Rational CFT is parallel transport

Jens Fjelstad and Urs Schreiber

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Abstract

From the data of any semisimple modular tensor category $\mathcal{C}$ the prescription [2] constructs a 3-dimensional TFT by encoding 3-manifolds in terms of string diagrams in $\mathcal{C}$. From the additional data of a certain Frobenius algebra object internal to $\mathcal{C}$, the presciption [18, 4] obtains (the combinatorial aspect of) the corresponding full boundary CFT by decorating triangulations of surfaces with objects and morphisms in $\mathcal{C}$.

We show that these decoration prescriptions are "quantum differential cocycles" on the worldvolume for a 3-functorial extended QFT. The boundary CFT arises from a morphism between two chiral copies of the (locally trivialized) TFT 3-functor.

The crucial observation is that all 3-dimensional string diagrams in [18] are Poincaré-dual to cylinders in $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$ which arise as components of a lax-natural transformation between two 3-functors that factor through $\mathbf{B}\mathbf{B}\mathcal{C} \hookrightarrow \mathbf{B}\mathrm{Bimod}(\mathcal{C})$.

This exhibits the "holographic" relation between 3d TFT and 2d CFT as the hom-adjunction in 3Cat, which says that a transformation between two 3-functors is itself, in components, a 2-functor.

Contents

1 Preface

The general problem of quantum field theories of the kinds known as Σ-*models* and *gauge theories* is this:

On a target space X a differential cocycle encoding a higher bundle with connection is given. This comes with its parallel transport map $\nabla : \Pi_\omega(X) \to T$ which sends k-dimensional volume elements to the "phase" obtained by parallel transport over them. This assignment is the (interaction part of) the *classical action functional.*

Picking a"worldvolume" Σ, one wants to *quantize* this classical action functional by transgressing ∇ to the mapping space, the *space of fields* or *space of paths* $\mathrm{hom}(\Pi_\omega(\Sigma), \Pi_\omega(X))$ and then in some way integrate the result over that space. This (problematic) procedure is known as the *path integral.*

Whatever the procedure is, the result is supposed to be, in the *Schrödinger picture*, a linear representation of a category of n-dimensional cobordisms – a functor from that category to vector spaces –, which assigns *spaces of states* to codimension one manifolds and linear maps, the *correlators* or *evolution operators*, to cobordisms. One can take this to be part of the *definition* of what we are willing to address as a path integral. The functoriality of the cobordism respresentation is nothing but the famous gluing property supposed to be satisfied by the path integral.

Given the elegant definition of a quantum field theory as a cobordism representation, which emerged with the work of Atiyah and Segal, one can study these structures without worrying about whether or not – and how – these representations are obtained from path integrals over classical action functionals. This fact has lead – and is leading to – a large body of work on the entirely *algebraic* aspects of quantum field theory, concerned, essentially, with nothing but the classification of cobordism representations.

Among the most detailed results in this algebraic approach is the work [2] and [18] which characterizes all those 3-dimensional topological and all those 2-dimensional conformal quantum field theories whose algebraic structure is entirely encoded in the data provided by the choice of a modular tensor category $\mathcal{C}$.

Even though the quantum field theories constructed from this data are not manifestly obtained by a quantization procedure of a classical action functional, for most of them it is clear, from plenty of circumstantial evidence, which action functional they must be the correct quantization of: they correspond notably to 3-dimensional Chern-Simons theories and to 2-dimensional Wess-Zumino-Witten theories which are systems determined by higher bundles with connection associated to groups and their classifying spaces.

These particular higher bundles with connection are comparatively well understood. In particular, it is known how they are encoded entirely in terms of their parallel transport n-functors [1, 21, 23] – nonabelian differential cocycles –, which send pieces of target space to the corresponding parallel transport morphisms over them.

And there is a striking similarity: the diagrammatic formulas for this higher dimensional parallel transport in terms of local data assigned to small patches has, if one simply subjects it to Poincaré duality, an appearance entirely analogous to the decorated diagrams which [2] and [18] use to construct their 3-dimensional and 2-dimensional QFTs.

This suggests two things:

- The combinatorial prescription of [2] and [18] of 3-dimensional TFT and 2-dimensional

CFT is, secretly, itself the data of a differential cocycle on the worldvolume: the local data defining a "parallel transport" n-functor – only that the "phases" assigned by this parallel transport are now thought of as *correlators.*

- Quantization of Σ-models and of gauge theories in general, and the path integral in particular, should be an operation on the space of all differential cocycles (of all transport n-functors) if these are conceived suitably: it sends differential cocycles on a target space X, whose parallel transport computes classical phases, to differential cocycles on worldvolumes, whose parallel transport computes correlators.

Our aim here is to demonstrate the first point.

1.1 Statement of the main result

We demonstrate that the decoration presciption of [18] of triangulated surfaces with objects and morphisms in a modular tensor category $\mathcal{C}$ is the expression of surface holonomy in local data obtained from a parallel transport 2-functor with values in

$$\mathrm{Cyl}(\mathbf{B}\mathrm{Bimod}(\mathcal{C}))$$

locally trivialized along the inclusion

$$\mathrm{Cyl}(\mathbf{BB}\mathcal{C}) \hookrightarrow \mathrm{Cyl}(\mathbf{B}\mathrm{Bimod}(\mathcal{C}))\,.$$

Here $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$ is the 3-category with a single object, with algebras internal to $\mathcal{C}$ as morphisms, bimodules for these algebras as 2-morphisms and bimodule homomorphisms as 3-morphisms.

$\mathrm{Cyl}(\mathbf{B}\mathrm{Bimod}(\mathcal{C})) \subset (\mathbf{B}\mathrm{Bimod}(\mathcal{C}))^I$ is the full sub2-category of the 2-category of cylinders in $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$ whose top and bottom face factor through the inclusion $\mathbf{B}\mathcal{C} \hookrightarrow \mathrm{Bimod}(\mathcal{C})$.

Such a parallel transport 2-functor is therefore the component map of a pseudonatural transformation between two 3-functors with values in $\mathbf{BB}\mathcal{C} \hookrightarrow \mathbf{B}\mathrm{Bimod}(\mathcal{C})$.

We indicate how this can be understood as defining a morphism between two "chiral copies" of TFT 3-functors which are gauge trivial when restrited to the 2-dimensional boundary that the CFT lives on.

Outline

- In 2 we describe the 3- and 2-categories which our differential cocycles take values in.
- In 3 we describe Frobenius algebras as the cocycles which arise from local trivialization by means of special ambidextrous adjunctions.
- In 4 we describe the local trivialization of 2-functors by special ambidextrous adjunctions and derive the structure of the corresponding differential cocycles which express them in terms of local data.
- In 5 we describe the main result, showing how the various diagrams describing RCFT correlators according to [18] arise from local differential cocycle data of parallel 2-transport.

1.2 Differential cocycles and cobordism representations

The Schrödinger picture of quantum field theory was originally formalized in terms of linear representations of cobordism categories: functors $\mathrm{QFT} : n\mathrm{Cob}_S \to \mathrm{Vect}$ on n-dimensional cobordisms equipped with S-structure (for instance, conformal, or Riemannian structure) which send $(n-1)$-dimensional manifolds to vector spaces "of states" and cobordisms between these to linear maps. More recently, e.g. [28] and [Hopkins et al.], it was noticed that this picture deserves refinement: one wants to assign k-categorical data to codimension $(k+1)$-manifolds in a way compatible under all possible gluing. This is known as *extended QFT*.

n-dimensional QFT	**n-category of "spaces of states"**
quantum mechanics	1-category of Hilbert spaces
2d RCFT	2-category $\mathrm{Bimod}(\mathcal{C})$ of algebras and bimodules
3d CS TFT	3-category $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$

Table 1: **The dimension of quantum field theories is reflected by the categorical degree of their "spaces of states" assigned to 0-dimensional manifolds.**

Here we realize this extended picture in a way adapted to the notion of nonabelian differential cocycles: for any cobordism Σ we consider a representation of an n-category of (glubular) k-paths *in* Σ. Obtaining from such a piecewise representation a representation of the full cobordism in the original sense is then a matter of *taking traces*. This is completely analogous to how (higher) holonomies are obtained from (higher) parallel transport.

$$\begin{tikzcd}
\text{parallel transport} \arrow[r, mapsto, "\text{tracing}"] & \text{holonomy}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\text{representation of}\\ \text{paths in cobordisms}\end{array} \arrow[r, mapsto, "\text{tracing}"] & \begin{array}{c}\text{representation of}\\ \text{cobordisms}\end{array}
\end{tikzcd}$$

Table 2: **From parallel transport to cobordism representations.**

1.3 List of notions

Quantum field theory notions

- By *RT 3d TFT functor* we refer to the functor from 3-dimensional cobordisms to vector spaces which can be cosntructed from a modular tensor category $\mathcal{C}$ as in [2].
- *combinatorial CFT* shall be our term for the description of 2-dimensional rational conformal field theory correlators as developed in [18].

In the existing literature this is often addressed as “the TFT approach”, since in its last step it makes use of the 3D RT functor to define the 2d CFT correlators.

From our point of view here, however, it is, while important, not the crucial aspect of [18] that it exploits the (familiar) relation between 2d CFT and 3d TFT. What is crucial is that the approach does so using combinatorial data can be interpreted as descent data for parallel 2-transport.

The n-categories that appear

- $\mathrm{pt} := \{\bullet\}$ – the terminal 2-category, which is also the tensor unit with respect to the (Gray) tensor product.
- $I := \{ \bullet \longrightarrow \circ \}$ for the 2-category with 2-objects, one nontrivial 1-morphism and no nontrivial 2-morphism.
- $\mathcal{C}$ – a braided monoidal category
- $\mathrm{Bimod}(\mathcal{C})$ – the bicategory whose objects are algebras and whose morphisms are bimodules internal to $\mathcal{C}$. This is a *proarrow equipment* and hence a *framed bicategory* in the sense of [26] (appendix C), the framing given by the inclusion

  $$\mathrm{Algebras}(\mathcal{C}) \hookrightarrow \mathrm{Bimod}(\mathcal{C}) \; .$$

  Since $\mathcal{C}$ is braided, $\mathrm{Bimod}(\mathcal{C})$ is monoidal.
- $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$ – the one-object 3-category corresponding to the monoidal 2-category $\mathrm{Bimod}(\mathcal{C})$
- $\mathrm{Cyl}(\mathbf{B}\mathrm{Bimod}(\mathcal{C})) := (\mathbf{B}\mathrm{Bimod}(\mathcal{C}))^I$ – the 2-category of cylinder in $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$
- $\mathrm{TwBimodules}(\mathcal{C})$ – the 2-category of algebras and twisted bimodules in $\mathcal{C}$, being the restriction of $\mathrm{Cyl}(\mathrm{Bim}(\mathcal{C}))$ to cylinders whose top and bottom face lie in $\mathbf{B}\mathcal{C}$:

  $$\begin{tikzcd}
  \mathrm{TwBimodules}(\mathcal{C}) \arrow[r] \arrow[d] & (\mathbf{B}\mathrm{Bimod}(\mathcal{C}))^I \arrow[d, "\mathrm{dom}\times\mathrm{codom}"] \\
  \mathbf{B}\mathcal{C} \times \mathbf{B}\mathcal{C} \arrow[r, hook] & \mathrm{Bimod}(\mathcal{C}) \times \mathrm{Bimod}(\mathcal{C})
  \end{tikzcd}$$

We will be dealing with monoidal weak 2-categories (bicategories) regarded as weak 3-categories (tricategories) with a single object. However, since the bicategory in question is *framed* or *proarrow equipped* in the sense of [26] it turns out that we can get away with behaving essentially as if we were dealing with *strict* 3-categories. We shall therefore mostly suppress, notationally, all structure morphisms such as compositors, associators, pentagonators, etc.

1.4 The FRS prescription for rational conformal fied theory

In this section we deal with a rational conformal field theory (RCFT) with chiral algebra $\mathcal{V}$, a rational conformal vertex algebra. We demand that we can consistently assign correlators of our theory to any compact oriented two-dimensional manifold with Riemannian metric, in particular we allow surfaces both with and without boundary, and refer to such a theory as an open/closed RCFT. We use string-inspired notation, and call the surfaces on which the RCFT is defined *world sheets.* By solving the theory we mean, as is conventional, to assign a consistent correlator to any allowed world sheet.

The main reason why RCFT is (at least in principle) solvable, is the fact that solving a theory can be done in a two-stage process, the first of which is complex analytic in nature, and the second of which is an algebraic/combinatorial problem.

The first step is better known as constructing a *chiral* CFT, a gadget that lives on complex curves with marked points labelled by representations of a rational conformal vertex algebra. Given a complex curve Y of genus g with m marked points labelled by representations $\lambda_1, \dots, \lambda_m$ of $\mathcal{V}$, one defines the subspace

$$\mathcal{B}_{\mathrm{Y}}(\lambda_1, \dots, \lambda_m) \subset (\lambda_1 \otimes_{\mathbb{C}} \lambda_2 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \lambda_m)^*$$

of invariants with respect to a certain action of $\mathcal{V}$ [25]. The vector spaces $\mathcal{B}_{\mathrm{Y}}(\lambda_1, \dots, \lambda_m)$, called spaces of *conformal blocks*, turn out to be finite dimensional, and combine to finite rank vector bundles $\mathcal{E}^g_{\vec{\lambda}}$ over the moduli spaces $\mathcal{M}_{g,m}$ of m-pointed curves of genus g. Furthermore, the action of $\mathcal{V}$ provides this bundle with a projectively flat connection. The chiral correlators $\{\langle (v_1, p_1), \dots, (v_m, p_m) \rangle_\alpha\}_{\alpha=1}^{\dim(\mathcal{B}_{\mathrm{Y}}(\lambda_1,\dots,\lambda_m))}$, also called conformal blocks, are obtained by inserting elements $v_1 \in \lambda_1, \dots, v_m \in \lambda_m$ in (a basis of) flat sections of $\mathcal{E}^g_{\vec{\lambda}}$, thus leading to the conventional appearance of conformal blocks as multivalued functions on $\mathcal{M}_{g,m}$.

The assignment of spaces of conformal blocks to every complex curve with a finite number of marked points is conveniently described as a *modular functor* from a suitably defined geometric category $\mathcal{E}xt_{\mathrm{Rep}_{\mathcal{V}}}$ of "$\mathrm{Rep}_{\mathcal{V}}$-extended" complex curves to $\mathrm{Vect}_{\mathbb{C}}$. This is a modular functor of the type compatible with gluing of surfaces along boundary circles or, alternatively, extended to compactifications of $\mathcal{M}_{g,n}$ corresponding to including curves with double point singularities. It is worth mentioning that although it is known for any rational conformal vertex algebra how to construct the spaces of conformal blocks, to actually carry this out explicitly is a highly non-trivial problem. Chiral correlators are therefore known in detail only for a few cases.

The second step amounts to choosing, for each world sheet X, (i) a suitable complex curve $\widehat{\mathrm{X}}$ with marked points, and (ii) an element $\mathrm{Cor}(\mathrm{X}) \in \mathcal{B}_{\widehat{\mathrm{X}}}$. As a complex curve, $\widehat{\mathrm{X}}$ is taken to be the *complex double* of X. If the world sheet X has p field insertions in the interior and q field insertions on the boundary, the curve $\widehat{\mathrm{X}}$ will have $2p+q$ marked points, labelled by elements in representations of $\mathcal{V}$ specified by the corresponding field insertions. The elements Cor(X) are required to satisfy two types of constraints: sewing constraints relating correlation functions on curves of different genus, and invariance under the mapping class group of X. Given these elements Cor(X) one finally obtains the correlation *functions* by inserting vectors specified by the field insertions, and letting insertion points and complex structure vary such that one gets a chiral corrrelator on $\widehat{\mathrm{X}}$. The constraint of invariance under the mapping class group implies that these chiral correlators are genuine functions on $\mathcal{M}_{g,2p+q}$, thus deserving the name correlation functions.

As already indicated above, although the first step is understood in principle, there are significant technical obstacles to overcome before it is feasible to attack the second problem in any generality. However, a similar structure to that described above is obtained by forgetting the vertex algebra $\mathcal{V}$ and retaining only the combinatorial data of its representation category, i.e. an abstract modular tensor category[1], while at the same time forgetting the complex structure of the complex curves, retaining only the structure of oriented topological surfaces (with marked points). Given any modular category $\mathcal{C}$, one obtains a modular functor on a certain category of $\mathcal{C}$-extended topological surfaces via 3-dimensional TQFT as constructed by Reshetikhin and Turaev [16, 17]. In this case, elements in the vector spaces associated to surfaces (which we will continue to call spaces of conformal blocks) have concrete interpretations in terms of $\mathcal{C}$-coloured ribbon graphs in 3-manifolds, and properties of $\mathcal{C}$ make calculations amenable. In this simplified situation we can consider the same two-step procedure to construct correlators. The first step is immediately given by the TQFT, and the second step has been solved in a series of papers [6, 7, 8, 3, 4] (see also [15] for previous work). The solution turns out to be given in terms of a symmetric special Frobenius algebra in $\mathcal{C}$ (the result only depends on the Morita class of the algebra, but the methods used require the choice of a particular representative).

We proceed by summarize the prescription given in [3] for obtaining RCFT correlators through a TQFT based on a modular category $\mathcal{C}$ (there is a slightly more sofisticated verson given in [4], defined similarly to a modular functor). Due to the simplified setting as compared to a genuine conformal field theory, the assignment of correlators will in the present paper be referred to as a *combinatorial* CFT, reflecting that we retain only the combinatorial data of the representations of the chiral algebra of a RCFT. It is worth stressing that although we are using a 3-dimensional TQFT to construct correlators on topological world sheets, the resulting combinatorial CFT is *not* a topological field theory. At the end of this section we sketch how it is in principle possible to obtain correlation functions of a RCFT from the correlators of a combinatorial CFT.

To define a combinatorial CFT we first define the relevant class of (topological) world sheets such a theory is defined on, and then use 3-d TQFT to construct the assignment of a correlator to any world sheet. Fix for this purpose a modular tensor category $\mathcal{C}$ and a symmetric special Frobenius algebra A in $\mathcal{C}$. We denote by $\{U_i\}_{i\in I}$ a set of representatives of simple objects in $\mathcal{C}$, and the two (left and right respectively) α-induced bimodules on the object U are denoted $A \otimes^{\pm} U$ resp. $U \otimes^{\pm} A$, where $+$ refers to using the braiding $-$ to its inverse.

By a *world sheet* is meant a tuple $(\mathrm{X}, \mathrm{or}_2(\mathrm{X}), \{\Phi_s\}_{s=1}^m, \{\Psi_t\}_{t=1}^n, \{M_u\}_{u=1}^p)$ where

- X is a compact 2-dimensional topological manifold with orientation $\mathrm{or}_2(\mathrm{X})$, where

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[1] See the recent proof [27] that the representation category of any rational conformal vertex algebra satisfying the C_2-cofiniteness condition, is modular.

the (possibly empty) boundary $\partial \mathrm{X}$ is equipepd with an orientation $\mathrm{or}_1(\partial \mathrm{X})$ obtained from $\mathrm{or}_2(\mathrm{X})$ by the inward pointing normal.

- There are $m \geq 0$ marked points in the interior $\mathrm{X} \backslash \partial \mathrm{X}$. An interior marked point is a *bulk insertion* $\Phi = (i, j, \phi, p, [\gamma])$ where $i, j \in I$, $\phi \in \mathrm{Hom}_{A|A}(U_i \otimes^+ A, A \otimes^- U_j)$, $p \in \mathrm{X} \backslash \partial \mathrm{X}$, and $[\gamma]$ is an arc-germ with $\gamma(0) = p$. For any two bulk insertions Φ_1 and Φ_2 we require $p_1 \neq p_2$.
- There are $n \geq 0$ marked points on $\partial \mathrm{X}$. A marked point on the boundary is a *boundary insertion* $\Psi = (M, N, V, \psi, p, [\gamma])$ where M and N are left A modules, $V \in \mathrm{Obj}(\mathcal{C})$, $\psi \in \mathrm{Hom}_A(M \otimes V, N)$, $p \in \partial \mathrm{X}$, and $[\gamma]$ is an arc-germ with $\gamma(0) = p$ such that every representative has a restriction to $\partial \mathrm{X}$. For any two distinct boundary insertions Ψ_1 and Ψ_2 we require $p_1 \neq p_2$. Further, if $\Psi_1 = (M_1, N_1, V_1, \psi_1, p_1, [\gamma_1])$ and $\Psi_2 = (M_2, N_2, V_2, \psi_2, p_2, [\gamma_2])$ are adjacent on the same conncted component of $\partial \mathrm{X}$, and p_1 is located after p_2 by following $\mathrm{or}_1(\partial \mathrm{X})$, it is required that $N_1 = M_2$.
- Each of the $p \geq 0$ connected components of $\partial \mathrm{X}$ that contains no marked point is labelled by a left A-module M_u, $u = 1, \ldots, p$.

The correlator of a world sheet X is an element in the finite dimensional vector space $\mathcal{H}(\widehat{\mathrm{X}})$ associated by the TQFT based on $\mathcal{C}$ to the *marked double* $\widehat{\mathrm{X}}$ of the world sheet X. The marked double of $(\mathrm{X}, \mathrm{or}_2(\mathrm{X}), \{\Phi_s\}_{s=1}^m, \{\Psi_t\}_{t=1}^n, \{M_u\}_{u=1}^p)$ is a tuple $(\widehat{X}, \{W_q\}_{q=1}^{2m+n}, \lambda)$, where

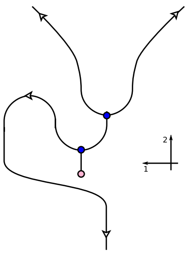
- The topological double $\widehat{\mathrm{X}}$ is a closed oriented surface obtained from the total space of the orientation bundle over X as $\widehat{\mathrm{X}} := \mathrm{Or}(\mathrm{X}) / \sim$ where $(p, \mathrm{or}_2) \sim (p, -\mathrm{or}_2)$ for $p \in \partial \mathrm{X}$, equipped with the natural orientation.
- There are $2m + n$ marked points $W_q = (\tilde{p}_q, [\tilde{\gamma}_q], Y_q, +)$, $q = 1, \ldots 2m + n$. For $q \in \{1, \ldots, 2m\}$ we define $\tilde{p}_{2s} := (p_s, \mathrm{or}_2(\mathrm{X}))$, resp. $\tilde{p}_{2s-1} := (p_s, -\mathrm{or}_2(\mathrm{X})$. $[\tilde{\gamma}_{2s}] := [(\gamma_s, \mathrm{or}_2(\mathrm{X}))]$, $[\tilde{\gamma}_{2s-1}] := [(\gamma_s, -\mathrm{or}_2(\mathrm{X}))]$. $Y_{2s} := U_{i_s}$, $Y_{2s-1} := U_{j_s}^\vee$. Here, p_s, $[\gamma_s]$ U_{i_s}, U_{j_s} are parts of the bulk insertion data of the world sheet.
  For $q \in \{2m + 1, \ldots, 2m + n\}$ we define $\tilde{p}_{2m+t} := [p_t, \pm \mathrm{or}_2(\mathrm{X})]$, $[\tilde{\gamma}_{2m+t}] := [\gamma_t]$, $Y_{2m+t} := V_t$, where the data now comes from the boundary insertion data of the world sheet.
- The last datum, $\lambda \subset \mathrm{H}_1(\widehat{\mathrm{X}}, \mathbb{R})$ is a Lagrangian subspace defined in the following way. First, define the *connecting manifold* $\mathrm{M_X}$ of X as $\mathrm{M_X} := \left(\widehat{\mathrm{X}} \times [-1, 1]\right) / \sim$, where $([p, \mathrm{or}_2], t) \sim ([p, -\mathrm{or}_2], -t)$. $\mathrm{M_X}$ is a 3-manifold such that there is a natural identification $\partial \mathrm{M_X} \cong \widehat{\mathrm{X}}$. We can think of this identification as an inclusion $f : \widehat{\mathrm{X}} \hookrightarrow \mathrm{M_X}$, and define $\lambda := \mathrm{Ker}(f_*)$.

The final step is to equip the connecting manifold $\mathrm{M_X}$ with a ribbon graph R to get an extended cobordism $\mathrm{M_X}[\mathrm{R}]$. Note that there is a natural embedding of X into $\mathrm{M_X}$ given by

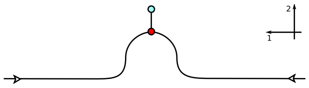
$$\iota_{\mathrm{X}} : p \mapsto [p, \pm \mathrm{or}_2(\mathrm{X}), 0]. \tag{1}$$

The ribbon graph $\mathrm{R}_{T,A} \subset \mathrm{M_X}$ is constructed in the following manner.

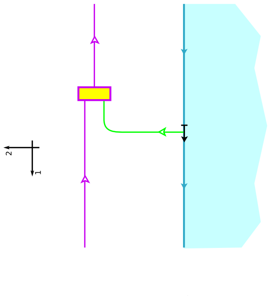
- Choose a dual triangulation T of X with only two-valent and tri-valent vertices, such that
  1. any connected component of ∂X is contained in T
  2. the two-valent vertices are located precisely at the marked points of X
  3. for any bulk insertion, there must be a representative γ of $[\gamma]$ that is contained in T
- On every trivalent vertex of T in $\iota_{\mathrm{X}}(\mathrm{X}\backslash\partial\mathrm{X})$, place the following graph with three out-going A-ribbons

  such that the pictured 2-orientation matches $\mathrm{or}_2(\mathrm{X})$. There are three possible ways to place this graph, choose one possibility.
- On every edge of T in $\iota_{\mathrm{X}}(\mathrm{X}\backslash\partial\mathrm{X})$, place the following piece of ribbon graph connecting

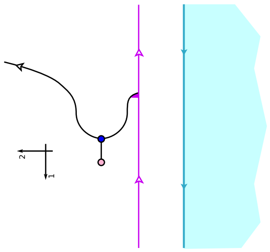
  

  the graphs covering two adjacent vertices, in such a way that the depicted 2-orientations match $\mathrm{or}_2(\mathrm{X})$.
- Every connected component of $\iota_{\mathrm{X}}(\partial\mathrm{X})$, cover the edges of T by left A-modules according to the labelling of the world sheet. The orientation and core-orientation of such a ribbon is taken to be opposite of $\mathrm{or}_2(\mathrm{X})$ and the orientation of $\iota_{\mathrm{X}}(\partial\mathrm{X})$ induced from $\mathrm{or}_2(\mathrm{X})$ respectively.
- On a two-valent vertex of T in $\iota_{\mathrm{X}}(\partial\mathrm{X})$ corresponding to the boundary insertion $(M, N, V, \psi, p, [\gamma])$, place the ribbon graph

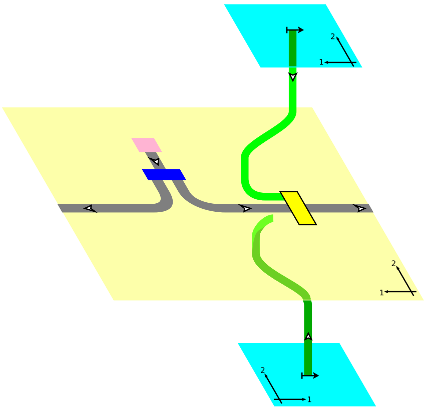
  such that the depicted orientation coincides with $\mathrm{or}_2(\mathrm{X})$. Note that the ribbons labelled M resp. N are covering $\iota_{\mathrm{X}}(\partial\mathrm{X})$, and the rightmost part of the boundary is therefore indicating a horisontal section of the boundary of M_{X}.

- On every trivalent vertex of T in $\iota_{\mathrm{X}}(\partial\mathrm{X})$, place the following graph

  such that the indicated bulk and boundary orientations match $\mathrm{or}_2(\mathrm{X})$ and $\mathrm{or}(\partial\mathrm{X})$ respectively.

- On a two-valent vertex in the interior of the embedded world sheet corresponding to the bulk insertion $(i, j, \phi, p, [\gamma])$, place the following ribbon graph where the indicated orientations are to match the orientations of M_{X}, $\iota_{\mathrm{X}}(\mathrm{X})$, and $f(\widehat{\mathrm{X}})$ respectively.

Denote the resulting extended cobordism by $\mathrm{M}_{\mathrm{X}}[\mathrm{R}_{T,A}]$.

Definition 1 *The combinatorial CFT given by the modular category $\mathcal{C}$ and the symmetric special Frobenius algebra A in $\mathcal{C}$ is given by the assignment $\mathrm{X} \mapsto \mathrm{Cor}_A(\mathrm{X})$ defined as*

$$\mathrm{Cor}_A(\mathrm{X}) := Z(\mathrm{M}_{\mathrm{X}}[\mathrm{R}_{T,A}]). \tag{2}$$

As indicated, there are various choices involved in constructing the ribbon graph. We refer to [3] for references to various parts of the proof that the correlators, for a given algebra A, are independent of all arbitrary choices.

It is known for the case where $\mathcal{C}$ is equivalent to the representation category of the conformal vertex algebra associated to an untwisted affine Lie algebra $\hat{\mathfrak{g}}$ at positive integral level k, that the spaces of conformal blocks of the corresponding chiral CFT are naturally isomorphic to the vector spaces given by the modular functor associated to the TQFT based on $\mathcal{C}$, which we will also refer to as spaces of conformal blocks. Accepting the conjecture that this

2 The coefficients for B$\mathcal{C}$-descent

Given a braided monoidal category $\mathcal{C}$, we canonically have an inclusion of (weak) 3-categories

$$\begin{tikzcd}\mathbf{B}\mathbf{B}\mathcal{C} \arrow[r, hook, "i"] & \mathbf{B}\mathrm{Bimod}(\mathcal{C})\end{tikzcd} .$$

We say a 3-functor

$$F : S \to \mathbf{B}\mathrm{Bimod}(\mathcal{C})$$

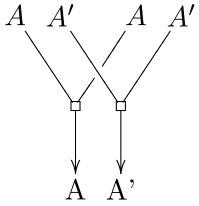
is i-trivial if it factors through this inclusion i. For F and G two such i-trivial 3-functors, a transformation

$$\begin{tikzcd}F \arrow[r, Rightarrow, "\eta"] & G\end{tikzcd}$$

2.1 $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$

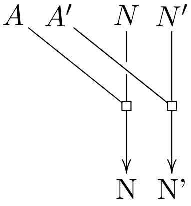
2.1.1 The monoidal structure on $\mathrm{Bimod}(\mathcal{C})$

For A and A' two algebras, their tensor product $A \otimes A'$ is the algebra which is $A \otimes A'$ as an object in $\mathcal{C}$ and equipped with the product obtained by using the braiding to exchange A with A':



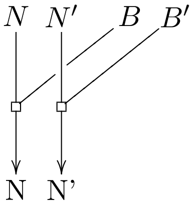
.

Accordingly, the left A-module N and the left A'-module N' are tensored to form the $A \otimes A'$-module $N \otimes N'$ with the action given by using the braiding:



.

Similarly, if N is a right B-module and N' is a right A'-module, the right action of $B \otimes B'$ on $N \otimes N'$ is

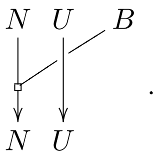


.

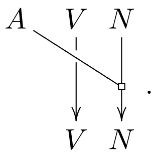
A simple special case of this turns out to be interesting in applications. The tensor unit $\mathbb{1}$ of $\mathcal{C}$ with the trivial algebra structure on it is always an algebra internal to $\mathcal{C}$. Any object of $\mathcal{C}$ is a $\mathbb{1}$-$\mathbb{1}$ bimodule. This yields a canonical inclusion

$$\begin{tikzcd}\mathbf{B}(\mathcal{C}) \arrow[r, "\subset"] & \mathrm{Bimod}(\mathcal{C})\end{tikzcd} .$$

This means that for any A-B bimodule N, and any object U in $\mathcal{C}$, we may consider $N \otimes U$ as another A-B bimodule, with the obvious left action and with the right action given by



Similarly, for V any object of $\mathcal{C}$, we obtain the A-B bimodule $V \otimes N$ with the obvious right action and the left action given by



Quite literally, we can think of the tensor structure on $\mathrm{Bim}(\mathcal{C})$ as obtained from arranging bimodules in front of each other.

2.1.2 The 3-category structure

The formal expression of this geometric intuition is that from the monoidal 2-category $\mathrm{Bim}(\mathcal{C})$ we can form the suspension, $\Sigma(\mathrm{Bim}(\mathcal{C}))$, which is the 3-category with a single object $\bullet$, such that $\mathrm{End}(\bullet) = \mathrm{Bim}(\mathcal{C})$, and such that composition across that single object is the tensor product on $\mathrm{Bim}(\mathcal{C})$.

If

$$\begin{tikzcd}
A \arrow[rr, bend left, "N"] \arrow[rr, bend right, "N'"'] & \Downarrow \rho & B
\end{tikzcd}$$

is a 2-morphism in $\mathrm{Bim}(\mathcal{C})$, we draw the corresponding 3-morphism in $\Sigma(\mathrm{Bim}(\mathcal{C}))$ as

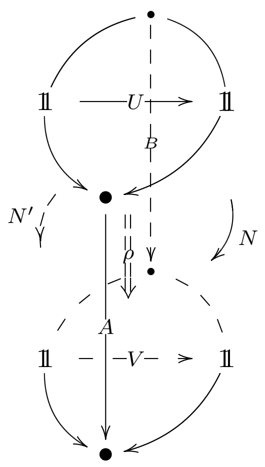
$$\begin{tikzcd}
 & \bullet \arrow[dr, bend left] & \\
A \arrow[ur, bend left] \arrow[rr, bend left, "N"] \arrow[rr, dashed, bend right, "N'"'] \arrow[dr, bend right] & \Downarrow \rho & B \arrow[dl, bend left] \\
 & \bullet &
\end{tikzcd} .$$

Since $\mathcal{C}$ is braided, by assumption, it can itself be regarded as a 3-category with a single object and a single morphism. This is the double suspension $\Sigma(\Sigma(\mathcal{C}))$ of $\mathcal{C}$. As before, we have a canonical inclusion

$$\mathbf{B}(\mathbf{B}(\mathcal{C})) \xrightarrow{\subset} \mathbf{B}(\mathrm{Bimod}(\mathcal{C}))\,.$$

2.2 Twisted bimodules in $\mathcal{C}$

Consider a clyinder in $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$, i.e. a 2-morphism in $(\mathbf{B}\mathrm{Bimod}(\mathcal{C}))^I$, whose top and bottom face are labelled by the trivial algebra.



in $\Sigma(\mathrm{Bim}(\mathcal{C}))$.

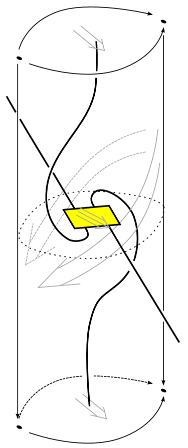


Figure 1: **The fundamental diagram** in combinatorial CFT has A-B bimodules N and N' running on the worldsheet, interpolating between the A and the B-phase of the CFT. Where they tranform into each other a bulk field, labelled by objects $U, V \in \mathcal{C}$ may be inserted. The insertion point is labelled by a homomorphism of induced bimodules. The figure indicates how the Poincaré-dual of the corresponding string diagram in $\mathcal{C}$ is precisely a cylinder in $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$, hence a 2-morphism in $\mathrm{TwBim}(\mathcal{C})$.

Cutting this open, this is a 3-morphism ρ from

$$\begin{tikzcd}
\bullet \arrow[r, "\mathbb{1}" description] \arrow[r, bend left=60, "\mathbb{1}"] \arrow[r, phantom, bend left=30, "\Downarrow U"] \arrow[d, "B"'] \arrow[dr, phantom, "\swarrow N"] & \bullet \arrow[d, "A"] \\
\bullet \arrow[r, "\mathbb{1}"'] & \bullet
\end{tikzcd},$$

to

$$\begin{tikzcd}
\bullet \arrow[r, "\mathbb{1}"] \arrow[d, "B"'] \arrow[dr, phantom, "\swarrow N'"] & \bullet \arrow[d, "A"] \\
\bullet \arrow[r, "\mathbb{1}" description] \arrow[r, bend right=60, "\mathbb{1}"'] \arrow[r, phantom, bend right=30, "\Downarrow V"] & \bullet
\end{tikzcd}.$$

In other words, ρ is a morphism from the $A \otimes \mathbb{1}$-$B \otimes \mathbb{1}$-bimodule $N \otimes U$ to the $\mathbb{1} \otimes A$-$\mathbb{1} \otimes B$-bimodule $V \otimes N'$:

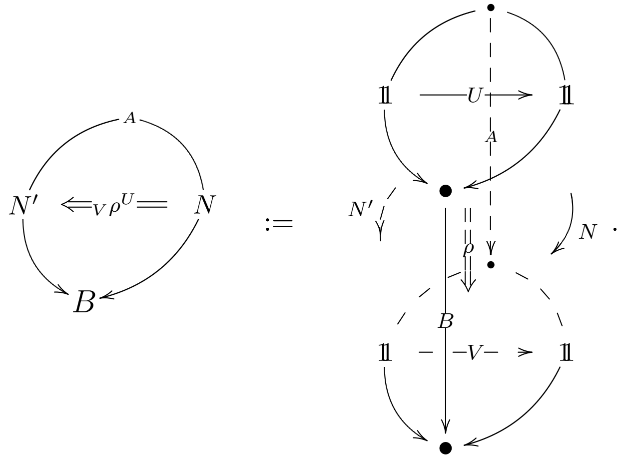
$$\begin{tikzcd}
& N \arrow[d, dash] & U \arrow[dl, dash] \\
& \rho \arrow[dl] \arrow[d] & \\
V & N' &
\end{tikzcd}$$

All tin cans ρ in $\Sigma(\mathrm{Bim}(\mathcal{C}))$ of this kind, with top and bottom a $\mathbb{1}$-$\mathbb{1}$ bimodule, form a 2-category in the obvious way. We will address this as

Definition 2 *The 2-category* $\mathrm{TwBim}(\mathcal{C})$ *of* **twisted bimodules** *is the 2-category of tin cans in* $\mathbf{B}(\mathrm{TwBim}(\mathcal{C}))$ *whose top and bottom are* $\mathbb{1}$-$\mathbb{1}$*-bimodules,*

$$\mathrm{TwBimod}(\mathcal{C}) := \left\{ \begin{tikzcd} A \arrow[r, bend left=50, "N"] \arrow[r, bend right=50, "N'"'] \arrow[r, phantom, "\Downarrow_V \rho^U"] & B \end{tikzcd} \right\}.$$

Here the 2-morphism is to be regarded as a "squashed cylinder" in **B**Bimod($\mathcal{C}$).



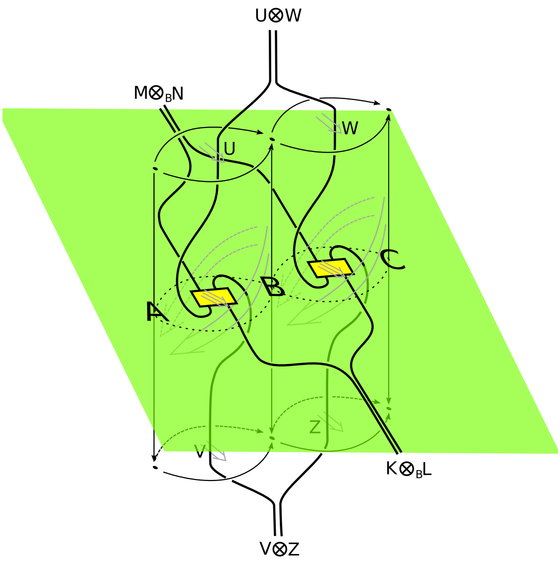


Figure 2: **Horizontal composition of 2-morphisms in** TwBimod($\mathcal{C}$)**.** Each 2-morphism is really a cylinder in **B**Bimod($\mathcal{C}$), as displayed. A, B and C are algebras in $\mathcal{C}$ and M, N, K L are bimodules in $\mathcal{C}$. The top and bottom rims are restricted to be labelled with the trivial algebra $\mathbb{1}$, so that U, W, V, Z are $\mathbb{1}$-bimodules and hence can be identified with plain objects in $\mathcal{C}$, under the inclusion $\mathbf{B}\mathcal{C} \hookrightarrow \mathrm{Bimod}(\mathcal{C})$.

2.2.1 The definition of TwBimod($\mathcal{C}$)

Write

$$A \xrightarrow{N} B$$

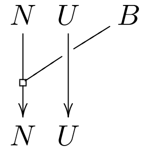
for an object N of $\mathcal{C}$ with the structure of an A-B bimodule, for A and B algebra objects internal to $\mathcal{C}$.

If we assume that all algebras are special Frobenius, then the relations we need for checking the exchange law below are easily obtained.

Write

$$\begin{tikzcd} A \arrow[r, bend left=50, "N"] \arrow[r, bend right=50, "N"'] \arrow[r, phantom, "\Downarrow \mathrm{Id}^U"] & B \end{tikzcd}$$

for the **twisted** or **induced** bimodule object $N \otimes U$ whose action is that of N combined with braiding under U

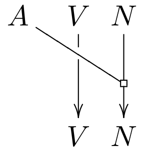


.

Similarly, write

$$\begin{tikzcd} A \arrow[r, bend left=50, "N"] \arrow[r, bend right=50, "N"'] \arrow[r, phantom, "\Downarrow {}_V\mathrm{Id}"] & B \end{tikzcd}$$

for the bimodule obtained by braiding over V

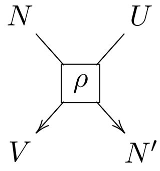


.

Write

$$\begin{tikzcd} A \arrow[r, bend left=50, "N"] \arrow[r, bend right=50, "N'"'] \arrow[r, phantom, "\Downarrow {}_V\rho^U"] & B \end{tikzcd}$$

for a morphism

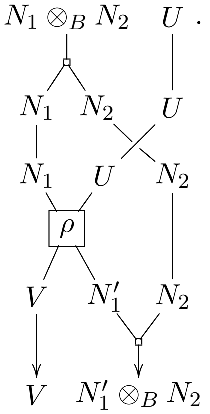


of such induced bimodules.

Write

$$\begin{tikzcd}
A \arrow[r, bend left=50, "N_1"] \arrow[r, bend right=50, "N_1'"'] \arrow[r, phantom, "\Downarrow {}_V\rho^U"] & B \arrow[r, "N_2"] & C
\end{tikzcd}$$

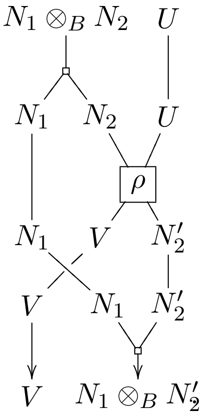
for



Write

$$\begin{tikzcd}
A \arrow[r, "N_1"] & B \arrow[r, bend left=50, "N_2"] \arrow[r, bend right=50, "N_2'"'] \arrow[r, phantom, "\Downarrow {}_V\rho^U"] & C
\end{tikzcd}$$

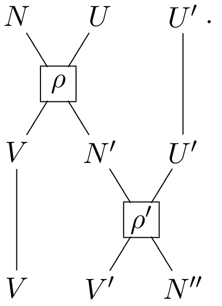
for



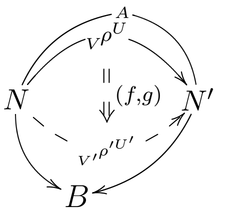
Write

$$\begin{tikzcd}
A \arrow[r, bend left=70, "N"] \arrow[r, "N'" description] \arrow[r, bend right=70, "N''"'] \arrow[r, phantom, bend left=35, "\Downarrow {}_V\rho^U"] \arrow[r, phantom, bend right=35, "\Downarrow {}_{V'}\rho'^{U'}"] & B
\end{tikzcd}$$

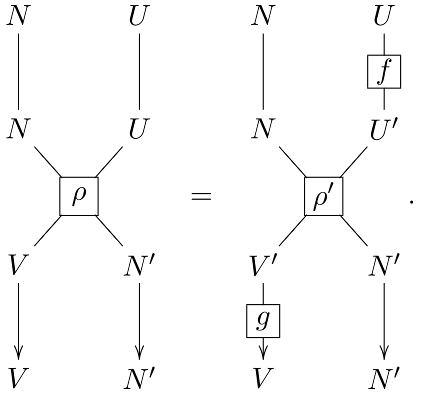
for



Write



for



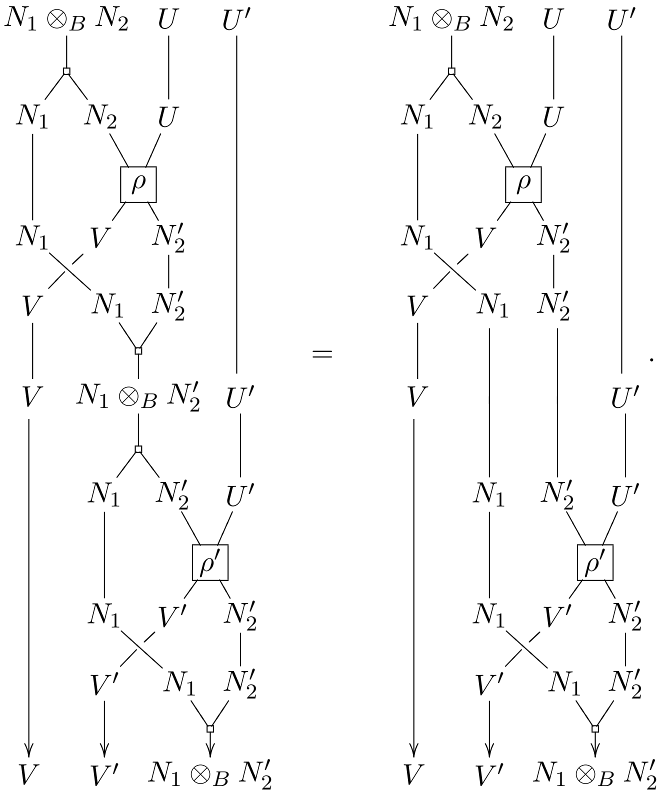
Proposition 1 *From the above definitions it follows that horizontal composition with identity 2-morphisms satisfies the exchange law strictly:*

$$\begin{tikzcd}
A \arrow[r, "N_1'" description, ""{name=M}] \arrow[r, bend left=60, "N_1", ""'{name=T}] \arrow[r, bend right=60, "N_1''"', ""{name=D}] & B \arrow[r, "N_2"] & C
\arrow[Rightarrow, from=T, to=M, "{}_V\rho^U"]
\arrow[Rightarrow, from=M, to=D, "{}_{V'}\rho'^{U'}"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "N_1'" description, ""{name=M1}] \arrow[r, bend left=60, "N_1", ""'{name=T}] & B \arrow[r, "N_2"] & C \\
A \arrow[r, "N_1'" description, ""'{name=M2}] \arrow[r, bend right=60, "N_1''"', ""{name=D}] & B \arrow[r, "N_2"'] & C
\arrow[Rightarrow, from=T, to=M1, "{}_V\rho^U"]
\arrow[Rightarrow, from=M2, to=D, "{}_{V'}\rho'^{U'}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "N_1"] & B \arrow[r, bend left=60, "N_2"] \arrow[r, "N_2'" description] \arrow[r, bend right=60, "N_2''"'] \arrow[r, phantom, bend left=30, "\Downarrow{}_V\rho^U"] \arrow[r, phantom, bend right=30, "\Downarrow{}_{V'}\rho'^{U'}"] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "N_1"] & B \arrow[r, bend left=60, "N_2"] \arrow[r, "N_2'" description] \arrow[r, phantom, bend left=30, "\Downarrow{}_V\rho^U"] & C \\
A \arrow[r, "N_1"'] & B \arrow[r, "N_2'" description] \arrow[r, bend right=60, "N_2''"'] \arrow[r, phantom, bend right=30, "\Downarrow{}_{V'}\rho'^{U'}"] & C
\end{tikzcd}$$

Proof.



□

Proposition 2 *The composition*

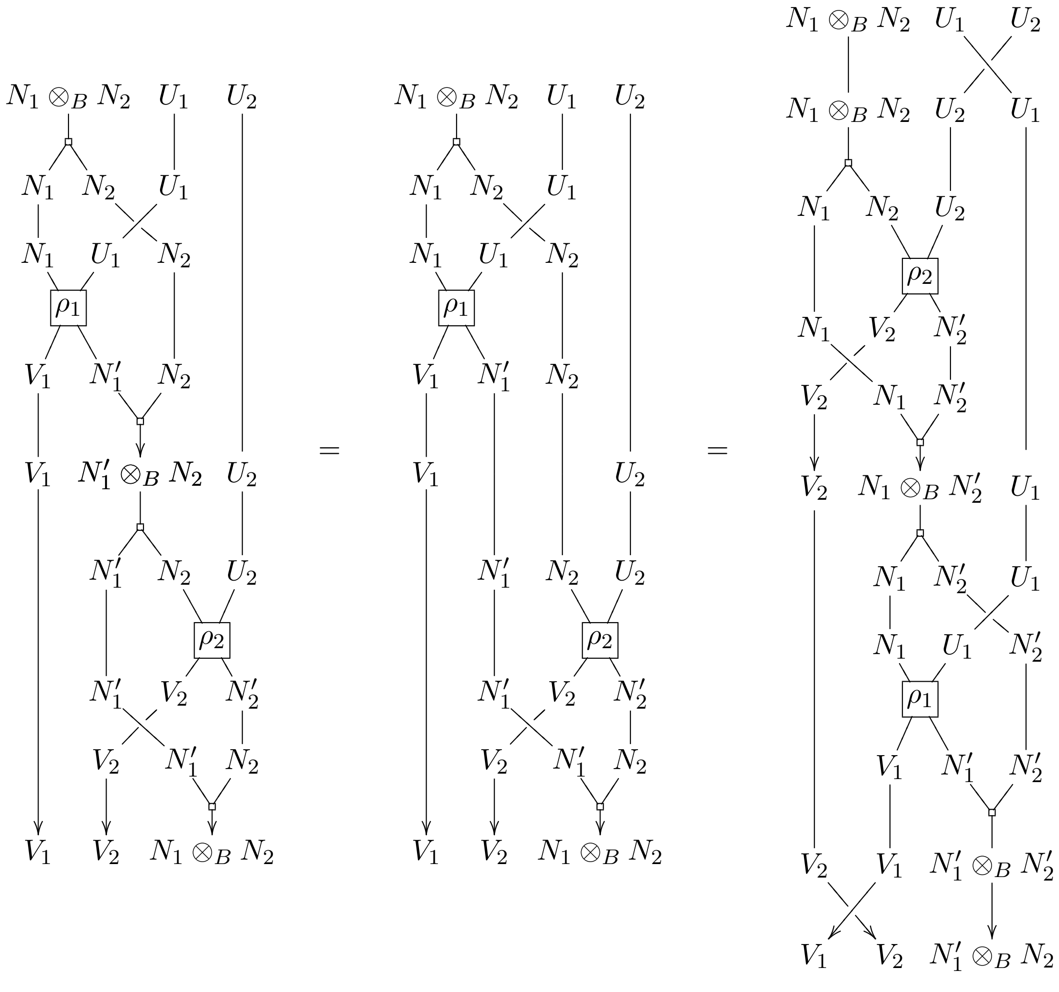
$$\begin{tikzcd}
A \arrow[r, bend left=50, "N_1"] \arrow[r, bend right=50, "N_1'"'] \arrow[r, phantom, "\Downarrow{}_{V_1}\rho_1^{U_1}"] & B \arrow[r, bend left=50, "N_2"] & C \\
A \arrow[r, bend right=50, "N_1'"'] & B \arrow[r, bend left=50, "N_2"] \arrow[r, bend right=50, "N_2'"'] \arrow[r, phantom, "\Downarrow{}_{V_2}\rho_2^{U_2}"] & C
\end{tikzcd}$$

is isomorphic to

$$\begin{tikzcd}
A \arrow[r, bend left, "N_1"] & B \arrow[r, bend left, "N_2"] \arrow[r, bend right, "N_2'"'] \arrow[r, phantom, "\Downarrow {}_{V_2}\rho_2^{U_2}"] & C \, . \\
A \arrow[r, bend left, "N_1"] \arrow[r, bend right, "N_1'"'] \arrow[r, phantom, "\Downarrow {}_{V_1}\rho_1^{U_1}"] & B \arrow[r, bend right, "N_2'"'] & C
\end{tikzcd}$$

The isomorphism is is given by the braiding on $U_1 \otimes U_2$ and $V_1 \otimes V_2$, respectively. It is unique.

Proof.



□

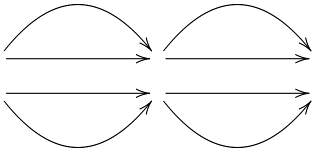
Definition 3 *Given the above, there are different but isomorphic ways to define the hor-*

izontal composition of 2-morphisms. For definiteness, we set

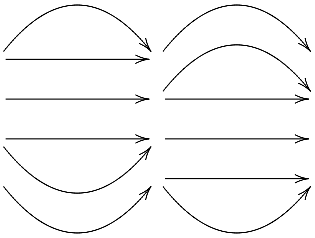
$$\begin{tikzcd}
A \arrow[r, bend left, "N_1"] \arrow[r, bend right, "N_1'"'] \arrow[r, phantom, "\Downarrow_{V_1}\rho_1^{U_1}"] & B \arrow[r, bend left, "N_2"] \arrow[r, bend right, "N_2'"'] \arrow[r, phantom, "\Downarrow_{V_2}\rho_2^{U_2}"] & C
\end{tikzcd}
:=
\begin{tikzcd}
A \arrow[r, bend left, "N_1"] \arrow[r, bend right, "N_1'"'] \arrow[r, phantom, "\Downarrow_{V_1}\rho_1^{U_1}"] & B \arrow[r, bend left, "N_2"] & C \\
A \arrow[r, bend right, "N_1'"'] & B \arrow[r, bend left, "N_2"] \arrow[r, bend right, "N_2'"'] \arrow[r, phantom, "\Downarrow_{V_2}\rho_2^{U_2}"] & C
\end{tikzcd}$$

Proposition 3 *Composition of 2-morphisms satisfies the exchange law up to isomorphism.*

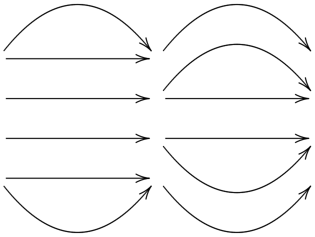
Proof.



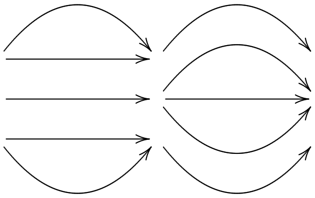
$$=$$



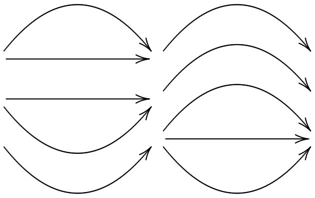
$$\simeq$$



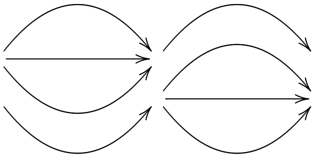
$$=$$



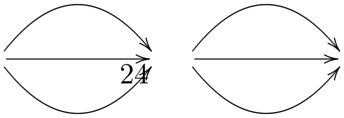
$$\simeq$$



$$=$$



$$=$$



□

3 Frobenius Algebras and Adjunctions

3.1 Adjunctions

In a 2-categorical context invertibility of morphisms is in general replaced by *equivalence*, i.e. by invertibility up to 2-isomorphism. In some situations however, even the notion of equivalence is too strong, and one is left merely with *adjunctions.* For applications as those to be presented in the following, an *ambidextrous adjunction* which satisfies a *bubble move equation* will be seen to provide sufficiently many features of a true equivalence to admit the inversion operations needed here.

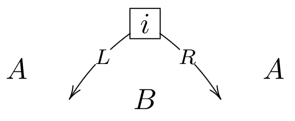
Definition 4 *An* **adjunction**

$$\begin{tikzcd} A \arrow[rr, bend left=50, "L"] & \overset{i}{\Rightarrow} \\ & \underset{e}{\Rightarrow} & B \arrow[ll, bend left=50, "R"] \end{tikzcd}$$

in a 2-category $\mathcal{K}$ is a collection of

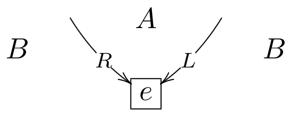
1. *1-morphisms* $A \xrightarrow{L} B$ *and* $B \xrightarrow{R} A$ *in* $\mathrm{Mor}_1(\mathcal{K})$
2. *2-morphisms*

$$\begin{tikzcd} A \arrow[rr, bend left, "\mathrm{Id}"] \arrow[dr, bend right, "L"'] & \Downarrow i & A \\ & B \arrow[ur, bend right, "R"'] & \end{tikzcd}$$



and

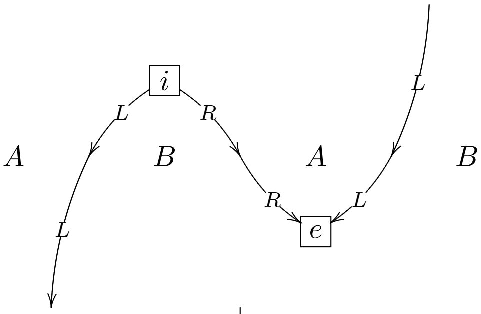
$$\begin{tikzcd} & A \arrow[dr, bend left, "L"] & \\ B \arrow[ur, bend left, "R"] \arrow[rr, bend right, "\mathrm{Id}"'] & \Downarrow e & B \end{tikzcd}$$



in $\mathrm{Mor}_2(\mathcal{K})$

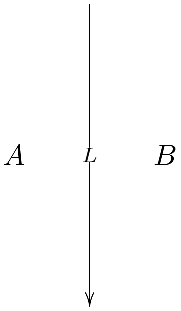
satisfying the **zig-zag idenities**, *which look like*

$$\begin{tikzcd} A \arrow[r, "L"] \arrow[rr, bend left=60, "\mathrm{Id}"] \arrow[rr, phantom, bend left=30, "\Downarrow i"] & B \arrow[r, "R"] \arrow[rr, bend right=60, "\mathrm{Id}"'] \arrow[rr, phantom, bend right=30, "\Downarrow e"'] & A \arrow[r, "L"] & B \end{tikzcd}$$



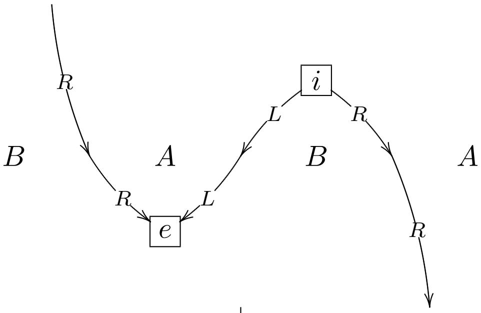
$$= \quad \begin{tikzcd} A \arrow[r, "L"] & B \end{tikzcd}$$

$=$



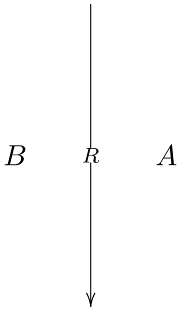
and

$$\begin{tikzcd} B \arrow[r, "R"] \arrow[rr, bend right=60, "\mathrm{Id}"'] \arrow[rr, phantom, bend right=30, "\Downarrow e"'] & A \arrow[r, "L"] \arrow[rr, bend left=60, "\mathrm{Id}"] \arrow[rr, phantom, bend left=30, "\Downarrow i"] & B \arrow[r, "R"] & A \end{tikzcd}$$



$$= \quad \begin{tikzcd} B \arrow[r, "R"] & A \end{tikzcd}$$

$=$



Definition 5 ([12, 13]) *A pair of adjunctions*

$$\begin{tikzcd} A \arrow[r, bend left=50, "L"] \arrow[r, phantom, shift left=2, "\overset{i}{\Leftrightarrow}"] \arrow[r, phantom, shift right=2, "\underset{e}{\Leftrightarrow}"'] & B \arrow[l, bend left=50, "R"] \end{tikzcd} \equiv \begin{tikzcd} A \arrow[r, bend left=50, "L"] \arrow[r, phantom, shift left=2, "\overset{i}{\Rightarrow}"] \arrow[r, phantom, shift right=2, "\underset{e}{\Rightarrow}"'] & B \arrow[l, bend left=50, "R"] \arrow[r, bend left=50, "R"] \arrow[r, phantom, shift left=2, "\overset{\tilde{i}}{\Rightarrow}"] \arrow[r, phantom, shift right=2, "\underset{\tilde{e}}{\Rightarrow}"'] & A \arrow[l, bend left=50, "L"] \end{tikzcd}$$

is called an **ambidextrous adjunction**.

Definition 6 *We call an ambidextrous adjunction* **special** *iff*

$$\begin{tikzcd}
A \arrow[r, "L"] \arrow[rr, bend left=70, "\mathrm{Id}", "\Downarrow i"'] \arrow[rr, bend right=70, "\mathrm{Id}"', "\Downarrow \tilde{e}"] & B \arrow[r, "R"] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
B \arrow[r, "R"] \arrow[rr, bend left=70, "\mathrm{Id}", "\Downarrow \tilde{i}"'] \arrow[rr, bend right=70, "\mathrm{Id}"', "\Downarrow e"] & A \arrow[r, "L"] & B
\end{tikzcd} .$$

both are 2-*iso*morphisms.

A special ambidextrous adjunction is hence similar to an equivalence, but weaker.

We will mostly be interested in special cases where all the 1-morphisms sets are vector spaces. (More precisely, we will be intersted in the case where our 2-category $\mathcal{K}$ is the 2-category of biomodules of a modular tensor category $\mathcal{C}$.) In this case we shall be more specific about the precise nature of the above 2-isomorphisms.

Definition 7 *When the 2-morphism sets of $\mathcal{K}$ are vector spaces, we call an ambidextrous adjunction in $\mathcal{K}$* **special** *iff*

$$\begin{tikzcd}
A \arrow[r, "L"] \arrow[rr, bend left=70, "\mathrm{Id}", "\Downarrow i"'] \arrow[rr, bend right=70, "\mathrm{Id}"', "\Downarrow \tilde{e}"] & B \arrow[r, "R"] & A
\end{tikzcd} = \beta_{LR} \cdot \left(A \xrightarrow{\mathrm{Id}} A \right)$$

and

$$\begin{tikzcd}
B \arrow[r, "R"] \arrow[rr, bend left=70, "\mathrm{Id}", "\Downarrow \tilde{i}"'] \arrow[rr, bend right=70, "\mathrm{Id}"', "\Downarrow e"] & A \arrow[r, "L"] & B
\end{tikzcd} = \beta_{RL} \cdot \left(B \xrightarrow{\mathrm{Id}} B \right) .$$

for β_{LR} and β_{RL} elements of the ground field.

Remark. Below we will see (prop. 14) how speciality of ambidextrous adjunctions translates into speciality of the Frobenius algebras that they give rise to. Speciality for Frobenius algebras is an established concept (def. 9 below) which hence motivated our choice of the term *special* for the above property of ambidextrous adjunctions.

3.2 Frobenius Algebras

Definition 8 *A* **Frobenius algebra** *in a monoidal category $\mathcal{C}$ is an object $A \in \mathrm{Obj}(\mathcal{C})$ togther with morphisms*

1. *product*

$$A \otimes A \xrightarrow{m} A$$

2. *unit*

$$\mathbb{1} \xrightarrow{i} A$$

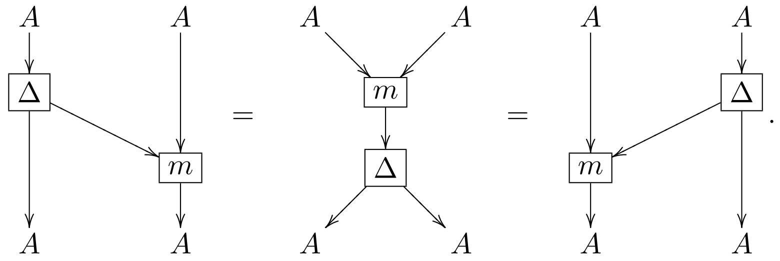
3. *coproduct*

$$A \xrightarrow{\Delta} A \otimes A$$

4. *counit*

$$A \xrightarrow{e} \mathbb{1}$$

such that (m, i) is an algebra, (Δ, e) is a coalgebra and such that product and coproduct satisfy the **Frobenius property**



Remark. For manipulations of diagrams as in the following it is often helpful to think of the Frobenius property as saying that, with A regarded as a bimodule over itself, the coproduct is a bimodule homomorphism form ${}_A(A)_A$ to ${}_A(A \otimes A)_A$

We will be interested in Frobenius algebras with additional properties. The Frobenius algebras of relevance here are

- special (def. 9)
- symmetric (def. 10) .

Unfortunately, while standard, the terms "special" and "symmetric" are rather unsuggestive of the phenomena they are suppposed to describe.

1. Speciality says that the two "bubble diagrams" in a Frobenius algebra are proportional to identity morphisms.
2. Symmetry of a Frobenius algebra says that the two obvious isomorphisms of A with its dual object $A^\vee$ are equal.

The reader should in particular be warned that symmetry, in this sense, of a Frobenius algebra is not directly related to whether or not that algebra is (braided) *commutative.* (But in modular tensor categories braided commutativity together with triviality of the twist implies symmetry.)

Definition 9 ([6], def. 3.4) *Let A be a Frobenius algebra object in an abelian tensor category. A is* **special** *precisely if*

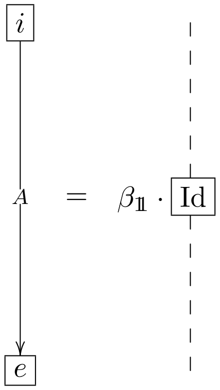
$$\begin{tikzcd}
\mathbb{1} \arrow[dr, "i"] \arrow[dd, "\beta_{\mathbb{1}}\cdot\mathrm{Id}"'] & \\
& A \arrow[dl, "e"] \\
\mathbb{1} &
\end{tikzcd}$$

and

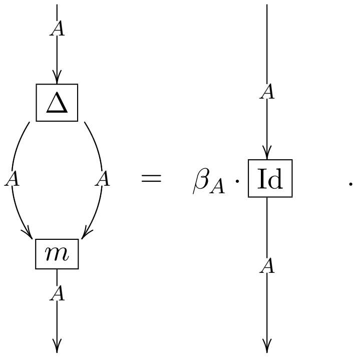
$$\begin{tikzcd}
A \arrow[dr, "\Delta"] \arrow[dd, "\beta_A\cdot\mathrm{Id}"'] & \\
& A \otimes A \arrow[dl, "m"] \\
A &
\end{tikzcd}$$

for some constants $\beta_{\mathbb{1}}$ and β_A

In terms of string diagrams in the suspension of $\mathcal{C}$ these two conditions look like

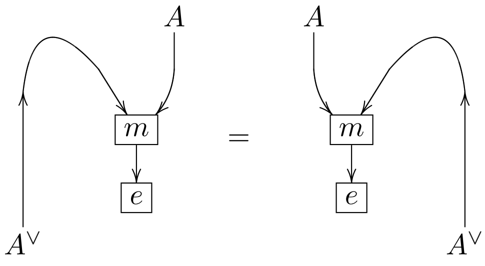


and



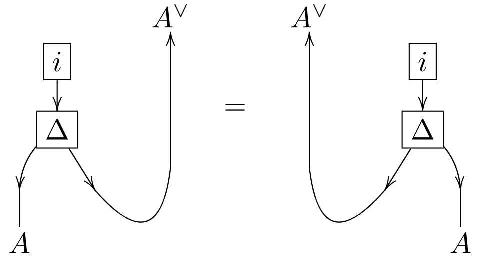
Speciality of Frobenius algebras will be related to speciality of ambidextrous adjunctions in prop. 14 on p. 14.

Definition 10 ([6], (3.33)) *A Frobenius algebra is called* **symmetric** *if the following two isomorphisms of A with its dual, $A^\vee$, are equal:*



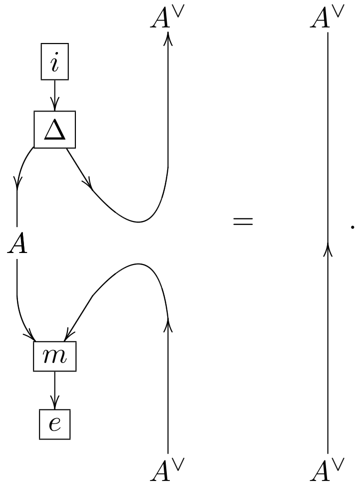
Proposition 4 ([6], (3.35)) *The morphisms in (10) are indeed isomorphisms.*

Proof. Using the Frobenius property, one checks that the inverse morphisms are



□

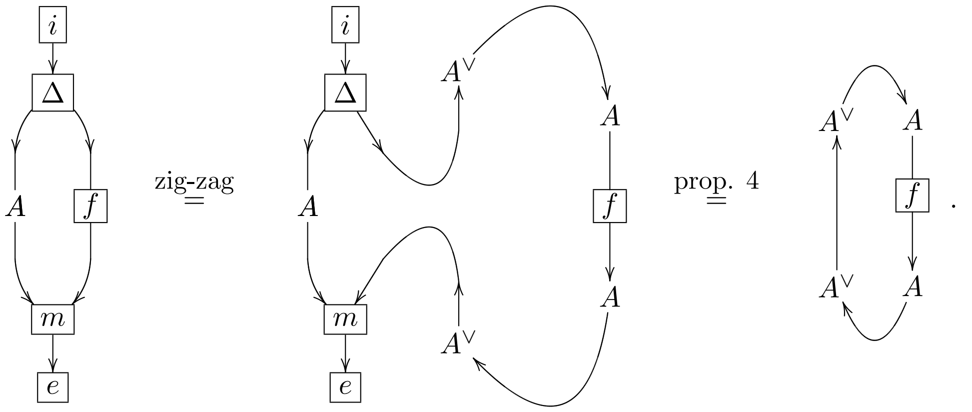
Hence we have in particular



Using relations like this we frequently pass back and forth between diagrams with or without occurence of the dual $A^\vee$ of A.

Example 1

When dealing with FRS diagrams for disk correlators (§**??**) one encounters the following situation. Let $A \xrightarrow{f} A$ be any morphism. Then



3.3 Bimodules

We can define bimodules in any abelian monoidal category.

Definition 11 *An* **abelian category** $\mathcal{C}$ *is a category with the following properties:*

1. *The* **hom-spaces** $\mathrm{Hom}(a,b)$ **are abelian groups** *for all* $a, b \in \mathrm{Obj}(\mathcal{C})$.

   The abelian group operation **'+' distributes over composition** *of morphisms. This means that for every diagram*

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & c \arrow[r, "h"] & d \end{tikzcd}$$

   we have

$$a \xrightarrow{f} b \circ \begin{pmatrix} \begin{tikzcd} b \arrow[rr, bend left, "g_1"] & & c \\ & + & \\ b \arrow[rr, bend right, "g_2"'] & & c \end{tikzcd} \end{pmatrix} \circ c \xrightarrow{h} d = \begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, bend left, "g_1"] & c \arrow[r, "h"] & d \\ & & + & \\ a \arrow[r, "f"'] & b \arrow[r, bend right, "g_2"'] & c \arrow[r, "h"'] & d \end{tikzcd} .$$

2. $\mathcal{C}$ **contains a zero-object** 0, *(an object which is both initial and terminal).*

3. *For all* $a, b \in \mathrm{Obj}(\mathcal{C})$ *the* **direct product** $a \times b$ **exists.**

4. *Every morphism* f *in* $\mathcal{C}$ *has* **kernel and cokernel** $\ker(f)$, $\mathrm{coker}(f)$ *in* $\mathcal{C}$.

5. $\mathrm{coker}(\ker(f)) = f$ *for every* $f \in \mathrm{Mor}(\mathcal{C})$

6. $\ker(\mathrm{coker}(f)) = f$ *for every* $f \in \mathrm{Mor}(\mathcal{C})$

Definition 12 *Let $\mathcal{C}$ be any monoidal category and let A and B be algebra objects in $\mathcal{C}$. An A-B* **bimodule** *in $\mathcal{C}$ is an object ${}_AN_B \in \mathrm{Obj}(\mathcal{C})$ together with commuting left and right action morphisms*

$$A \otimes {}_AN_B \xrightarrow{\ell} A$$

and

$$ {}_AN_B \otimes B \xrightarrow{r} A$$

satisfying

1. <u>*compatibility with the product*</u>

$$\begin{tikzcd}
A \otimes A \otimes {}_AN_B \arrow[r, "m\otimes {}_AN_B"] \arrow[d, "A\otimes\ell"'] & A \otimes {}_AN_B \arrow[d, "\ell"] \\
A \otimes {}_AN_B \arrow[r, "\ell"'] & {}_AN_B
\end{tikzcd}$$

$$\begin{tikzcd}
{}_AN_B \otimes B \otimes B \arrow[r, "{}_AN_B\otimes m"] \arrow[d, "r\otimes B"'] & {}_AN_B \otimes B \arrow[d, "r"] \\
{}_AN_B \otimes B \arrow[r, "r"'] & {}_AN_B
\end{tikzcd}$$

2. <u>*compatibility with the unit*</u>

$$\begin{tikzcd}
\mathbb{1} \otimes {}_AN_B \arrow[rr] \arrow[dr, "i_A\otimes {}_AN_B"'] & & {}_AN_B \\
& A \otimes {}_AN_B \arrow[ur, "\ell"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
{}_AN_B \otimes \mathbb{1} \arrow[rr] \arrow[dr, "{}_AN_B\otimes i_B"'] & & {}_AN_B \\
& A \otimes {}_AN_B \arrow[ur, "r"'] &
\end{tikzcd}$$

Definition 13 *Let $\mathcal{C}$ be an abelian monoidal category. Let ${}_AM_B$ and ${}_BN_C$ be bimodules in $\mathcal{C}$. Then the* **bimodule tensor product** *is the cokernel*

$$ {}_AM_B \otimes {}_BN_C \xrightarrow{\otimes_B(M,N)} {}_AM_B \otimes_B {}_BN_C \equiv \mathrm{coker}\left({}_AM_B \otimes B \otimes {}_BN_C \xrightarrow{r\otimes N - M\otimes \ell} {}_AM_B \otimes {}_BN_C \right).$$

Example 2

Let $A \in \mathrm{Obj}(\mathcal{C})$ be a special Frobenius algebra. Let ${}_AA_A$ be A but regarded as a bimodule over itself with $\ell = m = r$.

Set

$$
{}_AA_A \otimes {}_AA_A \xrightarrow{\otimes_B(M,N)} {}_AA_A \otimes_A {}_AA_A = {}_AA_A \otimes {}_AA_A \xrightarrow{\quad m \quad} {}_AA_A \;.
$$

We have that

$$
{}_AA_A \otimes A \otimes {}_AA_A \xrightarrow{m\otimes A - A\otimes m} {}_AA_A \otimes {}_AA_A \xrightarrow{\quad m \quad} {}_AA_A
$$

is the 0-arrow, due to the associativity of m. Given any other arrow ϕ with this property we have

$$
\begin{tikzcd}
{}_AA_A \otimes A \otimes {}_AA_A \arrow[r, "m\otimes A - A\otimes m"] & {}_AA_A \otimes {}_AA_A \arrow[r, "m"] \arrow[d, "\phi"] & {}_AA_A \arrow[dl, dashed, "\Delta\circ\phi"] \;. \\
& {}_AA_A &
\end{tikzcd}
$$

The fact that the morphism $\Delta \circ \phi$ makes this diagram commute depends on the special Frobenius property of A as well as on the fact that $(m \otimes A - A \otimes m) \circ \phi = 0$.

Definition 14 *Let $\mathcal{C}$ be any monoidal category. The* **2-category of (Frobenius) algebra bi-modules internal to** $\mathcal{C}$*, denoted* $\mathbf{BiMod}(\mathcal{C})$*, is defined as follows:*

1. *objects are all (Frobenius) algebras A internal to $\mathcal{C}$*
2. *1-morphisms $A \xrightarrow{{}_AM_B} B$ are all internal $A - B$ bimodules ${}_AM_B$*
3. *2-morphisms*

$$
\begin{tikzcd}
A \arrow[r, bend left, "{}_AM_B"{name=U}] \arrow[r, bend right, "{}_AN_B"'{name=D}] & B
\arrow[Rightarrow, from=U, to=D, "\phi"]
\end{tikzcd}
$$

are all internal bimodule homomorphisms (intertwiners) ${}_AM_B \xrightarrow{\phi} {}_AN_B$.

Horizontal composition in $\mathbf{BiMod}(\mathcal{C})$ *is the tensor product of bimodules. Vertical composition is the composition of bimodule homomorphisms.*

Remark.

1. $\mathbf{BiMod}(\mathcal{C})$ is really a *weak* 2-category (a bicategory) with nontrivial associator. As usual, we here consider its strictification and suppress all appearances of the associator.

2. The tensor unit $\mathbb{1} \in \mathcal{C}$ equipped with the trivial (co)product is always a (Frobenius) algebra internal to $\mathcal{C}$. The sub-2-category $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ of $\mathbf{BiMod}(\mathcal{C})$ is $\mathcal{C}$ itself:

$$
\mathrm{Hom}_{\mathbf{BiMod}(\mathcal{C})}(\mathbb{1}, \mathbb{1}) \simeq \mathcal{C} \,.
$$

3.3.1 Left-induced Bimodules

A particularly important role for our construction is played by left-induced bimodules.

Definition 15 *A* **left-induced bimodule** *in* **BiMod**$(\mathcal{C})$ *is a bimodule of the form*

$$\begin{tikzcd} {}_A N_B \equiv A \arrow[r, dashed, "m"] & A \otimes V & B \arrow[l, dashed, "\phi \circ m"'] \end{tikzcd}$$

for $V \in \mathrm{Obj}(\mathcal{C})$*, where the left action by* A *comes from the action of* A *on itself and where the right action by* B *comes from composing the morphism*

$$\begin{tikzcd} V \otimes B \arrow[r, "\phi"] & A \otimes V \end{tikzcd} \in \mathrm{Mor}_1(\mathcal{C})$$

with the right action of A *on itself.* ϕ *is required to make the following diagrams commute:*

1. *compatibility with the product*

$$\begin{tikzcd} V \otimes B \otimes B \arrow[r, "\phi\otimes B"] \arrow[d, "V\otimes m"'] & A \otimes V \otimes B \arrow[r, "A\otimes\phi"] & A \otimes A \otimes V \arrow[d, "m\otimes V"] \\ V \otimes B \arrow[rr, "\phi"'] & & A \otimes V \end{tikzcd}$$

2. *compatibility with the unit*

$$\begin{tikzcd} & V \arrow[dl, "V\otimes i_B"'] \arrow[dr, "i_A\otimes V"] & \\ V \otimes B \arrow[rr, "\phi"'] & & A \otimes V \end{tikzcd}$$

When we discuss trivializability of 2-functors with values in left-induced bimodules, we will be focusing on those which satisfy in addition the following two conditions:

1. compatibility with the coproduct

$$\begin{tikzcd} V \otimes B \otimes B \arrow[r, "\phi\otimes B"] & A \otimes V \otimes B \arrow[r, "A\otimes\phi"] & A \otimes A \otimes V \\ V \otimes B \arrow[rr, "\phi"'] \arrow[u, "V\otimes\Delta"] & & A \otimes V \arrow[u, "\Delta\otimes V"'] \end{tikzcd}$$

2. compatibility with the counit

$$\begin{tikzcd} & V & \\ V \otimes B \arrow[ur, "V\otimes e_B"] \arrow[rr, "\phi"'] & & A \otimes V \arrow[ul, "e_A\otimes V"'] \end{tikzcd}$$

Proposition 5 *For* special *Frobenius algebras the four conditions in def 15 may not be independent. For A and B special Frobenius algebras*

- *compatibility with the coproduct is* implied *by compatibility with the product if* $\beta_A = \beta_B$
- *compatibility with the counit is* implied *by compatibility with the product if the constants* $\beta_{\mathbb{1}}$ *agree.*

Proof. From the commuting diagram describing the compatibility with the product

$$\begin{tikzcd}
V\otimes B \arrow[rr, "\phi"] & & A\otimes V \\
V\otimes B\otimes B \arrow[u, "V\otimes m"] \arrow[r, "\phi\otimes B"'] & A\otimes V\otimes B \arrow[r, "A\otimes\phi"'] & A\otimes A\otimes V \arrow[u, "m\otimes V"']
\end{tikzcd}$$

we obtain, by definition 9, the commuting diagram

$$\begin{tikzcd}
V\otimes B \arrow[rr, "\phi"] & & A\otimes V \arrow[dd, bend left=60, "\frac{1}{\beta_A}\cdot\mathrm{Id}"] \\
V\otimes B\otimes B \arrow[u, "V\otimes m"] \arrow[r, "\phi\otimes B"'] & A\otimes V\otimes B \arrow[r, "A\otimes\phi"'] & A\otimes A\otimes V \arrow[u, "m\otimes V"'] \\
V\otimes B \arrow[u, "V\otimes\Delta"] \arrow[uu, bend left=60, "\beta_B\cdot\mathrm{Id}"] & & A\otimes V \arrow[u, "\Delta\otimes V"']
\end{tikzcd} .$$

This immediately implies the diagram which expresses compatibility with the coproduct iff $\beta_A = \beta_B$.

Similarly, from the commuting diagram describing the compatibility with the unit

$$\begin{tikzcd}
 & V \arrow[dl, "V\otimes i_B"'] \arrow[dr, "i_A\otimes V"] & \\
V\otimes B \arrow[rr, "\phi"'] & & A\otimes V
\end{tikzcd}$$

we obtain, by definition 9, the commuting diagram

$$\begin{tikzcd}
 & V \arrow[dl, "V\otimes i_B"] \arrow[dr, "i_A\otimes V"'] & \\
V\otimes B \arrow[rr, "\phi"'] \arrow[d, "V\otimes e_B"] & & A\otimes V \arrow[d, "e_A\otimes V"'] \\
V \arrow[uur, bend left=70, "\frac{1}{(\beta_{\mathbb{1}})_B}\cdot\mathrm{Id}"] & & V \arrow[uul, bend right=70, "\frac{1}{(\beta_{\mathbb{1}})_A}\cdot\mathrm{Id}"']
\end{tikzcd} .$$

This immediately implies the diagram which expresses compatibility with the counit if $(\beta_{\mathbb{1}})_A = (\beta_{\mathbb{1}})_B$.

□

Example 3 *We get a left-induced $A - A$ bimodule $(A \otimes V, \phi = c^{\pm}_{A,V})$, where*

$$V \otimes A \xrightarrow{\phi} A \otimes V \ = \ V \otimes A \xrightarrow{c^{\pm}_{A,V}} A \otimes V$$

is the left or right **braiding** *in $\mathcal{C}$. This is the crucial example for the application to FRS formalism, where modules of this form describe* **field insertions** *with V being interpreted as the chiral data of the field.*

Proposition 6 *A* **morphism of left-induced bimodules**

$$\begin{tikzcd}
A \arrow[r, bend left=50, "{}_A(A\otimes V_1,\phi_1)_B"] \arrow[r, bend right=50, "{}_A(A\otimes V_2,\phi_2)_B"'] \arrow[r, phantom, "\Downarrow \rho"] & B
\end{tikzcd}$$

is specified by a morphism

$$\begin{tikzcd}
V_1 \arrow[d, "\rho"] \\
A \otimes V_2
\end{tikzcd}$$

as

$$\begin{tikzcd}
A \otimes V_1 \arrow[d, "A\otimes\rho"] \\
A \otimes A \otimes V_2 \arrow[d, "m\otimes V_2"] \\
A \otimes V_2
\end{tikzcd} .$$

This ρ has to make the diagrams

$$\begin{tikzcd}
V_1 \otimes B \arrow[r, "\phi_1"] \arrow[d, "\rho\otimes B"'] & A \otimes V_1 \arrow[d, "A\otimes\rho"] \\
A \otimes V_2 \otimes B \arrow[r, "A\otimes\phi_2"] & A \otimes A \otimes V_2
\end{tikzcd}$$

commute.

Remark. Note that, in general, ρ is not unique.

Definition 16 *We denote the sub-2-category of left-induced bimodules by*

$$\mathbf{LFBiMod}(\mathcal{C}) \subset \mathbf{BiMod}(\mathcal{C}) \ .$$

Proposition 7 *The bimodule tensor product $A \xrightarrow{{}_AN_B} B \xrightarrow{{}_BN'_C} C$ of two left-induced bimodules is the left-induced bimodule*

$$\begin{tikzcd}
{}_AN_B \otimes_B {}_BN'_C \equiv \ A \arrow[r, dashed, "m"] & A \otimes V \otimes V' & C \arrow[l, dashed, "\phi'\circ\phi\circ m"']
\end{tikzcd} .$$

Proof.

We claim that the map

$$\begin{tikzcd}
A\otimes V \;\otimes\; B\otimes V' \arrow[rr, "f"] \arrow[dr, "A\otimes\phi\otimes V'"'] & & A\otimes V\otimes V' \\
& A\otimes A\otimes V\otimes V' \arrow[ur, "m_A\otimes V\otimes V'"'] &
\end{tikzcd}$$

is a cokernel for

$$\begin{tikzcd}
A\otimes V \;\otimes\; B \;\otimes\; B\otimes V' \arrow[r, "r\otimes(B\otimes V')-(A\otimes V)\otimes\ell"] & A\otimes V \;\otimes\; B\otimes V' .
\end{tikzcd}$$

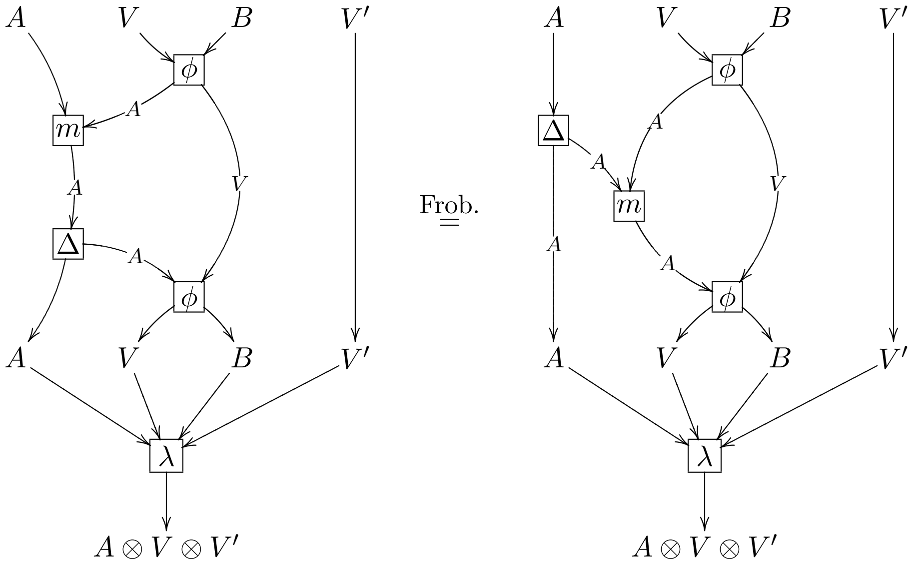
Consider the sequence

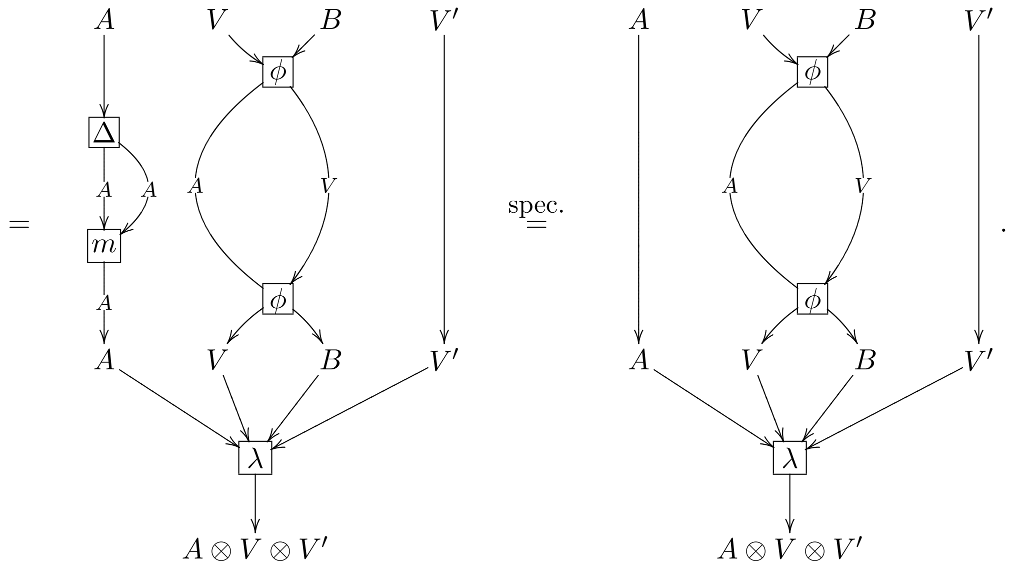
$$\begin{tikzcd}
A\otimes V \;\otimes B \;\otimes B\otimes V' \arrow[r, "r\otimes(B\otimes V')-(A\otimes V)\otimes\ell"] & (A\otimes V)\otimes(B\otimes V') \arrow[r, "f"] \arrow[d, "\lambda"] & (A\otimes V\otimes V') \arrow[dl, dashed, "g"] , \\
& A\otimes V\otimes V' &
\end{tikzcd}$$

and set

$$\begin{tikzcd}
A\otimes V\otimes V' \arrow[r, "g"] \arrow[d, "\Delta\otimes V\otimes V'"'] & A\otimes V\otimes V' \\
A\otimes A\otimes V\otimes V' \arrow[r, "A\otimes\bar{\phi}\otimes V'"'] & (A\otimes V)\otimes(B\otimes V') \arrow[u, "\lambda"']
\end{tikzcd} .$$

One sees that g really makes the diagram commute by the following computation.





In the first step we have used the Frobenius property of A, in the second the compatibility of ϕ with the product and of λ with the left and right action. Finally, in the third step we have used speciality of A, assuming that $\beta_A = 1$. The resulting morphism is clearly equal to λ.

(UNIQUENESS OF g REMAINS TO BE SHOWN) ☐

Proposition 8 *Every algebra homomorphism* $B \xrightarrow{\rho} A$ *defines a left-induced bimodule*

$$ {}_A\rho_B \equiv \; A \overset{m}{\dashrightarrow} A\otimes \mathbb{1} \overset{\rho\circ m}{\dashleftarrow} B \; . $$

Proposition 9 *The bimodule tensor product of bimodules coming from algebra homomorphisms corresponds to the composition of the respective morphisms. More precisely, given algebra homomorphisms* $C \xrightarrow{\rho'} B \xrightarrow{\rho} A$ *we have*

$$ {}_A\rho_B \otimes_B {}_B\rho'_C = \; A \overset{m}{\dashrightarrow} A\otimes \mathbb{1} \overset{\rho'\circ\rho\circ m}{\dashleftarrow} C \; . $$

Proposition 10

1. *The bimodule tensor product of left-induced bimodules* ${}_A(A\otimes V_1,\phi_1)_B$ *and* ${}_B(C\otimes V_2,\phi_2)_C$ *is*
$$ {}_A(A\otimes V_1,\phi_1)_B \otimes_B {}_B(B\otimes V_2,\phi_2)_C = {}_A(A\otimes V_1\otimes V_2,\phi_2\circ\phi_1)_C $$
2. *The horizontal product in* **LFBiMod**$(\mathcal{C})$ *is given by the following expression:*

$$\begin{tikzcd}
A \arrow[r, bend left, "{}_A(A\otimes V_1,\phi_1)_B"] \arrow[r, bend right, "{}_A(A\otimes V_2,\phi_2)_B"'] \arrow[r, phantom, "\Downarrow\rho"] & B \arrow[r, bend left, "{}_B(B\otimes V_1',\phi_1')_C"] \arrow[r, bend right, "{}_B(B\otimes V_2',\phi_2')_C"'] \arrow[r, phantom, "\Downarrow\rho'"] & C = A \arrow[r, bend left, "{}_A(A\otimes V_1\otimes V_1',\phi_1'\circ\phi_1)_B"] \arrow[r, bend right, "{}_A(A\otimes V_2\otimes V_2',\phi_2'\circ\phi_2)_B"'] \arrow[r, phantom, "\Downarrow\rho\cdot\rho'"] & C \; ,
\end{tikzcd}$$

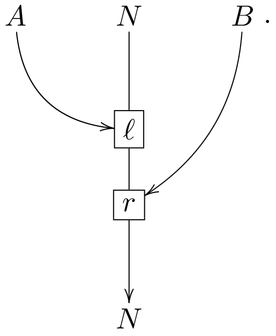
where

$$\begin{tikzcd}
V_1 \otimes V_1' \arrow[r, "\rho\otimes\rho'"] \arrow[d, "\rho\cdot\rho'"] & A \otimes V_2 \otimes B \otimes V_2' \arrow[d, "A\otimes\phi\otimes V_2'"] \\
A \otimes V_2 \otimes V_2' & A \otimes A \otimes V_2 \otimes V_2' \arrow[l, "m\otimes V_2\otimes V_2'"]
\end{tikzcd} .$$

3.3.2 Conjugation of Bimodules

There are three kinds of conjugation operations on bimodules.

Proposition 11 ([7], prop. 2.10) *Let ${}_AN_B$ be a bimodule with action*



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1.

3.4 Expressing Frobenius Algebras in Terms of Adjunctions

Every Frobenius algebra object in $\mathcal{C}$ can be expressed in terms of an adjunction in **BiMod**$(\mathcal{C})$.

In the literature one can find (at least) two slightly different realizations of this fact.

- From the general perspective of Eilenberg-Moore objects (and actually in more generality than we need here) in [12] (extending a similar construction in [14]) a construction using left-induced bimodules is given, where the two units and and counits of the ambijunction are built directly from the action of the Frobenius algebra's (co)product and (co)unit.
- In def. 2.12 of [7] a construction in terms of left A modules and their duals is given, where, implicitly, the units and counits of the ambijunction are constructed from the unit and counit of the *duality* on objects, composed with a projection operation.

Both these sources do not make all the details explicit that we will need. For instance [7] does not mention adjunctions at all. Therefore we spell out the details in the following two subsections.

3.4.1 Adjunctions using left-induced Bimodules

The algebra A may trivially be taken as a left, right or a bimodule over itself. We write ${}_A A_{\mathbb{1}}$, ${}_{\mathbb{1}} A_A$ and ${}_A A_A$, respectively, for the object A equipped with an A-module structure this way.

All three of these are left-induced bimodules. In order to be able to make full use of the rules for tensor products of left-induced bimodules, the following definition spells out the left-induced bimodule structure on

$$ {}_A L_{\mathbb{1}} \equiv {}_A A_{\mathbb{1}} $$

and

$$ {}_{\mathbb{1}} R_A \equiv {}_{\mathbb{1}} A_A $$

according to def. 15.

Definition 17 *Given a Frobenius algebra A in $\mathcal{C}$, we define the following left-induced bimodules.*

1.

$$ {}_{\mathbb{1}} L_A \equiv {}_{\mathbb{1}}(\mathbb{1} \otimes A, \phi)_A \equiv \; \mathbb{1} \overset{m}{\dashrightarrow} \mathbb{1} \otimes A \overset{\phi \circ m}{\dashleftarrow} A $$

with

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\phi"] \arrow[dr, "m"'] & & \mathbb{1} \otimes A \\
& A \arrow[ur] &
\end{tikzcd}$$

2.

$$ {}_A R_{\mathbb{1}} \equiv {}_A(A \otimes \mathbb{1}, \phi)_{\mathbb{1}} \equiv \; A \overset{m}{\dashrightarrow} A \otimes \mathbb{1} \overset{\phi \circ m}{\dashleftarrow} \mathbb{1} $$

with

$$\begin{tikzcd}
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\phi"] \arrow[d] & A \otimes \mathbb{1} \\
\mathbb{1} \arrow[r, "i"'] & A \arrow[u]
\end{tikzcd}$$

and the following bimodule morphisms

1. *left unit*

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, bend left, "\mathbb{1}"] \arrow[dr, bend right, "{}_{\mathbb{1}}L_A"'] & \Downarrow i & \mathbb{1} \\
& A \arrow[ur, bend right, "{}_A R_{\mathbb{1}}"'] &
\end{tikzcd}
\quad \textit{given by} \quad
\begin{tikzcd}
\mathbb{1} \arrow[d, "i"] \\
\mathbb{1} \otimes A
\end{tikzcd}
\quad \textit{and hence inducing} \quad
\mathbb{1} \xrightarrow{i} A$$

2. *left counit*

$$\begin{tikzcd} & \mathbb{1} \arrow[dr, bend left, "{}_{\mathbb{1}}L_A"] & \\ A \arrow[ur, bend left, "{}_A R_{\mathbb{1}}"] \arrow[rr, bend right, "A"'] \arrow[rr, phantom, "\Downarrow e"] & & A \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} A \arrow[d] \\ A \otimes \mathbb{1} \end{tikzcd} \quad \text{and hence inducing} \quad A \otimes A \xrightarrow{m} A$$

3. *right unit*

$$\begin{tikzcd} A \arrow[rr, bend left, "A"] \arrow[dr, bend right, "{}_A R_{\mathbb{1}}"'] \arrow[rr, phantom, "\Downarrow \tilde{i}"] & & A \\ & \mathbb{1} \arrow[ur, bend right, "{}_{\mathbb{1}}L_A"'] & \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} \mathbb{1} \arrow[d, "i"] \\ A \arrow[d, "\Delta"] \\ A \otimes A \end{tikzcd} \quad \text{and hence inducing} \quad A \xrightarrow{\Delta} A \otimes A$$

4. *right counit*

$$\begin{tikzcd} & A \arrow[dr, bend left, "{}_A R_{\mathbb{1}}"] & \\ \mathbb{1} \arrow[ur, bend left, "{}_{\mathbb{1}}L_A"] \arrow[rr, bend right, "\mathbb{1}"'] \arrow[rr, phantom, "\Downarrow \tilde{e}"] & & \mathbb{1} \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} A \arrow[d, "e"] \\ \mathbb{1} \otimes \mathbb{1} \end{tikzcd} \quad \text{and hence inducing} \quad A \xrightarrow{e} \mathbb{1}$$

Proposition 12 *For A a special Frobenius algebra, this defines a special ambidextrous adjunction* $\mathrm{Adj}\,(A)$.

Proof. Using the rules for horizontal and vertical composition of left-induced bimodules given in §3.3.1 one straightforwardly checks the required zig-zag identities as well as the specialty property. ☐

Proposition 13 *The Frobenius algebra* $\mathrm{Frob}\,(\mathrm{Adj}(A))$ *obtained from this ambidextrous adjunction is A itself*

$$\mathrm{Frob}\,(\mathrm{Adj}(A)) = A\,.$$

Proof. Applying the rules for horizontal and vertical composition of left-induced bimodules yields the following identities.

1. product

$$\begin{tikzcd} \mathbb{1} \arrow[r, "L"] \arrow[rr, bend left=60, "A"] \arrow[rrrr, bend right=70, "A"'] & A \arrow[r, "R"] \arrow[dr] \arrow[u, phantom, "\Downarrow \mathrm{Id}"] & \mathbb{1} \arrow[r, "L"] \arrow[rr, bend left=60, "A"] \arrow[d, phantom, "\Downarrow e"] & A \arrow[r, "R"] \arrow[u, phantom, "\Downarrow \mathrm{Id}"] & \mathbb{1} \\ & & A \arrow[ur] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & & \\ & & {} & & \end{tikzcd} = \begin{tikzcd} \mathbb{1} \arrow[r, bend left, "A"] \arrow[rr, bend right=60, "A"'] & \mathbb{1} \arrow[r, bend left, "A"] \arrow[d, phantom, "\Downarrow m"] & \mathbb{1} \\ & {} & \end{tikzcd}$$

2. coproduct

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "L"] \arrow[rr, bend left=70, "A"] \arrow[rrrr, bend right=80, "A"'] & A \arrow[r, "R"] \arrow[d, phantom, "\Uparrow \mathrm{Id}" near start] \arrow[dr, bend right] & \mathbb{1} \arrow[r, "L"] \arrow[rr, bend left=70, "A"] \arrow[d, phantom, "\Uparrow \tilde{i}"] & A \arrow[r, "R"] & \mathbb{1} \\
& & A \arrow[ur, bend right] \arrow[d, phantom, "\Uparrow \mathrm{Id}"] & & \\
& & & & \\
& & & & \\
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{1} \arrow[r, bend left=70, "A"] \arrow[rr, bend right=80, "A"'] & \mathbb{1} \arrow[r, bend left=70, "A"] \arrow[d, phantom, "\Delta \Uparrow"] & \mathbb{1} \\
& &
\end{tikzcd}$$

3. unit

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "L"] \arrow[rr, bend left=70, "\mathbb{1}"] \arrow[rr, bend right=70, "A"'] & A \arrow[r, "R"] \arrow[u, phantom, "\Downarrow i"] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \mathbb{1}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{1} \arrow[rr, bend left=70, "\mathbb{1}"] \arrow[rr, bend right=70, "A"'] \arrow[rr, phantom, "\Downarrow i"] & & \mathbb{1}
\end{tikzcd}$$

4. counit

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "L"] \arrow[rr, bend left=70, "A"] \arrow[rr, bend right=70, "\mathbb{1}"'] & A \arrow[r, "R"] \arrow[u, phantom, "\Downarrow \mathrm{Id}"] \arrow[d, phantom, "\Downarrow \tilde{e}"] & \mathbb{1}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{1} \arrow[rr, bend left=70, "A"] \arrow[rr, bend right=70, "\mathbb{1}"'] \arrow[rr, phantom, "\Downarrow e"] & & \mathbb{1}
\end{tikzcd}$$

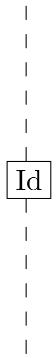
□

Proposition 14 *Under the relation of Frobenius algebras with ambidextrous adjunctions (prop. 12 and 13)* special *Frobenius algebras (def. 9) correspond to* special *ambijunctions (def. 7). The constants are related by*

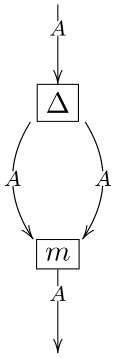
$$\begin{aligned}
\beta_{\mathbb{1}} &= \beta_{LR} \\
\beta_A &= \beta_{RL}\,.
\end{aligned}$$

Proof. We have

$$= \begin{tikzcd} \mathbb{1} \arrow[r, "L"] \arrow[rr, bend left=60, "\mathrm{Id}"{name=U}] \arrow[rr, bend right=60, "\mathrm{Id}"'{name=D}] & A \arrow[r, "R"] & \mathbb{1} \arrow[from=U, to=1-2, phantom, "\Downarrow i"] \arrow[from=1-2, to=D, phantom, "\Downarrow \tilde{e}"] \end{tikzcd} \overset{\text{def. 7}}{=} \beta_{LR} \cdot \left(\begin{tikzcd} \mathbb{1} \arrow[r, "\mathrm{Id}"] & \mathbb{1} \end{tikzcd} \right) = \beta_{\mathbb{1}} \cdot$$

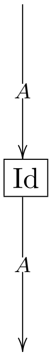


and



$$= \begin{tikzcd} & & A \arrow[ddrr, "R"] & & \\ \mathbb{1} \arrow[urr, "L"] \arrow[drr, "L"'] \arrow[r, "L"] & A \arrow[r, "R"] \arrow[rr, bend left=50, "\mathrm{Id}"{name=T}] \arrow[rr, bend right=50, "\mathrm{Id}"'{name=B}] & \mathbb{1} \arrow[r, "L"] & A \arrow[r, "R"] & \mathbb{1} \\ & & A \arrow[urr, "R"'] & & \arrow[from=1-3, to=T, phantom, "\Downarrow \mathrm{Id}"] \arrow[from=T, to=2-3, phantom, "\Downarrow \tilde{i}"] \arrow[from=2-3, to=B, phantom, "\Downarrow e"] \arrow[from=B, to=3-3, phantom, "\Downarrow \mathrm{Id}"] \end{tikzcd}$$

$$\overset{\text{def. 7}}{=} \beta_{RL} \cdot \left(\begin{tikzcd} & & A \arrow[ddrr, "R"] & & \\ \mathbb{1} \arrow[urr, "L"] \arrow[drr, "L"'] \arrow[r, "L"] & A \arrow[rr, "\mathrm{Id}"] & {} & A \arrow[r, "R"] & \mathbb{1} \\ & & A \arrow[urr, "R"'] & & \arrow[from=1-3, to=2-3, phantom, "\Downarrow \mathrm{Id}"] \arrow[from=2-3, to=3-3, phantom, "\Downarrow \mathrm{Id}"] \end{tikzcd} \right) = \beta_A \cdot$$



□

3.4.2 Adjunctions using Duality and Projection

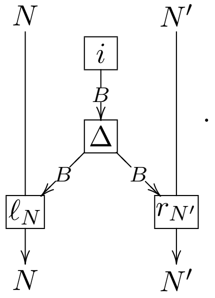
Due to the extra structure present on Frobenius algebras, bimodules for Frobenius algebras are easier to handle than general bimodules. Using the coproduct one can built the following projector, which sends objects in the ordinary tensor product of two bimodules to their images in the bimodule tensor product.

Definition 18 ([7], (2.45)) *Let A, B, C be special, symmetric Frobenius algebras in the rigid (meaning that all duals exist) monoidal category $\mathcal{C}$. For every pair ${}_AN_B$, ${}_BN'_C$ of bimodules, let* $N \otimes N' \xrightarrow{P_{N,N'}} N \otimes N'$ *be given by*

$$\begin{tikzcd}
N \otimes N' \arrow[r, "P_{N,N'}"] \arrow[d, "N\otimes i\otimes N'"'] & N \otimes N' \\
N \otimes B \otimes N' \arrow[r, "\Delta"'] & N \otimes B \otimes B \otimes N' \arrow[u, "\ell_N\otimes r_{N'}"']
\end{tikzcd} .$$

Note that the tensor product '$\otimes$' is that of $\mathcal{C}$. The bimodule tensor product (over B, say), will be denoted $\otimes_B$.

In string diagrams $P_{N,N'}$ looks like



Using associativity and the Frobenius property, one readily checks that $P_{N,N'}$ is a projector and, when $\mathcal{C}$ is abelian, that it annihilates elements that vanish in the bimodule tensor product.

Hence the image of $P_{N,N'}$ is indeed the bimodule tensor product of N with N',

$$N \otimes_B N' = \operatorname{im}\left(P_{N,N'}\right) .$$

More precisely, we have the following general definition of images

Definition 19 (compare [7], def. 2.12) *The object* $\operatorname{im}\left(P_{N,N'}\right) \in \operatorname{Obj}(\mathcal{C})$ *is called the* **image** *of* $N \otimes N' \xrightarrow{P_{N,N'}} N \otimes N'$ *if there are morphisms*

$$\operatorname{im}\left(P_{N,N'}\right) \xrightarrow{e} N \otimes N'$$

(the **injection of the image into the domain***) and*

$$N \otimes N' \xrightarrow{r} \operatorname{im}\left(P_{N,N'}\right)$$

(the **projection of the domain onto the image***) such that*

$$\begin{tikzcd}
N \otimes N' \arrow[rr, "P_{N,N'}"] \arrow[dr, "r"'] & & N \otimes N' \\
& \operatorname{im}\left(P_{N,N'}\right) \arrow[ur, "e"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\operatorname{im}\left(P_{N,N'}\right) \arrow[rr, "\mathrm{Id}"] \arrow[dr, "e"'] & & \operatorname{im}\left(P_{N,N'}\right) \\
& N \otimes N' \arrow[ur, "r"'] &
\end{tikzcd} .$$

In the cases of interest here, where $\mathcal{C}$ is abelian and semisimple, such morphisms e and r do exist for every projector.

Next we construct ambijunctions, using left A-modules N and the projector $P_{N^\vee,N}$, along the lines of [7], prop. 2.13

1. <u>left unit</u>

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, bend left, "\mathbb{1}"] \arrow[dr, bend right, "N^\vee"'] & \arrow[d, phantom, "\Downarrow i"] & \mathbb{1} \\
& A \arrow[ur, bend right, "N"'] &
\end{tikzcd} \quad \text{given by} \quad \begin{tikzcd}
\mathbb{1} \arrow[d] \\
N^\vee \otimes N \arrow[d, "r"] \\
N^\vee \otimes_A N
\end{tikzcd}$$

2. <u>left counit</u>

$$\begin{tikzcd}
& \mathbb{1} \arrow[dr, bend left, "N^\vee"] \arrow[d, phantom, "\Downarrow e"] & \\
A \arrow[ur, bend left, "N"] \arrow[rr, bend right, "A"'] & & A
\end{tikzcd} \quad \text{given by} \quad \begin{tikzcd}
N \otimes N^\vee \arrow[d] \\
A \otimes_A N \otimes N^\vee \otimes_A A \arrow[d, "e \otimes e"] \\
A \otimes N \otimes N^\vee \otimes A \arrow[d, "A \otimes b \otimes A"] \\
\mathbb{1}
\end{tikzcd}$$

3. <u>right unit</u>

$$\begin{tikzcd}
A \arrow[rr, bend left, "A"] \arrow[dr, bend right, "N"'] & \arrow[d, phantom, "\Downarrow \tilde{i}"] & A \\
& \mathbb{1} \arrow[ur, bend right, "N^\vee"'] &
\end{tikzcd}$$

given by

$$\begin{tikzcd}
A \arrow[d] \\
A \otimes_A A \arrow[d, "e"] \\
A \otimes A \arrow[d] \\
A \otimes N \otimes N^\vee \otimes A \arrow[d, "r \otimes r"] \\
N \otimes N^\vee \arrow[d, "r"] \\
N \otimes_A N^\vee
\end{tikzcd}$$

4. <u>right counit</u>

$$\begin{tikzcd}
& A \arrow[dr, bend left, "N"] \arrow[d, phantom, "\Downarrow \tilde{e}"] & \\
\mathbb{1} \arrow[ur, bend left, "N^\vee"] \arrow[rr, bend right, "\mathbb{1}"'] & & \mathbb{1}
\end{tikzcd}$$

given by

$$\begin{tikzcd}
N^\vee \otimes_A N \arrow[d, "e"] \\
N^\vee \otimes N \arrow[d] \\
\mathbb{1}
\end{tikzcd}$$

[...]

3.4.3 Relation between the two Constructions

The two constructions described above are closely related whenever the left A module N is A itself, regarded as a left module over itself, i.e. whenever

$$ {}_A N_{\mathbb{1}} = {}_A R_{\mathbb{1}} \,, $$

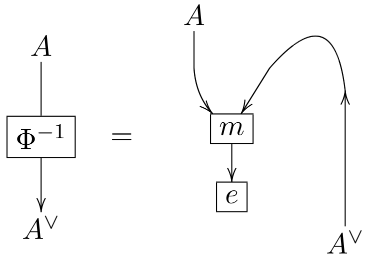
where ${}_A R_{\mathbb{1}}$ was defined in def. 17.

Here we will relate the construction of special ambidextrous adjunctions from §3.4.1 to the constructions used in [7], section 2.4.

What relates the two constructions is the isomorphism between a special symmetric Frobenius algebra A and its dual from def. 10.

We need this simple

Proposition 15 *The morphism*



is in particular also a morphism of right A-modules.

Proof. Attach an A-line to the incoming A-line, use associativity to pass it past the product, observe that the result is the right A-action on $A^\vee$. ☐

Let ${}_{\mathbb{1}}L_A$ be, as before, A regarded as a right A-module over itself, and let $N^\vee = A^\vee$ be $A^\vee$ regarded as a right A-module. This means we have isomorphisms of $\mathbb{1} - A$ bimodules

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left=50, "{}_{\mathbb{1}}L_A", ""{name=U, below}] \arrow[r, bend right=50, "N^\vee"', ""{name=D}] & A \arrow[Rightarrow, from=U, to=D, "\Phi^{-1}"] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left=50, "N^\vee", ""{name=U, below}] \arrow[r, bend right=50, "{}_{\mathbb{1}}L_A"', ""{name=D}] & A \arrow[Rightarrow, from=U, to=D, "\Phi"] \end{tikzcd} .$$

By pasting these 2-cells wherever appropriate, we can transform the diagrams corresponding to the adjunction on ${}_{\mathbb{1}}L_A$ and ${}_A R_{\mathbb{1}}$ to those of the adjunction in N and $N^\vee$, where $N = A$ as a right A-module over itself. And vice versa.

3.5 Opposite Algebras

3.5.1 Definitions

We recall some facts and definitions on opposite algebras in ribbon categories from section 3.5 of [6] and section 2.1 of [7].

In the following, let $\mathcal{C}$ be a *ribbon category.* Denote its braiding morphisms by

$$\begin{tikzcd} U\otimes V \arrow[r, "c_{U,V}", "\simeq"'] & V\otimes U \end{tikzcd}$$

and its twist morphisms by

$$\begin{tikzcd} U \arrow[r, "\theta_U", "\simeq"'] & U \end{tikzcd} .$$

Definition 20 *Let A be an algebra with product*

$$\begin{tikzcd} A\otimes A \arrow[r, "m"] & A \end{tikzcd}$$

internal to some ribbon category $\mathcal{C}$*.*

The **opposite algebra** A_{op} *is the internal algebra based on the same object, $A_{\mathrm{op}} = A$, but with product m_{op} given by*

$$\begin{tikzcd} A\otimes A \arrow[rr, "m_{\mathrm{op}}"] \arrow[dr, "c_{A,A}"'] & & A \\ & A\otimes A \arrow[ur, "m"'] & \end{tikzcd} .$$

Remark. In general $(A_{\mathrm{op}})_{\mathrm{op}}$ is not isomorphic to A. We write

$$A = A^{(0)}$$

and

$$A^{(n+1)} \equiv (A^{(n)})_{\mathrm{op}} .$$

Definition 21 *A morphism*

$$\begin{tikzcd} A \arrow[r, "\sigma"] & A \end{tikzcd}$$

is called an **algebra antihomomorphism** *if regarded as a morphism*

$$\begin{tikzcd} A \arrow[r, "\sigma"] & A^{\mathrm{op}} \end{tikzcd}$$

it is an ordinary algebra homomorphism.

Hence σ is an algebra antihomomorphism iff

$$\begin{tikzcd} A\otimes A \arrow[r, "\sigma\otimes\sigma"] \arrow[d, "m"'] & A\otimes A \arrow[d, "m^{\mathrm{op}}"] \\ A \arrow[r, "\sigma"'] & A \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "i"] \arrow[dr, "i"'] & & A \\
& A \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

commute.

Definition 22 *An algebra antihomomorphism is called a* **reversion** *if it squares to the twist, i.e. if*

$$\begin{tikzcd}
A \arrow[rr, "\theta"] \arrow[dr, "\sigma"'] & & A \\
& A \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Definition 23 *If A is also a coalgebra we let the coproduct $\Delta^{\rm op}$ on $A^{\rm op}$ be given by*

$$\begin{tikzcd}
A \arrow[rr, "\Delta^{\rm op}"] \arrow[dr, "\Delta"'] & & A \otimes A \ . \\
& A \otimes A \arrow[ur, "\bar{c}_{AA}"'] &
\end{tikzcd}$$

An antihomomorphism of coalgebras then has to satisfy

$$\begin{tikzcd}
A \arrow[r, "\sigma"] \arrow[d, "\Delta^{\rm op}"'] & A \arrow[d, "\Delta"] \\
A \otimes A \arrow[r, "\sigma\otimes\sigma"'] & A \otimes A
\end{tikzcd} \quad .$$

Proposition 16 *An algebra A with reversion is Morita equivalent to its opposite algebra $A^{\rm op}$. If A is special Frobenius then so is $A^{\rm op}$, and we have*

$$\begin{aligned} \beta_A &= \beta_{A^{\rm op}} \\ (\beta_{\mathbb{1}})_A &= (\beta_{\mathbb{1}})_{A^{\rm op}} \, . \end{aligned}$$

Proof. A reversion is an algebra homomorphism and hence induces invertible left-induced bimodules

$${}_A N_{\sigma A^{\rm op}} = \ A \overset{m}{\dashrightarrow} A \overset{m\circ\sigma}{\dashleftarrow} A^{\rm op}$$

and

$${}_{A^{\rm op}} N_{\sigma A} = \ A^{\rm op} \overset{m^{\rm op}}{\dashrightarrow} A^{\rm op} \overset{m\circ\bar{\sigma}}{\dashleftarrow} A$$

whose product is, according to prop. 9 (p. 38),

$${}_A N_{\sigma A^{\rm op}} \ \otimes_{A^{\rm op}} \ {}_{A^{\rm op}} N_{\sigma A} = {}_A A_A$$

and

$${}_{A^{\rm op}} N_{\sigma A} \ \otimes_A \ {}_A N_{\sigma A^{\rm op}} = {}_{A^{\rm op}} A^{\rm op}{}_{A^{\rm op}} \, .$$

The fact that A^{op} is special with the same constants as A follows from the commutativity of the diagram

$$\begin{tikzcd}
 & A \otimes A \arrow[dr, "m^{\mathrm{op}}"] \arrow[d, bend left, "c_{AA}"] & \\
A \arrow[ur, "\Delta^{\mathrm{op}}"] \arrow[r, "\Delta"] \arrow[rr, bend right=40, "\beta_A \cdot \mathrm{Id}"'] & A \otimes A \arrow[r, "m"] \arrow[u, bend left, "\bar{c}_{AA}"] & A
\end{tikzcd} .$$

That the $\beta_{\mathbb{1}}$ coincide is trivial, since unit and counit of A and A^{op} coincide. ☐

Example 4

Consider a bimodule which relates an A-phase with an A^{op}-phase

$$\begin{tikzcd}
 & A \arrow[r, "{(A,\sigma)}"] \arrow[dd, "\mathrm{Id}"'] \arrow[ddr, phantom, "\Swarrow_{\mathrm{Id}}"] & A^{\mathrm{op}} \arrow[dd, "\mathrm{Id}"] & \\
A\text{-phase} & & & A^{\mathrm{op}}\text{-phase} \\
 & A \arrow[r, "{(A,\sigma)}"'] & A^{\mathrm{op}} &
\end{tikzcd}$$

According to §18, local trivialization turns this defect locally into

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\mathbb{1}"] \arrow[d, "{}_{\mathbb{1}}L_A"'] \arrow[dr, phantom, "\Swarrow_{\sigma}"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_{A^{\mathrm{op}}}"] \\
A \arrow[r, "{(A,\sigma)}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow_{\mathrm{Id}}"] & A^{\mathrm{op}} \arrow[d, "\mathrm{Id}"] \\
A \arrow[r, "{(A,\sigma)}"] \arrow[d, "{}_A R_{\mathbb{1}}"'] \arrow[dr, phantom, "\Swarrow_{\mathrm{Id}}"] & A^{\mathrm{op}} \arrow[d, "{}_{A^{\mathrm{op}}} R_{\mathbb{1}}"] \\
\mathbb{1} \arrow[r, "\mathbb{1}"'] & \mathbb{1}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{1} \arrow[d, bend right=60, "A"'] \arrow[d, bend left=60, "A"] \arrow[d, phantom, "\Leftarrow_{\sigma}"] \\
\mathbb{1}
\end{tikzcd}
=
\begin{tikzcd}
{} & \boxed{\sigma} \arrow[l, "A"] & {} \arrow[l, dash, "A"]
\end{tikzcd} .$$

On the left we have here the 2-morphism in $\Sigma(\mathcal{C})$ which is obtained from the above 2-morphism in $\mathbf{BiMod}(\mathcal{C})$ by the trivialization procedure described in §18. In the middle the same 2-morphism is depicted, now with the products of bimodules explicitly evaluated. On the right the Poincaé-dual string diagram of this 2-morphism is given. This is simply an A-line with a reversion. Compare section 3 of [7].

3.5.2 The Half-Twist in terms of Adjunctions

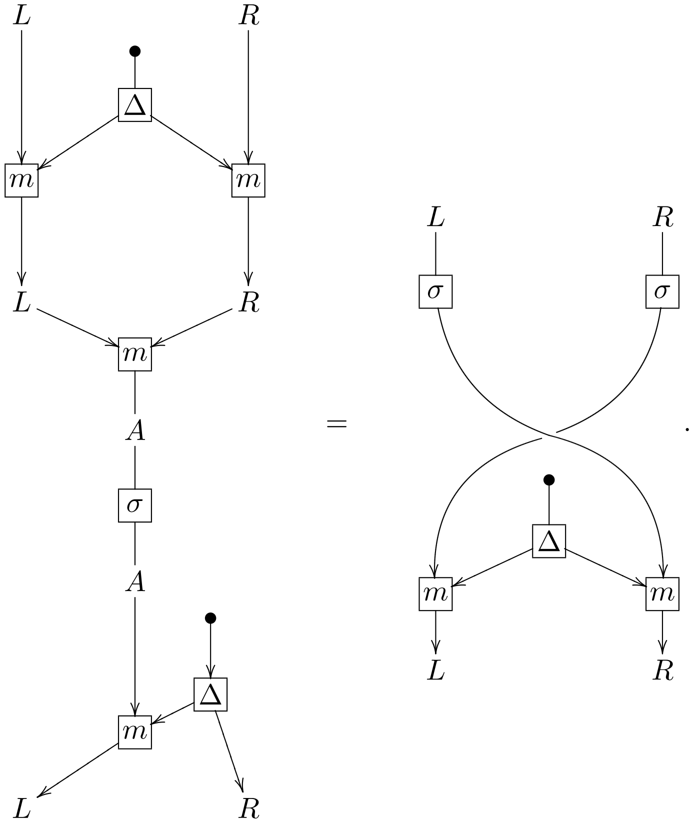
We would like to understand how the reversion

$$A \xrightarrow{\sigma} A$$

looks like in terms of the ambidextrous adjunction that A is made of. Using the formalism from [7], section 2 (def. 19) we can resolve A as the image of the map $L \otimes R \to L \otimes_A R$ and push σ through to the right A-module ${}_{\mathbb{1}}L_A = (A, m)$ and the left module ${}_A R_{\mathbb{1}} = (A, m)$.

Doing so, one finds

Proposition 17



Proof. Use the defining properties of σ as an (invertible) algebra antihomomorphism. □

Notice how both L and R are, as objects, nothing but A itself.

If we think of two lines, one labelled by L, one by R, as the boundary of a ribbon which is decorated by $L \otimes_A A \simeq A$, then the above proposition says that a reversion acts on these ribbons by acting as σ on L and R and by performing a half-twist of that ribbon.

3.5.3 $\mathbf{Vect}_1$

$$\begin{tikzcd}\mathbb{C} \arrow[r, "\sigma"] & \mathbb{C}^{\mathrm{op}}\end{tikzcd}$$

$$V \otimes \mathbb{C}_\sigma \simeq V_\sigma$$

$$vc \simeq (v,1)c = (v,\bar{c}) = (v\bar{c},1) = \bar{c}(v,1) \simeq \bar{c}v$$

$$\mathbb{C}_\sigma \otimes V \simeq {}_\sigma V$$

$$cv \simeq c(1,v) = (c,v) = (1\bar{c},v) = (1,\bar{c}v) = (1,v)\bar{c} \simeq v\bar{c}$$

3.5.4 Involutions

For any real vector space V with complex structure, let $\bar{V}$ be the same real vector space, but with opposite complex structure.

Denote by

$$\begin{array}{ccccc} \sigma & : & \mathbb{C} & \to & \mathbb{C} \\ & & c & \mapsto & \bar{c} \end{array}$$

the conjugation involution on $\mathbb{C}$ and by $\mathbb{C}_\sigma$ the $\mathbb{C}$-$\mathbb{C}$ bimodule which, as an object, is $\mathbb{C}$ itself, with the left $\mathbb{C}$-action being multiplication in $\mathbb{C}$ and the right $\mathbb{C}$ action given by first acting with σ and then multiplying in $\mathbb{C}$:

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}_\sigma & \xrightarrow{l} & \mathbb{C}_\sigma \\ (c,d) & \mapsto & cd \end{array}$$

$$\begin{array}{ccc} \mathbb{C}_\sigma \times \mathbb{C} & \xrightarrow{r} & \mathbb{C}_\sigma \\ (d,c) & \mapsto & \bar{c}d \end{array}.$$

Similarly, for any complex vector space V, let

$$V_\sigma \simeq V \otimes \mathbb{C}_\sigma$$

and

$${}_\sigma V \simeq \mathbb{C}_\sigma \otimes V$$

be the $\mathbb{C}$-$\mathbb{C}$-bimodule V, as an object, but with the left or right $\mathbb{C}$ action twisted, as indicated.

Notice that we have the canonical isomorphism

$${}_\sigma V_\sigma \simeq \bar{V}$$

and in particular the canonical identitfication

$$\mathbb{C}_\sigma \otimes \mathbb{C}_\sigma \simeq \bar{\mathbb{C}} \simeq \mathbb{C}\,.$$

Denote by $\mathbf{BiMod}_{\mathbb{C}}$ the 2-category of $\mathbb{C}$-$\mathbb{C}$-bimodules, with single object $\mathbb{C}$, bimodules up to canonical isomorphism as 1-morphisms and bimodule intertwiners as 2-morphisms.

We write

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "\bar{V}"] \arrow[r, bend right=50, "\bar{W}"'] \arrow[r, phantom, "\Downarrow \bar{\phi}"] & \bullet & \equiv & \bullet \arrow[r, "\mathbb{C}_\sigma"] & \bullet \arrow[r, bend left=50, "V"] \arrow[r, bend right=50, "W"'] \arrow[r, phantom, "\Downarrow \phi"] & \bullet \arrow[r, "\mathbb{C}_\sigma"] & \bullet \end{tikzcd}$$

and find in particular

$$\begin{tikzcd} \bullet \arrow[r, "\mathbb{C}_\sigma"] & \bullet \arrow[r, bend left=50, "\mathbb{C}"] \arrow[r, bend right=50, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \arrow[r, "\mathbb{C}_\sigma"] & \bullet & = & \bullet \arrow[r, bend left=50, "\mathbb{C}"] \arrow[r, bend right=50, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow \bar{c}"] & \bullet & . \end{tikzcd}$$

It follows that we obtain a representation of the automorphism 2-group $\mathrm{Aut}(U(1))$ of $U(1)$, on $\mathbf{BiMod}_{\mathbb{C}}$ by setting

$$\rho \quad : \quad \Sigma(\mathrm{Aut}(U(1))) \quad \to \qquad \mathbf{BiMod}_{\mathbb{C}}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow g"] & \bullet & \mapsto & \bullet \arrow[r, bend left=50, "\mathbb{C}"] \arrow[r, bend right=50, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow g"] & \bullet & . \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "\sigma"] \arrow[r, bend right=50, "\sigma"'] \arrow[r, phantom, "\Downarrow g"] & \bullet & \mapsto & \bullet \arrow[r, bend left=50, "\mathbb{C}_\sigma"] \arrow[r, bend right=50, "\mathbb{C}_\sigma"'] \arrow[r, phantom, "\Downarrow g"] & \bullet \end{tikzcd}$$

Here we have denoted the nontrivial element in the automorphism group $\mathbb{Z}_2$ of $U(1)$ also by σ.

We are interested in transition morphisms in $\mathbf{Trans}(\mathcal{P}, \mathbf{BiMod}_{\mathbb{C}})$. Consider the case where such a morphism involves $\mathbb{C}_\sigma$ in its defining tin can equation as follows

$$\begin{tikzcd} & \bullet \arrow[dr, "p_{23}^*L"] \arrow[d, phantom, "\Downarrow f"] & & & & \bullet \arrow[d, "\mathbb{C}_\sigma"] \arrow[dr, "p_{23}^*L"] & & \\ \bullet \arrow[ur, "p_{12}^*L"] \arrow[rr, "p_{13}^*L"] \arrow[d, "\mathbb{C}_\sigma"'] \arrow[drr, phantom, "\Swarrow \mathrm{Id}"] & & \bullet \arrow[d, "\mathbb{C}_\sigma"] & = & \bullet \arrow[ur, "p_{12}^*L"] \arrow[d, "\mathbb{C}_\sigma"'] \arrow[r, phantom, "\Swarrow \mathrm{Id}"] & \bullet \arrow[dr, "p_{23}^*L'"] \arrow[r, phantom, "\Swarrow \mathrm{Id}"] \arrow[d, phantom, "\Downarrow f'"] & \bullet \arrow[d, "\mathbb{C}_\sigma"] & . \\ \bullet \arrow[rr, "p_{13}^*L'"'] & & \bullet & & \bullet \arrow[ur, "p_{12}^*L'"] \arrow[rr, "p_{13}^*L'"'] & & \bullet & \end{tikzcd}$$

The existence of the identity-2-morphisms here says that the transition line bundles are related by $L' = \bar{L}$.

When we equivalently rewrite this equation as

$$\begin{tikzcd} & \bullet \arrow[dr, "p_{23}^*L'"] \arrow[d, phantom, "\Downarrow f'"] & \\ \bullet \arrow[ur, "p_{12}^*L'"] \arrow[rr, "p_{13}^*L'" description] & & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} & & \bullet \arrow[dr, "p_{23}^*L"] \arrow[d, phantom, "\Downarrow f"] & & \\ \bullet \arrow[r, "\mathbb{C}_\sigma"] & \bullet \arrow[ur, "p_{12}^*L"] \arrow[rr, "p_{13}^*L" description] & & \bullet \arrow[r, "\mathbb{C}_\sigma"] & \bullet \end{tikzcd} \,,$$

which says that

$$f' = \bar{f}\,.$$

4 Differential cocycles and local trivialization

Fix once and for all some monoidal category $\mathcal{C}$. We are interested in 2-functors

$$\mathrm{tra} : \mathcal{P}_2 \to \mathbf{BiMod}(\mathcal{C})$$

from some geometric 2-category $\mathcal{P}_2$ to the 2-category of algebra bimodules (of special symmetric Frobenius algebras) internal to $\mathcal{C}$ (def. 14). In the context of the present discussion these 2-functors shall be called **transport 2-functors**.

There is a general theory of transport 2-functors and in particular of local trivialization of 2-transport. All of the following constructions are just special instances of that general theory. For more details see [**?**].

4.1 Trivial 2-Transport

Definition 24 *We say a transport 2-functor* $\mathrm{tra} : \mathcal{P}_2 \to \mathbf{BiMod}(\mathcal{C})$ *is* **trivial** *precisely if it takes values only in* $\mathcal{C} \simeq \mathrm{Hom}_{\mathbf{BiMod}(\mathcal{C})}(\mathbb{1}, \mathbb{1}) \subset \mathbf{BiMod}(\mathcal{C})$.

We write

$$\mathrm{tra}_{\mathbb{1}} : \mathcal{P}_2 \to \mathcal{C} \subset \mathbf{BiMod}(\mathcal{C})$$

for a trivial transport 2-functor $\mathrm{tra}_{\mathbb{1}}$.

Remark. The terminology "trivial" here is motivated from a similar condition on 2-transport in 2-bundles. It is not supposed to suggest that a trivial transport 2-functor encodes no interesting information. Rather, one should think of a general transport 2-functor as defining an algebra-bundle over the space of objects of $\mathcal{P}_2$. For a trivial transport 2-functor this bundle is trivial in that all its fibers are identified with the (trivial) algebra $\mathbb{1}$.

Definition 25 *A* **trivialization** *of a transport 2-functor*

$$\mathrm{tra} : \mathcal{P}_2(X) \to \mathbf{BiMod}(\mathcal{C})$$

is a choice *of a trivial transport 2-functor*

$$\mathrm{tra}_{\mathbb{1}} : \mathcal{P}_2 \to \mathcal{C} \subset \mathbf{BiMod}(\mathcal{C})$$

together with a choice *of special ambidextrous adjunction (defs. 4, 5, 7)*

$$\begin{tikzcd}
\mathrm{tra}_{\mathbb{1}} \arrow[r, bend left=50, "\bar{t}"] \arrow[r, phantom, shift left=2ex, "\overset{i}{\Leftrightarrow}"] \arrow[r, phantom, shift right=2ex, "\underset{e}{\Leftrightarrow}"] & \mathrm{tra} \arrow[l, bend left=50, "t"]
\end{tikzcd}$$

A transport 2-functor is called **trivializable** if it admits a trivialization.

The most general condition under which a transport 2-functor is trivializable is not investigated here. We shall be content with showing that all transport 2-functors of the following form are trivializable.

Theorem 1 *Transport 2-functors to left-induced bimodules*

$$\mathrm{tra} : \mathcal{P}_2 \to \mathbf{LFBiMod}(\mathcal{C}) \subset \mathbf{BiMod}(\mathcal{C})$$

are trivializable if *every 2-morphism*

$$\mathrm{tra}\left(\begin{tikzcd} x \arrow[r, bend left=50, "\gamma_1"{name=U}] \arrow[r, bend right=50, "\gamma_2"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "S"] \end{tikzcd}\right) = \begin{tikzcd} A_x \arrow[r, bend left=50, "{(A_x \otimes V_{\gamma_1}, \phi_{\gamma_1})}"{name=U}] \arrow[r, bend right=50, "{(A_x \otimes V_{\gamma_2}, \phi_{\gamma_2})}"'{name=D}] & A_y \arrow[from=U, to=D, Rightarrow, "\mathrm{tra}(S)"'] \end{tikzcd} \quad \in \mathrm{Mor}_2(\mathbf{LFBiMod}(\mathcal{C}))$$

for all $S \in \mathrm{Mor}_2(\mathcal{P}_2)$ *is of the form*

$$\begin{tikzcd}
A_x \otimes V_{\gamma_1} \arrow[d, "\mathrm{tra}(S)"'] & A_x \otimes V_{\gamma_1} \arrow[d, "A_x \otimes \lambda_S"] \\
A_x \otimes V_{\gamma_2} \arrow[ur, phantom, "="] & A_x \otimes V_{\gamma_2}
\end{tikzcd}$$

for some $\lambda_S \in \mathrm{Mor}(\mathcal{C})$*.*

For the proof of prop. 15 below it is crucial to note by prop. 6 (p. 36) $\mathrm{tra}(S)$ being a morphism of bimodules implies that λ_S is such that the diagrams

$$\begin{tikzcd}
V_{\gamma_1} \otimes A_y \arrow[r, "\phi_{\gamma_1}"] \arrow[d, "\lambda_s \otimes A_y"'] & A_x \otimes V_{\gamma_1} \arrow[d, "A_x \otimes \lambda_S"] \\
V_{\gamma_2} \otimes A_y \arrow[r, "\phi_{\gamma_2}"] & A_x \otimes V_{\gamma_2}
\end{tikzcd} \tag{3}$$

commute.

Remark. In the application to the FRS formalism $\mathrm{tra}(S)$ plays the role of the morphism which connects the field insertions on the *connecting 3-manifold.* Hence the curious restriction on the nature of $\mathrm{tra}(S)$, which is crucial for the above theorem to be true, is precisely the property that $\mathrm{tra}(S)$ is assumed to have in FRS formalism.

The proof of proposition 1 amounts to constructing a trivialization and checking its properties. This is the content of the following subsection.

4.2 Trivialization of trivializable 2-Transport

In order to prove theorem 1 we need to construct a trivial transport 2-functor as well as all ingredients of a special ambidextrous adjunction such that all the required conditions are satisfied.

1. the trivial transport functor

   Define

   $$\mathrm{tra}_{\mathbb{1}} : \mathcal{P}_2 \to \mathcal{C}$$

   by

   $$\mathrm{tra}\left(\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"{name=U}] \arrow[r, bend right, "\gamma_2"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "S"] \end{tikzcd}\right) = \begin{tikzcd} \mathbb{1} \arrow[r, bend left, "V_{\gamma_1}"{name=U}] \arrow[r, bend right, "V_{\gamma_2}"'{name=D}] & \mathbb{1} \arrow[from=U, to=D, Rightarrow, "\lambda_S"] \end{tikzcd} \quad \in \mathrm{Mor}_2(\mathcal{C})$$

   for all $S \in \mathrm{Mor}_2(\mathcal{P}_2)$.

2. the trivialization morphism

   Define the pseuodnatural transformation $\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathbb{1}}$ to be that given by the map

   $$x \xrightarrow{\gamma} y \quad \mapsto \quad \begin{tikzcd} \mathrm{tra}(x) \arrow[r, "\mathrm{tra}(\gamma)"] \arrow[d, "t(x)"] & \mathrm{tra}(y) \arrow[d, "t(y)"] \arrow[dl, phantom, "\swarrow t(\gamma)"] \\ \mathrm{tra}(x) \arrow[r, "\mathrm{tra}(\gamma)"] & \mathrm{tra}_i(y) \end{tikzcd} \equiv \begin{tikzcd} A_x \arrow[r, "(A_x \otimes V_\gamma, \phi_\gamma)"] \arrow[d, "{}_{A_x}R_{\mathbb{1}}"] & A_y \arrow[d, "{}_{A_y}R_{\mathbb{1}}"] \arrow[dl, phantom, "\swarrow \mathrm{Id}"] \\ \mathbb{1} \arrow[r, "V_\gamma"] & \mathbb{1} \end{tikzcd} .$$

   That we really have an identity 2-morphism on the right hand side follows from def

17. This is readily seen to satisfy the required tin can equation

$$\begin{tikzcd}[column sep=huge, row sep=large]
A_x \arrow[r, bend left=50, "{(A_x\otimes V_{\gamma_1},\phi_{\gamma_1})}"] \arrow[r, "{(A_x\otimes V_{\gamma_2},\phi_{\gamma_2})}"'] \arrow[r, phantom, shift left=3, "\overset{\Vert}{\Downarrow}\mathrm{tra}(S)"] \arrow[d, "{{}_{A_x}R_{\mathbb{1}}}"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & A_y \arrow[d, "{{}_{A_y}R_{\mathbb{1}}}"] \\
\mathbb{1} \arrow[r, "V_\gamma"] & \mathbb{1}
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=large]
A_x \arrow[r, "{(A_x\otimes V_{\gamma_1},\phi_{\gamma_1})}"] \arrow[d, "{{}_{A_x}R_{\mathbb{1}}}"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & A_y \arrow[d, "{{}_{A_y}R_{\mathbb{1}}}"] \\
\mathbb{1} \arrow[r, "V_{\gamma_1}"] \arrow[r, bend right=50, "V_{\gamma_2}"'] \arrow[r, phantom, shift right=3, "\overset{\Vert}{\Downarrow}\lambda_S"] & \mathbb{1}
\end{tikzcd}.$$

The functoriality condition

$$\begin{tikzcd}[column sep=huge, row sep=large]
A_x \arrow[r, "{(A_x\otimes V_{\gamma_1\circ\gamma_2},\phi_{\gamma_1\circ\gamma_2})}"] \arrow[d, "{{}_{A_x}R_{\mathbb{1}}}"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & A_z \arrow[d, "{{}_{A_z}R_{\mathbb{1}}}"] \\
\mathbb{1} \arrow[r, "V_{\gamma_1\circ\gamma_2}"] & \mathbb{1}
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=large]
A_x \arrow[r, "{(A_x\otimes V_{\gamma_1},\phi_{\gamma_1})}"] \arrow[d, "{{}_{A_x}R_{\mathbb{1}}}"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & A_y \arrow[r, "{(A_y\otimes V_{\gamma_2},\phi_{\gamma_2})}"] \arrow[d, "{{}_{A_z}R_{\mathbb{1}}}"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & A_z \arrow[d, "{{}_{A_z}R_{\mathbb{1}}}"] \\
\mathbb{1} \arrow[r, "V_{\gamma_1}"] & \mathbb{1} \arrow[r, "V_{\gamma_2}"] & \mathbb{1}
\end{tikzcd}$$

follows from the functoriality of tra.

3. <u>the adjoint of the trivialization morphism</u>

   Define the pseudonatural transformation $\mathrm{tra}_{\mathbb{1}} \xrightarrow{\bar{t}} \mathrm{tra}$ to be that given by

$$x \xrightarrow{\gamma} y \quad \mapsto \quad
\begin{tikzcd}[column sep=large, row sep=large]
\mathrm{tra}(x) \arrow[r, "\mathrm{tra}(\gamma)"] \arrow[d, "\bar{t}(x)"'] \arrow[dr, phantom, "\Swarrow \bar{t}(\gamma)"] & \mathrm{tra}(y) \arrow[d, "\bar{t}(y)"] \\
\mathrm{tra}(x) \arrow[r, "\mathrm{tra}(\gamma)"] & \mathrm{tra}(y)
\end{tikzcd}
\equiv
\begin{tikzcd}[column sep=large, row sep=large]
\mathbb{1} \arrow[r, "V_\gamma"] \arrow[d, "{{}_{\mathbb{1}}L_{A_x}}"'] \arrow[dr, phantom, "\Swarrow \phi_\gamma"] & \mathbb{1} \arrow[d, "{{}_{\mathbb{1}}L_{A_y}}"] \\
A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi_\gamma)}"] & A_y
\end{tikzcd}.$$

   This is well defined (i.e. this 2-morphism really gives a morphism of bimodules $V_\gamma \otimes_{\mathbb{1}} {}_{\mathbb{1}}L_{A_y} \xrightarrow{\phi_\gamma} {}_{\mathbb{1}}L_{A_x} \otimes_{A_x} (A_x \otimes V_\gamma, \phi_\gamma)$) due to the fact that ϕ_γ is compatible with the product (see def. 15).

The required tin can equation

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left=50, "V_{\gamma_1}", ""{name=U, below}] \arrow[r, "V_{\gamma_2}"', ""{name=D, above}] \arrow[d, "{}_{\mathbb{1}}L_{A_x}"'] \arrow[Rightarrow, from=U, to=D, "\lambda_S"] \arrow[dr, phantom, "\Swarrow \phi_{\gamma_2}"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_{A_y}"] & = & \mathbb{1} \arrow[r, "V_{\gamma_1}"] \arrow[d, "{}_{\mathbb{1}}L_{A_x}"'] \arrow[dr, phantom, "\Swarrow \phi_{\gamma_1}"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_{A_y}"] \\
A_x \arrow[r, "{(A_x\otimes V_{\gamma_2},\phi_{\gamma_2})}"'] & A_y & & A_x \arrow[r, "{(A_x\otimes V_{\gamma_1},\phi_{\gamma_1})}", ""{name=T, below}] \arrow[r, bend right=50, "{(A_x\otimes V_{\gamma_2},\phi_{\gamma_2})}"', ""{name=B, above}] \arrow[Rightarrow, from=T, to=B, "\mathrm{tra}(S)"] & A_y
\end{tikzcd}$$

holds by assumption on tra(S) (namely using the commutativity of the diagram (3), on p. 55).

The functoriality condition

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "V_{\gamma_1\circ\gamma_2}"] \arrow[d, "{}_{\mathbb{1}}L_{A_x}"'] \arrow[dr, phantom, "\Swarrow \phi_{\gamma_1\circ\gamma_2}"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_{A_z}"] & = & \mathbb{1} \arrow[r, "V_{\gamma_1}"] \arrow[d, "{}_{\mathbb{1}}L_{A_x}"'] \arrow[dr, phantom, "\Swarrow \phi_{\gamma_1}"] & \mathbb{1} \arrow[r, "V_{\gamma_2}"] \arrow[d, "{}_{\mathbb{1}}L_{A_z}"] \arrow[dr, phantom, "\Swarrow \phi_{\gamma_2}"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_{A_z}"] \\
A_x \arrow[r, "{(A_x\otimes V_{\gamma_1\circ\gamma_2},\phi_{\gamma_1\circ\gamma_2})}"] & A_y & & A_x \arrow[r, "{(A_x\otimes V_{\gamma_1},\phi_{\gamma_1})}"'] & A_y \arrow[r, "{(A_y\otimes V_{\gamma_2},\phi_{\gamma_2})}"'] & A_z
\end{tikzcd}$$

again follows from the functoriality of tra.

4. <u>the left unit</u>

   Define the modification

$$\begin{tikzcd}
\mathrm{tra}_{\mathbb{1}} \arrow[rr, bend left=30, "\mathrm{Id}", ""{name=I, below}] \arrow[dr, bend right, "\bar{t}"'] & & \mathrm{tra}_{\mathbb{1}} \\
& \mathrm{tra} \arrow[ur, bend right, "t"'] \arrow[Rightarrow, from=I, to=2-2, shorten >=12pt, "i"] &
\end{tikzcd}$$

   to be that given by the map

$$\mathrm{Obj}(\mathcal{P}_2) \ni \ x \ \mapsto \begin{tikzcd}
\mathbb{1} \arrow[rr, bend left=30, "\mathbb{1}", ""{name=J, below}] \arrow[dr, bend right, "{}_{\mathbb{1}}L_{A_x}"'] & & \mathbb{1} \\
& A_x \arrow[ur, bend right, "{}_{A_x}R_{\mathbb{1}}"'] \arrow[Rightarrow, from=J, to=2-2, shorten >=12pt, "i"] &
\end{tikzcd} \in \mathrm{Mor}_2(\mathbf{BiMod}(\mathcal{C}))$$

The required tin can equation

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "V_\gamma"] \arrow[d, "{}_{\mathbb{1}}L_{A_x}"'] \arrow[dr, phantom, "\overset{\phi_\gamma}{\swarrow}"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_{A_y}"] \arrow[dd, bend left=60, "\mathbb{1}"] \\
A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi_\gamma)}"] \arrow[d, "{}_{A_x}R_{\mathbb{1}}"'] \arrow[dr, phantom, "\overset{\mathrm{Id}}{\swarrow}"] & A_y \arrow[d, "{}_{A_y}R_{\mathbb{1}}"] \arrow[r, phantom, "\overset{i_{A_y}}{\Longleftarrow}"] & {} \\
\mathbb{1} \arrow[r, "V_\gamma"] & \mathbb{1}
\end{tikzcd}
=
\begin{tikzcd}
 & \mathbb{1} \arrow[r, "V_\gamma"] \arrow[dl, "{}_{\mathbb{1}}L_{A_x}"'] \arrow[dd, "\mathbb{1}"] \arrow[ddr, phantom, "\overset{\mathrm{Id}}{\swarrow}"] & \mathbb{1} \arrow[dd, "\mathbb{1}"] \\
A_x \arrow[dr, "{}_{A_x}R_{\mathbb{1}}"'] \arrow[r, phantom, "\overset{i_{A_x}}{\Longleftarrow}"] & {} & \\
 & \mathbb{1} \arrow[r, "V_\gamma"] & \mathbb{1}
\end{tikzcd}$$

follows from the compatibility of ϕ with the unit (see def. 15).

5. <u>the left counit</u>

   Define the modification

$$\begin{tikzcd}
 & \mathrm{tra}_{\mathbb{1}} \arrow[dr, bend left, "\bar{t}"] \arrow[d, phantom, "\Downarrow e"] & \\
\mathrm{tra} \arrow[ur, bend left, "t"] \arrow[rr, bend right, "\mathrm{Id}"'] & {} & \mathrm{tra}
\end{tikzcd}$$

   to be that given by the map

$$\mathrm{Obj}(\mathcal{P}_2) \ni x \;\; \mapsto \;\; \begin{tikzcd}
 & \mathbb{1} \arrow[dr, bend left, "{}_{\mathbb{1}}L_{A_x}"] \arrow[d, phantom, "\Downarrow e"] & \\
A_x \arrow[ur, bend left, "{}_{A_x}R_{\mathbb{1}}"] \arrow[rr, bend right, "\mathrm{Id}"'] & {} & A_x
\end{tikzcd} \;\; \in \mathrm{Mor}_2(\mathbf{BiMod}(\mathcal{C}))\ .$$

   The required tin can equation

$$\begin{tikzcd}
{} & A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi_\gamma)}"] \arrow[d, "{}_{A_x}R_{\mathbb{1}}"'] \arrow[dd, bend right=70, "\mathbb{1}"'] \arrow[dr, phantom, "\overset{\mathrm{Id}}{\swarrow}"] & A_y \arrow[d, "{}_{A_y}R_{\mathbb{1}}"] \\
{} & \mathbb{1} \arrow[l, phantom, "\overset{e_{A_x}}{\Longleftarrow}"] \arrow[r, "V_\gamma"] \arrow[d, "{}_{\mathbb{1}}L_{A_x}"'] \arrow[dr, phantom, "\overset{\phi_\gamma}{\swarrow}"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_{A_y}"] \\
{} & A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi)}"'] & A_y
\end{tikzcd}
=
\begin{tikzcd}
A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi_\gamma)}"] \arrow[dd, "A_x"'] \arrow[ddr, phantom, "\overset{\mathrm{Id}}{\swarrow}"] & A_y \arrow[dd, "A_y"'] \arrow[dr, "{}_{A_y}R_{\mathbb{1}}"] & \\
 & {} & \mathbb{1} \arrow[dl, "{}_{\mathbb{1}}L_{A_y}"] \arrow[l, phantom, "\overset{\tilde{e}_{A_y}}{\Longleftarrow}"] \\
A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi_\gamma)}"'] & A_y &
\end{tikzcd}$$

   holds due to the compatibility of ϕ_γ with the counit.

6. <u>the right unit</u>

   Define the modification

$$\begin{tikzcd}
\text{tra} \arrow[rr, bend left, "A"] \arrow[dr, bend right, "t"'] & \Downarrow \tilde{i} & \text{tra} \\
& \text{tra}_{\mathbb{1}} \arrow[ur, bend right, "\bar{t}"'] &
\end{tikzcd}$$

   to be that given by the map

$$\text{Obj}\,(\mathcal{P}_2) \ni x \;\; \mapsto \;\; \begin{tikzcd}
A_x \arrow[rr, bend left, "A_x"] \arrow[dr, bend right, "{_{A_x}R_{\mathbb{1}}}"'] & \Downarrow \tilde{i} & A_x \\
& \mathbb{1} \arrow[ur, bend right, "{_{\mathbb{1}}L_{A_x}}"'] &
\end{tikzcd} \;\; \in \text{Mor}_2\,(\textbf{BiMod}\,(\mathcal{C}))$$

   The required tin can equation

$$\begin{tikzcd}
A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi_\gamma)}"] \arrow[d, "{_{A_x}R_{\mathbb{1}}}"'] \arrow[dr, phantom, "\text{Id}"] & A_y \arrow[d, "{_{A_y}R_{\mathbb{1}}}"] \arrow[dd, bend left=60, "A_y"] \\
\mathbb{1} \arrow[r, "V_\gamma"] \arrow[d, "{_{\mathbb{1}}L_{A_x}}"'] \arrow[dr, phantom, "\phi_\gamma"] & \mathbb{1} \arrow[d, "{_{\mathbb{1}}L_{A_y}}"] \arrow[r, phantom, "\overset{\tilde{i}_{A_y}}{\Longleftarrow}"] & {} \\
A_x \arrow[r, "V_\gamma"] & A_y
\end{tikzcd} \quad = \quad \begin{tikzcd}
& A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi_\gamma)}"] \arrow[dl, "{_{A_x}R_{\mathbb{1}}}"'] \arrow[dd, "A_x"] \arrow[ddr, phantom, "\text{Id}"] & A_y \arrow[dd, "A_y"] \\
\mathbb{1} \arrow[dr, "{_{\mathbb{1}}L_{A_x}}"'] \arrow[r, phantom, "\overset{\tilde{i}_{A_x}}{\Longleftarrow}"] & {} & \\
& A_x \arrow[r, "{(A_x\otimes V_\gamma,\phi_\gamma)}"] & A_y
\end{tikzcd}$$

   holds due to the compatibility of ϕ_γ with the coproduct.

   Notice at this point that all these statements are straightforward to check, but require a careful application of all the definitions governing composition of morphisms of left free bimodules. The truth of the above statement for instance involves the commutativity of the diagram

$$\begin{tikzcd}
V_\gamma \arrow[d, "e_{A_y}"] \arrow[ddr, "e_{A_x}"] & \\
V_\gamma \otimes A_y \arrow[d, "V_\gamma \otimes \Delta_{A_y}"] \arrow[dr, "\phi_\gamma"] & \\
V_\gamma \otimes A_y \otimes A_y \arrow[d, "\phi_\gamma \otimes A_y"] & A_x \otimes V_\gamma \arrow[ddl, "\Delta_{A_x} \otimes V_\gamma"] \\
A_x \otimes V_\gamma \otimes A_y \arrow[d, "A_x \otimes \phi_\gamma"] & \\
A_x \otimes A_x \otimes V_\gamma &
\end{tikzcd}$$

which holds due to compatibility with counit (upper part) and coproduct (lower part).

7. <u>the right counit</u> Define the modification

$$\begin{tikzcd}
& \mathrm{tra} \arrow[dr, bend left, "t"] \arrow[d, Rightarrow, "\tilde{e}"] & \\
\mathrm{tra}_{\mathbb{1}} \arrow[ur, bend left, "\bar{t}"] \arrow[rr, bend right, "\mathrm{Id}"'] & {} & \mathrm{tra}_{\mathbb{1}}
\end{tikzcd}$$

to be that given by the map

$$\mathrm{Obj}(\mathcal{P}_2) \ni x \mapsto \begin{tikzcd}
& A_x \arrow[dr, bend left, "{}_{A_x}R_{\mathbb{1}}"] \arrow[d, Rightarrow, "\tilde{e}"] & \\
\mathbb{1} \arrow[ur, bend left, "{}_{\mathbb{1}}L_{A_x}"] \arrow[rr, bend right, "\mathrm{Id}"'] & {} & \mathbb{1}
\end{tikzcd} .$$

The required tin can equation

$$\begin{tikzcd}
& \mathbb{1} \arrow[r, "V_\gamma"] \arrow[d, "{}_{\mathbb{1}}L_{A_x}"] \arrow[dd, bend right=60, "\mathbb{1}"'] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_{A_y}"] \arrow[dl, Rightarrow, "\phi_\gamma"'] \\
{} & A_x \arrow[r, "-(A_x \otimes V_\gamma{,}\phi_\gamma)-"] \arrow[d, "{}_{A_x}R_{\mathbb{1}}"] \arrow[l, Rightarrow, "\tilde{e}_{A_x}"'] & A_y \arrow[d, "{}_{A_y}R_{\mathbb{1}}"] \arrow[dl, Rightarrow, "\mathrm{Id}"'] \\
& \mathbb{1} \arrow[r, "V_\gamma"'] & \mathbb{1}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{1} \arrow[r, "V_\gamma"] \arrow[dd, "A_x"] & \mathbb{1} \arrow[dd, "A_y"] \arrow[dr, "{}_{\mathbb{1}}L_{A_y}"] \arrow[ddl, Rightarrow, "\mathrm{Id}"'] & \\
& {} & A_y \arrow[dl, "{}_{A_y}R_{\mathbb{1}}"] \arrow[l, Rightarrow, "\tilde{e}_{A_y}"'] \\
\mathbb{1} \arrow[r, "V_\gamma"'] & \mathbb{1} &
\end{tikzcd}$$

holds due to the compatibility of ϕ_γ with the the counit.

Finally, we need to check that the zig-zag identities are satisfied. But this is automatic, since the composition of modifications of pseudonatural transformations corresponds to the compositon of the respective 2-morphisms in the target 2-category. But these 2-morphisms, as defined above, are precisely those of the underlying ambijunction itself.

This then completes the proof. ☐

4.3 Expressing 2-Transport in Terms of Trivial 2-Transport

The crucial aspect of a trivialization of a 2-transport 2-functor is that it allows to express tra completely in terms of $\mathrm{tra}_{\mathbb{1}}$, t and $\bar{t}$.

This involves two proposition, which are stated and proven in this section

1. Trivializable 2-transport is expressible completely in terms of trivial 2-transport (prop. 18).
2. Possibly non-trivializable 2-transport gives rise to transitions between trivializable 2-transport (prop. 19).

4.3.1 Trivializable 2-Transport

Proposition 18 *The image of a trivializable 2-transport* tra *trvialized by*

$$\begin{tikzcd}[column sep=huge]
\mathrm{tra}_{\mathbb{1}} \arrow[r, bend left=50, "\bar{t}"] \arrow[r, phantom, shift left=2, "\overset{i}{\Leftrightarrow}"] \arrow[r, phantom, shift right=2, "\underset{e}{\Leftrightarrow}"] & \mathrm{tra} \arrow[l, bend left=50, "t"]
\end{tikzcd}$$

can be expressed in terms of the trivialization as follows:

$$\mathrm{tra}\left(\begin{tikzcd} x \arrow[r, bend left=50, "\gamma_1"] \arrow[r, bend right=50, "\gamma_1"'] \arrow[r, phantom, "\Downarrow S"] & y \end{tikzcd}\right) = \frac{1}{\beta_A} \begin{tikzcd}[row sep=large, column sep=huge]
& A_x \arrow[r, "\mathrm{tra}(\gamma_1)"] \arrow[d, "t(x)"] \arrow[dd, bend right=60, "A_x"'] \arrow[dr, phantom, "t(\gamma_1)\,\Swarrow"] & A_y \arrow[d, "t(y)"] \arrow[dd, bend left=60, "A_y"] & \\
{} & \mathbb{1} \arrow[l, phantom, "\overset{\tilde{e}_{A_x}}{\Leftarrow}"] \arrow[r, bend left, "\mathrm{tra}_{\mathbb{1}}(\gamma_1)"] \arrow[r, bend right, "\mathrm{tra}_{\mathbb{1}}(\gamma_2)"'] \arrow[r, phantom, "\overset{\Vert}{\underset{\Downarrow}{\mathrm{tra}_{\mathbb{1}}(S)}}"] \arrow[d, "\bar{t}(x)"] \arrow[dr, phantom, "\bar{t}(\gamma_2)\,\Swarrow"] & \mathbb{1} \arrow[d, "\bar{t}(y)"] \arrow[r, phantom, "\overset{i_{A_y}}{\Leftarrow}"] & {} \\
& A_x \arrow[r, "\mathrm{tra}(\gamma_2)"] & A_y &
\end{tikzcd} \tag{4}$$

for all $S \in \mathrm{Mor}_2(\mathcal{P}_2)$.

Proof. Use the tin can equation for the pseudonatural transformation

$$\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathbb{1}}$$

which reads

$$\begin{tikzcd}[row sep=huge, column sep=huge]
A_x \arrow[r, bend left=60, "\mathrm{tra}(\gamma_1)"{name=U}] \arrow[r, "\mathrm{tra}(\gamma_2)"{name=D}] \arrow[d, "t(x)"'] \arrow[dr, phantom, "\swarrow t(\gamma_2)"] & A_y \arrow[d, "t(y)"] \\
\mathbb{1} \arrow[r, "\mathrm{tra}(\gamma_2)"] & \mathbb{1}
\arrow[Rightarrow, from=U, to=D, "\mathrm{tra}(S)"]
\end{tikzcd}
=
\begin{tikzcd}[row sep=huge, column sep=huge]
A_x \arrow[r, "\mathrm{tra}(\gamma_1)"] \arrow[d, "t(x)"'] \arrow[dr, phantom, "\swarrow t(\gamma_1)"] & A_y \arrow[d, "t(y)"] \\
\mathbb{1} \arrow[r, "\mathrm{tra}_{\mathbb{1}}(\gamma_1)"{name=A}] \arrow[r, bend right=60, "\mathrm{tra}_{\mathbb{1}}(\gamma_2)"'{name=B}] & \mathbb{1}
\arrow[Rightarrow, from=A, to=B, "\mathrm{tra}_{\mathbb{1}}(S)"]
\end{tikzcd},$$

as well as the tin can equation for the modification

$$\begin{tikzcd}
\mathrm{tra} \arrow[rr, bend left=40, "\mathrm{Id}"] \arrow[dr, "t"'] \arrow[rr, phantom, "\Downarrow i"] & & \mathrm{tra} \\
& \mathrm{tra}_{\mathbb{1}} \arrow[ur, "\bar{t}"'] &
\end{tikzcd}$$

, which reads

$$\begin{tikzcd}[row sep=huge, column sep=huge]
A_x \arrow[r, "\mathrm{tra}(\gamma)"] \arrow[d, "t(x)"'] \arrow[dr, phantom, "t(\gamma)\,\swarrow"] & A_y \arrow[d, "t(y)"] \arrow[dd, bend left=70, "\mathrm{tra}(y)"] & \\
\mathbb{1} \arrow[r, "\mathrm{tra}_{\mathbb{1}}(\gamma)"] \arrow[d, "\bar{t}(x)"'] \arrow[dr, phantom, "\bar{t}(\gamma)\,\swarrow"] & \mathbb{1} \arrow[d, "\bar{t}(y)"] \arrow[r, phantom, "\overset{e_{A_y}}{\Longleftarrow}"] & {} \\
A_x \arrow[r, "\mathrm{tra}(\gamma)"] & A_y &
\end{tikzcd}
=
\begin{tikzcd}[row sep=huge, column sep=huge]
& A_x \arrow[r, "\mathrm{tra}(\gamma)"] \arrow[dl, "t(x)"'] \arrow[dd, "\mathrm{tra}(x)"] & A_y \arrow[dd, "\mathrm{tra}(y)"] \\
\mathbb{1} \arrow[dr, "\bar{t}(x)"'] \arrow[r, phantom, "\overset{e_{A_x}}{\Longleftarrow}"] & {} \arrow[r, phantom, "\mathrm{Id}\,\swarrow"] & {} \\
& A_x \arrow[r, "\mathrm{tra}(\gamma)"] & A_y
\end{tikzcd}.$$

Finally, use the condition that the adjunction is *special*. □

Remark. It is precisely the above construction which makes us want to consider *special* ambidextrous adjunctios. In the following we will assume that we have arranged that

$$\beta_{A_x} = \beta_{A_y} = 1\,,$$

which can always be done.

By contracting the identity morphisms in (4) to a point, we can redraw this diagram more suggestively as

$$\mathrm{tra}\left(\begin{tikzcd} x \arrow[r, bend left=50, "\gamma_1"] \arrow[r, bend right=50, "\gamma_2"'] \arrow[r, phantom, "\Downarrow S"] & y \end{tikzcd}\right) = \begin{tikzcd}[column sep=large] A_x \arrow[rrr, bend left=60, "\mathrm{tra}(\gamma_1)"] \arrow[rrr, bend right=60, "\mathrm{tra}(\gamma_2)"'] \arrow[r, bend left, "t(x)"] \arrow[r, phantom, "\Leftarrow e(x) ="] & \mathbb{1} \arrow[l, bend left, "\bar{t}(x)"] \arrow[r, bend left=50, "\mathrm{tra}_{\mathbb{1}}(\gamma_1)"] \arrow[r, bend right=50, "\mathrm{tra}_{\mathbb{1}}(\gamma_2)"'] \arrow[r, phantom, "\Downarrow \mathrm{tra}_{\mathbb{1}}"] & \mathbb{1} \arrow[r, bend right, "\bar{t}(y)"'] \arrow[r, phantom, "\Leftarrow \tilde{i}(y) ="] & A_y \arrow[l, bend right, "t(y)"'] \end{tikzcd} .$$

with the 2-cells $\Downarrow t(\gamma_1)$ between $\mathrm{tra}(\gamma_1)$ and $\mathrm{tra}_{\mathbb{1}}(\gamma_1)$, and $\Downarrow t(\gamma_2)$ between $\mathrm{tra}_{\mathbb{1}}(\gamma_2)$ and $\mathrm{tra}(\gamma_2)$.

Often it is helpful to use transport over bigons which have a square-like appearance. Using the functoriality of pseudonatural transformations we can write

$$\mathrm{tra}\left(\begin{tikzcd} & {} \arrow[dr, "\gamma_2"] & \\ x \arrow[ur, "\gamma_1"] \arrow[dr, "\gamma_1'"'] & \Downarrow S & z \\ & {} \arrow[ur, "\gamma_2'"'] & \end{tikzcd}\right) = \begin{tikzcd} & & & {} \arrow[d, "t(y)"] \arrow[dddrrr, "\mathrm{tra}(\gamma_2)"] & & & \\ & t(\gamma_1) \Swarrow & & {} \arrow[ddrr, "\mathrm{tra}_{\mathbb{1}}(\gamma_2)"] & & \Leftarrow t(\gamma_2) & \\ A_x \arrow[uurrr, "\mathrm{tra}(\gamma_1)"] \arrow[r, bend left, "t(x)"] \arrow[r, phantom, "\Leftarrow e(x) ="] & \mathbb{1} \arrow[l, bend left, "\bar{t}(x)"] \arrow[urr, "\mathrm{tra}_{\mathbb{1}}(\gamma_1)"] \arrow[drr, "\mathrm{tra}_{\mathbb{1}}(\gamma_1')"'] & & \Downarrow \mathrm{tra}_{\mathbb{1}}(S) & & \mathbb{1} \arrow[r, bend right, "\bar{t}(z)"'] \arrow[r, phantom, "\Leftarrow \tilde{i}(z) ="] & A_y \arrow[l, bend right, "t(z)"'] \\ & \Leftarrow \bar{t}(\gamma_1) & & {} \arrow[urr, "\mathrm{tra}_{\mathbb{1}}(\gamma_2')"'] \arrow[d, "\bar{t}(y')"] & & \Downarrow \bar{t}(\gamma_2) & \\ & & & {} \arrow[uulll, "\mathrm{tra}(\gamma_1')"] \arrow[uurrr, "\mathrm{tra}(\gamma_2')"'] & & & \end{tikzcd}$$

4.3.2 Not-necessarily trivializable 2-transport

Definition 26 *We can use the trivialization morphism and its adjoint as defined in def. 4.2 to assign to every 2-morphism*

$$\begin{tikzcd} A \arrow[r, bend left=50, "(A\otimes V_1{,}\phi_1)"] \arrow[r, bend right=50, "(A\otimes V_2{,}\phi_2)"'] \arrow[r, phantom, "\Downarrow \rho"] & B \end{tikzcd}$$

in **LFBiMod**$(\mathcal{C})$ *the 2-morphism*

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left=50, "V_1\otimes B", ""{name=U, below}] \arrow[r, bend right=50, "A\otimes V_2"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "\tilde{\rho}"] & \mathbb{1} \end{tikzcd} \equiv \begin{tikzcd} \mathbb{1} \arrow[r, "V_1"] \arrow[d, "{}_{\mathbb{1}}L_A"'] \arrow[dr, phantom, "\overset{\phi_1}{\swarrow}"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_B"] \\ A \arrow[r, bend left=40, "{(A\otimes V_1,\phi_1)}", ""{name=P, below}] \arrow[r, bend right=40, "{(A\otimes V_2,\phi_2)}"', ""{name=Q}] \arrow[Rightarrow, from=P, to=Q, "\rho"] \arrow[d, "{}_A R_{\mathbb{1}}"'] \arrow[dr, phantom, "\overset{\mathrm{Id}}{\swarrow}"] & B \arrow[d, "{}_B R_{\mathbb{1}}"] \\ \mathbb{1} \arrow[r, "V_2"'] & \mathbb{1} \end{tikzcd}$$

in $\Sigma(\mathcal{C})$.

Definition 27 *We make*

$$\mathbb{1} \xrightarrow{V_1\otimes B} \mathbb{1} \;=\; \mathbb{1} \xrightarrow{V_1} \mathbb{1} \xrightarrow{{}_{\mathbb{1}}L_B} B \xrightarrow{{}_B R_{\mathbb{1}}} \mathbb{1}$$

into an internal A-B-bimodule by using the obvious right action by B and left action by A. More precisely, the left A-action is that given by

$$\begin{tikzcd}[column sep=small] A \arrow[dr] & & V_1\otimes B \arrow[dl] \\ & A\otimes V_1\otimes B \arrow[dl, "\bar{\phi_1}\otimes B"] & \\ V_1\otimes B\otimes B \arrow[d, "V_1\otimes m_B"] & & \\ V_1\otimes B & & \end{tikzcd} = \begin{tikzcd}[column sep=small] \mathbb{1} \arrow[rrrr, bend left=40, "A", ""{name=T, below}] \arrow[rr, "{}_{\mathbb{1}}L_A"] \arrow[dr, "V_1"] \arrow[dddrrrrrrr, bend right=50, "V_1\otimes B"'] & & A \arrow[Rightarrow, from=T, "\mathrm{Id}"] \arrow[rr, "{}_A R_{\mathbb{1}}"] \arrow[dr, "{(A\otimes V_1,\phi_1)}"] & & \mathbb{1} \arrow[dr, "V_1"] \arrow[dddrrr, bend left=40, "V_1\otimes B"] & & & \\ & \mathbb{1} \arrow[u, phantom, "\overset{\bar{\phi}_1}{\swarrow}"] \arrow[rr, "{}_{\mathbb{1}}L_B"'] & & B \arrow[u, phantom, "\overset{\mathrm{Id}}{\swarrow}"] \arrow[rr, "{}_B R_{\mathbb{1}}"'] \arrow[drrr, bend right, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{e_B}{\swarrow}"] \arrow[dd, phantom, "\overset{\mathrm{Id}}{\swarrow}"] & & \mathbb{1} \arrow[dr, "{}_{\mathbb{1}}L_B"] \arrow[r, phantom, "\overset{\mathrm{Id}}{\swarrow}"] & & \\ & & & & & & B \arrow[dr, "{}_B R_{\mathbb{1}}"] & \\ & & & & & & & \mathbb{1} \end{tikzcd}$$

We make

$$\mathbb{1} \xrightarrow{A\otimes V_2} \mathbb{1} \;=\; \mathbb{1} \xrightarrow{{}_{\mathbb{1}}L_A} A \xrightarrow{{}_A R_{\mathbb{1}}} \mathbb{1} \xrightarrow{V_1} \mathbb{1}$$

into an internal A-B-bimodule by using the obvious right action by B and left action by A. More precisely, the right B-action is that given by [. . .].

Proposition 19 *With the A-B-bimodule structure on $V_1\otimes B$ and $A\otimes V_2$ as defined above,* $\begin{tikzcd} \mathbb{1} \arrow[r, bend left=50, "V_1\otimes B", ""{name=U, below}] \arrow[r, bend right=50, "A\otimes V_2"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "\tilde{\rho}"] & \mathbb{1} \end{tikzcd}$ *(def. 26) is indeed a bimodule homomorphism.*

Proof. We need the equality

$$\begin{tikzcd}
A \arrow[r, "{}_{\mathbb{1}}R_A"] \arrow[dr, "\mathrm{Id}"'] & \mathbb{1} \arrow[r, "V_1"] \arrow[d, "{}_A L_{\mathbb{1}}"'] \arrow[dl, phantom, "\Swarrow e"] & \mathbb{1} \arrow[d, "{}_B L_{\mathbb{1}}"] \arrow[dl, phantom, "\Swarrow \phi_1"] \\
& A \arrow[r, "(A\otimes V_1{,}\phi_1)"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "{}_{\mathbb{1}}R_A"] \arrow[d, "(A\otimes V_1{,}\phi_1)"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & \mathbb{1} \arrow[d, "V_1"] \\
B \arrow[r, "{}_{\mathbb{1}}R_B"] \arrow[dr, "\mathrm{Id}"'] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_B"] \arrow[dl, phantom, "\Swarrow e"] \\
& B
\end{tikzcd}$$

which is readily seen to be equivalent to the tin can equation in item 5 on p. 59. Using this equation, we get

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "{}_{\mathbb{1}}L_A"] & A \arrow[r, "{}_A R_{\mathbb{1}}"] \arrow[dr, "\mathrm{Id}"'] & \mathbb{1} \arrow[r, "V_1"] \arrow[d, "{}_{\mathbb{1}}L_A"'] \arrow[dl, phantom, "e\,\Swarrow"] & \mathbb{1} \arrow[d, "{}_{\mathbb{1}}L_B"] \arrow[dl, phantom, "\phi_1\,\Swarrow"] \\
& & A \arrow[r, bend left, "(A\otimes V_1{,}\phi_1)"{name=U}] \arrow[r, bend right, "(A\otimes V_2{,}\phi_2)"'{name=D}] \arrow[d, "{}_A R_{\mathbb{1}}"'] \arrow[Rightarrow, from=U, to=D, "\rho"] & B \arrow[d, "{}_B R_{\mathbb{1}}"] \\
& & \mathbb{1} \arrow[r, "V_2"'] \arrow[ur, phantom, "\mathrm{Id}\,\Swarrow"] & \mathbb{1}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
\mathbb{1} \arrow[rr, "{}_{\mathbb{1}}L_A"] \arrow[dr, "V_1"] \arrow[dd, "{}_{\mathbb{1}}L_A"'] & & A \arrow[rr, "{}_A R_{\mathbb{1}}"] \arrow[dr, "(A\otimes V_1{,}\phi_1)"] \arrow[d, phantom, "\Swarrow \bar{\phi}_1"] & & \mathbb{1} \arrow[dr, "V_1"] \arrow[d, phantom, "\Swarrow \mathrm{Id}"] & & \\
& \mathbb{1} \arrow[rr, "{}_{\mathbb{1}}L_B"] \arrow[dl, phantom, "\phi_1\,\Swarrow"] & & B \arrow[rr, "{}_B R_{\mathbb{1}}"] \arrow[drrr, bend right, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow \rho"] & & \mathbb{1} \arrow[dr, "{}_{\mathbb{1}}L_B"] \arrow[d, phantom, "e_B\,\Swarrow"] & \\
A \arrow[urrr, bend right, "(A\otimes V_1{,}\phi_1)"] \arrow[rrrrrr, bend right, "(A\otimes V_2{,}\phi_2)"] \arrow[d, "{}_A R_{\mathbb{1}}"'] & & & {} & & {} & B \arrow[d, "{}_B R_{\mathbb{1}}"] \\
\mathbb{1} \arrow[rrrrrr, "V_2"'] & & & \Swarrow \mathrm{Id} & & & \mathbb{1}
\end{tikzcd} ,$$

where in the top left corner we have inserted the identity 2-morphism

$$\begin{tikzcd}
& A \arrow[dr, "(A\otimes V_1{,}\phi_1)"] \arrow[d, phantom, "\Downarrow \bar{\phi}_1"] & \\
\mathbb{1} \arrow[ur, "{}_{\mathbb{1}}L_A"] \arrow[r, "V_1"] \arrow[dr, "{}_{\mathbb{1}}L_A"'] & \mathbb{1} \arrow[r, "{}_{\mathbb{1}}L_B"] \arrow[d, phantom, "\Downarrow \phi_1"] & B \\
& A \arrow[ur, "(A\otimes V_1{,}\phi_1)"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{1} \arrow[r, "{}_{\mathbb{1}}L_A"] & \mathbb{1} \arrow[r, "(A\otimes V_1{,}\phi_1)"] & B
\end{tikzcd} .$$

But according to the definition def. 27 of the left A-action on $V_1 \otimes B$ this says nothing but that $\tilde{\rho}$ respects the left A-action.

A precisely analogous argument applies to the right B-action. $\square$

Remark. By the very definition of **BiMod**$(\mathcal{C})$, the 2-morphism

$$\begin{tikzcd}
A \arrow[r, bend left=50, "{(A\otimes V_1,\phi_1)}", ""{name=U, below}] \arrow[r, bend right=50, "{(A\otimes V_2,\phi_2)}"', ""{name=D}] & B
\arrow[Rightarrow, from=U, to=D, "\rho"]
\end{tikzcd}$$

is an internal homomorphism of bimodules internal to $\mathcal{C}$. Above we have constructed (def. 26) a 2-morphism

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left=50, "V_1\otimes B", ""{name=U, below}] \arrow[r, bend right=50, "A\otimes V_2"', ""{name=D}] & \mathbb{1}
\arrow[Rightarrow, from=U, to=D, "\tilde{\rho}"]
\end{tikzcd}$$

in $\Sigma(\mathcal{C})$ by composing ρ with the trivialization data obtained in §4.2. Remarkably, $\tilde{\rho}$ is *not* quite the same internal bimodule homomorphism as ρ, it does not even relate the same internal bimodules. But the difference between the two is small. The source bimodule of $\tilde{\rho}$ is obtained from the source bimodule of ρ by acting with the isomorphism $\bar{\phi}_1$. This is a direct consequence of the fact that our trivialization data had to contain this isomorphism in order to constitute a trivialization of the transport 2-functors in theorem refproposition on trivializations of 2-transport.

This has the following interesting consequence. Recall that we used only left-induced bimodules in **BiMod**$(\mathcal{C})$, because **LFBiMod**$(\mathcal{C})$ is precisely large enough to accomodate an ambidextrous adjunction realizing every Frobenius algebra in $\mathcal{C}$. But by sending these left-induced bimodules and their homomorphisms to $\mathcal{C}$ by means of our trivialization, some *right*-free bimodules appear automatically. In particular, all homomorphisms of left-induced bimodules become, as shown above, homomorphism between one right-free and one left-induced bimdoule.

This is important, because precisely these latter types of bimodule homomorphisms do appear in FRS formalism (e.g. p.5 of [6]), where they encode the insertion of bulk fields. We will see in example §**??** that this is precisely reproduced by locally trivialized 2-transport.

4.4 Boundary Trivialization of 2-Transport

Precisely at the boundary of the surface whose 2-transport we want to compute there is another possibility to express it in terms of trivial 2-transport.

Suppose we are given an adjunction

$$\begin{tikzcd}
\mathrm{tra}_{\mathbb{1}} \arrow[r, bend left=50, "\bar{b}"] \arrow[r, phantom, shift left=2, "\overset{i}{\Rightarrow}"] \arrow[r, phantom, shift right=2, "\underset{e}{\Rightarrow}"] & \mathrm{tra} \arrow[l, bend left=50, "b"]
\end{tikzcd}$$

not necessarily a special ambedextrous one.

The pseudonatural transformation b gives rise to a tin can equation

$$\begin{tikzcd}
A_x \arrow[r, bend left=60, "\mathrm{tra}(\gamma_1)"] \arrow[r, phantom, bend left=30, "\Downarrow \mathrm{tra}(S)"] \arrow[r, "\mathrm{tra}(\gamma_2)" description] \arrow[d, "{}_{A_x}b(x)_{\mathbb{1}}"'] & A_y \arrow[d, "{}_{A_y}b(y)_{\mathbb{1}}"] \arrow[dl, phantom, "\swarrow b(\gamma_2)"] \\
\mathbb{1} \arrow[r, "\mathrm{tra}(\gamma_2)" description] & \mathbb{1}
\end{tikzcd}
=
\begin{tikzcd}
A_x \arrow[r, "\mathrm{tra}(\gamma_1)" description] \arrow[d, "{}_{A_x}b(x)_{\mathbb{1}}"'] & A_y \arrow[d, "{}_{A_y}b(y)_{\mathbb{1}}"] \arrow[dl, phantom, "\swarrow b(\gamma_1)"] \\
\mathbb{1} \arrow[r, "\mathrm{tra}_{\mathbb{1}}(\gamma_1)" description] \arrow[r, phantom, bend right=30, "\Downarrow \mathrm{tra}_{\mathbb{1}}(S)"] \arrow[r, bend right=60, "\mathrm{tra}_{\mathbb{1}}(\gamma_2)"'] & \mathbb{1}
\end{tikzcd}
.$$

An analogous statement holds for non-invertible morphism

$$\mathrm{tra}_{\mathbb{1}} \xrightarrow{\bar{b}} \mathrm{tra} \; .$$

Therefore, assigning 1-sided A-modules to the boundaries of a surface allows to completely express the surface transport, which originally takes values in $\mathbf{BiMod}(\mathcal{C})$, in terms of 2-morphisms in $\mathcal{C}$.

4.5 Defect lines from trivialized bimodules

A *defect line* in 2-dimensional quantum field theory is a line labeled by an object in a monoidal category $\mathcal{C}$ which is equipped with the structure of an internal bimodule.

For our purposes it is crucial to carefully distinguish the defect line itself from the bimodule object $\dot{K}$ it is labeled by, and to distinguish that object, in turn, from the corresponding morphism K in the category $\mathrm{Bim}(\mathcal{C})$.

We now discuss how the defect line is to be thought of as arising from the $(\,\Sigma\mathcal{C} \hookrightarrow \mathrm{Bim}(\mathcal{C}))$ -trivialization of the functor which assigns the bimodule K to edges.

Proposition 20 *Let $D = \mathrm{par}_2$ be the 1-categorical interval*

$$\mathrm{par}_2 := \{\, \bullet_1 \longrightarrow \bullet_2 \,\}$$

and let $\mathcal{C}$ be any monoidal category. Then a sufficient condition for a 2-functor

$$\mathrm{tra}_q : D \to \mathrm{Bim}(\mathcal{C})$$

to be $(\,\Sigma\mathcal{C} \overset{i}{\hookrightarrow} \mathrm{Bim}(\mathcal{C})\,)$-trivializable is that it sends $\bullet_1$ and $\bullet_2$ to special Frobenius algebra objects in $\mathcal{C}$.

Proof. Write

$$\mathrm{tra}_q(\, \bullet_1 \longrightarrow \bullet_2 \,) = \; A \xrightarrow{N} B$$

for the value of our 2-functor on the edge. *Choose* two special ambidextrous adjunctions

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "L_A"] \arrow[rr, bend left=60, "\mathrm{Id}", ""{name=U, below}] \arrow[rr, bend right=60, "\mathrm{Id}"', ""{name=D, above}] & A \arrow[r, "R_A"] & \mathbb{1} & = & \mathbb{1} \arrow[r, "\mathrm{Id}"] & \mathbb{1}
\arrow[from=U, to=1-2, Rightarrow]
\arrow[from=1-2, to=D, Rightarrow]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "L_B"] \arrow[rr, bend left=60, "\mathrm{Id}", ""{name=U, below}] \arrow[rr, bend right=60, "\mathrm{Id}"', ""{name=D, above}] & B \arrow[r, "R_B"] & \mathbb{1} & = & \mathbb{1} \arrow[r, "\mathrm{Id}"] & \mathbb{1}
\arrow[from=U, to=1-2, Rightarrow]
\arrow[from=1-2, to=D, Rightarrow]
\end{tikzcd}$$

corresponding to the special Frobenius algebras A and B, respectively. We write

$$\dot{K}_{A,B} := \dot{K} := \mathbb{1} \xrightarrow{L_A} A \xrightarrow{K} B \xrightarrow{R_B} \mathbb{1} \ ,$$

for the corresponding object part of K relative to these chosen ambijunctions.

Let the i-trivial 2-functor be given by

$$\mathrm{triv}_q : (\ \bullet_1 \longrightarrow \bullet_2\ ) \ \mapsto \ \mathbb{1} \xrightarrow{\dot{K}} \mathbb{1} \ .$$

Take the component map of $t : \mathrm{tra}_q \to \mathrm{triv}_q$ to be

$$\begin{tikzcd}
A \arrow[r, "K"] \arrow[d, "R_A"'] & B \arrow[d, "R_B"] \arrow[dl, Rightarrow, "t(\bullet_1\to\bullet_2)"' description] & := & A \arrow[r, "\mathrm{Id}", ""{name=I, below}] \arrow[d, "R_A"'] & A \arrow[r, "K", ""{name=K1, below}] & B \arrow[d, "R_B"] \\
\mathbb{1} \arrow[r, "\dot{K}"'] & \mathbb{1} & & \mathbb{1} \arrow[ur, "L_A"] \arrow[rr, "\dot{K}"', ""{name=K2, above}] & & \mathbb{1}
\arrow[from=I, to=2-4, Rightarrow]
\arrow[from=K1, to=K2, Rightarrow, "="]
\end{tikzcd}$$

and that of $\tilde{t} : \mathrm{triv}_q \to \mathrm{tra}_q$ to be

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\dot{K}"] \arrow[d, "L_A"'] & \mathbb{1} \arrow[d, "L_B"] \arrow[dl, Rightarrow, "\tilde{t}(\bullet_1\to\bullet_2)"' description] & := & \mathbb{1} \arrow[rr, "\dot{K}", ""{name=K1, below}] \arrow[d, "L_A"'] & & \mathbb{1} \arrow[d, "L_B"] \\
A \arrow[r, "K"'] & B & & A \arrow[r, "K"', ""{name=K2, above}] & B \arrow[ur, "R_B"] \arrow[r, "\mathrm{Id}"'] & B
\arrow[from=K1, to=K2, Rightarrow, "="]
\arrow[from=1-6, to=2-6, Rightarrow, shift right=3]
\end{tikzcd} \ .$$

Then we have

$$\begin{tikzcd}
A \arrow[r, "\mathrm{Id}"] \arrow[d, "R_A"'] & A \arrow[rr, "K"] & {} & B \arrow[d, "R_B"] \arrow[dd, bend left=60, "\mathrm{Id}"] & & & A \arrow[r, "K"] \arrow[dl, "L_A"'] \arrow[dd, "\mathrm{Id}" description] & B \arrow[dd, "\mathrm{Id}"] \arrow[ddl, Rightarrow, "="] \\
\mathbb{1} \arrow[ur, "L_A"] \arrow[rrr, "\dot{K}" description] \arrow[d, "L_A"'] & {} & & \mathbb{1} \arrow[d, "L_B"] \arrow[r, phantom, "\Leftarrow"] & {=} & \mathbb{1} \arrow[dr, "L_A"'] \arrow[r, phantom, "\Leftarrow"] & {} & \\
A \arrow[rr, "K"'] & {} & B \arrow[r, "\mathrm{Id}"'] \arrow[ur, "R_B"] & B & & & A \arrow[r, "K"'] & B
\arrow[from=1-2, to=2-1, Rightarrow, shorten=3mm]
\arrow[from=1-3, to=2-2, Rightarrow, "="]
\arrow[from=2-2, to=3-2, Rightarrow, "="]
\arrow[from=2-4, to=3-4, Rightarrow, shift right=2, shorten=2mm]
\end{tikzcd}$$

and

$$\begin{tikzcd}
{} & A \arrow[r, "\mathrm{Id}"] \arrow[d, "R_A"'] \arrow[dd, bend right=60, "\mathrm{Id}"'] & A \arrow[rr, "K"] & {} & B \arrow[d, "R_B"] & & A \arrow[r, "K"] \arrow[dd, "\mathrm{Id}"'] & B \arrow[dd, "\mathrm{Id}" description] \arrow[dr, "R_b"] \arrow[ddl, Rightarrow, "="] & \\
{} \arrow[r, phantom, "\Leftarrow"] & \mathbb{1} \arrow[ur, "L_A"] \arrow[rrr, "\dot{K}" description] \arrow[d, "L_A"'] & {} & & \mathbb{1} \arrow[d, "L_B"] & {=} & & {} \arrow[r, phantom, "\Leftarrow"] & \mathbb{1} \arrow[dl, "L_B"] \\
{} & A \arrow[rr, "K"'] & {} & B \arrow[r, "\mathrm{Id}"'] \arrow[ur, "R_B"] & B & & A \arrow[r, "K"'] & B &
\arrow[from=1-3, to=2-2, Rightarrow, shorten=3mm]
\arrow[from=1-4, to=2-3, Rightarrow, "="]
\arrow[from=2-3, to=3-3, Rightarrow, "="]
\arrow[from=2-5, to=3-5, Rightarrow, shift right=2, shorten=2mm]
\end{tikzcd}$$

□

Corollary 1 *The identity on K may be re-expressed as*

$$\begin{tikzcd}
A \arrow[r, "K"] \arrow[dd, "\mathrm{Id}"'] & B \arrow[dd, "\mathrm{Id}"] \arrow[ddl, Rightarrow, "="] & & {} & A \arrow[r, "\mathrm{Id}"] \arrow[d, "R_A"'] \arrow[dd, bend right=60, "\mathrm{Id}"'] & A \arrow[rr, "K"] & {} & B \arrow[d, "R_B"] \arrow[dd, bend left=60, "\mathrm{Id}"] & {} \\
& & {=} & {} \arrow[r, phantom, "\Leftarrow"] & \mathbb{1} \arrow[ur, "L_A"] \arrow[rrr, "\dot{K}" description] \arrow[d, "L_A"'] & {} & & \mathbb{1} \arrow[d, "L_B"] \arrow[r, phantom, "\Leftarrow"] & {} \\
A \arrow[r, "K"'] & B & & {} & A \arrow[rr, "K"'] & {} & B \arrow[r, "\mathrm{Id"'] \arrow[ur, "R_B"] & B & {}
\arrow[from=1-6, to=2-5, Rightarrow, shorten=3mm]
\arrow[from=1-7, to=2-6, Rightarrow, "="]
\arrow[from=2-6, to=3-6, Rightarrow, "="]
\arrow[from=2-8, to=3-8, Rightarrow, shift right=2, shorten=2mm]
\end{tikzcd} .$$

5 Correlators from differential B$\mathcal{C}$-cocycles

5.1 Field Insertions passing Triangulation Lines

When locally trivializing transport 2-functors one frequently encounters 2-morphisms of the form

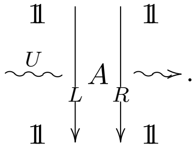
$$\begin{tikzcd}
\mathbb{1} \arrow[r, "U"] \arrow[d, "L"'] \arrow[dr, phantom, "\swarrow \phi"] & \mathbb{1} \arrow[d, "L"] \\
A \arrow[r, "{(A\otimes U,\phi)}"] \arrow[d, "R"'] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & A \arrow[d, "R"] \\
\mathbb{1} \arrow[r, "U"'] & \mathbb{1}
\end{tikzcd} .$$

These come from the tin can faces of the pseudonatural transformations $\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathbb{1}}$ and $\mathrm{tra}_{\mathbb{1}} \xrightarrow{\bar{t}} \mathrm{tra}$ which have been introduced in items 2 and 3 of §4.2.

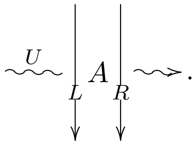
Assume that ϕ is the braiding

$$\phi = c_{U,A}$$

with U passing beneath A. Then the Poincaré dual to this diagram looks as follows (when rotated by $\pi/2$)



Here the bimodules $L \equiv {}_{\mathbb{1}}L_A$ and $R \equiv {}_A R_{\mathbb{1}}$ (introduced in def. 17) combine to yield the algebra object A regarded as a $\mathbb{1}-\mathbb{1}$-bimodule, as described in §3.4. We shall often suppress the symbol "$\mathbb{1}$" from string diagrams, such that all unlabelled regions are implicitly to be thought of as labelled by $\mathbb{1}$:



Here the U-line is supposed to pass beneath the A-line, depicting the braiding morphism

$$\begin{tikzcd}
U \otimes A \arrow[d, "c_{U,A}"] \\
A \otimes U
\end{tikzcd} .$$

That this does indeed represent the above globular diagram follows by applying the rules for horizontal and vertical compositon of morphism of left-induced bimodules.

5.2 Traces

Of particular interest is n-transport over n-paths of **nontrivial topology**, those which are not isomorphic to an n-disk. Describing transport $\mathrm{tra} : \mathcal{P} \to T$ over such n-paths in terms of n-morphisms of a geometric n-category requires certain structure at least on the codomain T, possibly also on the domain $\mathcal{P}$.

The structure needed on T is the existence of **partial traces** which implement the gluing of n-paths along $(n-1)$-paths. This gluing may, or may not, be already present in $\mathcal{P}$.

In Segal's description of n-dimensional QFT in terms of 1-functors on 1-categories of n-cobordisms this is not a seperate issue, since the cobordisms may have arbitrary topology. The n-categorical refinement which we are considering here, however, requires a framework which allows to construct topologically nontrivial n-cobordisms by gluing topologically trivial n-morphisms.

Dimension $n = 2$. Let $\mathcal{P}$ be some geometric 2-category. Assume that $\mathcal{P}$ has the following special properties

1. Every 1-morphism $x \xrightarrow{\gamma} y$ is part of an ambidextrous adjunction.
2. All the monoidal 1-categories $\mathrm{Hom}_{\mathcal{P}}(x,x)$ are braided.

5.2.1 Sphere

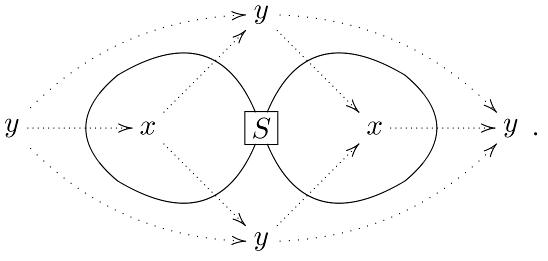
Consider a 2-morphism

$$\begin{tikzcd}
 & y \arrow[dr, "B"] & \\
x \arrow[ur, "A"] \arrow[dr, "A"'] & \Downarrow S & x \\
 & y \arrow[ur, "B"'] &
\end{tikzcd}$$

in $\mathcal{P}$. Glue the two copies of A and the two copies of B by composing with unit and counit of the respective adjunctions.

$$\begin{tikzcd}
 & & y \arrow[dr, "B"] \arrow[drr, bend left, "\mathrm{Id}"] & & \\
y \arrow[r, "\bar{A}"] \arrow[urr, bend left, "\mathrm{Id}"] \arrow[drr, bend right, "\mathrm{Id}"'] & x \arrow[ur, "A"] \arrow[dr, "A"'] & \Downarrow S & x \arrow[r, "\bar{B}"] & y \\
 & & y \arrow[ur, "B"'] \arrow[urr, bend right, "\mathrm{Id}"'] & &
\end{tikzcd}$$

The Poincaré-dual string diagram is



5.2.2 Torus

Consider a 2-morphism

$$\begin{tikzcd}
 & x \arrow[dr, "B"] & \\
x \arrow[ur, "A"] \arrow[dr, "B"'] & \Downarrow S & x \\
 & x \arrow[ur, "A"'] &
\end{tikzcd}$$

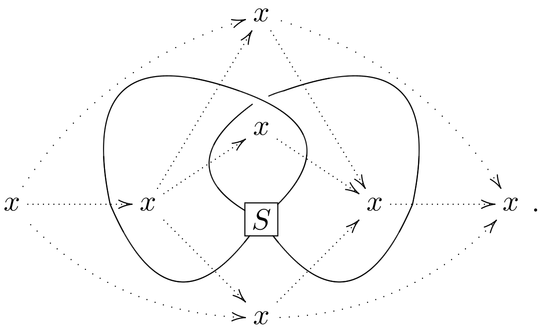
in $\mathcal{P}$. In order to be able to glue A with A and B with B, first move them on the same side by composing with a braiding

$$\begin{tikzcd}
 & x \arrow[dddr, "A"] & \\
 & \Downarrow & \\
 & x \arrow[dr, "B"] & \\
x \arrow[uuur, "B"] \arrow[ur, "A"] \arrow[dr, "B"'] & \Downarrow S & x\ . \\
 & x \arrow[ur, "A"'] &
\end{tikzcd}$$

Then glue by composing with unit and counit of the respective adjunctions.

$$\begin{tikzcd}
 & & x \arrow[dddrr, bend left, "\mathrm{Id}"] \arrow[dddr, "A"] & & \\
 & \Searrow & \Downarrow & \Swarrow & \\
 & & x \arrow[dr, "B"] & & \\
x \arrow[uuurr, bend left, "\mathrm{Id}"] \arrow[r, "\bar{B}"] \arrow[ddrr, bend right, "\mathrm{Id}"'] & x \arrow[uuur, "B"] \arrow[ur, "A"] \arrow[ddr, "B"'] & \Downarrow S & x \arrow[r, "\bar{A}"] & x \\
 & \Swarrow & & \Searrow & \\
 & & x \arrow[uur, "A"'] \arrow[uurr, bend right, "\mathrm{Id}"'] & &
\end{tikzcd}$$

The Poincaré-dual string diagram is

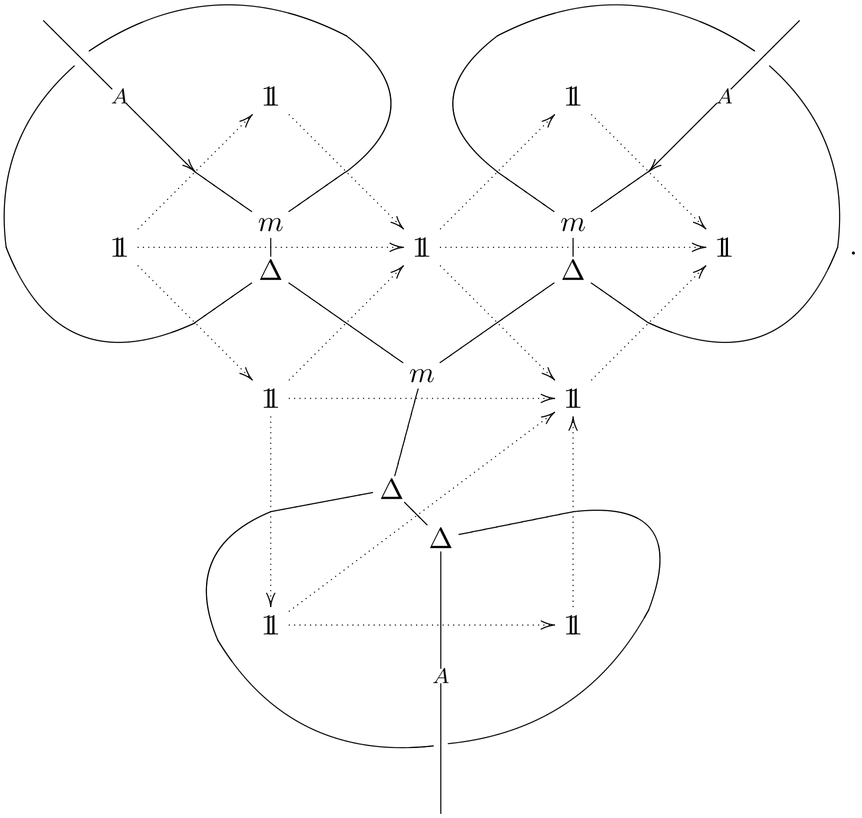


5.2.3 Trinion (Pair of Pants)

Consider the pair of pants

$$\begin{tikzcd}
 & x \arrow[dr, "A"] & & z \arrow[dr, "\gamma_2"] & \\
x \arrow[ur, "\gamma_1"] \arrow[dr, "A"'] & \Downarrow & w \arrow[ur, "B"] \arrow[dr] & \Downarrow & z \; . \\
 & w \arrow[ur] \arrow[rr] \arrow[d, "C"'] \arrow[drr, phantom, "\Downarrow"] & \Downarrow & w \arrow[ur, "B"'] & \\
 & z \arrow[rr] & & z \arrow[u, "\bar{C}"'] &
\end{tikzcd}$$

With the structure described above we cannot do the required braiding in order to contract identitfied boundaries. But we may consider the image under some 2-transport of this 2-morphism in a braided tensor category (possibly obtained by first locally trivializing) and then braid and trace in that image. For instance, for a trivialization as in [**?**] this yields



5.3 The disk

the structure of a boundary field insertion in *combinatorial CFT*, as a string diagram in $\mathcal{C}$:

- A and B are Frobenius algebras labelling the "phases" of the CFT;
- $U \in \mathcal{C}$ labels the boundary field;
- the left B-module N and the left A-module N' label the boundary conditions ("D-branes");
- M is a defect line interpolating between a B- and an A-phase;
- the yellow coupon is a module homomorphism from the left A-module $M \otimes_B N \otimes U$ to the left A-module N'

noticing that a left B-module is a B-$\mathbb{1}$ bimodule and passing to the Poincaré-dual of the above diagram exhibits it is a 2-morphism in $\mathrm{Bimod}(\mathcal{C}) \subset \mathrm{TwBimod}(\mathcal{C})$.

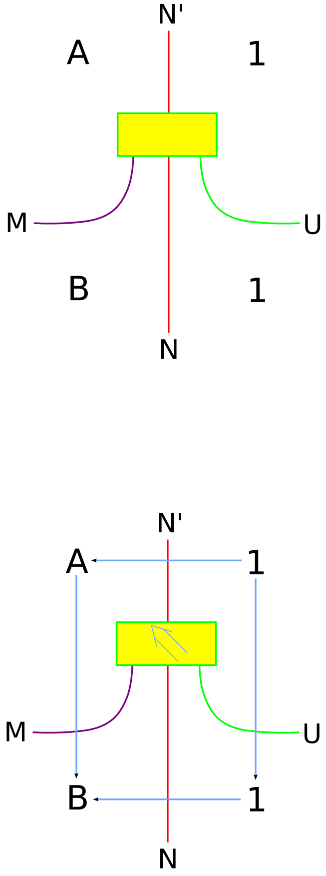


Figure 3: **A boundary field insertion** as a string diagram in $\mathcal{C}$ and as a 2-morphism in $\mathrm{TwBimod}(\mathcal{C})$.

5.3.1 Quantum n-Particle concept formation

We establish some terminology, useful for describing the situation which we want to look at.

The open 2-particle is the 2-category $\mathrm{par}_2 = \{ \bullet_1 \longrightarrow \bullet_2 \}$. A 2-functor

$$\mathrm{tra}_q : \mathrm{par}_2 \to \mathrm{Bim}(\mathcal{C}) \subset 2\mathrm{Vect}_{\mathcal{C}}$$

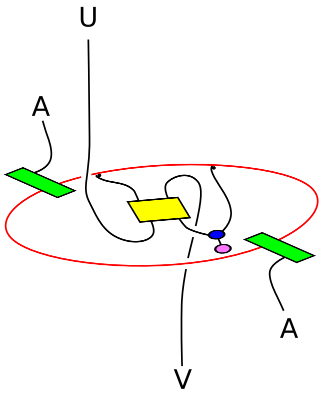
is a *2-space of states* of the 2-particle, where the algebras

$$A := \mathrm{tra}_q(\bullet_1)$$

and

$$B := \mathrm{tra}_q(\bullet_2)$$

are to be thought of, under the embedding $\mathrm{Bim}(\mathcal{C}) \hookrightarrow 2\mathrm{Vect}_{\mathcal{C}}$ as the 2-vector space of states over the endpoints of the string.



- the black line in the center plane is a transition line labelled by a Forbenius algebra A;
- U and V are the two chiral labels of a bulk field insertion;
- the yellow coupon is a homomorphism of induced A-A bimodules;
- the red line, encoding the boundary condition at the boundary of the disk (the "D-brane") is labelled by an A-module.

Figure 4: **The disk correlator** as a string diagram in $\mathcal{C}$.

A morphism

$$U : \mathrm{tra}_q \to \mathrm{tra}_q$$

or, more generally

$$U : \mathrm{tra}_q \to \mathrm{tra}'_q$$

is a *propagator* of the 2-particle over a strip ("time evolution operator").

We shall concentrate on a propagator whose component map is a bigon

$$U(\bullet_1 \longrightarrow \bullet_2) \;:=\; \begin{tikzcd} A \arrow[rr, bend left=50, "K"{name=U}] \arrow[rr, bend right=50, "K'"'{name=D}] & & B \arrow[Rightarrow, from=U, to=D, "\rho"] \end{tikzcd}$$

in $\mathrm{Bim}(\mathcal{C})$ with A and B special Frobenius algebras.

For $I : \mathrm{par}_2 \to \mathrm{Bim}(\mathcal{C})$ an $(\,\Sigma\mathcal{C} \xhookrightarrow{i} \mathrm{Bim}(\mathcal{C})\,)$-trivial transport, a (Schrö)dinger *state* of a the 2-particle is a morphism

$$|\psi\rangle : I \to \mathrm{tra}_q\,.$$

Its component map is hence a 2-cell in $\mathrm{Bim}(\mathcal{C})$ of the form

$$|\psi\rangle(\bullet_1 \longrightarrow \bullet_2) \;=\; \begin{tikzcd} \mathbb{1} \arrow[r, "H"] \arrow[d, "N_A"'] & \mathbb{1} \arrow[d, "N_B"] \arrow[dl, Rightarrow, "\psi"] \\ A \arrow[r, "K"'] & B \end{tikzcd}\;.$$

The modules N_A and N_B are called the *D-branes* to which the endpoints of the 2-particle in this state are attached.

We assume $\mathcal{C}$ to be rigid and hence to have duals on objects. Then the state of the 2-particle has an adjoint state

$$\langle\psi| : \mathrm{tra}_q \to I$$

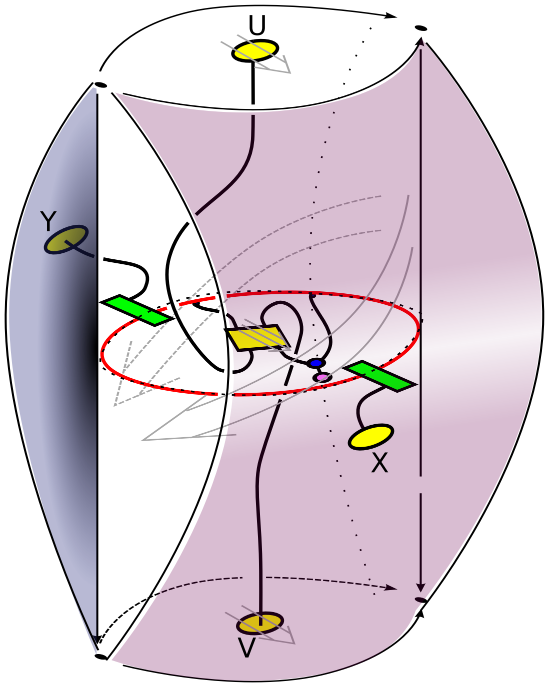


Figure 5: **The disk correlator** schematically as a pasting diagram of cylinders in $\mathbf{B}\mathrm{Bimod}(\mathcal{C})$ which we may interpret as a pasting diagram of 2-morphism in $\mathrm{TwBimod}(\mathcal{C})$ by projecting the cylinders onto their equatorial plane.

given by the component map

$$\langle\psi|\left(\bullet_1 \longrightarrow \bullet_2 \right) \;=\; \begin{tikzcd} A \arrow[r, "K"] \arrow[d, "N_A^\vee"'] & B \arrow[d, "N_B^\vee"] \arrow[dl, Rightarrow, "\psi^\dagger"] \\ \mathbb{1} \arrow[r, "H^\vee"'] & \mathbb{1} \end{tikzcd} .$$

A *Heisenberg state* is a functor

$$\phi : \mathrm{End}(\mathrm{tra}_q) \to \mathrm{End}(I)$$

which sends operators on the space of states to correlators.

The Hom-functor sends Schrödinger states to the corresponding Heisenberg states.

$$\mathrm{Hom}\left(\begin{tikzcd} \mathbb{1} \arrow[d, "|\psi_1\rangle"'] \\ \mathrm{tra}_q \end{tikzcd} , \begin{tikzcd} \mathbb{1} \\ \mathrm{tra}_q \arrow[u, "\langle\psi_2|"] \end{tikzcd} \right) \;:\; \mathrm{End}(\mathrm{tra}_q) \to \mathrm{End}(I)\,.$$

(Here we are, for simplicity, ignoring some details, like the freedom to have different tra_q and I and the issue of whether and how to identify $H \simeq H^\vee$).

We take

$$\begin{tikzcd} \mathrm{End}_0(I) \arrow[r, hook, "j"] & \mathrm{End}(I) \end{tikzcd}$$

to be the sub-category whose objects are bigons (instead of rectangles) and demand that the Hom-pairing has a j-trivialization

$$\begin{tikzcd} \mathrm{End}(\mathrm{tra}_q) \arrow[r, "="] \arrow[d, "\mathrm{corr}(\psi_1,\psi_2)"'] & \mathrm{End}(\mathrm{tra}_q) \arrow[d, "\mathrm{Hom}(\psi_1,\psi_2)"] \arrow[dl, Rightarrow, "b"'] \\ \mathrm{End}_0(I) \arrow[r, hook, "j"] & \mathrm{End}(I) \end{tikzcd}$$

allowing to solve $\mathrm{corr}(\psi_1, \psi_2)$ for $\mathrm{Hom}(\psi_1, \psi_2)$. The trivializing morphism here is the *boundary condition* on the 2-particle.

5.3.2 The disk correlator from a pairing of 2-states

This way the *correlator* of two 2-states of the 2-particle over the strip in the above setup is a 2-cell in $\mathrm{Bimod}(\mathcal{C})$ of the form

$$\mathrm{corr}(\psi_1,\psi_2) := \langle \psi_2 | U | \psi_1 \rangle_b := \begin{tikzcd}
 & \mathbb{1} \arrow[r, "H"] \arrow[d, "N_A"'] \arrow[dd, bend right=70] & \mathbb{1} \arrow[d, "N_B"] \arrow[dd, bend left=70] \arrow[dl, Rightarrow, "\phi_1"'] & \\
\mathrm{Id} & A \arrow[l, Rightarrow] \arrow[r, bend left, "K", ""{name=U, below}] \arrow[r, bend right, "K'"', ""{name=D, above}] \arrow[d, "N_A^\vee"'] & B \arrow[d, "N_B^\vee"] \arrow[dl, Rightarrow, "\phi_2^*"] & \mathrm{Id} \arrow[l, Rightarrow] \\
 & \mathbb{1} \arrow[r, "H"'] & \mathbb{1} &
\arrow[Rightarrow, from=U, to=D, "\rho"]
\end{tikzcd} . \tag{5}$$

5.3.3 The local trivialization of the disk correlator

We now use local ($\mathbf{B}\mathcal{C} \hookrightarrow \mathrm{Bimod}(\mathcal{C})$)-trivialization of the correlator (5) to rewrite it identically such that its interior becomes a pasting diagram entirely in $\Sigma\mathcal{C}$. The string diagram Poincaré-dual to this globular pasting diagram is the FRS disk diagram for a disk correlator with a bulk field insertion, two boundary field insertions and a defect line.

Given any morphism of bimodules

$$\begin{tikzcd}
A \arrow[r, bend left=50, "K", ""{name=U, below}] \arrow[r, bend right=50, "K'"', ""{name=D, above}] & B
\arrow[Rightarrow, from=U, to=D, "\rho"]
\end{tikzcd}$$

we may, using corollary 1, identically rewrite it as

$$\begin{tikzcd}
A \arrow[rr, bend left=50, "K", ""{name=P}] \arrow[rr, bend right=50, "K'"', ""{name=Q}] & & B
\arrow[Rightarrow, from=P, to=Q, "\rho"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & A & A & & B & \\
& \Longleftarrow & \mathbb{1} & & & \mathbb{1} & \Longleftarrow \\
& A & A & & B & B & \\
\Longleftarrow & \mathbb{1} & & & \mathbb{1} & \Longleftarrow & \\
& A & & B & B & &
\arrow[from=1-3, to=1-4, "\mathrm{Id}", ""{name=I1}]
\arrow[from=1-4, to=1-6, "K", ""{name=T}]
\arrow[from=1-3, to=2-3, "R_A"]
\arrow[from=2-3, to=1-4, "L_A"']
\arrow[from=1-6, to=2-6, "R_B"]
\arrow[from=2-3, to=2-6, "\dot K", ""{name=D1}]
\arrow[from=1-3, to=3-2, bend right=40, "\mathrm{Id}"']
\arrow[from=1-6, to=3-6, bend left=60, "\mathrm{Id}"]
\arrow[from=2-3, to=3-3, "L_A"]
\arrow[from=2-6, to=3-6, "L_B"]
\arrow[from=3-5, to=2-6, "R_B"]
\arrow[from=3-2, to=3-3, "\mathrm{Id}", ""{name=I3}]
\arrow[from=3-3, to=3-5, bend left=40, "K", ""{name=U}]
\arrow[from=3-3, to=3-5, bend right=40, "K'"', ""{name=L}]
\arrow[from=3-5, to=3-6, "\mathrm{Id}"', ""{name=I2}]
\arrow[from=3-2, to=4-2, "R_A"]
\arrow[from=4-2, to=3-3, "L_A"']
\arrow[from=3-5, to=4-5, "R_B"]
\arrow[from=4-2, to=4-5, "\dot K", ""{name=D2}]
\arrow[from=3-2, to=5-2, bend right=50, "\mathrm{Id}"']
\arrow[from=3-6, to=5-5, bend left=30, "\mathrm{Id}"]
\arrow[from=4-2, to=5-2, "L_A"]
\arrow[from=4-5, to=5-5, "L_B"]
\arrow[from=5-4, to=4-5, "R_B"]
\arrow[from=5-2, to=5-4, "K"', ""{name=Bk}]
\arrow[from=5-4, to=5-5, "\mathrm{Id}"', ""{name=I4}]
\arrow[Rightarrow, from=I1, to=2-3]
\arrow[Rightarrow, from=T, to=D1, "="]
\arrow[Rightarrow, from=D1, to=U, "="]
\arrow[Rightarrow, from=U, to=L, "\rho"]
\arrow[Rightarrow, from=L, to=D2, "="]
\arrow[Rightarrow, from=D2, to=Bk, "="]
\arrow[Rightarrow, from=2-6, to=I2]
\arrow[Rightarrow, from=I3, to=4-2]
\arrow[Rightarrow, from=4-5, to=I4]
\end{tikzcd}$$

After performing the same operation inside the correlator (5), one can merge the incoming and outgoing states each with one half of the inserted identity 2-cells:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "H"] \arrow[d, "N_A"'] \arrow[dd, bend right=70, "\mathrm{Id}"'{name=L}] & \mathbb{1} \arrow[d, "N_B"] \arrow[dd, bend left=70, "\mathrm{Id}"{name=R}] \arrow[dl, Rightarrow, "\phi_1"'] \\
A \arrow[r, bend left, "K"{name=U}] \arrow[r, bend right, "K'"'{name=D}] \arrow[d, "N_A^\vee"'] & B \arrow[d, "N_B^\vee"] \arrow[dl, Rightarrow, "\phi_2^*"] \\
\mathbb{1} \arrow[r, "H"'] & \mathbb{1}
\arrow[Rightarrow, from=U, to=D, "\rho"]
\arrow[Rightarrow, from=2-1, to=L]
\arrow[Rightarrow, from=R, to=2-2]
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
& \mathbb{1} \arrow[rrr, "H"] \arrow[d, "N_A"'] \arrow[ddddddl, bend right=70, "\mathrm{Id}"'{name=L0}] & & & \mathbb{1} \arrow[d, "N_B"] \arrow[ddddddl, bend left=50, "\mathrm{Id}"{name=R0}] \arrow[dlll, Rightarrow, "\phi_1"] \\
& A \arrow[r, "\mathrm{Id}"{name=I1}] \arrow[d, "R_A"'] \arrow[ddl, bend right=40, "\mathrm{Id}"'{name=L1}] & A \arrow[rr, "K"{name=K1}] & & B \arrow[d, "R_B"] \arrow[dd, bend left=60, "\mathrm{Id}"{name=R1}] \\
& \mathbb{1} \arrow[ru, "L_A"'] \arrow[rrr, "\dot{K}"{description, name=DK1}] \arrow[d, "L_A"'] & & & \mathbb{1} \arrow[d, "L_B"] \\
A \arrow[r, "\mathrm{Id}"{name=I2}] \arrow[d, "R_A"'] \arrow[dd, bend right=50, "\mathrm{Id}"'{name=L2}] & A \arrow[rr, bend left, "K"{description, name=U}] \arrow[rr, bend right, "K'"{description, name=D}] & & B \arrow[r, "\mathrm{Id}"'{name=I3}] \arrow[d, "R_B"] \arrow[ru, "R_B"] & B \arrow[ddl, bend left=30, "\mathrm{Id}"{name=R2}] \\
\mathbb{1} \arrow[ru, "L_A"'] \arrow[rrr, "\dot{K}"{description, name=DK2}] \arrow[d, "L_A"'] & & & \mathbb{1} \arrow[d, "L_B"] \\
A \arrow[rr, "K"'{name=K2}] \arrow[d, "N_A^\vee"'] & & B \arrow[r, "\mathrm{Id}"'{name=I4}] \arrow[ru, "R_B"] & B \arrow[d, "N_B^\vee"] \arrow[dlll, Rightarrow, "\phi_2^*"] \\
\mathbb{1} \arrow[rrr, "H"'] & & & \mathbb{1}
\arrow[Rightarrow, from=I1, to=3-2]
\arrow[Rightarrow, from=K1, to=DK1, "="]
\arrow[Rightarrow, from=DK1, to=U, "="]
\arrow[Rightarrow, from=3-5, to=I3]
\arrow[Rightarrow, from=U, to=D, "\rho"]
\arrow[Rightarrow, from=I2, to=5-1]
\arrow[Rightarrow, from=D, to=DK2, "="]
\arrow[Rightarrow, from=DK2, to=K2, "="]
\arrow[Rightarrow, from=5-4, to=I4]
\arrow[Rightarrow, from=4-1, to=L0]
\arrow[Rightarrow, from=3-2, to=L1]
\arrow[Rightarrow, from=5-1, to=L2]
\arrow[Rightarrow, from=R1, to=3-5]
\arrow[Rightarrow, from=R0, to=4-5]
\arrow[Rightarrow, from=R2, to=5-4]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{1} \arrow[rr, "H"] \arrow[d, "N_A" description] \arrow[ddddddl, bend right=60, "\mathrm{Id}"', "\Leftarrow"] & & \mathbb{1} \arrow[d, "N_B" description] \arrow[ddll, Rightarrow, "\Phi_1"] \arrow[ddddddl, bend left=60, "\mathrm{Id}", "\Leftarrow"'] \\
 & A \arrow[d, "R_A" description] \arrow[ddl, bend right, "\mathrm{Id}"', "\Leftarrow"] & & B \arrow[d, "R_B" description] \arrow[dd, bend left, "\mathrm{Id}", "\Leftarrow"'] \\
 & \mathbb{1} \arrow[rr, "\dot{K}" description] \arrow[dl, "L_A" description] & {} \arrow[ddl, Rightarrow, "\tilde{\rho}"] & \mathbb{1} \arrow[d, "L_B" description] \\
A \arrow[d, "R_A" description] \arrow[dd, bend right, "\mathrm{Id}"', "\Leftarrow"] & & & B \arrow[dl, "R_B" description] \arrow[ddl, bend left, "\mathrm{Id}", "\Leftarrow"'] \\
\mathbb{1} \arrow[rr, "\dot{K}'" description] \arrow[d, "L_A" description] & & \mathbb{1} \arrow[d, "L_B" description] \arrow[ddll, Rightarrow, "\Phi_2^*"] & \\
A \arrow[d, "N_A^\vee" description] & & B \arrow[d, "N_B^\vee" description] & \\
\mathbb{1} \arrow[rr, "H"'] & & \mathbb{1} &
\end{tikzcd} .$$

This way a pasting diagram entirely internal to $\Sigma\mathcal{C}$ is obtained. Accordingly, its Poincaré-dual string diagram is a tangle diagram in $\mathcal{C}$. This is shown in figure 6. The Poncaré-dual string diagram obtained this way is just the one, shown in figure 3, that encodes the disk correlator in combinatorial CFT,

5.4 The torus

5.5 The sphere

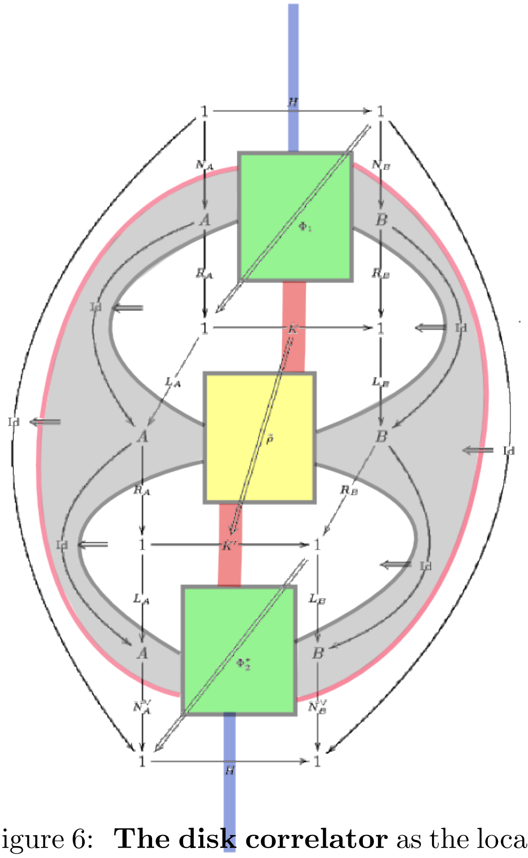


Figure 6: **The disk correlator** as the locally ($\mathbf{B}\mathcal{C} \overset{i}{\hookrightarrow} \mathrm{TwBimod}(\mathcal{C})$)-trivialized of the disk holonomy of a TwBim($\mathcal{C}$)-valued 2-functor. The thin black lines indicate the 2-morphisms in TwBimod($\mathcal{C}$). The coloring indicates the Poincaré-dual string diagram in $\mathcal{C}$, which reproduces the string diagram shown in figure 3.

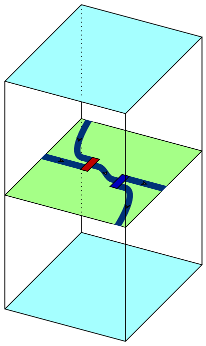


Figure 7: **Components of the torus partition function**. The torus is cut open to a polygon, the 2-dimensional cylinder-valued parallel transport is applied to it and paired with an incoming (from the 3d TFT perspective) and an outgoing state containing nothing but a “defect line” labelled by $U_i, U_j \in \mathcal{C}$. The result is then traced as in 5.2.2, yielding the component Z_{ij} of the torus partition function.

A Morphisms of 2-Functors

Definition 28 *Let* $S \xrightarrow{F_1} T$ *and* $S \xrightarrow{F_2} T$ *be two 2-functors. A* **pseudonatural transformation**

$$\begin{tikzcd}
S \arrow[r, bend left=50, "F_1", ""{name=U, below}] \arrow[r, bend right=50, "F_2"', ""{name=D}] & T
\arrow[Rightarrow, from=U, to=D, "\rho"]
\end{tikzcd}$$

is a map

$$\begin{tikzcd}
 & & & & F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[dd, "\rho(x)"'] & F_1(y) \arrow[dd, "\rho(y)"] & \\
\mathrm{Mor}_1(S) \ni & x \arrow[r, "\gamma"] & y & \mapsto & & & \in \mathrm{Mor}_2(T) \\
 & & & & F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y) &
\arrow[Rightarrow, from=1-6, to=3-5, shorten=12mm, "\rho(\gamma)"]
\end{tikzcd}$$

which is functorial in the sense that

$$\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma_1)"] \arrow[d, "\rho(x)"] & F_1(y) \arrow[r, "F_1(\gamma_2)"] \arrow[d, "\rho(y)"] & F_1(z) \arrow[d, "\rho(z)"] \\
F_2(x) \arrow[r, "F_2(\gamma_1)"'] & F_2(y) \arrow[r, "F_2(\gamma_2)"'] & F_2(z)
\arrow[Rightarrow, from=1-2, to=2-1, shorten=8mm, "\rho(\gamma_1)"]
\arrow[Rightarrow, from=1-3, to=2-2, shorten=8mm, "\rho(\gamma_2)"]
\end{tikzcd}
=
\begin{tikzcd}
F_1(x) \arrow[rr, "F_1(\gamma_1 \cdot \gamma_2)"] \arrow[d, "\rho(x)"] & & F_1(z) \arrow[d, "\rho(z)"] \\
F_2(x) \arrow[rr, "F_2(\gamma_1 \cdot \gamma_2)"'] & & F_2(z)
\arrow[Rightarrow, from=1-3, to=2-1, shorten=14mm, "\rho(\gamma_1 \cdot \gamma_2)"]
\end{tikzcd}$$

and which makes the pseudonaturality tin can 2-commute

$$\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma_1)"] \arrow[d, "\rho(x)"'] & F_1(y) \arrow[d, "\rho(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma_1)"{name=A}] \arrow[r, bend right=60, "F_2(\gamma_2)"'{name=B}] & F_2(y)
\arrow[Rightarrow, from=1-2, to=2-1, shorten=8mm, "\rho(\gamma_1)"]
\arrow[Rightarrow, from=A, to=B, "F_2(S)"]
\end{tikzcd}
=
\begin{tikzcd}
F_1(x) \arrow[r, bend left=60, "F_1(\gamma_1)"{name=C}] \arrow[r, "F_1(\gamma_2)"'{name=E}] \arrow[d, "\rho(x)"'] & F_1(y) \arrow[d, "\rho(y)"] \\
F_2(x) \arrow[r, "F_2(\gamma_2)"] & F_2(y)
\arrow[Rightarrow, from=C, to=E, "F_2(S)"]
\arrow[Rightarrow, from=1-2, to=2-1, shorten=8mm, "\rho(\gamma_2)"]
\end{tikzcd}$$

for all $\begin{tikzcd} x \arrow[r, bend left=50, "\gamma_1", ""{name=U, below}] \arrow[r, bend right=50, "\gamma_2"', ""{name=D}] & y \arrow[Rightarrow, from=U, to=D, "S"] \end{tikzcd} \in \mathrm{Mor}_2(S)$.

Definition 29 *The vertical* **composition of pseudonatural transformations**

$$\begin{tikzcd}
S \arrow[rr, bend left=60, "F_1"{name=U}] \arrow[rr, bend right=60, "F_3"'{name=D}] & & T
\arrow[Rightarrow, from=U, to=D, "\rho"]
\end{tikzcd}
\equiv
\begin{tikzcd}
S \arrow[rr, bend left=60, "F_1"{name=U}] \arrow[rr, "F_2" description, ""{name=M}] \arrow[rr, bend right=60, "F_3"'{name=D}] & & T
\arrow[Rightarrow, from=U, to=M, "\rho_1"] \arrow[Rightarrow, from=M, to=D, "\rho_2"]
\end{tikzcd}$$

is given by

$$\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)" description] \arrow[d, "\rho(x)"'] & F_1(y) \arrow[d, "\rho(y)"] \arrow[dl, Rightarrow, shorten=12mm, "\rho(\gamma)"] \\
F_3(x) \arrow[r, "F_3(\gamma)" description] & F_3(y)
\end{tikzcd}
\equiv
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)" description] \arrow[d, "\rho_1(x)"'] & F_1(y) \arrow[d, "\rho_1(y)"] \arrow[dl, Rightarrow, shorten=8mm, "\rho_1(\gamma)"] \\
F_2(x) \arrow[r, "F_2(\gamma)" description] \arrow[d, "\rho_2(x)"'] & F_2(y) \arrow[d, "\rho_2(y)"] \arrow[dl, Rightarrow, shorten=8mm, "\rho_2(\gamma)"] \\
F_3(x) \arrow[r, "F_3(\gamma)" description] & F_3(y)
\end{tikzcd}$$

Definition 30 *Let* $F_1 \xrightarrow{\rho_1} F_2 \quad F_1 \xrightarrow{\rho_2} F_2$ *be two pseudonatural transformations. A* **modification** *(of pseudonatural transformations)*

$$\begin{tikzcd}
F_1 \arrow[rr, bend left=40, "\rho_1"{name=U}] \arrow[rr, bend right=40, "\rho_2"'{name=D}] & & F_2
\arrow[Rightarrow, from=U, to=D, "\mathcal{A}"]
\end{tikzcd}$$

is a map

$$\mathrm{Obj}(S) \ni x \mapsto
\begin{tikzcd}
F_1(x) \arrow[rr, bend left=40, "\rho_1(x)"{name=U}] \arrow[rr, bend right=40, "\rho_2(x)"'{name=D}] & & F_2(x)
\arrow[Rightarrow, from=U, to=D, "\mathcal{A}(x)"]
\end{tikzcd}
\in \mathrm{Mor}_2(T)$$

such that

$$\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho_1(x)" description, ""{name=A}] \arrow[d, bend right=70, "\rho_2(x)"', ""{name=B}] & F_1(y) \arrow[d, "\rho_1(y)" description] \arrow[dl, Rightarrow, shorten=8mm, "\rho_1(\gamma)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y)
\arrow[Rightarrow, from=A, to=B, "\mathcal{A}(x)"]
\end{tikzcd}
=
\begin{tikzcd}
F_1(x) \arrow[r, "F_1(\gamma)"] \arrow[d, "\rho_2(x)" description] & F_1(y) \arrow[d, "\rho_2(y)" description, ""{name=A}] \arrow[d, bend left=70, "\rho_1(y)", ""{name=B}] \arrow[dl, Rightarrow, shorten=8mm, "\rho_2(\gamma)"] \\
F_2(x) \arrow[r, "F_2(\gamma)"'] & F_2(y)
\arrow[Rightarrow, from=B, to=A, "\mathcal{A}(y)"]
\end{tikzcd}$$

for all $x \xrightarrow{\gamma} y \in \mathrm{Mor}_1(S)$.

Definition 31 *The horizontal and vertical composite of modifications is, respectively, given by the horizontal and vertical composites of the maps to 2-morphisms in* $\mathrm{Mor}_2(T)$*.*

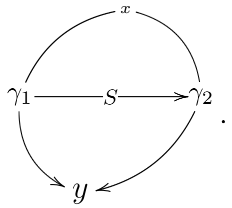
Definition 32 *Let S and T be two 2-categories. The* **2-functor 2-category** T^S *is the 2-category*

1. *whose objects are functors $F : S \to T$*
2. *whose 1-morphisms are pseudonatural transformations* $F_1 \xrightarrow{\rho} F_2$
3. *whose 2-morphisms are modifications*

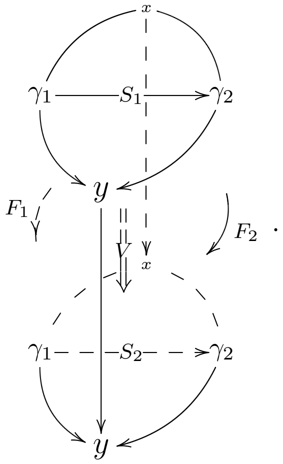
$$\begin{tikzcd} F_1 \arrow[r, bend left, "\rho_1"{name=U}] \arrow[r, bend right, "\rho_2"'{name=D}] & F_2 \arrow[from=U, to=D, Rightarrow, phantom, "\Downarrow \mathcal{A}"] \end{tikzcd} .$$

B Morphisms of 3-Functors

We shall regard 3-categories as special categories internal to 2Cat. From this point of view, a 3-category has a 2-category of objects S, each of which looks like



In a general category internal to 2Cat, we similarly have a 2-category of morphisms $S_1 \xrightarrow{V} S_2$, that look like



We shall restrict attention to the special case where the vertical faces here are identities. Then the above shape looks like

$$\begin{tikzcd}
& x \arrow[dr, bend left] & \\
\gamma_1 \arrow[ur, bend left, no head] \arrow[rr, bend left, "S_1"] \arrow[rr, dashed, bend right, "S_2"'] \arrow[dr, bend right] & \overset{\shortparallel}{\Downarrow} V & \gamma_2 \arrow[dl, bend left] \\
& y &
\end{tikzcd} .$$

Instead of saying that V is a morphism of a category internal to 2Cat, we say V is a 3-morphism. Similarly, S_1, S_2 are 2-morphisms, γ_1, γ_2 are 1-morphisms and x and y are objects.

We would have arrived at the same picture had we regarded categories enriched over 2Cat. However, we find that thinking of 3-morphisms as morphisms of a category internal to 2Cat facilitates handling morphisms of 3-functors, to which we now turn.

A 3-functor $F : S \to T$ between 3-categories S and T is a functor internal to 2Cat, hence a map

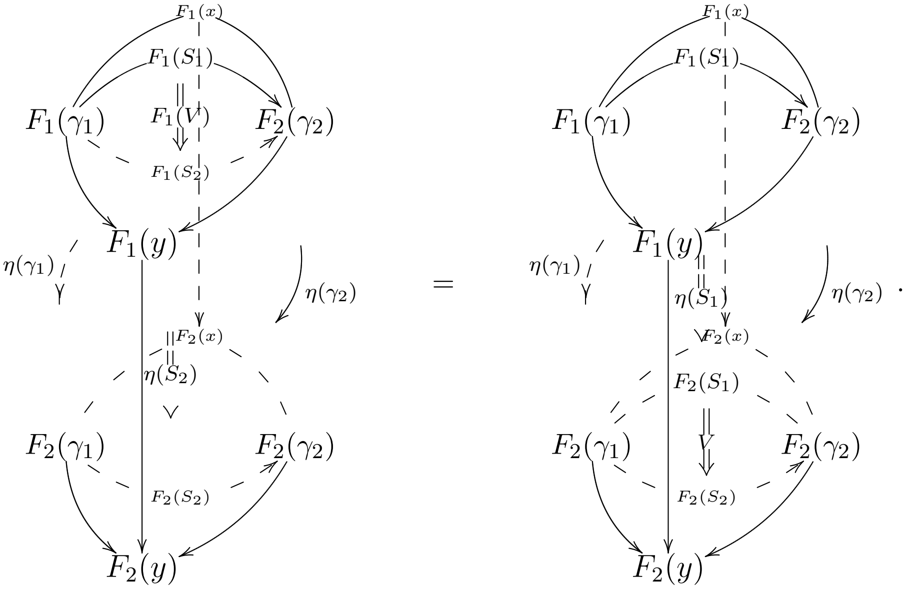
$$\begin{tikzcd}
& & x \arrow[dr, bend left] & & & & F(x) \arrow[dr, bend left] & \\
F : & \gamma_1 \arrow[ur, bend left, no head] \arrow[rr, bend left, "S_1"] \arrow[rr, dashed, bend right, "S_2"'] \arrow[dr, bend right] & \overset{\shortparallel}{\Downarrow} V & \gamma_2 \arrow[dl, bend left] & \mapsto & F(\gamma_1) \arrow[ur, bend left, no head] \arrow[rr, bend left, "F(S_1)"] \arrow[rr, dashed, bend right, "F(S_2)"'] \arrow[dr, bend right] & \overset{\shortparallel}{\Downarrow} F(V) & F(\gamma_2) \arrow[dl, bend left] \\
& & y & & & & F(y) &
\end{tikzcd}$$

that respects vertical composition strictly and is 2-functorial up to coherent 3-isomorphisms with respect to the composition perpendicular to that.

A 1-morphism $F_1 \xrightarrow{\eta} F_2$ between two such 3-functors is a natural transformation internal to 2Cat, hence a 2-functor from the object 2-category to the morphism 2-category, hence a 2-functorial assignment

$$\begin{tikzcd}
& & & & & & F_1(x) \arrow[dl, bend right, no head] \arrow[dr, bend left] \arrow[ddd, dashed] & \\
& & & & & F_1(\gamma_1) \arrow[rr, "F_1(S)"] \arrow[dr, bend right] \arrow[ddd, dashed, "\eta(\gamma_1)"'] & & F_1(\gamma_2) \arrow[dl, bend left] \arrow[ddd, "\eta(\gamma_2)"] \\
& & x \arrow[dl, bend right, no head] \arrow[dr, bend left] & & & & F_1(y) \arrow[ddd, "\overset{\shortparallel}{\Downarrow} \eta(S)"] & \\
\eta : & \gamma_1 \arrow[rr, "S"] \arrow[dr, bend right] & & \gamma_2 \arrow[dl, bend left] & \mapsto & & F_2(x) \arrow[dl, dashed, bend right, no head] \arrow[dr, dashed, bend left] & \\
& & y & & & F_2(\gamma_1) \arrow[rr, "F_2(S)"] \arrow[dr, bend right] & & F_2(\gamma_2) \arrow[dl, bend left] \\
& & & & & & F_2(y) &
\end{tikzcd}$$

that satisfies the naturality condition

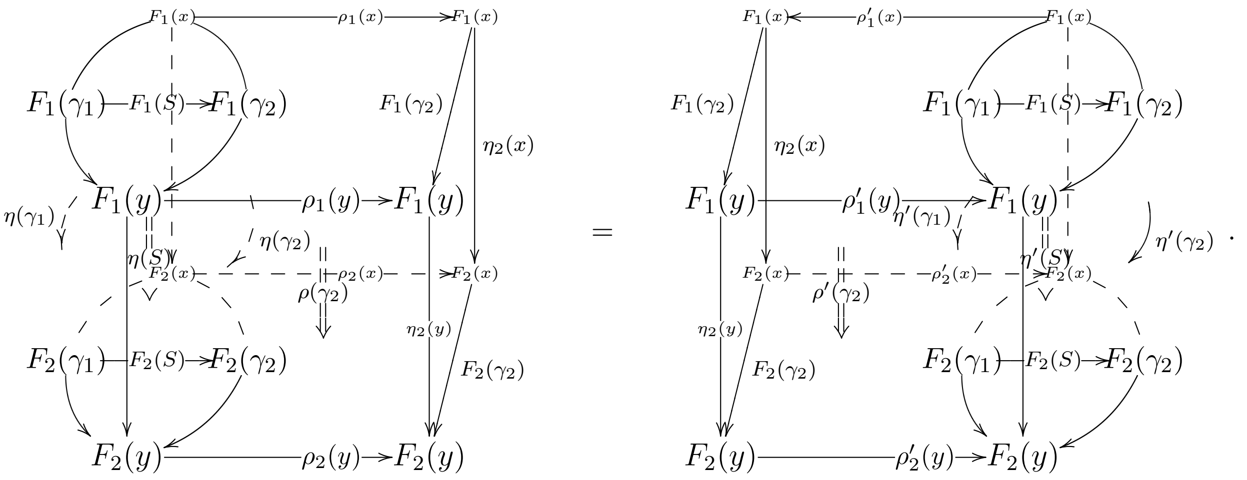


Accordingly, 2-morphisms and 3-morphisms of our 3-functors are 1-morphisms and 2-morphisms of these 2-functors η.

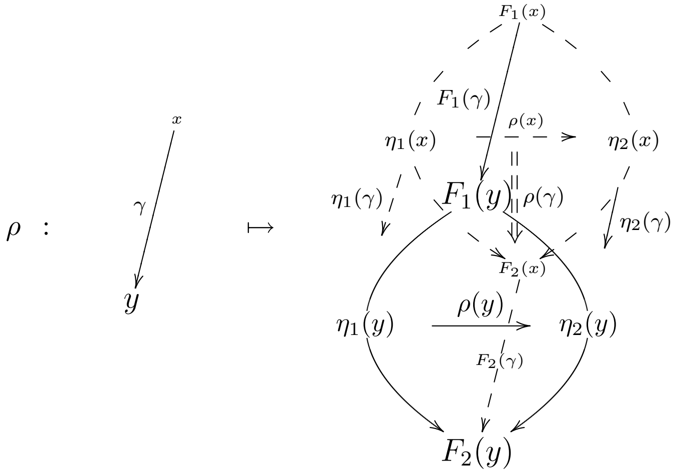
Hence a 2-morphism $\eta \xrightarrow{\rho} \eta'$ of our 3-functors is a 1-functorial assignment

$$\rho \;:\; \begin{tikzcd} x \arrow[d, "\gamma"'] \\ y \end{tikzcd} \quad \mapsto \quad \begin{tikzcd}
 & F_1(x) \arrow[rr, "\rho_1(x)"] \arrow[dl, "F_1(\gamma)"'] \arrow[dd, dashed, "\eta_1(x)"] & & F_1(x) \arrow[dl, "F_1(\gamma)"'] \arrow[dd, "\eta_2(x)"] \\
F_1(y) \arrow[rr, "\rho_1(y)"] \arrow[dd, "\eta_1(y)"'] & {} \arrow[d, Rightarrow, "\rho(\gamma)"] & F_1(y) \arrow[dd, "\eta_2(y)"] & \\
 & F_2(x) \arrow[rr, dashed, "\rho_2(x)"] \arrow[dl, dashed, "F_2(\gamma)"] & & F_2(x) \arrow[dl, "F_2(\gamma)"] \\
F_2(y) \arrow[rr, "\rho_2(y)"'] & & F_2(y) &
\end{tikzcd}$$

such that



We want to restrict attention to those ρ for which the horizontal 1-morphisms $\rho_1(x)$, $\rho_2(x)$, etc. are identities.



Proceeding this way, a modification $\lambda : \rho_1 \to \rho_2$ of transformations ρ gives us a 3-morphisms of 3-functors. This now is a map

$$\lambda \;:\; x \quad\mapsto\quad \begin{tikzcd}
& F_1(x) \arrow[dl, bend right, no head] \arrow[dr, bend left, no head] & \\
\eta_1(x) \arrow[rr, dashed, bend left, "\rho_1(x)"{name=U}] \arrow[rr, bend right, "\rho_2(x)"'{name=D}] \arrow[dr, bend right] & & \eta_2(x) \arrow[dl, bend left] \\
& F_2(x) &
\arrow[Rightarrow, from=U, to=D, "\lambda(x)"]
\end{tikzcd}$$

such that

$$\begin{tikzcd}[row sep=huge, column sep=huge]
F_1(x) \arrow[r, dashed, bend right=60, "\eta_1(x)"'] \arrow[r, dashed, bend left=60, "\eta_2(x)"] \arrow[r, phantom, "\rho_1(x)\overset{\lambda(x)}{\Rrightarrow}\rho_2(x)"] \arrow[d, "F_1(\gamma)"'] \arrow[dr, phantom, "\eta_1(\gamma)\quad\overset{\rho_2(\gamma)}{\Rrightarrow}\quad\eta_2(\gamma)"] & F_2(x) \arrow[d, dashed, "F_2(\gamma)"] \\
F_1(y) \arrow[r, bend right=60, "\eta_1(y)"'] \arrow[r, bend left=60, "\eta_2(y)"] \arrow[r, phantom, "\rho_2(y)"] & F_2(y)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[row sep=huge, column sep=huge]
F_1(x) \arrow[r, dashed, bend right=60, "\eta_1(x)"'] \arrow[r, dashed, bend left=60, "\eta_2(x)"] \arrow[r, phantom, "\rho_1(x)"] \arrow[d, "F_1(\gamma)"'] \arrow[dr, phantom, "\eta_1(\gamma)\quad\overset{\rho_1(\gamma)}{\Rrightarrow}\quad\eta_2(\gamma)"] & F_2(x) \arrow[d, dashed, "F_2(\gamma)"] \\
F_1(y) \arrow[r, bend right=60, "\eta_1(y)"'] \arrow[r, bend left=60, "\eta_2(y)"] \arrow[r, phantom, "\rho_1(y)\overset{\lambda(y)}{\Rrightarrow}\rho_2(y)"] & F_2(y)
\end{tikzcd} .$$

We thus get a 3-category of 3-morphisms of 3-functors.

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