[l] \end{tikzcd}$$

*for all $k \in \mathrm{inv}(\mathfrak{g}_{(n)})$, or equivalently*

$$\begin{tikzcd} \Omega^\bullet(X) & & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[ll, bend right, "(A,F_A)"'] \arrow[ll, bend left, "(A',F_{A'})"] \arrow[ll, phantom, "\Downarrow"] & b\mathfrak{g}_{(n)}^* \arrow[l] \end{tikzcd} \quad = \quad \begin{tikzcd} \Omega^\bullet(X) & \mathrm{inn}(\mathfrak{g}_{(n)})^* \arrow[l, "(A,F_A)"'] & b\mathfrak{g}_{(n)}^* \arrow[l] \end{tikzcd}\,.$$

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