Lie ∞-algebra connections and their application to String- and Chern-Simons n-Transport Part II

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Abstract

We give a generalization of the notion of a Cartan-Ehresmann connection from Lie algebras to L_∞-algebras and use it to study the obstruction theory of lifts through higher String-like extensions of Lie algebras.

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1 Statement of the main results

We define, for any L_∞-algebra $\mathfrak{g}$ and any smooth space X, a notion of

- $\mathfrak{g}$-descent objects over X;

and an extension of these to

- $\mathfrak{g}$-connection descent objects over X .

These descent objects are to be thought of as the data obtained from locally trivializing an n-bundle (with connection) whose structure n-group has the Lie n-algebra $\mathfrak{g}$. Being differential versions of n-functorial descent data of such n-bundles, they consist of morphisms of quasi free differential graded-commutative algebras (qDGCAs).

We define for each L_∞-algebra $\mathfrak{g}$ a dg-algebra $\mathrm{inv}(\mathfrak{g})$ of *invariant polynomials* on $\mathfrak{g}$. We show that every $\mathfrak{g}$-connection descent object gives rise to a collection of deRham classes on X: its *characteristic classes*. These are images of the elements of $\mathrm{inv}(\mathfrak{g})$.

Two descent objects are taken to be equivalent if they are *concordant* in a natural sense.

Our first main result is

Theorem 1 *Characteristic classes are indeed characteristic of $\mathfrak{g}$-descent objects (but do not necessarily fully characterize them) in the following sense:*

- *Concordant $\mathfrak{g}$-connection descent objects have the same characteristic classes.*

This is our proposition 22.

Remark. We expect that this result can be strengthened. Currently our characteristic classes are just in deRham cohomology. One would expect that these are images of classes in integral cohomology. While we do not attempt here to discuss integral characteristic classes in general, we discuss some aspects of this for the case of abelian Lie n-algebras $\mathfrak{g} = b^{n-1}\mathfrak{u}(1)$ in 3.2.1 by relating $\mathfrak{g}$-descent objects to Deligne cohomology.

We define String-like extensions $\mathfrak{g}_\mu$ of L_∞-algebras coming from any L_∞-algebra cocycle μ: a closed element in the Chevalley-Eilenberg dg-algebra $\mathrm{CE}(\mathfrak{g})$ .corresponding to $\mathfrak{g}$: $\mu \in \mathrm{CE}(\mathfrak{g})$. These generalize the String Lie 2-algebra which governs the dynamics of (heterotic) superstrings.

Our second main result is

Theorem 2 *For $\mu \in \mathrm{CE}(\mathfrak{g})$ any degree $n+1$ $\mathfrak{g}$-cocycle that transgresses to an invariant polynomial $P \in \mathrm{inv}(\mathfrak{g})$, the obstruction to lifting a $\mathfrak{g}$-descent object to a $\mathfrak{g}_\mu$-descent object is a $(b^n\mathfrak{u}(1))$-descent object whose single characteristic class is the class corresponding to P of the original $\mathfrak{g}$-descent object.*

This is our proposition 30.

We discuss the following **applications**.

- For $\mathfrak{g}$ an ordinary semisimple Lie algebra and μ its canonical 3-cocycle, the obstruction to lifting a $\mathfrak{g}$-bundle to a String 2-bundle is a Chern-Simons 3-bundle. This is a special case of our proposition 30 which is spelled out in detail in in 4.3.1.

  The vanishing of this obstruction is known as a String structure [36]. In categorical language, this issue was first discussed in [43].

- This result generalizes to all String-like extensions. Using the 7-cocycle on $\mathrm{so}(n)$ we obtain lifts through extensions by a Lie 6-algebra, which we call the Fivebrane Lie 6-algebra. Accordingly, fivebrane structures on string structures are obstructed by the second Pontrjagin class.

  This pattern continues and one would expect our obstruction theory for lifts through string-like extensions with respect to the 11-cocycle on $\mathrm{so}(n)$ to correspond to *Ninebrane* structure.

The issue of p-brane structures for higher p was discussed before in [34]. In contrast to the discussion there, we here see p-brane structures only for $p = 4n+1$, corresponding to the list of invariant polynomials and cocycles for so(n). While our entire obstruction theory applies to all cocycles on all Lie ∞-algebras, it is only for those on $\mathfrak{so}(n)$ and maybe $\mathfrak{e}_8$ for which the physical interpretation in the sense of p-brane structures is understood.

- We discuss how the action functional of the topological field theory known as BF-theory arises from an invariant polynomial on a strict Lie 2-algebra, in a generalization of the integrated Pontrjagin 4-form of the topological term in Yang-Mills theory. See proposition 10 and the example in 2.6.1.

  This is similar to but different from the Lie 2-algebraic interpretation of BF theory indicated in [20, 21], where the "cosmological" bilinear in the connection 2-form is not considered and a constraint on the admissable strict Lie 2-algebras is imposed.

- We briefly indicate the parallel transport induced by a $\mathfrak{g}$-connection, relate it to the n-functorial parallel transport of [40, 41, 42] and point out how this leads to σ-model actions in terms of dg-algebra morphisms. See section 5.

2 L_∞-algebras and their String-like extensions

2.1 L_∞-algebras

Definition 1 *Given a graded vector space V, the* tensor space $T^\bullet(V) := \bigoplus_{n=0} V^{\otimes n}$ *with V^0 being the ground field. We will denote by $T^a(V)$ the* tensor algebra *with the concatenation product on $T^\bullet(V)$:*

$$x_1 \otimes x_2 \otimes \cdots \otimes x_p \bigotimes x_{p+1} \otimes \cdots \otimes x_n \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_n \tag{1}$$

and by $T^c(V)$ the tensor coalgebra *with the deconcatenation product on $T^\bullet(V)$:*

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto \sum_{p+q=n} x_1 \otimes x_2 \otimes \cdots \otimes x_p \bigotimes x_{p+1} \otimes \cdots \otimes x_n. \tag{2}$$

The graded symmetric algebra $\wedge^\bullet(V)$ *is the quotient of the tensor algebra $T^a(V)$ by the graded action of the symmetric groups $\mathbf{S}_n$ on the components $V^{\otimes n}$. The* graded symmetric coalgebra $\vee^\bullet(V)$ *is the sub-coalgebra of the tensor coalgebra $T^c(V)$ fixed by the graded action of the symmetric groups $\mathbf{S}_n$ on the components $V^{\otimes n}$.*

Remark. $\vee^\bullet(V)$ is spanned by graded symmetric tensors

$$x_1 \vee x_2 \vee \cdots \vee x_p \tag{3}$$

for $x_i \in V$ and $p \geq 0$, where we use $\vee$ rather than $\wedge$ to emphasize the coalgebra aspect, e.g.

$$x \vee y = x \otimes y \pm y \otimes x. \tag{4}$$

In characteristic zero, the graded symmetric algebra can be identified with a sub-algebra of $T^a(V)$ but that is unnatural and we will try to avoid doing so.

The coproduct on $\vee^\bullet(V)$ is given by

$$\Delta(x_1 \vee x_2 \cdots \vee x_n) = \sum_{p+q=n} \sum_{\sigma \in \mathrm{Sh}(p,q)} \epsilon(\sigma)(x_{\sigma(1)} \vee x_{\sigma(2)} \cdots x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \vee \cdots x_{\sigma(n)})\,. \tag{5}$$

Here

- $\mathrm{Sh}(p,q)$ is the subset of all those bijections (the "unshuffles") of $\{1, 2, \cdots, p+q\}$ that have the property that $\sigma(i) < \sigma(i+1)$ whenever $i \neq p$;

- $\epsilon(\sigma)$, which is shorthand for $\epsilon(\sigma, x_1 \vee x_2, \cdots x_{p+q})$, the Koszul sign, defined by

$$x_1 \vee \cdots \vee x_n = \epsilon(\sigma) x_{\sigma(1)} \vee \cdots x_{\sigma(n)} \,. \tag{6}$$

Definition 2 (L_∞-algebra) *An L_∞-algebra $\mathfrak{g} = (\mathfrak{g}, D)$ is a $\mathbb{N}_+$-graded vector space $\mathfrak{g}$ equipped with a degree -1 differential coderivation*

$$D : \vee^\bullet \mathfrak{g} \to \vee^\bullet \mathfrak{g} \tag{7}$$

on the graded co-commutative coalgebra generated by $\mathfrak{g}$, such that $D^2 = 0$. This induces a differential

$$d_{\mathrm{CE}(\mathfrak{g})} : \mathrm{Sym}^\bullet(\mathfrak{g}) \to \mathrm{Sym}^{\bullet+1}(\mathfrak{g}) \tag{8}$$

on graded-symmetric multilinear functions on $\mathfrak{g}$. When $\mathfrak{g}$ is finite dimensional this yields a degree +1 differential

$$d_{\mathrm{CE}(\mathfrak{g})} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^* \tag{9}$$

on the graded-commutative algebra generated from $\mathfrak{g}^*$. This is the Chevalley-Eilenberg dg-algebra corresponding to the L_∞-algebra $\mathfrak{g}$.

Remark. That the original definition of L_∞-algebras in terms of multibrackets yields a codifferential coalgebra as above was shown in [31]. That every such codifferential comes from a collection of multibrackets this way is due to [32].

Example For $(\mathfrak{g}[-1], [\cdot, \cdot])$ an ordinary Lie algebra (meaning that we regard the vector space $\mathfrak{g}$ to be in degree 1), the corresponding Chevalley-Eilenberg qDGCA is

$$\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d_{\mathrm{CE}(\mathfrak{g})}) \tag{10}$$

with

$$d_{\mathrm{CE}(\mathfrak{g})} : \; \mathfrak{g}^* \xrightarrow{[\cdot,\cdot]^*} \mathfrak{g}^* \wedge \mathfrak{g}^* \;\; . \tag{11}$$

If we let $\{t_a\}$ be a basis of $\mathfrak{g}$ and $\{C^a{}_{bc}\}$ the corresponding structure constants of the Lie bracket $[\cdot, \cdot]$, and if we denote by $\{t^a\}$ the corresponding basis of $\mathfrak{g}^*$, then we get

$$d_{\mathrm{CE}(\mathfrak{g})} t^a = -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c \,. \tag{12}$$

If $\mathfrak{g}$ is concentrated in degree $1, \ldots, n$, we also say that $\mathfrak{g}$ is a **Lie n-algebra**.

Notice that we have built in a shift of degree for convenience, which makes ordinary Lie 1-algebras be in degree 1 already. In much of the literature a Lie n-algebra would be based on a vector space concentratred in degrees 0 to $n - 1$.

An ordinary Lie algebra is a Lie 1-algebra. Here the coderivation differential $D = [\cdot, \cdot]$ is just the Lie bracket, extended as a coderivation to $\vee^\bullet \mathfrak{g}$, with $\mathfrak{g}$ regarded as being in degree 1.

In the rest of the paper we assume, just for simplicity and since it is sufficient for our applications, all $\mathfrak{g}$ to be finite-dimensional. Then, by the above, these L_∞-algebras are equivalently conceived of in terms of their dual Chevalley-Eilenberg algebras, $\mathrm{CE}(\mathfrak{g})$, as indeed every quasi-free differential graded commutative algebra ("qDGCA", meaning that it is free as a graded commutative algebra) corresponds to an L_∞-algebra. We will find it convenient to work entirely in terms of qDGCAs, which we will usually denote as $\mathrm{CE}(\mathfrak{g})$.

While not interesting in themselves, truly free differential algebras are a useful tool for handling quasi-free differential algebras.

Definition 3 *We say a qDGCA is* free *(even as a differential algebra) if it is of the form*

$$\mathrm{F}(V) := (\wedge^\bullet(V^* \oplus V^*[1]), d_{\mathrm{F}(V)}) \tag{13}$$

with

$$d_{\mathrm{F}(V)}|_{V^*} = \sigma : V^* \to V^*[1] \tag{14}$$

the canonical isomorphism and

$$d_{\mathrm{F}(V)}|_{V^*[1]} = 0\,. \tag{15}$$

Remark. Such algebras are indeed free in that they satisfy the universal property: given any linear map $V \to W$, it uniquely extends to a morphism of qDGCAs $F(V) \to (\bigwedge^\bullet(W^*), d)$ for any choice of differential d.

Example. The free qDGCA on a 1-dimensional vector space in degree 0 is the graded commutative algebra freely generated by two generators, t of degree 0 and dt of degree 1, with the differential acting as $d : t \mapsto dt$ and $d : dt \mapsto 0$. In rational homotopy theory, this models the interval $I = [0, 1]$. The fact that the qDGCA is free corresponds to the fact that the interval is homotopy equivalent to the point.

We will be interested in qDGCAs that arise as mapping cones of morphisms of L_∞-algebras.

Definition 4 ("mapping cone" of qDGCAs) *Let*

$$\mathrm{CE}(\mathfrak{h}) \xleftarrow{t^*} \mathrm{CE}(\mathfrak{g}) \tag{16}$$

be a morphism of qDGCAs. The mapping cone of t^*, which we write $\mathrm{CE}(\mathfrak{h} \xrightarrow{t} g)$, is the qDGCA whose underlying graded algebra is

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]) \tag{17}$$

and whose differential d_{t^*} is such that it acts as

$$d_{t^*} = \begin{pmatrix} d_{\mathfrak{g}} & 0 \\ t^* & d_{\mathfrak{h}} \end{pmatrix}\,. \tag{18}$$

We postpone a more detailed definition and discussion to 4.1, see definition 22 and proposition 23. Strictly speaking, the more usual notion of mapping cones of chain complexes applies to $t : \mathfrak{h} \to \mathfrak{g}$, but then is extended as a derivation differential to the entire qDGCA.

Definition 5 (Weil algebra of an L_∞-algebra) *The mapping cone of the identity on $\mathrm{CE}(\mathfrak{g})$ is the Weil algebra*

$$\mathrm{W}(\mathfrak{g}) := \mathrm{CE}(\mathfrak{g} \xrightarrow{\mathrm{Id}} \mathfrak{g}) \tag{19}$$

of $\mathfrak{g}$.

Proposition 1 *For $\mathfrak{g}$ an ordinary Lie algebra this does coincide with the ordinary Weil algebra of $\mathfrak{g}$.*

Proof. See the example in 2.1.1. □

The Weil algebra has two important properties.

Proposition 2 *The Weil algebra $\mathrm{W}(\mathfrak{g})$ of any L_∞-algebra $\mathfrak{g}$*

- *is isomorphic to a free differential algebra*

$$\mathrm{W}(\mathfrak{g}) \simeq \mathrm{F}(\mathfrak{g})\,, \tag{20}$$

  and hence is contractible;

- *has a canonical surjection*

$$\mathrm{CE}(\mathfrak{g}) \overset{i^*}{\twoheadleftarrow} \mathrm{W}(\mathfrak{g}) \; . \tag{21}$$

Proof. Define a morphism

$$f : \mathrm{F}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g}) \tag{22}$$

by setting

$$\begin{aligned} f &: a \mapsto a \\ f &: (d_{\mathrm{F}(V)}a = \sigma a) \mapsto (d_{\mathrm{W}(\mathfrak{g})}a = d_{\mathrm{CE}(\mathfrak{g})}a + \sigma a) \end{aligned} \tag{23}$$

for all $a \in \mathfrak{g}^*$ and extend as an algebra homomorphism. This clearly does respect the differentials: for all $a \in V^*$ we find

$$\begin{tikzcd} a \arrow[r, mapsto, "d_{F(\mathfrak{g})}"] \arrow[d, mapsto, "f"] & \sigma a \arrow[d, mapsto, "f"] \\ a \arrow[r, mapsto, "d_{\mathrm{W}(\mathfrak{g})}"'] & d_{\mathrm{CE}(\mathfrak{g})}a + \sigma a \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \sigma a \arrow[r, mapsto, "d_{F(\mathfrak{g})}"] \arrow[d, mapsto, "f"] & 0 \arrow[d, mapsto, "f"] \\ d_{\mathrm{W}(\mathfrak{g})}a \arrow[r, mapsto, "d_{\mathrm{W}(\mathfrak{g})}"'] & 0 \end{tikzcd} \; . \tag{24}$$

One checks that the strict inverse exists and is given by

$$f^{-1}|_{\mathfrak{g}^*} \; : \; a \mapsto a \tag{25}$$

$$f^{-1}|_{\mathfrak{g}^*[1]} \; : \; \sigma a \mapsto d_{F(\mathfrak{g})}a - d_{\mathrm{CE}(\mathfrak{g})}a \, . \tag{26}$$

Here $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*[1]$ is the canonical isomorphism that shifts the degree.

The surjection $\mathrm{CE}(\mathfrak{g}) \overset{i^*}{\twoheadleftarrow} \mathrm{W}(\mathfrak{g})$ simply projects out all elements in the shifted copy of $\mathfrak{g}$:

$$i^*|_{\wedge^\bullet \mathfrak{g}^*} = \mathrm{id} \tag{27}$$

$$i^*|_{\mathfrak{g}^*[1]} = 0 \, . \tag{28}$$

This is an algebra homomorphism that respects the differential. □

As a corollary we obtain

Corollary 1 *For $\mathfrak{g}$ any L_∞-algebra, the cohomology of $\mathrm{W}(\mathfrak{g})$ is trivial.*

Remark. As we will shortly see, $\mathrm{W}(\mathfrak{g})$ plays the role of the algebra of differential forms on the universal $\mathfrak{g}$-bundle. The surjection $CE(\mathfrak{g}) \overset{i^*}{\twoheadleftarrow} W(\mathfrak{g})$ plays the role of the restriction to the differential forms on the fiber of the universal $\mathfrak{g}$-bundle.

2.1.1 Examples

In 2.4 we construct large families of examples of L_∞-algebras, based on the first two of the following examples:

1. Ordinary Weil algebras as Lie 2-algebras. What is ordinarily addressed as the Weil algebra $\mathrm{W}(\mathfrak{g})$ of a Lie algebra $(\mathfrak{g}[-1], [\cdot,\cdot])$ can, since it is again a DGCA, also be interpreted as the Chevalley-Eilenberg algebra of a Lie 2-algebra. This Lie 2-algebra we call $\mathrm{inn}(\mathfrak{g})$. It corresponds to the Lie 2-group $\mathrm{INN}(G)$ discussed in [38]:

$$\mathrm{W}(\mathfrak{g}) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) \, . \tag{29}$$

We have

$$\mathrm{W}(\mathfrak{g}) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1]), d_{\mathrm{W}(\mathfrak{g})}) \, . \tag{30}$$

Denoting by $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*[1]$ the canonical isomorphism, extended as a derivation to all of $\mathrm{W}(\mathfrak{g})$, we have

$$d_{\mathrm{W}(\mathfrak{g})} : \ \mathfrak{g}^* \xrightarrow{[\cdot,\cdot]^* + \sigma} \mathfrak{g}^* \wedge \mathfrak{g}^* \oplus \mathfrak{g}^*[1] \tag{31}$$

and

$$d_{\mathrm{W}(\mathfrak{g})} : \ \mathfrak{g}^*[1] \xrightarrow{-\sigma \circ d_{\mathrm{CE}(\mathfrak{g})} \circ \sigma^{-1}} \mathfrak{g}^* \otimes \mathfrak{g}^*[1] \ . \tag{32}$$

With $\{t^a\}$ a basis for $\mathfrak{g}^*$ as above, and $\{\sigma t^a\}$ the corresponding basis of $\mathfrak{g}^*[1]$ we find

$$d_{\mathrm{W}(\mathfrak{g})} : t^a \mapsto -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + \sigma t^a \tag{33}$$

and

$$d_{\mathrm{W}(\mathfrak{g})} : \sigma t^a \mapsto -C^a{}_{bc} t^b \sigma t^c \, . \tag{34}$$

The Lie 2-algebra $\mathrm{inn}(\mathfrak{g})$ is, in turn, nothing but the strict Lie 2-algebra as in the third example below, which comes from the infinitesimal crossed module $(\mathfrak{g} \xrightarrow{\mathrm{Id}} \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{der}(\mathfrak{g}))$.

2. Shifted $\mathfrak{u}(1)$. By the above, the qDGCA corresponding to the Lie algebra $\mathfrak{u}(1)$ is simply

$$\mathrm{CE}(\mathfrak{u}(1)) = (\wedge^\bullet \mathbb{R}[1], d_{\mathrm{CE}(\mathfrak{u}(1))} = 0) \, . \tag{35}$$

We write

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) = (\wedge^\bullet \mathbb{R}[n], d_{\mathrm{CE}(b^n \mathfrak{u}(1))} = 0) \tag{36}$$

for the Chevalley-Eilenberg algebras corresponding to the Lie n-algebras $b^{n-1}\mathfrak{u}(1)$.

3. Infinitesimal crossed modules and strict Lie 2-algebras. An *infinitesimal crossed module* is a diagram

$$(\ \mathfrak{h} \xrightarrow{t} \mathfrak{g} \xrightarrow{\alpha} \mathrm{der}(\mathfrak{h})\ ) \tag{37}$$

of Lie algebras where t and α satisfy two compatibility conditions. These conditions are equivalent to the nilpotency of the differential on

$$\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) := (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]), d_t) \tag{38}$$

defined by

$$d_t|_{\mathfrak{g}^*} = [\cdot,\cdot]^*_{\mathfrak{g}} + t^* \tag{39}$$

$$d_t|_{\mathfrak{h}^*[1]} = \alpha^* \, , \tag{40}$$

where we consider the vector spaces underlying both $\mathfrak{g}$ and $\mathfrak{h}$ to be in degree 1. Here in the last line we regard α as a linear map $\alpha : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$. The Lie 2-algebras $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ thus defined are called strict Lie 2-algebras: these are precisely those Lie 2-algebras whose Chevalley-Eilenberg differential contains at most co-binary components.

2.2 L_∞-algebra homotopy and concordance

There are two different but related notions of higher morphisms between qDGCAs. One of them is infinitesimal in nature and the other is finite.

The infinitesimal notion is simply that induced from cochain complexes: a homotopy between two qDGCA morphisms, which are in particular cochain maps, is a cochain homotopy. But for qDGCAs, homomorphisms are specified by specifying them *on generators*. Accordingly, one wants to characterize homotopies between them by their action *on generators*.

We now define transformations (2-morphisms) between morphisms of qDGCAs by first defining them for the case when the domain is a Weil algebra, and then extending the definition to arbitrary qDGCAs.

	name	nature
infinitesimal	transformation	chain homotopy $\begin{tikzcd} \mathrm{CE}(\mathfrak{g}) & \mathrm{CE}(\mathfrak{h}) \arrow[l, bend right, "f^*"'] \arrow[l, bend left, "g^*"] \arrow[l, phantom, "\Downarrow\eta"] \end{tikzcd}$
finite	homotopy/concordance	extension over interval $\begin{tikzcd} \mathrm{CE}(\mathfrak{g}) & \mathrm{CE}(\mathfrak{g}) \otimes \Omega^\bullet(I) \arrow[l, shift right, "\mathrm{Id}\otimes s^*"'] \arrow[l, shift left, "\mathrm{Id}\otimes t^*"] & \mathrm{CE}(\mathfrak{h}) \arrow[l, "\eta^*"'] \arrow[ll, bend right, "g^*"'] \arrow[ll, bend left, "f^*"] \end{tikzcd}$

Table 1: The two different notions of **higher morphisms** of qDGCAs.

Definition 6 (transformation of morphisms of L_∞-algebras) *We define transformations between qDGCA morphisms in two steps*

- *A 2-morphism*

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{g}) & \mathrm{F}(\mathfrak{h}) \arrow[l, bend right, "f^*"'] \arrow[l, bend left, "g^*"] \arrow[l, phantom, "\Downarrow\eta"] \end{tikzcd} \tag{41}$$

is defined by a degree -1 map $\eta : \mathfrak{h}^* \oplus \mathfrak{h}^*[1] \to \mathrm{CE}(\mathfrak{g})$ which is extended to a linear degree -1 map $\eta : \wedge^\bullet(\mathfrak{h}^* \oplus \mathfrak{h}^*[1]) \to \mathrm{CE}(\mathfrak{g})$ by defining it on all monomials by the formula

$$\eta : x_1 \wedge \cdots \wedge x_n \mapsto$$

$$\frac{1}{n!} \sum_\sigma \epsilon(\sigma) \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k-1} |x_{\sigma(i)}|} g^*(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k-1)}) \wedge \eta(x_{\sigma(k)}) \wedge f^*(x_{\sigma(k+1)} \wedge \cdots \wedge x_{\sigma(n)}) \tag{42}$$

for all $x_1, \cdots, x_n \in \mathfrak{h}^* \oplus \mathfrak{h}^*[1]$, such that this is a chain homotopy from f^* to g^*:

$$g^* = f^* + [d, \eta]\,.$$

- *A general 2-morphism*

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{g}) & \mathrm{CE}(\mathfrak{h}) \arrow[l, bend right, "f^*"'] \arrow[l, bend left, "g^*"] \arrow[l, phantom, "\Downarrow\eta"] \end{tikzcd} \tag{43}$$

is a 2-morphism

$$\begin{tikzcd}
 & \mathrm{CE}(\mathfrak{h}) \arrow[dl, bend right, "g^*"'] & & \\
\mathrm{CE}(\mathfrak{g}) & \arrow[l, phantom, "\Swarrow"] & \mathrm{W}(\mathfrak{h}) \arrow[ul, "i^*"'] \arrow[dl, "i^*"] & \mathrm{F}(\mathfrak{h}) \arrow[l, "\simeq"'] \\
 & \mathrm{CE}(\mathfrak{g}) \arrow[ul, bend left, "f^*"] & &
\end{tikzcd} \qquad (44)$$

of the above kind that vanishes on the shifted generators, i.e. such that

$$\begin{tikzcd}
 & \mathrm{CE}(\mathfrak{h}) \arrow[dl, bend right, "g^*"'] & & \\
\mathrm{CE}(\mathfrak{g}) & \arrow[l, phantom, "\Swarrow"] & \mathrm{W}(\mathfrak{h}) \arrow[ul, "i^*"'] \arrow[dl, "i^*"] & \mathfrak{h}^*[1] \arrow[l, hook'] \\
 & \mathrm{CE}(\mathfrak{g}) \arrow[ul, bend left, "f^*"] & &
\end{tikzcd} \qquad (45)$$

vanishes.

Proposition 3 *Formula 42 is consistent in that* $g^*|_{\mathfrak{h}^*\oplus\mathfrak{h}^*[1]} = (f^*+[d,\eta])|_{\mathfrak{h}^*\oplus\mathfrak{h}^*[1]}$ *implies that* $g^* = f^*+[d,\eta]$ *on all elements of* $F(\mathfrak{h})$*.*

Remark. Definition 6, which may look ad hoc at this point, has a practical and a deep conceptual motivation.

- **Practical motivation.** While it is clear that 2-morphisms of qDGCAs should be chain homotopies, it is not straightforward, in general, to characterize these by their action on generators. Except when the domain qDGCA is free, in which case our formula 6 makes sense. The prescription 44 then provides a systematic algorithm for extending this to arbitrary qDGCAs.

  In particular, using the isomorphism $\mathrm{W}(\mathfrak{g}) \simeq \mathrm{F}(\mathfrak{g})$ from proposition 2, the above yields the usual explicit description of the homotopy operator $\tau : \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g})$ with $\mathrm{Id}_{\mathrm{W}(\mathfrak{g})} = [d_{\mathrm{W}(\mathfrak{g})}, \tau]$. Among other things, this computes for us the transgression elements ("Chern-Simons elements") for L_∞-algbras in 2.3.

- **Conceptual motivation.** As we will see in 2.3 and 2.5, the qDGCA $\mathrm{W}(\mathfrak{g})$ plays an important twofold role: it is both the algebra of differential forms on the total space of the universal $\mathfrak{g}$-bundle – while $\mathrm{CE}(\mathfrak{g})$ is that of forms on the fiber –, as well as the domain for $\mathfrak{g}$-valued differential forms, where the shifted component, that in $\mathfrak{h}^*[1]$, is the home of the corresponding curvature.

  In the light of this, the above restriction 45 can be understood as saying either that

  – transformations on the fiber are *vertical* transformations on the total bundle;

  or

  – gauge transformations of $\mathfrak{g}$-valued forms are transformations under which the curvatures transform covariantly.

Example (transgression forms). As an example, we show how the usual Chern-Simons transgression form is computed using formula 42. The reader may want to first skip to our discussion of Lie ∞-algebra cohomology in 2.3 for more background.

So let $\mathfrak{g}$ be an ordinary Lie algebra with invariant bilinear form P, which we regard as a $d_{\mathrm{W}(\mathfrak{g})}$-closed element $P \in \wedge^2 \mathfrak{g}^*[1] \subset \mathrm{W}(\mathfrak{g})$. We want to compute τP, where τ is the contracting homotopy of $\mathrm{W}(\mathfrak{g})$, such that

$$[d, \tau] = \mathrm{Id}_{\mathrm{W}(\mathfrak{g})} ,$$

which according to proposition 2 is given on generators by

$$\tau : a \mapsto 0$$

$$\tau : d_{\mathrm{W}(\mathfrak{g})} a \mapsto a$$

for all $a \in \mathfrak{g}^*$. Let $\{t^a\}$ be a chosen basis of $\mathfrak{g}^*$ and let $\{P_{ab}\}$ be the components of P in that basis, then

$$P = P_{ab}(\sigma t^a) \wedge (\sigma t^b) .$$

In order to apply formula 42 we need to first rewrite this in terms of monomials in $\{t^a\}$ and $\{d_{\mathrm{W}(\mathfrak{g})} t^a\}$. Hence, using $\sigma t^a = d_{\mathrm{W}(\mathfrak{g})} t^a + \frac{1}{2} C^a{}_{bc} t^b \wedge t^c$, we get

$$\tau P = \tau \left(P_{ab}(d_{\mathrm{W}(\mathfrak{g})} t^a) \wedge (d_{\mathrm{W}(\mathfrak{g})} t^a) - P_{ab}(d_{\mathrm{W}(\mathfrak{g})} t^a) \wedge C^b{}_{cd} t^c \wedge t^d + \frac{1}{4} P_{ab} C^a{}_{cd} C^b_{ef} t^c \wedge t^d \wedge t^c \wedge t^d \right) .$$

Now equation 42 can be applied to each term. Noticing the combinatorial prefactor $\frac{1}{n!}$, which depends on the number of factors in the above terms, and noticing the sum over all permutations, we find

$$\tau \left(P_{ab}(d_{\mathrm{W}(\mathfrak{g})} t^a) \wedge (d_{\mathrm{W}(\mathfrak{g})} t^a) \right) = P_{ab}(d_{\mathrm{W}(\mathfrak{g})} t^a) \wedge t^b$$

$$\tau \left(-P_{ab}(d_{\mathrm{W}(\mathfrak{g})} t^a) \wedge C^b{}_{cd} t^c \wedge t^d \right) = \frac{1}{3!} \cdot 2 \, P_{ab} C^b_{cd} t^b \wedge t^c \wedge t^d = \frac{1}{3} C_{abc} t^a \wedge t^b \wedge t^c ,$$

where we write $C_{abc} := P_{ad} C^d{}_{bc}$ as usual. Finally $\tau \left(\frac{1}{4} P_{ab} C^a{}_{cd} C^b_{ef} t^c \wedge t^d \wedge t^c \wedge t^d \right) = 0$. In total this yields

$$\tau P = P_{ab}(d_{\mathrm{W}(\mathfrak{g})} t^a) \wedge t^b + \frac{1}{3} C_{abc} t^a \wedge t^b \wedge t^c .$$

By again using $d_{\mathrm{W}(\mathfrak{g})} t^a = -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + \sigma t^a$ together with the invariance of P (hence the $d_{\mathrm{W}(\mathfrak{g})}$-closedness of P which implies that the constants C_{abc} are skew symmetric in all three indices) one checks that this does indeed satisfy

$$d_{\mathrm{W}(\mathfrak{g})} \tau P = P .$$

In 2.5 we will see that after choosing a $\mathfrak{g}$-valued connection on space space Y the generators t^a here will get sent to components of a $\mathfrak{g}$-valued 1-form A, while the $d_{\mathrm{W}(\mathfrak{g})} t^a$ will get sent to the components of dA. Under this map the element $\tau P \in \mathrm{W}(\mathfrak{g})$ maps to the familiar Chern-Simons 3-form

$$\mathrm{CS}_P(A) := P(A \wedge dA) + \frac{1}{3} P(A \wedge [A \wedge A])$$

whose differential is the characteristic form of A with respect to P:

$$d\mathrm{CS}_P(A) = P(F_A \wedge F_A) .$$

Characteristic forms, for arbitrary Lie ∞-algebra valued forms, is discussed further in 2.6.

Proposition 4 *For the special case that $\mathfrak{g}$ is any Lie 2-algebra (any L_∞-algebra concentrated in the first two degrees) the 2-morphisms defined by definition 6 reproduce the 2-morphisms of Lie 2-algebras as stated in [2] and used in [3].*

This implies in particular that with the 1- and 2-morphisms as defined above, Lie 2-algebras do form a 2-category. There is an rather straightforward generalization of definition 6 to higher morphisms, which one would expect yields correspondingly n-categories of Lie n-algebras. But this we shall not try to discuss here.

We now come to the finite transformations of morphisms of DGCAs.

What we called 2-morphisms or transformations for qDGCAs above would in other contexts possibly be called a homotopy. Also the following concept is a kind of homotopy, and appears as such in [Stasheff et al]. Here we want to clearly distinguish these different kinds of homotopies and address the following concept as *concordance* – a finite notion of 2-morphism between dg-algebra morphisms.

Denote by I^* either of the following two models for differential forms on the interval:

$$I^* = \begin{cases} \Omega^\bullet([0,1]) & \text{deRham complex of forms on the standard interval} \\ \mathrm{F}(\mathbb{R}) & \text{the free qDGCA on a single degree 0 generator} \end{cases}$$

Definition 7 (concordance) *We say that two qDGCA morphisms*

$$\mathrm{CE}(\mathfrak{g}) \xleftarrow{g^*} \mathrm{CE}(\mathfrak{h}) \tag{46}$$

and

$$\mathrm{CE}(\mathfrak{g}) \xleftarrow{h^*} \mathrm{CE}(\mathfrak{h}) \tag{47}$$

are concordant, if there exists a dg-algebra homomorphism

$$\mathrm{CE}(\mathfrak{g}) \otimes I^* \xleftarrow{\eta^*} \mathrm{CE}(\mathfrak{h}) \tag{48}$$

from the source $\mathrm{CE}(\mathfrak{h})$ *to the the target* $\mathrm{CE}(\mathfrak{g})$ *tensored with forms on the interval, which restricts to the two given homomorphisms when pulled back along the two boundary inclusions*

$$\{\bullet\} \underset{t}{\overset{s}{\rightrightarrows}} I \ , \tag{49}$$

so that

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{g}) & \mathrm{CE}(\mathfrak{g}) \otimes \Omega^\bullet(I) \arrow[l, shift right, "\mathrm{Id}\otimes s^*"'] \arrow[l, shift left, "\mathrm{Id}\otimes t^*"] & \mathrm{CE}(\mathfrak{h}) \arrow[l, "\eta^*"'] \arrow[ll, bend right, "g^*"'] \arrow[ll, bend left, "f^*"] \end{tikzcd} \ . \tag{50}$$

See also table 1.

We can make precise the statement that definition 6 is the infinitesimal version of definition 7, as follows.

Proposition 5 *Concordances*

$$\mathrm{CE}(\mathfrak{g}) \otimes I^* \xleftarrow{\eta^*} \mathrm{CE}(\mathfrak{h}) \tag{51}$$

are in bijection with 1-parameter families

$$\alpha : [0,1] \to \mathrm{Hom}_{\mathrm{dg-Alg}}(\mathrm{CE}(\mathfrak{h}), \mathrm{CE}(\mathfrak{g})) \tag{52}$$

of morphisms whose derivatives with respect to the parameter is a chain homotopy

$$\forall t \in [0,1] \ : \quad \begin{tikzcd} \mathrm{CE}(\mathfrak{g}) \arrow[r, phantom, "\Downarrow\rho"] & \mathrm{CE}(\mathfrak{h}) \arrow[l, bend right, "0"'] \arrow[l, bend left, "\frac{d}{dt}\alpha(t)=[d,\rho]"] \end{tikzcd} \ . \tag{53}$$

For any such α, the morphisms f^* and g^* between which it defines a concordance are defined by the value of α on the boundary of the interval.

Proof. Writing $t : [0,1] \to \mathbb{R}$ for the canonical coordinate function on the interval $I = [0,1]$ we can decompose the dg-algebra homomorphism η^* as

$$\eta^* : \omega \mapsto (t \mapsto \alpha(\omega)(t) + dt \wedge \rho(\omega)(t)) \,. \tag{54}$$

α is itself a degree 0 dg-algebra homomorphism, while ρ is degree -1 map.

Then the fact that η^* respects the differentials implies that for all $\omega \in \mathrm{CE}(\mathfrak{h})$ we have

$$\begin{tikzcd} \omega \arrow[r, mapsto, "d_{\mathfrak{h}}"] \arrow[d, mapsto, "\eta^*"] & d_{\mathfrak{h}}\omega \arrow[d, mapsto, "\eta^*"] \\ (t \mapsto (\alpha(\omega)(t) + dt \wedge \rho(\omega)(t))) \arrow[r, mapsto, "d_{\mathfrak{g}}+d_t"] & \begin{array}{l} (t \mapsto (\alpha(d_{\mathfrak{h}}\omega)(t) + dt \wedge \rho(d_{\mathfrak{h}}\omega)(t)) \\ = (t \mapsto (d_{\mathfrak{g}}(\alpha(\omega))(t) + dt \wedge (\frac{d}{dt}\alpha(\omega) - d_{\mathfrak{g}}\rho(\omega))(t)) \end{array} \end{tikzcd} \tag{55}$$

The equality in the bottom right corner says that

$$\alpha \circ d_{\mathfrak{h}} - d_{\mathfrak{g}} \circ \alpha = 0 \tag{56}$$

and

$$\forall \omega \in \mathrm{CE}(\mathfrak{g}) : \frac{d}{dt}\alpha(\omega) = \rho(d_{\mathfrak{h}}\omega) + d_{\mathfrak{g}}(\rho(\omega)) \,. \tag{57}$$

But this means that α is a chain homomorphism whose derivative is given by a chain homotopy. □

2.3 L_∞-algebra cohomology

The study of ordinary Lie algebra cohomology and of invariant polynomials on the Lie algebra has a simple formulation in terms of the qDGCAs $\mathrm{CE}(\mathfrak{g})$ and $\mathrm{W}(\mathfrak{g})$. And this has a straightforward generalization to arbitrary L_∞-algebras, which we state now.

For

$$\mathrm{CE}(\mathfrak{g}) \overset{i^*}{\twoheadleftarrow} \mathrm{W}(\mathfrak{g}) \tag{58}$$

the canonical morphism from proposition 2, notice that

$$\mathrm{CE}(\mathfrak{g}) \simeq \mathrm{W}(\mathfrak{g})/\ker(i^*) \tag{59}$$

and that

$$\ker(i^*) = \langle \mathfrak{g}^*[1] \rangle_{\mathrm{W}(\mathfrak{g})} \,, \tag{60}$$

the ideal in $\mathrm{W}(\mathfrak{g})$ generated by $\mathfrak{g}^*[1]$.

Algebra derivations

$$\iota_X : \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g})$$

for $X \in \mathfrak{g}$ are like (contractions with) vector fields on the space on which $\mathrm{W}(\mathfrak{g})$ is like differential forms. In the case of an ordinary Lie algebra $\mathfrak{g}$, the corresponding inner derivations $[d_{\mathrm{W}(\mathfrak{g})}, \iota_X]$ for $X \in \mathfrak{g}$ are of degree -1 and are known as the Lie derivative L_X. They generate flows $\exp([d_{\mathrm{W}(\mathfrak{g})}, \iota_X]) : \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g})$ along these vector fields.

Definition 8 (vertical derivations) *We say an algebra derivation* $\tau : \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g})$ *is* vertical *if it vanishes on the shifted copy* $\mathfrak{g}^*[1]$ *of* $\mathfrak{g}^*$ *inside* $\mathrm{W}(\mathfrak{g})$*,*

$$\tau|_{\mathfrak{g}^*[1]} = 0\,.$$

The contractions ι_X are vertical derivations.

The reader should compare this and the following definitions to the theory of vertical Lie derivatives and basic differential forms with respect to any surjective submersion $\pi : Y \to X$. This is discussed in 3.1.

Definition 9 (basic forms and invariant polynomials) *The algebra* $\mathrm{W}(\mathfrak{g})_{\mathrm{basic}}$ *of* **basic forms** *in* $\mathrm{W}(\mathfrak{g})$ *is the intersection of the kernels of all the contractions* ι_X *and Lie derivatives* L_X *for* $X \in \mathfrak{g}$*. Since* $L_X = [d_{\mathrm{W}(\mathfrak{g})}, \iota_X]$*, it follows that in the kernel of* ι_X*, the Lie derivative vanishes only if* $\iota_X d_{\mathrm{W}(\mathfrak{g})}$ *vanishes.*

We define $\mathrm{inv}(\mathfrak{g})$ *to be the dg-algebra generated from all indecomposable monomials in* $\mathrm{W}(\mathfrak{g})_{\mathrm{basic}}$*, modulo those which are exact in* $\ker(i^*)$*.*

Using the obvious inclusion $\mathrm{W}(\mathfrak{g}) \xhookleftarrow{p^*} \mathrm{inv}(\mathfrak{g})$ we obtain the sequence

$$\mathrm{CE}(\mathfrak{g}) \twoheadleftarrow{i^*} \mathrm{W}(\mathfrak{g}) \xhookleftarrow{p^*} \mathrm{inv}(\mathfrak{g}) \tag{61}$$

of dg-algebras that plays a major role in our analysis.

Definition 10 (cocycles, invariant polynomials and transgression elements) *Let* $\mathfrak{g}$ *be an* L_∞*-algebra. Then*

- *An* L_∞*-algebra* **cocycle** *on* $\mathfrak{g}$ *is a* $d_{\mathrm{CE}(\mathfrak{g})}$*-closed element of* $\mathrm{CE}(\mathfrak{g})$*.*

$$\mu \in \mathrm{CE}(\mathfrak{g})\,, \qquad d_{\mathrm{CE}(\mathfrak{g})}\mu = 0\,. \tag{62}$$

- *An* L_∞*-algebra* **invariant polynomial** *on* $\mathfrak{g}$ *is an element* $P \in \mathrm{inv}(\mathfrak{g}) := \mathrm{W}(\mathfrak{g})_{basic}$*.*

- *An* L_∞*-algebra* **$\mathfrak{g}$-transgression element** *for a given cocycle* μ *and an invariant polynomial* P *is an element* $\mathrm{cs} \in \mathrm{W}(\mathfrak{g})$ *such that*

$$d_{\mathrm{W}(\mathfrak{g})}\mathrm{cs} = p^*P \tag{63}$$

$$i^*\mathrm{cs} = \mu\,. \tag{64}$$

If a transgression element for μ and P exists, we say that μ **transgresses** *to* P and that P **suspends** *to* μ. If $\mu = 0$ we say that P **suspends to** 0.

The situation is illustrated diagrammatically in figure 1 and figure 2.

Proposition 6 *For the case that* $\mathfrak{g}$ *is an ordinary Lie algebra, the above definition reproduces the ordinary definition of Lie algebra cocycles, invariant polynomials and of transgression elements. Moreover, all elements in* $\mathrm{inv}(\mathfrak{g})$ *are closed.*

Proof. That the definition of Lie algebra cocycles and transgression elements coincides is clear. It reamins to be checked that $\mathrm{inv}(\mathfrak{g})$ really contains the invariant polynomials. In the ordinary definition a $\mathfrak{g}$-invariant polynomial is a $d_{\mathrm{W}(\mathfrak{g})}$-closed element in $\wedge^\bullet(\mathfrak{g}^*[1])$. Hence one only needs to check that all elements in $\wedge^\bullet(\mathfrak{g}^*[1])$ with the property that their image under $d_{\mathrm{W}(\mathfrak{g})}$ is again in $\wedge^\bullet(\mathfrak{g}^*[1])$ are in fact already closed. This can be seen for instance in components, using the description of $\mathrm{W}(\mathfrak{g})$ given in 2.1.1. $\square$

cocycle | transgression element | inv. polynomial

$$\begin{tikzcd} G \arrow[r, hook, "i"] & EG \arrow[r, two heads, "p"] & BG \end{tikzcd}$$

$$\begin{tikzcd} & 0 & \\ 0 & p^*P \arrow[u, mapsto, "d"] & P \arrow[l, mapsto, "p^*"'] \\ \mu \arrow[u, mapsto, "d"] & \mathrm{cs} \arrow[l, mapsto, "i^*"'] \arrow[u, mapsto, "d"] & \end{tikzcd}$$

Figure 1: **Lie algebra cocycles, invariant polynomials and transgression forms** in terms of cohomology of the universal G-bundle. Let G be a simply connected compact Lie group with Lie algebra $\mathfrak{g}$. Then invariant polynomials P on $\mathfrak{g}$ correspond to elements in the cohomology $H^\bullet(BG)$ of the classifying space of G. When pulled back to the total space of the universal G-bundle $EG \to BG$, these classes become trivial, due to the contractability of EG: $p^*P = d(\mathrm{cs})$. Lie algebra cocycles, on the other hand, correspond to elements in the cohomology $H^\bullet(G)$ of G itself. A cocycle $\mu \in H^\bullet(G)$ is in transgression with an invariant polynomial $P \in H^\bullet(BG)$ if $\mu = i^*\mathrm{cs}$.

Remark. For ordinary Lie algebras $\mathfrak{g}$ corresponding to a simply connected compact Lie group G, the situation is often discussed in terms of the cohomology of the universal G-bundle. This is recalled in figure 1 and in 2.3.1. The general definition above is a precise analog of that familiar situation: $\mathrm{W}(\mathfrak{g})$ plays the role of the algebra of (left invariant) differential forms on the universal $\mathfrak{g}$-bundle and $\mathrm{CE}(\mathfrak{g})$ plays the role of the algebra of (left invariant) differential forms on its fiber. Then $\mathrm{inv}(\mathfrak{g})$ plays the role of differential forms on the base, $BG = EG/G$.

In fact, for G a compact and simply connected Lie group and $\mathfrak{g}$ its Lie algebra, we have

$$H^\bullet(\mathrm{inv}(\mathfrak{g})) \simeq H^\bullet(BG, \mathbb{R})\,. \tag{65}$$

In summary, the situation we thus obtain is that depicted in figure **??**.

Compare this to the following fact.

Proposition 7 *For $p : P \to X$ a principal G-bundle, let $\mathrm{vert}(P) \subset \Gamma(TP)$ be the vertical vector fields on P. The horizontal differential forms on P which are invariant under $\mathrm{vert}(P)$ are precisely those that are pulled back along p from X.*

These are called the **basic differential forms** in [23].

Proposition 8 *For every invariant polynomial $P \in \wedge^\bullet \mathfrak{g}[1] \subset \mathrm{W}(\mathfrak{g})$ on an L_∞-algebra $\mathfrak{g}$ such that $d_{\mathrm{W}(\mathfrak{g})} p^*P = 0$, there exists an L_∞-algebra cocycle $\mu \in \mathrm{CS}(\mathfrak{g})$ that transgresses to P.*

Proof. This is a consequence of proposition 2 and proposition 1. Let $P \in W(\mathfrak{g})$ be a characteristic polynomial. By proposition 2, p^*P is in the kernel of the restriction homomorphism $\mathrm{CE}(\mathfrak{g}) \xleftarrow{i^*} \mathrm{W}(\mathfrak{g})$: $i^*P = 0$. By proposition 1, p^*P is the image under $d_{\mathrm{W}(\mathfrak{g})}$ of an element $\mathrm{cs} := \tau(p^*P)$ and by the algebra homomorphism property of i^* we know that its restriction, $\mu := i^*\mathrm{cs}$, to the fiber is closed, because

$$d_{\mathrm{CE}(\mathfrak{g})} i^*\mathrm{cs} = i^* d_{\mathrm{W}(\mathfrak{g})}\mathrm{cs} = i^* p^* P = 0\,. \tag{66}$$

Therefore μ is an L_∞-algebra cocycle for $\mathfrak{g}$ that transgresses to the invariant polynomial P. $\square$

$$\begin{tikzcd}
\text{cocycle} & \text{transgression element} & \text{inv. polynomial} \\
\mathrm{CE}(\mathfrak{g}) & \mathrm{W}(\mathfrak{g}) \arrow[l, two heads, "i^*"'] & \mathrm{inv}(\mathfrak{g}) \arrow[l, hook', "p^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 & \\
0 & p^*P \arrow[u, mapsto, "d_{\mathrm{W}(\mathfrak{g})}"'] \arrow[d, bend right, "\tau"'] & P \arrow[l, mapsto, "p^*"] \\
\mu \arrow[u, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"'] & \mathrm{cs} \arrow[l, mapsto, "i^*"'] \arrow[u, mapsto, "d_{\mathrm{W}(\mathfrak{g})}"'] &
\end{tikzcd}$$

Figure 2: **The homotopy operator** τ is a contraction homotopy for $\mathrm{W}(\mathfrak{g})$. Acting with it on a closed invariant polynomial $P \in \mathrm{inv}(\mathfrak{g}) \subset \wedge^\bullet \mathfrak{g}[1] \subset \mathrm{W}(\mathfrak{g})$ produces an element $\mathrm{cs} \in W(\mathfrak{g})$ whose "restriction to the fiber" $\mu := i^*\mathrm{cs}$ is necessarily closed and hence a cocycle. We say that cs induces the *transgression* from μ to P, or that P *suspends* to μ.

Remark. Notice that this statement is useful only for *indecomposable* invariant polynomials. All others trivially suspend to the 0 cocycle.

Proposition 9 *No nontrivial indecomposable invariant polynomial suspends to a Lie algebra cocycle which is a coboundary.*

Proof. Let P be an indecomposable invariant polynomial, cs the corresponding transgression element and $\mu = i^*\mathrm{cs}$ the corresponding cocycle. Assume that μ is a coboundary in that $\mu = d_{\mathrm{CE}(\mathfrak{g})}b$ for some $b \in \mathrm{CE}(\mathfrak{g})$. Then by the definition of $d_{\mathrm{W}(\mathfrak{g})}$ it follows that $\mu = i^*(d_{\mathrm{W}(\mathfrak{g})}b)$.

Now notice that

$$\mathrm{cs}' := \mathrm{cs} - d_{\mathrm{W}(\mathfrak{g})}b$$

is another transgression element for P, since

$$d_{\mathrm{W}(\mathfrak{g})}\mathrm{cs}' = p^*P\,.$$

But now

$$i^*(\mathrm{cs}') = i^*(\mathrm{cs} - d_{\mathrm{W}(\mathfrak{g})}b) = 0\,,$$

which means that p^*P is a coboundary in $\ker(i^*)$. By definition 9 this means that P is 0. $\square$

Remark. This fact is the reason for removing exact elements in $\ker(i^*)$ from the definition of $\mathrm{inv}(\mathfrak{g})$ in definition 9.

2.3.1 Examples

To put our general considerations for L_∞-algebras into perspective, it is useful to keep the following classical results for ordinary Lie algebras in mind.

The cohomologies of G and of BG in terms of qDGCAs. A classical result of E. Cartan says that for a connected finite dimensional Lie group G, the cohomology $H^\bullet(G)$ of the group is isomorphic to that of the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$:

$$H^\bullet(G) \simeq H^\bullet(\mathrm{CE}(\mathfrak{g}))\,, \tag{67}$$

namely to the algebra of Lie algebra cocycles on $\mathfrak{g}$. If we denote by Q_G the space of *indecomposable* such cocycles, and form the qDGCA $\wedge^\bullet Q_G = H^\bullet(\wedge^\bullet Q_G)$ with trivial differential, the above says that we have an isomorphism in cohomology

$$H^\bullet(G) \simeq H^\bullet(\wedge^\bullet Q_G) = \wedge^\bullet Q_G \tag{68}$$

which is realized by the canonical inclusion

$$i: \wedge^\bullet Q_G \hookrightarrow \mathrm{CE}(\mathfrak{g}) \tag{69}$$

of all cocycles into the Chevalley-Eilenberg algebra.

Subsequently, we have the classical result of Borel: For a connected finite dimensional Lie group G, the cohomology of its classifying space BG is a finitely generated polynomial algebra on even dimensional generators:

$$H^\bullet(BG) \simeq \wedge^\bullet P_G\,. \tag{70}$$

Here P_G is the space of *indecomposable* invariant polynomials on $\mathfrak{g}$, hence

$$H^\bullet(BG) \simeq H^\bullet(\mathrm{inv}(\mathfrak{g}))\,. \tag{71}$$

In fact, P_G and Q_G are isomorphic after a shift:

$$P_G \simeq Q_G[1] \tag{72}$$

and this isomorphism is induced by *transgression* between indecomposable cocycles $\mu \in \mathrm{CE}(\mathfrak{g})$ and indecomposable invariant polynomials $P \in \mathrm{inv}(\mathfrak{g})$ via a transgression element $\mathrm{cs} = \tau P \in \mathrm{W}(\mathfrak{g})$.

Invariant polynomials on strict Lie 2-algebras. Let $\mathfrak{g}_{(2)} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g} \xrightarrow{\alpha} \mathrm{der}(\mathfrak{h}))$ be a strict Lie 2-algebra as described in section 2.1. Notice that there is a canonical projection homomorphism

$$\mathrm{CE}(\mathfrak{g}) \xleftarrow{\;j^*\;} \mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \tag{73}$$

which, of course, extends to the Weil algebras

$$\mathrm{W}(\mathfrak{g}) \xleftarrow{\;j^*\;} \mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})\ . \tag{74}$$

Here j^* is simply the identity on $\mathfrak{g}^*$ and on $\mathfrak{g}^*[1]$ and vanishes on $\mathfrak{h}^*[1]$ and $\mathfrak{h}^*[2]$.

Proposition 10 *Every invariant polynomial $P \in \mathrm{inv}(\mathfrak{g})$ of the ordinary Lie algebra $\mathfrak{g}$ lifts to an invariant polynomial on the Lie 2-algebra $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$:*

$$\begin{tikzcd} \mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \arrow[d, two heads, "i^*"] & \\ \mathrm{W}(\mathfrak{g}) & \mathrm{inv}(\mathfrak{g}) \arrow[l, hook'] \arrow[ul, dashed, hook'] \end{tikzcd}\ . \tag{75}$$

However, a closed invariant polynomial will not necessarily lift to a closed one.

Proof. Recall that $d_t := d_{\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$ acts on $\mathfrak{g}^*$ as

$$d_t|_{\mathfrak{g}^*} = [\cdot,\cdot]^*_{\mathfrak{g}} + t^* \,. \tag{76}$$

By definition 4 and definition 5 it follows that $d_{\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$ acts on $\mathfrak{g}^*[1]$ as

$$d_{\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}|_{\mathfrak{g}^*[1]} = -\sigma \circ [\cdot,\cdot]^*_{\mathfrak{g}} - \sigma \circ t^* \tag{77}$$

and on $\mathfrak{h}^*[1]$ as

$$d_{\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}|_{\mathfrak{h}^*[1]} = -\sigma \circ \alpha^* \,. \tag{78}$$

Then notice that

$$(\sigma \circ t^*) : \mathfrak{g}^*[1] \to \mathfrak{h}^*[2] \,. \tag{79}$$

But this means that $d_{\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$ differs from $d_{\mathrm{W}(\mathfrak{g})}$ on $\wedge^\bullet(\mathfrak{g}^*[1])$ only by elements that are annihilated by vertical ι_X. This proves the claim. □

It may be easier to appreciate this proof by looking at what it does in terms of a chosen basis.

Same discussion in terms of a basis. Let $\{t^a\}$ be a basis of $\mathfrak{g}^*$ and $\{b^i\}$ be a basis of $\mathfrak{h}^*[1]$. Let $\{C^a{}_{bc}\}$, $\{\alpha^i{}_{aj}\}$, and $\{t^a{}_i\}$, respectively, be the components of $[\cdot,\cdot]_{\mathfrak{g}}$, α and t in that basis. Then corresponding to $\mathrm{CE}(\mathfrak{g})$, $\mathrm{W}(\mathfrak{g})$, $\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$, and $\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$, respectively, we have the differentials

$$d_{\mathrm{CE}(\mathfrak{g})} : t^a \mapsto -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c, \tag{80}$$

$$d_{\mathrm{W}(\mathfrak{g})} : t^a \mapsto -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + \sigma t^a, \tag{81}$$

$$d_{\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} : t^a \mapsto -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + t^a{}_i b^i, \tag{82}$$

and

$$d_{\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} : t^a \mapsto -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + t^a{}_i b^i + \sigma t^a. \tag{83}$$

Hence we get

$$d_{\mathrm{W}(\mathfrak{g})} : \sigma t^a \mapsto -\sigma(-\frac{1}{2} C^a{}_{bc} t^b \wedge t^c) = C^a{}_{bc} (\sigma t^b) \wedge t^c \tag{84}$$

as well as

$$d_{\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} : \sigma t^a \mapsto -\sigma(-\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + t^a{}_i b^i) = C^a{}_{bc} (\sigma t^b) \wedge t^c + t^a{}_i \sigma b^i \,. \tag{85}$$

Then if

$$P = P_{a_1 \cdots a_n} (\sigma t^{a_1}) \wedge \cdots \wedge (\sigma t^{a_n}) \tag{86}$$

is $d_{\mathrm{W}(\mathfrak{g})}$-closed, i.e. an invariant polynomial on $\mathfrak{g}$, then it follows that

$$d_{\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} P = n P_{a_1, a_2, \cdots a_n} (t^{a_1}{}_i \sigma b^i) \wedge (\sigma t^{a_n}) \wedge \cdots \wedge (\sigma t^{a_n}) \,. \tag{87}$$

The right hand side is annihilated by vertical τ (all terms appearing are in the image of the shifting isomorphism σ), hence P is also an invariant polynomial on $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$. □

We will see a physical application of this fact in 2.6.

Remark. Notice that the invariant polynomials P lifted from $\mathfrak{g}$ to $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ this way are no longer *closed*, in general. This is a new phenomenon we encounter for higher L_∞-algebras. While, according to proposition 6, for $\mathfrak{g}$ an ordinary Lie algebra all elements in $\mathrm{inv}(\mathfrak{g})$ are closed, this is no longer the case here: the lifted elements P above vanish only after we hit with them with both $d_{\mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$ *and* a vertical τ.

2.4 L_∞-algebras from cocycles: String-like extensions

We now consider the main object of interest here: families of L_∞-algebras that are induced from L_∞-cocycles and invariant polynomials. First we need the following

Definition 11 (String-like extensions of L_∞-algebras) *Let $\mathfrak{g}$ be an L_∞-algebra.*

- *For each degree $(n+1)$-cocycle μ on $\mathfrak{g}$, let $\mathfrak{g}_\mu$ be the L_∞-algebra defined by*

$$\mathrm{CE}(\mathfrak{g}_\mu) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathbb{R}[n]), d_{\mathrm{CE}(\mathfrak{g}_\mu)}) \tag{88}$$

  with differential given by

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)}|_{\mathfrak{g}^*} := d_{\mathrm{CE}(\mathfrak{g})}, \tag{89}$$

  and

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)})|_{\mathbb{R}[n]} : b \mapsto -\mu\,, \tag{90}$$

  where $\{b\}$ denotes the canonical basis of $\mathbb{R}[n]$. This we call the **String-like extension** of $\mathfrak{g}$ with respecto to μ, because, as described below in 2.4.1, it generalizes the construction of the String Lie 2-algebra.

- *For each degree n invariant polynomial P on $\mathfrak{g}$, let $\mathrm{ch}_P(\mathfrak{g})$ be the L_∞-algebra defined by*

$$\mathrm{CE}(\mathrm{ch}_P(\mathfrak{g})) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus \mathbb{R}[2n-1]), d_{\mathrm{CE}(\mathrm{ch}_P(\mathfrak{g}))}) \tag{91}$$

  with the differential given by

$$d_{\mathrm{CE}(\mathrm{ch}_P(\mathfrak{g}))}|_{\mathfrak{g}^* \oplus \mathfrak{g}^*[1]} := d_{\mathrm{W}(\mathfrak{g})} \tag{92}$$

  and

$$d_{\mathrm{CE}(\mathrm{ch}_P(\mathfrak{g}))})|_{\mathbb{R}[2n-1]} : c \mapsto P\,, \tag{93}$$

  where $\{c\}$ denotes the canonical basis of $\mathbb{R}[2n-1]$. This we call the **Chern L_∞-algebra** corresponding to the invariant polynomial P, because, as described below in 2.5.1, connections with values in it pick out the Chern-form corresponding to P.

- *For each degree $2n-1$ transgression element* cs, *let $\mathrm{cs}_P(\mathfrak{g})$ be the L_∞-algebra defined by*

$$\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus \mathbb{R}[2n-2] \oplus \mathbb{R}[2n-1]), d_{\mathrm{CE}(\mathrm{ch}_P(\mathfrak{g}))}) \tag{94}$$

  with

$$d_{\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))}|_{\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])} = d_{\mathrm{W}(\mathfrak{g})} \tag{95}$$

$$d_{\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))}|_{\mathbb{R}[2n-2]} : b \mapsto -\mathrm{cs} + c \tag{96}$$

$$d_{\mathrm{CE}(\mathrm{ch}_p(\mathfrak{g}))}|_{\mathbb{R}[2n-1]} : c \mapsto P\,, \tag{97}$$

  where $\{b\}$ and $\{c\}$ denote the canonical bases of $\mathbb{R}[2n-2]$ and $\mathbb{R}[2n-1]$, respectively. This we call the **Chern-Simons L_∞-algebra** with respect to the transgression element cs, because, as described below in 2.5.1, connections with values in these come from (generalized) Chern-Simons forms.

The nilpotency of these differentials follows directly from the very definition of L_∞-algebra cocycles and invariant polynomials.

Proposition 11 (the string-like extensions) *For each L_∞-cocycle $\mu \in \wedge^n(\mathfrak{g}^*)$ of degree n, the corresponding String-like extension sits in an exact sequence*

$$0 \longleftarrow \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \twoheadleftarrow \mathrm{CE}(\mathfrak{g}_\mu) \hookleftarrow \mathrm{CE}(\mathfrak{g}) \longleftarrow 0$$

Proof. The morphisms arE the canonical inclusion and projection. ☐

Proposition 12 *For* $\mathrm{cs} \in \mathrm{W}(\mathfrak{g})$ *any transgression element interpolating between the cocycle* $\mu \in \mathrm{CE}(\mathfrak{g})$ *and the invariant polynomial* $P \in \wedge^\bullet(\mathfrak{g}[1]) \subset \mathrm{W}(\mathfrak{g})$*, we obtain a homotopy-exact sequence*

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{g}_\mu) & \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) \arrow[l, two heads] \arrow[d, dash, "\simeq"] & \mathrm{CE}(\mathrm{ch}_P(\mathfrak{g})) \arrow[l, hook'] \\ & \mathrm{W}(\mathfrak{g}_\mu) & \end{tikzcd} \; . \tag{98}$$

Here the isomorphism

$$f : \; \mathrm{W}(\mathfrak{g}_\mu) \xrightarrow{\simeq} \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) \tag{99}$$

is the identity on $\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus \mathbb{R}[n]$

$$f|_{\mathfrak{g}^*\oplus\mathfrak{g}^*[1]\oplus\mathbb{R}[n]} = \mathrm{Id} \tag{100}$$

and acts as

$$f|_{\mathbb{R}[n+1]} : b \mapsto c + \mu - \mathrm{cs} \tag{101}$$

for b the canonical basis of $\mathbb{R}[n]$ and c that of $\mathbb{R}[n+1]$. We check that this does respect the differentials

$$\begin{tikzcd} b \arrow[r, mapsto, "d_{\mathrm{W}(\mathfrak{g}_\mu)}"] \arrow[d, mapsto, "f"] & -\mu + c \arrow[d, mapsto, "f"] \\ b \arrow[r, mapsto, "d_{\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))}"] & -\mathrm{cs} + c \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, mapsto, "d_{\mathrm{W}(\mathfrak{g}_\mu)}"] \arrow[d, mapsto, "f"] & \sigma\mu \arrow[d, mapsto, "f"] \\ c + \mu - \mathrm{cs} \arrow[r, mapsto, "d_{\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))}"] & \sigma\mu \end{tikzcd} \; . \tag{102}$$

Recall from definition 22 that σ is the canonical isomorphism $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*[1]$ extended by 0 to $\mathfrak{g}^*[1]$ and then as a derivation to all of $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$.

Here the morphism between the Weil algebra of $\mathfrak{g}_\mu$ and the Chevalley-Eilenberg algebra of $\mathrm{cs}_P(\mathfrak{g})$ is indeed an *iso*morphism (not just an equivalence). This isomorphism exhibits one of the main points to be made here: it makes manifest that the invariant polynomial P that is related by transgression to the cocycle μ which induces $\mathfrak{g}_\mu$ becomes exact with respect to $\mathfrak{g}_\mu$. This is the statement of proposition 14 below.

L_∞-algebra cohomology and invariant polynomials of String-like extensions. The L_∞-algebra $\mathfrak{g}_\mu$ obtained from an L_∞-algebra $\mathfrak{g}$ with an L_∞-algebra cocycle $\mu \in H^\bullet(\mathrm{CE}(\mathfrak{g}))$ can be thought of as being obtained from $\mathfrak{g}$ by "killing" a cocycle μ. This is familiar from Sullivan models in rational homotopy theory.

Proposition 13 *Let $\mathfrak{g}$ be an ordinary semisimple Lie algebra and μ a cocycle on it. Then*

$$H^\bullet(\mathrm{CE}(\mathfrak{g}_\mu)) = H^\bullet(\mathrm{CE}(\mathfrak{g}))/\langle\mu\rangle \, . \tag{103}$$

Accordingly, one finds that, in cohomology, the invariant polynomials on $\mathfrak{g}_\mu$ are those of $\mathfrak{g}$, except for that one to which μ transgresses.

Proposition 14 *Let $\mathfrak{g}$ be an ordinay semisiple Lie algebra and $\mu \in H^\bullet(\mathrm{CE}(\mathfrak{g}))$ a class which is necessarily of odd degree, so that $\mu \wedge \mu = 0$ automatically. Let μ be in transgression with the invariant polynomial $P \in \mathrm{inv}(\mathfrak{g})$. Then*

$$H^\bullet(\mathrm{inv}(\mathfrak{g}_\mu)) = H^\bullet(\mathrm{inv}(\mathfrak{g}))/\langle P \rangle \,. \tag{104}$$

Proof. The point is that P is still closed in $\mathrm{W}(\mathfrak{g}_\mu)$, but now it is also exact in $\ker(i^*)$. This is a corollary of 9. $\square$

2.4.1 Examples

The String Lie 2-algebra.

Definition 12 *Let $\mathfrak{g}$ be a semisiple Lie algebra and $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ the canonical 3-cocycle on it. Then*

$$\mathrm{string}(\mathfrak{g}) \tag{105}$$

is defined to be the strict Lie 2-algebra coming from the crossed module

$$(\hat{\Omega}\mathfrak{g} \to P\mathfrak{g}) \,, \tag{106}$$

where $P\mathfrak{g}$ is the Lie algebra of based paths in $\mathfrak{g}$ and $\hat{\Omega}\mathfrak{g}$ the Lie algebra of based loops in $\mathfrak{g}$, with central extension induced by μ.

Proposition 15 ([3]) *The Lie 2-algebra $\mathfrak{g}_\mu$ obtained from $\mathfrak{g}$ and μ as in definition 11 is equivalent to the strict string Lie 2-algebra*

$$\mathfrak{g}_\mu \simeq \mathrm{string}(\mathfrak{g}) \,. \tag{107}$$

This means there are morphisms $\mathfrak{g}_\mu \to \mathrm{string}(\mathfrak{g})$ and $\mathrm{string}(\mathfrak{g}) \to \mathfrak{g}_\mu$ whose composite is the identity only up to homotopy

$$\begin{tikzcd} \mathfrak{g}_\mu \arrow[r] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathrm{string}(\mathfrak{g}) \arrow[r] \arrow[d, phantom, "\Downarrow="] & \mathfrak{g}_\mu \\ & {} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{string}(\mathfrak{g}) \arrow[r] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathfrak{g}_\mu \arrow[r] \arrow[d, phantom, "\Downarrow\eta"] & \mathrm{string}(\mathfrak{g}) \\ & {} & \end{tikzcd}$$

We call $\mathfrak{g}_\mu$ the *skeletal* and $\mathrm{string}(\mathfrak{g})$ the *strict* version of the String Lie 2-algebra.

The Fivebrane Lie 6-algebra

Definition 13 *Let $\mathfrak{g} = \mathrm{so}(n)$ and μ the canonical 7-cocycle on it. Then*

$$\mathrm{fivebrane}(\mathfrak{g}) \tag{108}$$

is defined to be the strict Lie 7-algebra which is equivalent to $\mathfrak{g}_\mu$

$$\mathfrak{g}_\mu \simeq \mathrm{fivebrane}(\mathfrak{g}) \,. \tag{109}$$

A Lie n-algebra is *strict* if it corresponds to a differential graded Lie algebra on a vector space in degree 1 to n. (Recall our grading conventions from 2.1.)

Remark. It is a major open problem to identify the strict fivebrane($\mathfrak{g}$). Proposition 15 suggests that it might involve hyperbolic Kac-Moody algebras and/or the torus algebra of $\mathfrak{g}$, since these would seem to be what comes beyond the affine Kac-Moody algebras relevant for string(n).

2.5 L_∞-algebra valued forms

Consider a connection form A regarded as a linear map

$$\mathfrak{g}^* \to \Omega^1(Y).$$

Since $CE(\mathfrak{g})$ is free as a graded commutative algebra, this linear map extends uniquely to a morphism of graded commutative algebras, though not in general of differential graded commutative algebra. In fact, the deviation is measured by the *curvature* F_A of the connection. However, the differential in $\mathrm{W}(\mathfrak{g})$ is precisely such that the connection does extend to a morphism of differential graded-commutative algebras

$$\mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(Y)\,.$$

A good notion of a $\mathfrak{g}$-valued differential form on a smooth space X is a morphism of differential graded-commutative algebras from the Weil algebra of $\mathfrak{g}$ to the algebra of differential forms on X.

Definition 14 *For Y a smooth space and $\mathfrak{g}$ an L_∞-algebra, we call*

$$\Omega^\bullet(Y,\mathfrak{g}) := \mathrm{Hom}_{\mathrm{dgc-Alg}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y)) \tag{110}$$

the space of **$\mathfrak{g}$-valued differential forms on** X *and*

$$\Omega^\bullet_{\mathrm{flat}}(Y,\mathfrak{g}) := \mathrm{Hom}_{\mathrm{dgc-Alg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y)) \tag{111}$$

the space of **flat $\mathfrak{g}$-valued differential forms on** Y*.*

Curvature. By pullback along the canonical surjection $\mathrm{W}(\mathfrak{g}) \twoheadrightarrow \mathrm{CE}(\mathfrak{g})$ the space of flat $\mathfrak{g}$-valued forms is injected into the space of all $\mathfrak{g}$-valued forms:

$$\Omega^\bullet_{\mathrm{flat}}(Y,\mathfrak{g}) \subset \Omega^\bullet(Y,\mathfrak{g})\,. \tag{112}$$

Usually we write $\mathfrak{g}$-valued differential forms as

$$(\ \Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g})\ ) \ \in\ \Omega^\bullet(Y,\mathfrak{g})\,, \tag{113}$$

where F_A denotes the restriction to the shifted copy $\mathfrak{g}^*[1]$ given by

$$\mathrm{curv} : (\ \Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g})\ ) \ \mapsto\ \begin{tikzcd} (\ \Omega^\bullet(Y) & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] & \mathfrak{g}^*[1]\ )\,. \arrow[l, hook'] \arrow[ll, bend right, "F_A"'] \end{tikzcd} \tag{114}$$

Since

$$(A,F_A) \in \Omega^\bullet_{\mathrm{flat}}(X,\mathfrak{g}) \ \Leftrightarrow\ F_A = 0 \tag{115}$$

we say that F_A **is the curvature** of the $\mathfrak{g}$-valued form (A,F_A).

Hence precisely when the curvature vanishes is the $\mathfrak{g}$-valued form flat. This is indicated by the following diagram.

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{g}) \arrow[d, dotted, "{(A',F_{A'}=0)}"] & \mathrm{W}(\mathfrak{g}) \arrow[l, two heads] \arrow[d, "{(A,F_A)}"] \\ \Omega^\bullet(Y) \arrow[r, no head, "="] & \Omega^\bullet(Y) \end{tikzcd}$$

Bianchi identity. Recall from 2.1 that the Weil algebra $\mathrm{W}(\mathfrak{g})$ of an L_∞-algebra $\mathfrak{g}$ is the same as the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathrm{inn}(\mathfrak{g}))$ of the L_∞-algebra of inner derivation of $\mathfrak{g}$.

It follows that $\mathfrak{g}$-valued differential forms on Y are the same as *flat* $\mathrm{inn}(\mathfrak{g})$-valued differential forms on Y:

$$\Omega^\bullet(Y,\mathfrak{g}) = \Omega^\bullet_{\mathrm{flat}}(\mathrm{inn}(\mathfrak{g}))\,. \tag{116}$$

By the above definition of curvature, this says that the curvature F_A of a $\mathfrak{g}$-valued connection (A, F_A) is itself a flat $\mathrm{inn}(\mathfrak{g})$-valued connection. This is the generalization of the ordinary *Bianchi identity* to L_∞-algebra valued forms.

Definition 15 *Two $\mathfrak{g}$-valued forms $A, A' \in \Omega^\bullet(Y,\mathfrak{g})$ are called* **(gauge) equivalent** *precisely if they are related by a* vertical *concordance, i.e. by a concordance, such that the corresponding derivation τ from proposition 5 is vertical.*

2.5.1 Examples

1. Ordinary Lie-algebra valued 1-forms. We have already mentioned ordinary Lie algebra valued 1-forms in this general context in **??**.

2. Crossed module valued forms. Let $\mathfrak{g}_{(2)} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ be a strict Lie 2-algebra coming from a crossed module. Then a $\mathfrak{g}_{(2)}$-valued form is an ordinary $\mathfrak{g}$-valued 1-form A and an ordinary $\mathfrak{h}$-valued 2-form B. The corresponding curvature is an ordinary $\mathfrak{g}$-valued 2-form $\beta = F_A + t(B)$ and an ordinary $\mathfrak{h}$-valued 3-form $H = d_A B$. This is denoted by the right vertical arrow in the following diagram.

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \arrow[d, dotted, "{(A,B)}"', "{F_A+t(B)=0}"] & & \mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \arrow[ll, two heads] \arrow[d, "{(A,B,\beta,H)}"', "{\beta=F_A+t(B),\ H=d_A B}"] \\
\Omega^\bullet(Y) \arrow[rr, dash, "="] & & \Omega^\bullet(Y)
\end{tikzcd}\,. \tag{117}$$

Precisely if the curvature components β and H vanish, does this morphism on the right factor through $\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$, which is indicated by the left vertical arrow of the above diagram.

3. String Lie n-algebra valued forms. For $\mathfrak{g}$ an ordinary Lie algebra and μ a degree $(2n+1)$-cocycle on $\mathfrak{g}$ the situation is captured by the following diagram

$$\begin{tikzcd}
 & \text{String-like} & \text{Chern-Simons} & \text{Chern} \\
1 & 2n & 2n+1 & 2n+1 \\
\mathrm{CE}(\mathfrak{g}) \arrow[r, hook] \arrow[d, "(A)"', "F_A=0"] & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[d, dotted, "{(A,B)}"', "{F_A=0,\ dB+\mathrm{CS}_k(A)=0}"] & \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) \arrow[l, two heads] \arrow[d, "{(A,B,C)}"', "{C=dB+\mathrm{CS}_P(A)}"] & \mathrm{CE}(\mathrm{ch}_P(\mathfrak{g})) \arrow[l, hook'] \arrow[d, "{(A,C)}"', "{dC=k((F_A)^{n+1})}"] \\
\Omega^\bullet(Y) \arrow[r, dash, "="] & \Omega^\bullet(Y) \arrow[r, dash, "="] & \Omega^\bullet(Y) \arrow[r, dash, "="] & \Omega^\bullet(Y)
\end{tikzcd}\,. \tag{118}$$

Here $\mathrm{CS}_P(A)$ denotes the Chern-Simons form such that $d\mathrm{CS}_P(A) = P(F_A)$, given by the specific contracting homotopy.

The standard example is that corresponding to the ordinary String-extension.

$$
\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) \arrow[r, hook] \arrow[d, equal] & \mathrm{CE}(\mathrm{string}(\mathfrak{g})) \arrow[d, equal, "\simeq"] & \mathrm{W}(\mathrm{string}_k(\mathfrak{g})) \arrow[l, two heads] \arrow[d, equal, "\simeq"] & \\
\mathrm{CE}(\mathfrak{g}) \arrow[r, hook] \arrow[d, "(A)"', "F_A=0"] & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[d, dotted, "(A,B)"', "{F_A=0 \atop dB+\mathrm{CS}_P(A)=0}"] & \mathrm{CE}(\mathrm{cs}_k(\mathfrak{g})) \arrow[l, two heads] \arrow[d, "(A,B,C)"', "C=dB+\mathrm{CS}_P(A)"] & \mathrm{CE}(\mathrm{ch}_P(\mathfrak{g})) \arrow[l, hook'] \arrow[d, "(A,C)"', "dC=\langle F_A\wedge F_A\rangle"] \\
\Omega^\bullet(Y) \arrow[r, dash, "="] & \Omega^\bullet(Y) \arrow[r, dash, "="] & \Omega^\bullet(Y) \arrow[r, dash, "="] & \Omega^\bullet(Y)
\end{tikzcd}
\tag{119}
$$

Here $\mathfrak{g}$ is semisimple with invariant bilinear form $P = \langle\cdot,\cdot,\cdot\rangle$ related by transgression to the 3-cocycle $\mu = \langle\cdot,[\cdot,\cdot]\rangle$. Then the Chern-Simons 3-form for any $\mathfrak{g}$-valued 1-form A is

$$
\mathrm{CS}_{\langle\cdot,\cdot\rangle}(A) = \langle A \wedge dA\rangle + \frac{1}{3}\langle A \wedge [A \wedge A]\rangle\,. \tag{120}
$$

2.6 L_∞-algebra characteristic forms

Definition 16 *For*

$$
\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \tag{121}
$$

any $\mathfrak{g}$-valued differential form, we call the composite

$$
\begin{tikzcd}
\Omega^\bullet(Y) & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] & \mathrm{inv}(\mathfrak{g}) \arrow[l, hook'] \arrow[ll, bend right, "{\{P(F_A)\}}"']
\end{tikzcd}
\tag{122}
$$

the tion of **characteristic forms** *of the $\mathfrak{g}$-valued form A. The deRham classes $[P(F_A)]$ of the characteristic forms arising as the image of closed invariant polynomials*

$$
\begin{tikzcd}
\Omega^\bullet(Y) & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] & \mathrm{inv}(\mathfrak{g}) \arrow[l, hook'] \arrow[ll, bend right, "{\{P_i(F_A)\}}"']
\end{tikzcd}
\tag{123}
$$

$$
H^\bullet_{\mathrm{dR}}(Y) \xleftarrow{\{[P(F_A)]\}} H^\bullet(\mathrm{inv}(\mathfrak{g}))
$$

we call the collection of **characteristic classes** *of the $\mathfrak{g}$-valued form A.*

Notice that Y will play the role of a cover of some space X soon, and that characteristic forms really live down on X. We will see shortly a constraint imposed which makes the characteristic forms descend down from the Y here to such an X.

Proposition 16 *Under gauge transformations as in definition 15, characteristic classes are invariant.*

Proof. This follows from proposition 5:

By that proposition, the derivative of the concordance form $\hat{A}$ along the interval $I = [0,1]$ is a chain homotopy

$$
\frac{d}{dt}\hat{A}(P) = [d,\iota_X]P = d\tau(P) + \iota_X(d_{\mathrm{W}(\mathfrak{g})}P)\,. \tag{124}
$$

By definition of gauge-transformations, ι_X is vertical. By definition of basic forms, P is both in the kernel of ι_X as well as in the kernel of $\iota_X \circ d$.

Hence the right hand vanishes. □

2.6.1 Examples

Proposition 17 *A $b^{n-1}\mathfrak{u}(1)$-valued form* $\Omega^\bullet(Y) \xleftarrow{A} \mathrm{W}(b^{n-1}\mathfrak{u}(1))$ *is precisely an n-form on Y:*

$$\Omega^\bullet(Y, b^{n-1}\mathfrak{u}(1)) \simeq \Omega^n(Y)\,. \tag{125}$$

If two such $b^{n-1}\mathfrak{u}(1)$-valued forms are gauge equivalent according to definition 15, then their curvatures coincide

$$(\,\Omega^\bullet(Y) \xleftarrow{A} \mathrm{W}(b^{n-1}\mathfrak{u}(1))\,) \sim (\,\Omega^\bullet(Y) \xleftarrow{A'} \mathrm{W}(b^{n-1}\mathfrak{u}(1))\,) \quad \Rightarrow \quad dA = dA'\,. \tag{126}$$

BF-theory. We demonstrate that the expression known in the literature as the *action functional for BF-theory with cosmological term* is the integral of an invariant polynomial for $\mathfrak{g}$-valued differential forms where $\mathfrak{g}$ is a Lie 2-algebra. Namely, let $\mathfrak{g}_{(2)} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ be any strict Lie 2-algebra as in 2.1. Let

$$P = \langle \cdot, \cdot \rangle \tag{127}$$

be an invariant bilinear form on $\mathfrak{g}$, hence a degree 2 invariant polynomial on $\mathfrak{g}$. According to proposition 10, P therefore also is an invariant polynomial on $\mathfrak{g}_{(2)}$.

Now for (A, B) a $\mathfrak{g}_{(2)}$-valued differential form on X, as in the example in 2.5,

$$\Omega^\bullet(Y) \xleftarrow{((A,B),(\beta,H))} \mathrm{W}(\mathfrak{g}_{(2)})\ , \tag{128}$$

one finds

$$\begin{tikzcd} \Omega^\bullet(Y) & \mathrm{W}(\mathfrak{g}_{(2)}) \arrow[l, "{((A,B),(\beta,H))}"'] & \mathrm{inv}(\mathfrak{g}_{(2)}) \arrow[l, hook'] \arrow[ll, bend left, "{P \mapsto \langle \beta, \beta \rangle}"] \end{tikzcd} \tag{129}$$

so that the corresponding characteristic form is the 4-form

$$P(\beta, H) = \langle \beta \wedge \beta \rangle = \langle (F_A + t(B)) \wedge (F_A + t(B)) \rangle\,. \tag{130}$$

Collecting terms as

$$P(\beta, H) = \underbrace{\langle F_A \wedge F_A \rangle}_{\text{Pontryagin term}} + 2\underbrace{\langle t(B) \wedge F_A \rangle}_{\text{BF-term}} + \underbrace{\langle t(B) \wedge t(B) \rangle}_{\text{“cosmological constant”}} \tag{131}$$

we recognize the Lagrangian for topological Yang-Mills theory and BF theory with cosmological term.

For X a compact 4-manifold, the corresponding action functional

$$S : \Omega^\bullet(X, \mathfrak{g}_{(2)}) \to \mathbb{R} \tag{132}$$

sends $\mathfrak{g}_{(2)}$-valued 2-forms to the intgral of this 4-form

$$(A, B) \mapsto \int_X \left(\langle F_A \wedge F_A \rangle + 2\langle t(B) \wedge F_A \rangle + \langle t(B) \wedge t(B) \rangle \right)\,. \tag{133}$$

The first term here is usually not considered an intrinsic part of BF-theory, but its presence does not affect the critical points of S.

The critical points of S, i.e. the $\mathfrak{g}_{(2)}$-valued differential forms on X that satisfy the equations of motion defined by the action S, are given by the equation

$$F_A + t(B) = 0\,. \tag{134}$$

Notice that this implies

$$d_A t(B) = 0 \tag{135}$$

but does not constrain the full 3-curvature

$$H = d_A B \tag{136}$$

to vanish. In other words, the critical points of S are precisely the *fake flat* $\mathfrak{g}_{(2)}$-valued forms that define strict parallel transport 2-functors [20, 41, 4].

Remark. Under the equivalence [3] of the skeletal String Lie 2-alggebra to its strict version, recalled in proposition 15, the characteristic forms for strict Lie 2-algebras apply also to one of our central objects of interest here, the String 2-connections. But a little care needs to be exercised here, because the strict version of the String Lie 2-algebra is no longer finite dimensional.

Remark. Our interpretation above of BF-theory as a gauge theory for Lie 2-algebras is not unrelated to, but different from the one considered in [20, 21]. There only the Lie 2-algebra coming from the infinitesimal crossed module $(|\mathfrak{g}| \stackrel{0}{\to} \mathfrak{g} \stackrel{\mathrm{ad}}{\to} \mathrm{der}(\mathfrak{g}))$ (for $\mathfrak{g}$ any ordinary Lie algebra and $|\mathfrak{g}|$ its underlying vector space, regarded as an abelian Lie algebra) is considered, and the action is restricted to the term $\int \langle F_A \wedge B\rangle$. We can regard the above discussion as a generalization of this approach to arbitrary Lie 2-algebras. Standard BF-theory (with "cosmological" term) is reproduced with the above Lagrangian by using the Lie 2-algebra $\mathrm{inn}(\mathfrak{g})$ corresponding to the infinitesimal crossed module $(\mathfrak{g} \stackrel{\mathrm{Id}}{\to} \mathfrak{g} \stackrel{\mathrm{ad}}{\to} \mathrm{der}(\mathfrak{g}))$ discussed in 2.1.1.

3 L_∞-algebra Cartan-Ehresmann connections

We will now combine all of the above ingredients to produce a definition of $\mathfrak{g}$-valued connections. As we shall explain, the construction we give may be thought of as a generalization of the notion of a Cartan-Ehresmann connection, which is given by a Lie algebra-valued 1-form on the total space of a bundle over base space satisfying two conditions:

- first Cartan-Ehresmann condition: on the fibers the connection form restricts to a *flat* canonical form
- second Cartan-Ehresmann condition: under vertical flows the connections transforms nicely, in such a way that its characteristic forms descend down to base space.

We will essentially interpret these two conditions as a pullback of the universal $\mathfrak{g}$-bundle, in its DGC-algebraic incarnation as given in equation 61.

The definition we give can also be seen as the Lie algebraic image of a similar construction involving locally trivializable transport n-functors [4, 42], but this shall not be further discussed here.

3.1 Surjective submersions and differential forms

We need the following standard definition.

Definition 17 *Let $\pi : Y \to X$ be a smooth map. The* **vertical deRham complex** , $\Omega^\bullet_{\mathrm{vert}}(Y)$*, with respect to Y is the deRham complex of Y modulo those forms that vanish when restricted in all arguments to vector fields in the kernel of $\pi_* : \Gamma(TY) \to \Gamma(TX)$, namely to vertical vector fields. The projection we denote*

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{\;i^*\;} \Omega^\bullet(Y)\ . \tag{137}$$

The elements in $\Omega^\bullet(Y)$ in the image of the pullback

$$\Omega^\bullet(Y) \xleftarrow{\pi^*} \Omega^\bullet(X) \tag{138}$$

are called the **basic forms** *on Y.*

discussion demanded that we first discuss all things related purely to L-infty algebras and then later the theory of their connections. Therefore I am not sure what to do here. But I have now added a pointer to the discussion here to the discussion of basic forms in W(g) .**)

Notice that if $\omega \in \Omega^\bullet(Y)$ vanishes when evalutated on vertical vector fields then obviously so does $\alpha \wedge \omega$, for any $\alpha \in \Omega^\bullet(Y)$. Moreover, due to the formula

$$d\omega(v_1, \cdots, v_{n+1}) = \sum_{\sigma \in \mathrm{Sh}(1,n+1)} \pm v_{\sigma_1}\omega(v_{\sigma_2}, \cdots, v_{\sigma_{n+1}}) + \sum_{\sigma \in \mathrm{Sh}(2,n+1)} \pm \omega([v_{\sigma_1}, v_{\sigma_2}], v_{\sigma_3}, \cdots, v_{\sigma_{n+1}}) \tag{139}$$

and the fact that for v, w vertical also $[v, w]$ is vertical also $d\omega$ is vertical. Hence vertical differential forms on Y indeed form a dg-subalgebra of all forms on Y.

Proposition 18 *For $\pi : Y \to X$ a surjective submersion with* connected *fibers, the basic forms on Y are precisely those forms in the kernel of i^* that are annihilated by all vertical Lie derivatives.*

Proof. The fact that $\pi : Y \to X$ is a submersion implies that around each point $x \in X$ there is a neighbourhood $x \in U_x \subset X$ over which Y looks like a cartesian product,

$$Y|_{U_x} \simeq U_x \times F \tag{140}$$

for some F. By assumption, this F is connected. Hence any function on F that is invariant under all vector fields along F has to be constant.

So the claim is clearly true over all such neighbourhoods U_x. This implies it is also true on all of X. □

3.1.1 Examples

The possibly most familiar kinds of sujective submersions are

- Fiber bundles.

  Indeed, the standard Cartan-Ehresmann theory of connections of principal bundles is obtained in our context by fixing a Lie group G and a principal G-bundle $p : P \to X$ and then using $Y = P$ itself as the surjective submersion.

  The definition of a connection on P in terms of a $\mathfrak{g}$-valued 1-form on P can be understood as the descent data for a connection on P obtained with respect to canonical trivialization of the pullback of P to $Y = P$.

  Using for the surjective submersion Y a principal G-bundle $P \to X$ is also most convenient for studying all kinds of higher n-bundles obstructing lifts of the given G-bundle. This is why we will often make use of this choice in the following.

- Covers by open subsets.

  The disjoint union of all sets in a cover of X by open subsets of X forms a surjective submersion $\pi : Y \to X$. In large parts of the literature on descent (locally trivialized bundles), these are the only kinds of surjective submersions that are considered.

  We will find here, that in order to characterize principal n-bundles entirely in terms of L_∞-algebraic data open covers are too restrictive and the full generality of surjective submersions is needed.

The reason is that, for $\pi : Y \to X$ a cover by open subsets, there are no nontrivial vertical vector fields

$$\ker(\pi) = 0 \tag{141}$$

hence

$$\Omega^\bullet_{\mathrm{vert}}(Y) = 0\,. \tag{142}$$

With the definition of $\mathfrak{g}$-descent objects in 3.2 this implies that all $\mathfrak{g}$-descent objects over a cover by open subsets are trivial.

There are two important subclasses of surjective submersions $\pi : Y \to X$:

- those for which Y is (smoothly) contractible;
- those for which the fibers of Y are connected.

Here we say Y is (smoothly) contractible if the identity map $\mathrm{Id} : Y \to Y$ is (smoothly) homotopic to a map $Y \to Y$ which is constant on each connected component. Hence Y is a disjoint union of spaces that are each (smoothly) contractible to a point.

In this case the Poincaré lemma says that the dg-algebra $\Omega^\bullet(Y)$ of differential forms on Y is contractible: each closed form is exact:

$$\begin{tikzcd} \Omega^\bullet(Y) & & \Omega^\bullet(Y) \arrow[ll, bend right, "0"'] \arrow[ll, bend left, "{[d,\tau]}"] \arrow[ll, phantom, "\Downarrow\tau"] \end{tikzcd}\ . \tag{143}$$

Here τ is the familiar homotopy operator that appears in the proof of the Poincaré lemma.

In practice, we often make use of the best of both worlds: surjective submersions that are (smoothly) contractible to a discrete set but still have a sufficiently rich collection of vertical vector fields.

The way to obtain these is by refinement: starting with any surjective submersion $\pi : Y \to X$ which has good vertical vector fields but might not be contractible, we can cover Y itself with open balls, whose disjoint union, Y', then forms a surjective submersion $Y' \to Y$ over Y. The composite π'

$$\begin{tikzcd} Y' \arrow[rr] \arrow[dr, "\pi"'] & & Y \arrow[dl] \\ & X & \end{tikzcd} \tag{144}$$

is then a contractible surjective submersion of X. We will see that all our descent objects can be pulled back along refinements of surjective submersions this way, so that it is possible, without restriction of generality, to always work on contractible surjective submersions. Notice that for these the structure of

$$\Omega^\bullet_{\mathrm{vert}}(Y) \twoheadleftarrow \Omega^\bullet(Y) \hookleftarrow \Omega^\bullet(X) \tag{145}$$

is rather similar to that of

$$\mathrm{CE}(\mathfrak{g}) \twoheadleftarrow \mathrm{W}(\mathfrak{g}) \hookleftarrow \mathrm{inv}(\mathfrak{g})\ , \tag{146}$$

since $\mathrm{W}(\mathfrak{g})$ is also contractible, according to proposition 2.

3.2 $\mathfrak{g}$-Bundle descent data

Definition 18 ($\mathfrak{g}$-bundle descent data) *Given a Lie n-algebra $\mathfrak{g}$, a $\mathfrak{g}$-bundle descent object on X is a pair (Y, A_{vert}) consisting of a choice of surjective submersion $\pi : Y \to X$ with* connected *fibers (this condition will be dropped when we extend to $\mathfrak{g}$-connection descent objects in 3.3) together with a morphism of dg-algebras*

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) \ . \tag{147}$$

Two such descent objects are taken to be equivalent

$$(\ \Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})\ ) \sim (\ \Omega^\bullet_{\mathrm{vert}}(Y') \xleftarrow{A'_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})\ ) \tag{148}$$

precisely if their pullbacks $\pi_1^* A_{\mathrm{vert}}$ and $\pi_2^* A'_{\mathrm{vert}}$ to the common refinement

$$\begin{tikzcd} Y \times_X Y' \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d, "\pi"] \\ Y' \arrow[r, "\pi'"] & X \end{tikzcd} \tag{149}$$

are concordant in the sense of definition 7.

Thus two such descent objects A_{vert}, A'_{vert} on the same Y are equivalent if there is η^*_{vert} such that

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \Omega^\bullet_{\mathrm{vert}}(Y \times I) \arrow[l, shift right, "s^*"'] \arrow[l, shift left, "t^*"] & \mathrm{CE}(\mathfrak{g}) \arrow[l, "\eta^*_{\mathrm{vert}}"'] \arrow[ll, bend right, "A_{\mathrm{vert}}"'] \arrow[ll, bend left, "A'_{\mathrm{vert}}"] \end{tikzcd} \ . \tag{150}$$

Recall from the discussion in **??** that the surjective submersions here play the role of open covers of X.

3.2.1 Examples

Example: ordinary G-bundles. The following example is meant to illustrate how the notion of descent data with respect to a Lie algebra $\mathfrak{g}$ as defined here can be related to the ordinary notion of descent data with respect to a Lie group G. Consider the case where $\mathfrak{g}$ is an ordinary Lie (1-)algebra. A $\mathfrak{g}$-cocycle then is a surjective submersion $\pi : Y \to X$ together with a $\mathfrak{g}$-valued flat vertical 1-form A_{vert} on Y. Assume the fiber of $\pi : Y \to X$ to be simply connected. Then for any two points $(y, y') \in Y \times_X Y$ in the same fiber we obtain an element $g(y, y') \in G$, where G is the simply connected Lie group integrating $\mathfrak{g}$, by choosing any path $y \xrightarrow{\gamma} y'$ in the fiber connecting y with y' and forming the parallel transport determined by A_{vert} along this path

$$g(y, y') := P \exp(\int_\gamma A_{\mathrm{vert}}) \,. \tag{151}$$

By the flatness of A_{vert} and the assumption that the fibers of Y are simply connected

- $g : Y \times_X Y \to G$ is well defined (does not depend on the choice of paths), and
- satisfies the cocycle condition for G-bundles

$$g : \begin{tikzcd} & y' \arrow[dr] & \\ y \arrow[ur] \arrow[rr] & & y'' \end{tikzcd} \mapsto \begin{tikzcd} & \bullet \arrow[dr, "g(y',y'')"] & \\ \bullet \arrow[ur, "g(y,y')"] \arrow[rr, "g(y,y'')"'] & & \bullet \end{tikzcd} \ . \tag{152}$$

Any such cocycle g defines a G-principal bundle. Conversely, every G-principal bundle $P \to X$ gives rise to a structure like this by choosing $Y := P$ and letting A_{vert} be the canonical invariant $\mathfrak{g}$-valued vertical 1-form on $Y = P$. Then suppose (Y, A_{vert}) and (Y, A'_{vert}) are two such cocycles defined on the same Y, and let $(\hat{Y} := Y \times I, \hat{A}_{\mathrm{vert}})$ be a concordance between them. Then, for every path

$$y \times \{0\} \xrightarrow{\gamma} y \times \{1\} \tag{153}$$

connecting the two copies of a point $y \in Y$ over the endpoints of the interval, we again obtain a group element

$$h(y) := P\exp(\int_\gamma \hat{A}_{\mathrm{vert}})\,. \tag{154}$$

By the flatness of $\hat{A}$, this is

- well defined in that it is independent of the choice of path;
- has the property that for all $(y, y') \in Y \times_X Y$ we have

$$h : \begin{tikzcd} y \times \{0\} \arrow[r] \arrow[d] & y \times \{1\} \arrow[d] \\ y' \times \{0\} \arrow[r] & y' \times \{1\} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, "h(y)"] \arrow[d, "g(y,y')"'] & \bullet \arrow[d, "g'(y,y')"] \\ \bullet \arrow[r, "h(y')"] & \bullet \end{tikzcd} \,. \tag{155}$$

Therefore h is a gauge transformation between g and g', as it should be.

Note that there is no holonomy since the fibers are assumed to be simply connected in this example.

Abelian gerbes, Deligne cohomology and $(b^{n-1}\mathfrak{u}(1))$-descent objects For the case that the L_∞-algebra in question is shifted $\mathfrak{u}(1)$, i.e. $\mathfrak{g} = b^{n-1}\mathfrak{u}(1)$, classes of $\mathfrak{g}$-descent objects on X should coincide with classes of "line n-bundles", i.e. with classes of abelian $(n-1)$-gerbes on X, hence with elements in $H^n(X, \mathbb{Z})$. In order to understand this, we relate classes of $b^{n-1}\mathfrak{u}(1))$-descent objects to Deligne cohomology. We recall Deligne cohomology for a fixed surjective submersion $\pi : Y \to X$. For comparison with some parts of the literature, the reader should choose Y to be the disjoint union of sets of a good cover of X. More discussion of this point is in 3.1.

The following definition should be thought of this way: a collection of p-forms on fiberwise intersections of a surjective submersion $Y \to X$ are given. The 0-form part defines an n-bundle (an $(n-1)$-gerbe) itself, while the higher forms encode a connection on that n-bundle.

Definition 19 (Deligne cohomology) *Given a surjective submersion $\pi : Y \to X$, we obtain the simplicial space*

$$Y^\bullet = \left(\cdots Y^{[3]} \overset{\pi_1}{\underset{\pi_3}{\overset{\pi_2}{\rightrightarrows}}} Y^{[2]} \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y \xrightarrow{\pi} Y^{[0]} \right) \tag{156}$$

of fiberwise cartesian powers of Y, $Y^{[n]} := \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n \text{ factors}}$, with $Y^{[0]} := X$. The double complex of differential forms

$$\Omega^\bullet(Y^\bullet) = \bigoplus_{n\in\mathbb{N}} \Omega^n(Y^\bullet) = \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{r,s\in\mathbb{N} \\ r+s=n}} \Omega^r(Y^{[s]}) \tag{157}$$

on $Y^\bullet$ has the differential $d \pm \delta$ coming from the deRham differential d and the alternating pullback operation

$$\begin{aligned} \delta : \Omega^r(Y^{[s]}) &\rightarrow \Omega^r(Y^{[s+1]}) \\ \delta : \omega &\mapsto \pi_1^*\omega - \pi_2^*\omega + \pi_3^*\omega + \cdots - (-1)^{s+1} . \end{aligned} \tag{158}$$

Here we take 0-forms to be valued in $\mathbb{R}/\mathbb{Z}$. Writing $\Omega_k^\bullet(Y^\bullet)$ for the space of forms that vanish on $Y^{[l]}$ for $l < k$ we define (everything with respect to Y):

- *A* **Deligne n-cocycle** *is a closed element in $\Omega^n(Y^\bullet)$;*
- *a* **flat Deligne n-cocycle** *is a closed element in $\Omega_1^n(Y^\bullet)$;*
- *a* **Deligne coboundary** *is an element in $(d \pm \delta)\Omega_1^\bullet(Y^\bullet)$;*
- *a* **shift of connection** *is an element in $(d \pm \delta)\Omega^\bullet(Y^\bullet)$.*

The 0-form part of a Deligne cocycle is like the transition function of a $U(1)$-bundle. Restricting to this part yields a group homomorphism

$$[\cdot] : \; H^n(\Omega^\bullet(Y^\bullet)) \twoheadrightarrow H^n(X, \mathbb{Z}) \tag{159}$$

to the integral cohomology on X. Addition of a Deligne coboundary is a gauge transformation. Using the fact [35] that the "fundamental complex"

$$\Omega^r(X) \xrightarrow{\delta} \Omega^r(Y) \xrightarrow{\delta} \Omega^r(Y^{[2]}) \cdots \tag{160}$$

is exact for all r, one sees that Deligne cocycles with the same class in $H^n(X, \mathbb{Z})$ differ by elements in $(d \pm \delta)\Omega_1^\bullet(Y^\bullet)$.

Let

$$v : \Omega^\bullet(Y^\bullet) \rightarrow \Omega^\bullet_{\text{vert}}(Y) \tag{161}$$

be the map which sends each Deligne n-cochain a with respect to Y to the vertical part of its $(n-1)$-form on $Y^{[1]}$

$$\nu : a \mapsto a|_{\Omega^{n-1}_{\text{vert}}(Y^{[1]})} . \tag{162}$$

Then we have

Proposition 19 *If two Deligne n-cocycles a and b over Y have the same class in $H^n(X, \mathbb{Z})$, then the classes of $\nu(a)$ and $\nu(b)$ coincide.*

Proof. As mentioned above, a and b have the same class in $H^n(X, \mathbb{Z})$ if and only if they differ by an element in $(d \pm \delta)(\Omega^\bullet(Y^\bullet))$. This means that on $Y^{[1]}$ they differ by an element of the form

$$d\alpha + \delta\beta = d\alpha + \pi^*\beta . \tag{163}$$

Since $\pi^*\beta$ is horizontal, this is exact in $\Omega^\bullet_{\text{vert}}(Y^{[1]})$. □

Proposition 20 *If the $(n-1)$-form parts $B, B' \in \Omega^{n-1}(Y)$ of two Deligne n-cocycles differ by a $d \pm \delta$-exact part, then the two Deligne cocycles have the same class in $H^n(X, \mathbb{Z})$.*

Proof.

If the surjective submersion is not yet contractible, we pull evereything back to a contractible refinement, as described in 3.1.1. So assume without restriction of generality that all $Y^{[n]}$ are contractible. This implies that $H^\bullet_{\mathrm{deRham}}(Y^{[n]}) = H^0(Y^{[n]})$, which is a vector space spanned by the connected components of $Y^{[n]}$. Now assume

$$B - B' = d\beta + \delta\alpha \tag{164}$$

on Y. We can immediately see that this implies that the real classes in $H^n(X, \mathbb{R})$ coincide: the Deligne cocycle property says

$$d(B - B') = \delta(H - H') \tag{165}$$

hence, by the exacness of the deRham complex we have now,

$$\delta(H - H') = \delta(d\alpha) \tag{166}$$

and by the exactness of δ we get $[H] = [H']$.

To see that also the integral classes coincide we do induction over k in $Y^{[k]}$. For instance on $Y^{[2]}$ we have

$$\delta(B - B') = d(A - A') \tag{167}$$

and hence

$$\delta d\beta = d(A - A')\,. \tag{168}$$

Now using again the exactness of the deRham differential d this implies

$$A - A' = \delta\beta + d\gamma\,. \tag{169}$$

This way we work our way up to $Y^{[n]}$, where it then follows that the 0-form cocycles are coboundant, hence that they have the same class in $H^n(X, \mathbb{Z})$. $\square$

Proposition 21 *$b^{n-1}\mathfrak{u}(1)$-descent objects with respect to a given surjective submersion Y are in bijection with closed vertical n-forms on Y:*

$$\left\{ \Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \right\} \;\leftrightarrow\; \{A_{\mathrm{vert}} \in \Omega^n_{\mathrm{vert}}(Y)\,,\;\; dA_{\mathrm{vert}} = 0\}\,. \tag{170}$$

Two such $b^{n-1}\mathfrak{u}(1)$ descent objects on Y are equivalent precisely if these forms represent the same cohomology class

$$(A_{\mathrm{vert}} \sim A'_{\mathrm{vert}}) \;\Leftrightarrow\; [A_{\mathrm{vert}}] = [A'_{\mathrm{vert}}] \in H^n(\Omega^\bullet_{\mathrm{vert}}(Y))\,. \tag{171}$$

Proof. The first statement is a direct consequence of the definition of $b^{n-1}\mathfrak{u}(1)$ in 2.1. The second statement follows from proposition 5 using the reasoning as in proposition 16. $\square$

Hence two Deligne cocycles with the same class in $H^n(X, \mathbb{Z})$ indeed specify the same class of $b^{n-1}\mathfrak{u}(1)$-descent data.

3.3 Connections on $\mathfrak{g}$-bundles: the extension problem

It turns out that a useful way to conceive of the curvature on a non-flat $\mathfrak{g}$ n-bundle is, essentially, as the $(n+1)$-*bundle with connection obstructing* the existence of a flat connection on the original $\mathfrak{g}$-bundle. This superficially trivial statement is crucial for our way of coming to grips with non-flat higher bundles with connection.

Definition 20 (descent object for $\mathfrak{g}$-connection) *Given $\mathfrak{g}$-bundle descent object*

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) \tag{172}$$

as above, a $\mathfrak{g}$-connection on it is a completion of this morphism to a diagram

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(Y) \arrow[u, two heads, "i^*"] & W(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[u, two heads] \\ \Omega^\bullet(X) \arrow[u, hook, "\pi^*"] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{K_i\}"] \arrow[u, hook] \end{tikzcd} \quad . \tag{173}$$

As before, two $\mathfrak{g}$-connection descent objects are taken to be equivalent, if their pullbacks to a common refinement are concordant.

The top square can always be completed: any representative $A \in \Omega^\bullet(Y)$ of $A_{\mathrm{vert}} \in \Omega^\bullet_{\mathrm{vert}}(Y)$ will do. The curvature F_A is then uniquely fixed by the dg-algebra homomorphism property. The commutativity of the lower square means that for all invariant polynomials P of $\mathfrak{g}$, the form $P(F_A)$ on Y is a form pulled back from X and is the differential of a form cs that vanishes on vertical vector fields

$$P(F_A) = \pi^* K\,. \tag{174}$$

The completion of the bottom square is hence an extra condition: it demands that A has been chosen such that its curvature F_A has the property that the form $P(F_A) \in \Omega^\bullet(Y)$ for all invariant polynomials P are lifted from base space, up to that exact part.

- The commutativity of the top square generalizes the **first Cartan-Ehresmann condition**: the connection form on the total space restricts to a nice form on the fibers.
- The commutativity of the lower square generalizes the **second Cartan-Ehresmann condition**: the connection form on the total space has to behave in such a way that the invariant polynomials applied to its curvature descend down to the base space.

The pullback

$$f^*(Y, (A, F_A)) = (Y', (f^*A, f^*F_A)) \tag{175}$$

of a $\mathfrak{g}$-connection descent object $(Y, (A, F_A))$ on a surjective submersion Y along a morphism

$$\begin{tikzcd} Y' \arrow[rr, "f"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{176}$$

is the $\mathfrak{g}$-connection descent object depicted in figure 3.

Notice that the characteristic forms remain unaffected by such a pullback. This way, any two $\mathfrak{g}$-connection descent objects may be pulled back to a common surjective submersion. A concordance between two $\mathfrak{g}$-connection descent objects on the same surjective submersion is depicted in figure 4.

Suppose (A, F_A) and $(A', F_{A'})$ are descent data for $\mathfrak{g}$-bundles with connection over the same Y (possibly after having pulled them back to a common refinement). Then a concordance between them is a diagram as in figure 4.

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y') & \Omega^\bullet_{\mathrm{vert}}(Y) \arrow[l, "f^*"'] & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \arrow[ll, bend left, "f^* A_{\mathrm{vert}}"] \\
\Omega^\bullet(Y') \arrow[u, two heads, "i'^*"] & \Omega^\bullet(Y) \arrow[l, "f^*"'] \arrow[u, two heads, "i^*"] & W(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[u, two heads] \arrow[ll, bend left, "{(f^*A, F_{f^*A}}"] \\
\Omega^\bullet(X) \arrow[u, hook, "\pi'^*"] & \Omega^\bullet(X) \arrow[l, "\mathrm{Id}"'] \arrow[u, hook, "\pi^*"] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{K_i\}"] \arrow[u, hook] \arrow[ll, bend left, "\{K_i\}"]
\end{tikzcd}
.
$$

Figure 3: **Pullback of a $\mathfrak{g}$-connection descent object** $(Y, (A, F_A))$ along a morphism $f : Y' \to Y$ of surjective submersions, to $f^*(Y, (A, F_A)) = (Y', (f^*A, F_{f^*A}))$.

3.3.1 Examples.

Example (ordinary Cartan-Ehresmann connection). Let $P \to X$ be a principal G-bundle and consider the descent object obtained by setting $Y = P$ and letting A_{vert} be the canonical invariant vertical flat 1-form on fibers P. Then finding the morphism

$$
\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \tag{177}
$$

such that the top square commutes amounts to finding a 1-form on the total space of the bundle which restricts to the canonical 1-form on the fibers. This is the first of the two conditions on a Cartan-Ehresmann connection. Then requiring the lower square to commute implies requiring that the $2n$-forms $P_i(F_A)$, formed from the curvature 2-form F_A and the degree n-invariant polynomials P_i of $\mathfrak{g}$, have to descend to $2n$-forms K_i on the base X. But that is precisely the case when $P_i(F_A)$ is invariant under flows along vertical vector fields. Hence it is true when A satisfies the second condition of a Cartan-Ehresmann connection, the one that says that the connection form transforms nicely under vertical flows.

Further examples appear in 4.3.1.

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \Omega^\bullet_{\mathrm{vert}}(Y \times I) \arrow[l, shift left, "s^*"'] \arrow[l, shift right, "t^*"] & \mathrm{CE}(\mathfrak{g}) \arrow[l, "\eta^*_{\mathrm{vert}}"'] \arrow[ll, bend right, "A_{\mathrm{vert}}"'] \arrow[ll, bend left, "A'_{\mathrm{vert}}"] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \Omega^\bullet(Y \times I) \arrow[l, shift left, "s^*"'] \arrow[l, shift right, "t^*"] \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "\eta^*"'] \arrow[u, two heads] \arrow[ll, bend right, "{(A,F_A)}"'] \arrow[ll, bend left, "{(A',F_{A'})}"] \\
\Omega^\bullet(X) \arrow[u, hook] & \Omega^\bullet(X \times I) \arrow[l, shift left, "s^*"'] \arrow[l, shift right, "t^*"] \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l] \arrow[u, hook] \arrow[ll, bend right, "\{K_i\}"'] \arrow[ll, bend left, "\{K'_i\}"]
\end{tikzcd}
$$

Figure 4: **Concordance between $\mathfrak{g}$-connection descent objects** $(Y,(A,F_A))$ and $(Y,(A',F_{A'}))$ defined on the same surjective submersion $\pi : Y \to X$. Concordance between descent objects not on the same surjective submersion is reduced to this case by pulling both back to a common refinement, as in figure 3.

3.4 Characteristic classes

Definition 21 *For any $\mathfrak{g}$-connection descent object $(Y,(A,F_A))$ we say that the deRham classes $[K_i] \in H^\bullet_{\mathrm{deRham}}(X)$ in*

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, two heads, "i^*"] & W(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook, "\pi^*"] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{K_i\}"'] \arrow[u, hook]
\end{tikzcd}
\tag{178}
$$

$$
H^\bullet_{\mathrm{dR}}(X) \xleftarrow[\{[K_i]\}]{} H^\bullet(\mathrm{inv}(\mathfrak{g}))
$$

are the **characteristic classes** *of* $(Y,(A,F_A))$.

We want to show that characteristic classes know about equivalence classes of $\mathfrak{g}$-descent objects.

Proposition 22 *If two $\mathfrak{g}$-connection descent objects $(Y,(A,F_A))$ and $(Y',(A',F_{A'}))$ are equivalent, then they have the same characteristic classes:*

$$
(Y,(A,F_A)) \sim (Y',(A',F_{A'})) \;\Rightarrow\; \{[K_i]\} = \{[K'_i]\}\,. \tag{179}
$$

Proof. By definition, the two objects are equivalent if their pullbacks to a common refinement

$$\begin{tikzcd} Y \times_X Y' \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d, "\pi"] \\ Y' \arrow[r, "\pi'"] & X \end{tikzcd} , \tag{180}$$

as in figure 3, are concordant, as in figure 4. We have seen that pullback does not change the characteristic forms. It follows from proposition 16 that the characteristic classes are invariant under concordance. □

4 Higher String- and Chern-Simons n-bundles: the lifting problem

We discuss the general concept of weak cokernels of morphisms of L_∞-algebras. Then we apply this to the special problem of lifts of differential $\mathfrak{g}$-cocycles through String-like extensions.

4.1 Weak cokernels of L_∞-morphisms

After introducing the notion of a mapping cone of qDGCAs, the main point here is proposition 26, which establishes the existence of the weak inverse f^{-1} that was mentioned in **??**. It will turn out to be that very weak inverse which picks up the information about the existence or non-existence of the lifts discussed in 4.3.

Definition 22 *(mapping cone of qDGCAs) Let* $\mathrm{CE}(\mathfrak{h}) \xleftarrow{t^*} \mathrm{CE}(\mathfrak{g})$ *be a morphism of qDGCAs such that* t^* *restricts to a surjective morphism on the underlying vector spaces, hence that it surjectively maps generators to generators. The mapping cone of* t^* *is the qDGCA whose underlying graded algebra is*

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]) \tag{181}$$

and whose differential d_t *is such that it acts on generators schematically as*

$$d_t = \begin{pmatrix} d_{\mathfrak{g}} & 0 \\ t^* & d_{\mathfrak{h}} \end{pmatrix} . \tag{182}$$

In more detail, d_{t^*} is defined as follows. We write σt^* for the degree $+1$ derivation on $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1])$ which acts on $\mathfrak{g}^*$ as t^* followed by a shift in degree and which acts on $\mathfrak{h}^*[1]$ as 0. Then, for any $a \in \mathfrak{g}^*$, we have

$$d_t a := d_{\mathrm{CE}(\mathfrak{g})} a + \sigma t^*(a) . \tag{183}$$

and

$$d_t \sigma t^*(a) := -\sigma t^*(d_{\mathrm{CE}(\mathfrak{g})} a) = -d_t d_{\mathrm{CE}(\mathfrak{g})} a . \tag{184}$$

Proposition 23 *The differential* d_t *defined this way indeed satisfies* $(d_t)^2 = 0$.

Proof. For $a \in \mathfrak{g}^*$ we have

$$d_t d_t a = d_t(d_{\mathrm{CE}(\mathfrak{g})} a + \sigma t^*(a)) = \sigma t^*(d_{\mathrm{CE}(\mathfrak{g})} a) - \sigma t^*(d_{\mathrm{CE}(\mathfrak{g})} a) = 0 . \tag{185}$$

Hence $(d_t)^2$ vanishes on $\wedge^\bullet(\mathfrak{g}^*)$. Since

$$d_t d_t \sigma t^*(a) = -d_t d_t d_{\mathrm{CE}(\mathfrak{g})} a \tag{186}$$

and since $d_{\mathrm{CE}(\mathfrak{g})}a \in \wedge^\bullet(\mathfrak{g}^*)$ this implies $(d_t)^2 = 0$. $\square$

We write $\mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g}) := (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]), d_t)$ for the resulting qDGCA and $(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g})$ for the corresponding L_∞-algebra.

The next proposition asserts that $\mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g})$ is indeed a (weak) kernel of t^*.

Proposition 24 *There is a canonical morphism* $\mathrm{CE}(\mathfrak{g}) \longleftarrow \mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g})$ *with the property that*

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{h}) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "t^*"'] \arrow[d, phantom, "\Uparrow\tau"] & \mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g}) \arrow[l] \arrow[ll, bend left, "0"] \\ & {} & \end{tikzcd} . \tag{187}$$

Proof. On components, this morphism is the identity on $\mathfrak{g}^*$ and 0 on $\mathfrak{h}^*[1]$. One checks that this respects the differentials. The homotopy to the 0-morphism sends

$$\tau : \sigma t^*(a) \mapsto t^*(a)\,. \tag{188}$$

Using definition 6 one checks that then indeed

$$[d, \tau] : a \mapsto \tau(d_{\mathrm{CE}(\mathfrak{g})}a + \sigma t^* a) = a$$

and

$$[d, \tau] : \sigma t^* a \mapsto d_{\mathrm{CE}(\mathfrak{g})}a + \tau(-\sigma t^*(d_{\mathrm{CE}(\mathfrak{g})}a)) = 0\,.$$

Here the last step makes crucial use of the condition 45 which demands that

$$\tau(d_{\mathrm{W}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g})}\sigma t^* a - d_{\mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g})}\sigma t^* a) = 0$$

and the formula (42) which induces precisely the right combinatorial factors. $\square$

But not only is $\mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g})$ in the kernel of t^*, it is indeed the universal object with this property, hence is *the* kernel of t^*.

Proposition 25 *Let* $\mathrm{CE}(\mathfrak{h}) \xleftarrow{t^*} \mathrm{CE}(\mathfrak{g}) \xleftarrow{u^*} \mathrm{CE}(\mathfrak{f})$ *be a sequence of qDGCAs with* t^* *as above and with the property that* u^* *restricts, on the underlying vector spaces of generators, to the kernel of the linear map underlying* t^**. Then there is a unique morphism* $f : \mathrm{CE}(\mathfrak{f}) \to \mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g})$ *such that*

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{h}) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "t^*"'] & \mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g}) \arrow[l] \\ & \mathrm{CE}(\mathfrak{f}) \arrow[u, "u^*"] \arrow[ur, dashed, "f"'] & \end{tikzcd} . \tag{189}$$

Proof. The morphism f has to be in components the same as $\mathrm{CE}(\mathfrak{g}) \leftarrow \mathrm{CE}(\mathfrak{f})$. By the assumption that this is in the kernel of t^*, the differentials are respected. $\square$

Remark. There should be a generalization of the entire discussion where u^* is not restricted to be the kernel of t^* on generators. However, for our application here, this simple situation is all we need.

Proposition 26 *In the case the the sequence*

$$\mathrm{CE}(\mathfrak{h}) \xleftarrow{t^*} \mathrm{CE}(\mathfrak{g}) \xleftarrow{u^*} \mathrm{CE}(\mathfrak{f}) \tag{190}$$

is a String-like extension

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{t^*} \mathrm{CE}(\mathfrak{g}_\mu) \xleftarrow{u^*} \mathrm{CE}(\mathfrak{g}) \tag{191}$$

from proposition 11 or the corresponding Weil-algebra version

$$\begin{tikzcd} \mathrm{W}(b^{n-1}\mathfrak{u}(1)) \arrow[d, dash, "="] & \mathrm{W}(\mathfrak{g}_\mu) \arrow[l, two heads, "t^*"'] \arrow[d, dash, "="] & \mathrm{W}(\mathfrak{g}) \arrow[l, hook', "u^*"'] \arrow[d, dash, "="] \\ \mathrm{CE}(\mathrm{inn}(b^{n-1}\mathfrak{u}(1))) & \mathrm{CE}(\mathrm{inn}(\mathfrak{g}_\mu)) \arrow[l, two heads, "t^*"'] & \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) \arrow[l, hook', "u^*"'] \end{tikzcd} \tag{192}$$

the morphism $f : \mathrm{CE}(\mathfrak{f}) \to \mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g})$ *has a weak inverse* $f^{-1} : \mathrm{CE}(\mathfrak{h} \stackrel{t}{\hookrightarrow} \mathfrak{g}) \to \mathrm{CE}(\mathfrak{f})$ *.*

Proof. We first construct a morphism f^{-1} and then show that it is weakly inverse to f. Start by choosing a splitting of the vector space V underlying $\mathfrak{g}^*$ as

$$V = \ker(t^*) \oplus V_1 \,. \tag{193}$$

This is the non-canonical choice we need to make. Then take the component map of f^{-1} to be the identity on $\ker(t^*)$ and 0 on V_1. Moreover, for $a \in V_1$ set

$$f^{-1} : \sigma t^*(a) \mapsto -(d_{\mathrm{CE}(\mathfrak{g})}a)|_{\wedge^\bullet \ker(t^*)} \,, \tag{194}$$

where the restriction is again with respect to the chosen splitting of V. We check that this assignment, extended as an algebra homomorphism, does respect the differentials.

For $a \in \ker(t^*)$ we have

$$\begin{tikzcd} a \arrow[r, mapsto, "d_t"] \arrow[d, mapsto, "f^{-1}"'] & d_{\mathrm{CE}(\mathfrak{g})}a \arrow[d, mapsto, "f^{-1}"'] \\ a \arrow[r, mapsto, "d_{\mathrm{CE}(\mathfrak{f})}"] & d_{\mathrm{CE}(\mathfrak{g})}a \end{tikzcd} \tag{195}$$

using the fact that, since t^* is a dg-morphism, $t^*a = 0$ implies that $t^*d_{\mathrm{CE}(\mathfrak{g})}a = 0$. For $a \in V_1$ we have

$$\begin{tikzcd} a \arrow[r, mapsto, "d_t"] \arrow[d, mapsto, "f^{-1}"'] & d_{\mathrm{CE}(\mathfrak{g})}a + \sigma t^*(a) \arrow[d, mapsto, "f^{-1}"'] \\ 0 \arrow[r, mapsto, "d_{\mathrm{CE}(\mathfrak{f})}"] & (d_{\mathrm{CE}(\mathfrak{g})}a)|_{\wedge^\bullet \ker(t^*)} - (d_{\mathrm{CE}(\mathfrak{g})}a)|_{\wedge^\bullet \ker(t^*)} \end{tikzcd} \,. \tag{196}$$

and

$$\begin{tikzcd} \sigma t^*(a) \arrow[r, mapsto, "d_t"] \arrow[d, mapsto, "f^{-1}"'] & -\sigma t^*(d_{\mathrm{CE}(\mathfrak{g})}a) \arrow[d, mapsto, "f^{-1}"'] \\ -(d_{\mathrm{CE}(\mathfrak{g})}a)|_{\wedge^\bullet \ker(t^*)} \arrow[r, mapsto, "d_{\mathrm{CE}(\mathfrak{f})}"] & -d_{\mathrm{CE}(\mathfrak{f})}((d_{\mathrm{CE}(\mathfrak{g})}a)|_{\ker(t^*)}) \end{tikzcd} \,. \tag{197}$$

This last condition happens to be satisfied for the examples stated in the proposition. The details for that are discussed in 4.1.1 below. By the above, f^{-1} is indeed a morphism of qDGCAs.

Next we check that f^{-1} is a weak inverse of f. Clearly

$$\mathrm{CE}(\mathfrak{f}) \longleftarrow \mathrm{CE}(\mathfrak{h} \overset{t}{\hookrightarrow} \mathfrak{g}) \longleftarrow \mathrm{CE}(\mathfrak{f}) \tag{198}$$

is the identity on $\mathrm{CE}(\mathfrak{f})$. What remains is to construct a homotopy

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{h} \overset{t}{\hookrightarrow} \mathfrak{g}) & \mathrm{CE}(\mathfrak{f}) \arrow[l] \arrow[d, phantom, "\Downarrow\tau"] & \mathrm{CE}(\mathfrak{h} \overset{t}{\hookrightarrow} \mathfrak{g}) \arrow[l] \arrow[ll, bend left, "\mathrm{Id}"] \\ & {} & \end{tikzcd} \; . \tag{199}$$

One checks that this is accomplished by taking τ to act on σV_1 as $\tau : \sigma V_1 \xrightarrow{\simeq} V_1$ and extended suitably.

□

4.1.1 Examples

Weak cokernel for the String-like extension. Let our sequence

$$\mathrm{CE}(\mathfrak{h}) \overset{t^*}{\twoheadleftarrow} \mathrm{CE}(\mathfrak{g}) \overset{u^*}{\hookleftarrow} \mathrm{CE}(\mathfrak{f}) \tag{200}$$

be a String-like extension

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \overset{t^*}{\twoheadleftarrow} \mathrm{CE}(\mathfrak{g}_\mu) \overset{u^*}{\hookleftarrow} \mathrm{CE}(\mathfrak{g}) \tag{201}$$

from proposition 11. Then the mapping cone Chevalley-Eilenberg algebra

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \tag{202}$$

is

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathbb{R}[n] \oplus \mathbb{R}[n+1]) \tag{203}$$

with differential given by

$$\begin{aligned} d_t|_{\mathfrak{g}^*} &= d_{\mathrm{CE}(\mathfrak{g})} \\ d_t|_{\mathbb{R}[n]} &= -\mu + \sigma \\ d_t|_{\mathbb{R}[n+1]} &= 0\,. \end{aligned} \tag{204}$$

(As always, σ is the canonical degree shifting isomorphism on generators extended as a derivation.) The morphism

$$\mathrm{CE}(\mathfrak{g}) \underset{\simeq}{\overset{f^{-1}}{\longleftarrow}} \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \tag{205}$$

acts as

$$\begin{aligned} f^{-1}|_{\mathfrak{g}^*} &= \mathrm{Id} \\ f^{-1}|_{\mathbb{R}[n]} &= 0 \\ f^{-1}|_{\mathbb{R}[n+1]} &= \mu\,. \end{aligned} \tag{206}$$

To check the condition in equation 197 explicitly in this case, let $b \in \mathbb{R}[n]$ and write $b := t^*b$ for simplicity (since t^* is the identity on $\mathbb{R}[n]$). Then

$$\begin{tikzcd} \sigma b \arrow[r, mapsto, "d_t"] \arrow[d, "f^{-1}"] & 0 \arrow[d, "f^{-1}"] \\ \mu \arrow[r, mapsto, "d_{\mathrm{CE}(\mathfrak{g})}"] & 0 \end{tikzcd} \tag{207}$$

does commute.

Weak cokernel for the String-like extension in terms of the Weil algebra. We will also need the analogous discussion not for the Chevalley-Eilenber algebras, but for the corresponding Weil algebras.

So consider now the sequence

$$\mathrm{W}(b^{n-1}\mathfrak{u}(1)) \overset{t^*}{\twoheadleftarrow} \mathrm{W}(\mathfrak{g}_\mu) \overset{u^*}{\hookleftarrow} \mathrm{W}(\mathfrak{g}) \ . \tag{208}$$

This is handled most conveniently by inserting the isomorphism

$$\mathrm{W}(\mathfrak{g}_\mu) \simeq \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) \tag{209}$$

from proposition 12 as well as the identitfcation

$$\mathrm{W}(\mathfrak{g}) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) \tag{210}$$

such that we get

$$\mathrm{CE}(\mathrm{inn}(b^{n-1}\mathfrak{u}(1))) \overset{t^*}{\twoheadleftarrow} \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) \overset{u^*}{\hookleftarrow} \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) \ . \tag{211}$$

Then we find that the mapping cone algebra $\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathrm{cs}_P(\mathfrak{g}))$ is

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus (\mathbb{R}[n] \oplus \mathbb{R}[n+1]) \oplus (\mathbb{R}[n+1] \oplus \mathbb{R}[n+2]))\,. \tag{212}$$

Write b and c for the canonical basis elements of $\mathbb{R}[n] \oplus \mathbb{R}[n+1]$, then the differential is characterized by

$$\begin{aligned} d_t|_{\mathfrak{g}^*\oplus\mathfrak{g}^*} &= d_{\mathrm{W}(\mathfrak{g})} \\ d_t &: b \mapsto c - \mathrm{cs} + \sigma b \\ d_t &: c \mapsto P + \sigma c \\ d_t &: \sigma b \mapsto -\sigma c \\ d_t &: \sigma c \mapsto 0\,. \end{aligned} \tag{213}$$

Notice the relative sign between σb and σc here. This implies that the canonical injection

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathrm{cs}_P(\mathfrak{g})) \overset{i}{\longleftarrow} \mathrm{W}(b^n\mathfrak{u}(1)) \tag{214}$$

also carries a sign: if we denote the degree $n+1$ and $n+2$ generators of $\mathrm{W}(b^n\mathfrak{u}(1))$ by h and dh, then

$$i \ : \ h \mapsto \sigma b \tag{215}$$

$$i \ : \ dh \mapsto -\sigma c\,. \tag{216}$$

This sign has no profound structural role, but we need to carefully keep track of it, for instance in order for our examples in 4.3.1 to come out right. The morphism

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathrm{cs}_P(\mathfrak{g})) \underset{\simeq}{\overset{f^{-1}}{\longleftarrow}} \mathrm{W}(\mathfrak{g}) \tag{217}$$

acts as

$$\begin{aligned} f^{-1}|_{\mathfrak{g}^*\oplus\mathfrak{g}^*[1]} &= \mathrm{Id} \\ f^{-1} : \sigma b &\mapsto \mathrm{cs} \\ f^{-1} : \sigma c &\mapsto -P\,. \end{aligned} \tag{218}$$

Again, notice the signs, as they follow from the general precription in proposition 26. We again check explicitly equation (197):

$$\begin{tikzcd} \sigma b \arrow[r, mapsto, "d_t"] \arrow[d, mapsto, "f^{-1}"] & -\sigma c \arrow[d, mapsto, "f^{-1}"] \\ \mathrm{cs} \arrow[r, mapsto, "d_{\mathrm{W}(\mathfrak{g})}"] & P \end{tikzcd} \ . \tag{219}$$

4.2 Lifts of $\mathfrak{g}$-descent objects through String-like extensions

We need the above general theory for the special case where we have the mapping cone $\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$ as the weak kernel of the left morphism in a String-like extension

$$\begin{tikzcd} \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \end{tikzcd} \tag{220}$$

coming from an $(n+1)$ cocycle μ on an ordinary Lie algebra $\mathfrak{g}$. In this case $\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$ looks like

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathbb{R}[n] \oplus \mathbb{R}[n+1]), d_t)\,. \tag{221}$$

By chasing this through the above definitions, we find

Proposition 27 *The morphism*

$$f^{-1} : \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \to \mathrm{CE}(\mathfrak{g}) \tag{222}$$

acts as the identity on $\mathfrak{g}^*$

$$f^{-1}|_{\mathfrak{g}^*} = \mathrm{Id}\,, \tag{223}$$

vanishes on $\mathbb{R}[n]$

$$f^{-1}|_{\mathbb{R}[n]} = 0, \tag{224}$$

and satisfies

$$f^{-1}|_{\mathbb{R}[n+1]} : 1 \mapsto \mu\,. \tag{225}$$

Therefore we find the $(n+1)$-cocycle

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l, "\hat{A}_{\mathrm{vert}}"'] \end{tikzcd} \tag{226}$$

obstructing the lift of a $\mathfrak{g}$-cocycle

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \end{tikzcd}\,, \tag{227}$$

according to **??** given by

$$\begin{tikzcd} & & & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[dll] \arrow[dl, "f^{-1}"] & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l, hook', "j"'] \arrow[ddlll, "\hat{A}_{\mathrm{vert}}"] \\ \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i^*"'] \arrow[d, dotted] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[dl, "A_{\mathrm{vert}}"] & & \\ & \Omega^\bullet_{\mathrm{vert}}(Y) & & & \end{tikzcd}\,, \tag{228}$$

to be the $(n+1)$-form

$$\mu(A_{\mathrm{vert}}) \in \Omega^{n+1}_{\mathrm{vert}}(Y)\,. \tag{229}$$

Proposition 28 *Let* $A_{\mathrm{vert}} \in \Omega^1_{\mathrm{vert}}(Y, \mathfrak{g})$ *be the cocycle of a* G*-bundle* $P \to X$ *for* $\mathfrak{g}$ *semisimple and let* $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ *be the canonical 3-cocycle. Then* $\mathfrak{g}_\mu$ *is the standard String Lie 3-algebra and the obstruction to lifting* P *to a String 2-bundle, i.e. lifitng to a* $\mathfrak{g}_\mu$*-cocycle, is the Chern-Simons 3-bundle with cocycle given by the vertical 3-form*

$$\langle A_{\mathrm{vert}} \wedge [A_{\mathrm{vert}} \wedge A_{\mathrm{vert}}] \rangle \in \Omega^3_{\mathrm{vert}}(Y)\,. \tag{230}$$

In the following we will express these obstruction in a more familiar way in terms of their characteristic classes. In order to do that, we first need to generalize the discussion to differential $\mathfrak{g}$-cocycle. But that is now straightforward.

4.2.1 Examples

The continuation of the discussion of 2.3.1 to coset spaces gives a classical illustration of the lifting construction considered here.

Cohomology of coset spaces. The above relation between the cohomology of groups and that of their Chevalley-Eilenberg qDGCAs generalizes to coset spaces. This also illustrates the constructions which are discussed later in 4.

Consider the case of an ordinary extension of (compact connected) Lie groups:

$$1 \to H \to G \to G/H \to 1 \tag{231}$$

or even the same sequence in which G/H is only a homogeneous space and not itself a group. For a closed connected subgroup $t : H \hookrightarrow G$, there is the induced map $Bt : BH \to BG$ and a commutative diagram

$$\begin{tikzcd} \mathrm{W}(\mathfrak{g}) \arrow[r, "dt^*"] & \mathrm{W}(\mathfrak{h}) \\ \wedge^\bullet P_G \arrow[u, hook] \arrow[r, "dt^*"] & \wedge^\bullet P_H \arrow[u, hook] \end{tikzcd} . \tag{232}$$

By analyzing the fibration sequence

$$G/H \to EG/H \simeq BH \to BG, \tag{233}$$

Halperin and Thomas [24] show there is a morphism

$$\wedge^\bullet(P_G \oplus Q_H) \to \Omega^\bullet(G/K) \tag{234}$$

inducing an isomorphism in cohomology. It is not hard to see that their morphism factors through

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]). \tag{235}$$

In general, the homogeneous space G/H itself is not a group, but in case of an extension $H \to G \to K$, we also have BK and the sequences $K \to BH \to BG$ and $BH \to BG \to BK$. Up to homotopy equivalence, the fibre of the bundle $BH \to BG$ is K and that of $BG \to BK$ is BH.

In particular, consider an extension of $\mathfrak{g}$ by a String-like Lie ∞-algebra

$$\begin{tikzcd} \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i"'] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \end{tikzcd}$$

Regard $\mathfrak{g}$ now as the quotient $\mathfrak{g}_\mu/b^{n-1}\mathfrak{u}(1)$ and recognize that corresponding to BH we have $b^n\mathfrak{u}(1)$. Thus we have a quasi-isomorphism

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \simeq \mathrm{CE}(\mathfrak{g})$$

and hence a morphism

$$\mathrm{CE}(b^n\mathfrak{u}(1)) \to \mathrm{CE}(\mathfrak{g}).$$

Given a $\mathfrak{g}$-bundle cocycle

$$\begin{tikzcd} & \mathrm{CE}(\mathfrak{g}) \arrow[dl, "A_{\mathrm{vert}}"] \\ \Omega^\bullet_{\mathrm{vert}}(Y) & \end{tikzcd}$$

and given an extension of $\mathfrak{g}$ by a String-like Lie ∞-algebra

$$
\begin{tikzcd}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i"'] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook']
\end{tikzcd}
$$

we ask if it is possible to *lift the cocycle* through this extension, i.e. to find a dotted arrow in

$$
\begin{tikzcd}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads] \arrow[dr, dotted] & & \mathrm{CE}(\mathfrak{g}) \arrow[ll, hook'] \arrow[dl, "A_{\mathrm{vert}}"] \\
& & \Omega^\bullet_{\mathrm{vert}}(Y) &
\end{tikzcd} .
$$

In general this is not possible. Indeed, consider the map A'_{vert} given by $\mathrm{CE}(b^n\mathfrak{u}(1)) \to CE(\mathfrak{g})$ composed with A_{vert}.

The nontriviality of the $b^n\mathfrak{u}(1)$-cocycle A'_{vert} is the obstruction to constructing the desired lift.

4.3 Lifts of $\mathfrak{g}$-connections through String-like extensions

In order to find the obstructing characteristic classes, we want to extend the above lift

$$
\begin{tikzcd}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i"'] \arrow[dr, dotted] & & \mathrm{CE}(\mathfrak{g}) \arrow[ll, hook'] \arrow[dl, "A_{\mathrm{vert}}"] \\
& & \Omega^\bullet_{\mathrm{vert}}(Y) &
\end{tikzcd} , \tag{236}
$$

of $\mathfrak{g}$-descent objects to a lift of $\mathfrak{g}$-connection descent objects extending them, according to 3.3. Hence we want to first *extend* A_{vert} to (A, F_A)

$$
\begin{tikzcd}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l] & & \mathrm{CE}(\mathfrak{g}) \arrow[ll] \arrow[dl, "A_{\mathrm{vert}}"] \\
& & \Omega^\bullet_{\mathrm{vert}}(Y) & \\
\mathrm{W}(b^{n-1}\mathfrak{u}(1)) \arrow[uu] & \mathrm{W}(\mathfrak{g}_\mu) \arrow[l] \arrow[uu] & & \mathrm{W}(\mathfrak{g}) \arrow[ll] \arrow[uu] \arrow[dl, "{(A,F_A)}"] \\
& & \Omega^\bullet(Y) \arrow[uu] & \\
\mathrm{inv}(b^{n-1}\mathfrak{u}(1)) \arrow[uu] & \mathrm{inv}(\mathfrak{g}_\mu) \arrow[l] \arrow[uu] & & \mathrm{inv}(\mathfrak{g}) \arrow[ll] \arrow[uu] \arrow[dl, "\{K_i\}"] \\
& & \Omega^\bullet(X) \arrow[uu] &
\end{tikzcd} \tag{237}
$$

and then *lift* the resulting $\mathfrak{g}$-connection descent object (A, F_A) to a $\mathfrak{g}_\mu$-connection object $(\hat{A}, F_{\hat{A}})$

$$
\begin{tikzcd}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l] \arrow[dr, dashed, "\hat{A}_{\mathrm{vert}}"] & & \mathrm{CE}(\mathfrak{g}) \arrow[ll] \arrow[dl, "A_{\mathrm{vert}}"'] \\
& & \Omega^\bullet_{\mathrm{vert}}(Y) & \\
\mathrm{W}(b^{n-1}\mathfrak{u}(1)) \arrow[uu] & \mathrm{W}(\mathfrak{g}_\mu) \arrow[l] \arrow[uu] \arrow[dr, dashed, "(\hat{A}_{\mathrm{vert}}, F_{\hat{A}_{\mathrm{vert}}})"] & & \mathrm{W}(\mathfrak{g}) \arrow[ll] \arrow[uu] \arrow[dl, "(A, F_A)"'] \\
& & \Omega^\bullet(Y) \arrow[uu] & \\
\mathrm{inv}(b^{n-1}\mathfrak{u}(1)) \arrow[uu] & \mathrm{inv}(\mathfrak{g}_\mu) \arrow[l] \arrow[uu] \arrow[dr, dashed, "\{\hat{K}_i\}"] & & \mathrm{inv}(\mathfrak{g}) \arrow[ll] \arrow[uu] \arrow[dl, "\{K_i\}"'] \\
& & \Omega^\bullet(X) \arrow[uu] &
\end{tikzcd}
\quad . \tag{238}
$$

The situation is essentially an obstruction problem as before, only that instead of single morphisms, we are now lifting an entiry sequence of morphisms. As before, we measure the obstruction to the existence of the lift by precomposing everything with the a map from a weak cokernel:

$$
\begin{tikzcd}
& & & & & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[dl] \\
& & & & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[dlll, "i"'] \arrow[dl, "\simeq"] & \\
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l] \arrow[dr, dashed] & & \mathrm{CE}(\mathfrak{g}) \arrow[ll] \arrow[dl] & & \mathrm{W}(b^n\mathfrak{u}(1)) \arrow[uu] \arrow[dl] \\
& & \Omega^\bullet_{\mathrm{vert}}(Y) & & \mathrm{W}(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[uu] \arrow[dl, "\simeq"] & \\
\mathrm{W}(b^{n-1}\mathfrak{u}(1)) \arrow[uu] & \mathrm{W}(\mathfrak{g}_\mu) \arrow[l] \arrow[uu] \arrow[dr, dashed, "(\hat{A}, F_{\hat{A}})"] & & \mathrm{W}(\mathfrak{g}) \arrow[ll] \arrow[uu] \arrow[dl, "(A, F_A)"'] & & \mathrm{inv}(b^n\mathfrak{u}(1)) \arrow[uu] \arrow[dl] \\
& & \Omega^\bullet(Y) \arrow[uu] & & \mathrm{inv}(b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[uu] \arrow[dl, "\simeq"] & \\
\mathrm{inv}(b^n U(1)) \arrow[uu] & \mathrm{inv}(\mathfrak{g}_\mu) \arrow[l] \arrow[uu] \arrow[dr, dashed] & & \mathrm{inv}(\mathfrak{g}) \arrow[ll] \arrow[uu] \arrow[dl, "\{K_i\}"'] & & \\
& & \Omega^\bullet(X) \arrow[uu] & & &
\end{tikzcd}
$$

The result is a $b^n\mathfrak{u}(1)$-connection object. We will call (the class of) this the **generalized Chern-Simons** $(n + 1)$**-bundle** obstructing the lift.

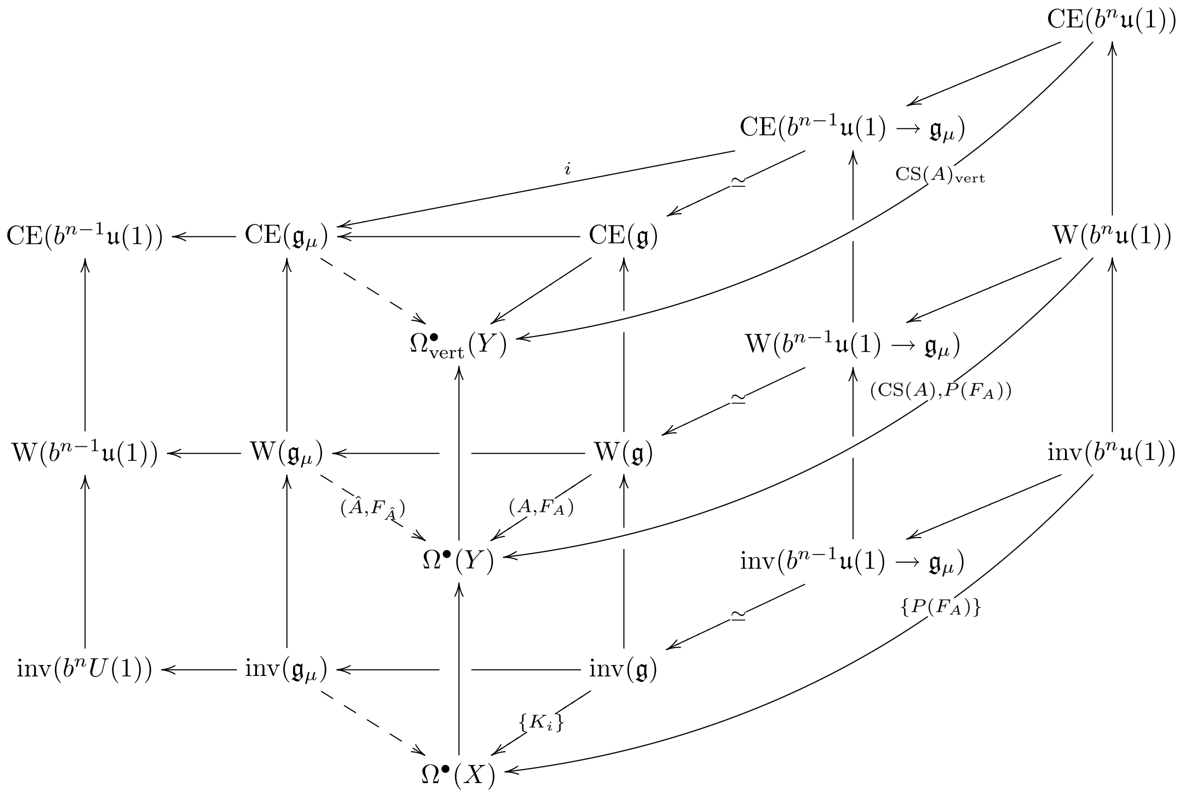

Figure 5: The **generalized Chern-Simons $b^n\mathfrak{u}(1)$-bundle that obstructs the lift of a given $\mathfrak{g}$-bundle to a $\mathfrak{g}_\mu$-bundle**, or rather the descent object representing it.

In order to construct the lift it is convenient, for similar reasons as in the proof of proposition 14, to work with $\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))$ instead of the isomorphic $\mathrm{W}(\mathfrak{g}_\mu)$, using the isomorphism from proposition 12. Furthermore, using the identity

$$\mathrm{W}(\mathfrak{g}) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) \tag{239}$$

mentioned in 2.1, we can hence consider instead of

$$\mathrm{W}(b^{n-1}) \twoheadleftarrow \mathrm{W}(\mathfrak{g}_\mu) \hookleftarrow \mathrm{W}(\mathfrak{g}) \tag{240}$$

the sequence

$$\mathrm{CE}(\mathrm{inn}(b^{n-1})) \twoheadleftarrow \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) \hookleftarrow \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) \ . \tag{241}$$

Luckily, this still satisfies the assumptions of proposition 25. So in complete analogy, we find the extension of proposition 27 from $\mathfrak{g}$-bundle cocyces to differential $\mathfrak{g}$-cocycles:

Proposition 29 *The morphism*

$$f^{-1} : \mathrm{CE}(\mathrm{inn}(b^{n-1}\mathfrak{u}(1)) \hookrightarrow \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) \to \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) \tag{242}$$

constructed as in proposition 27 acts as the identity on $\mathfrak{g}^* \oplus \mathfrak{g}^*[1]$

$$f^{-1}|_{\mathfrak{g}^*\oplus\mathfrak{g}^*[1]} = \mathrm{Id} \tag{243}$$

and satisfies

$$f^{-1}|_{\mathbb{R}[n+2]} : c \mapsto P \,. \tag{244}$$

This means that, as an extension of proposition 28, we find the differential $b^n\mathfrak{u}(1)$ $(n+1)$-cocycle

$$\Omega^\bullet(Y) \xleftarrow{\hat{A}} \mathrm{W}(b^n\mathfrak{u}(1)) \tag{245}$$

obstructing the lift of a differential $\mathfrak{g}$-cocycle

$$\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \;, \tag{246}$$

according to the above discussion

$$\begin{tikzcd}
 & & & \mathrm{CE}(\mathrm{inn}(b^{n-1}\mathfrak{u}(1)) \hookrightarrow \mathrm{inn}(\mathfrak{g}_\mu)) \arrow[dll] \arrow[dl, "f^{-1}"'] & \mathrm{W}(b^n\mathfrak{u}(1)) \;, \arrow[l, hook, "j"'] \arrow[ddlll, "(\hat{A},F_{\hat{A}})"] \\
\mathrm{W}(b^{n-1}\mathfrak{u}(1)) & \mathrm{W}(\mathfrak{g}_\mu) \arrow[l, two heads, "i^*"'] \arrow[d, dotted] & \mathrm{W}(\mathfrak{g}) \arrow[l, hook] \arrow[dl, "(A,F_A)"] & & \\
 & \Omega^\bullet(Y) & & &
\end{tikzcd} \tag{247}$$

to be the connection $(n+1)$-form

$$\hat{A} = \mathrm{CS}(A) \in \Omega^{n+1}(Y) \tag{248}$$

with the corresponding curvature $(n+2)$-form

$$F_{\hat{A}} = P(F_A) \in \Omega^{n+2}(Y)\,. \tag{249}$$

So we finally find, in particular,

Proposition 30 *For μ a cocycle on the ordinary Lie algebra $\mathfrak{g}$ in transgression with the invariant polynomial P, the obstruciton to lifting a $\mathfrak{g}$-bundle cocycle through the String-like extension determined by μ is the characteristic class given by P.*

Remark. Notice that, so far, all our statements about characteristic classes are in deRham cohomology. Possibly our construction actually obtains for integral cohomology classes, but if so, we have not extracted that yet. A more detailed consideration of this will be the subject of [45].

4.3.1 Examples

Chern-Simons 3-bundles obstructing lifts of G-bundles to $\mathrm{String}(G)$**-bundles.** Consider, on a base space X for some semisimple Lie group G, with Lie algebra $\mathfrak{g}$ a principal G-bundle $\pi : P \to X$. Identify our surjective submersion with the total space of this bundle

$$Y := P\,. \tag{250}$$

Let P be equipped with a connection, (P,∇), realized in terms of an Ehresmann connection 1-form

$$A \in \Omega^1(Y,\mathfrak{g}) \tag{251}$$

with curvature

$$F_A \in \Omega^2(Y,\mathfrak{g}) \tag{252}$$

i.e. a dg-algebra morphism

$$\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \tag{253}$$

satisfying the two Ehresmann conditions. By the discussion in 3.3.1 this yields a $\mathfrak{g}$-connection descent object $(Y, (A, F_A))$ in our sense.

We want to compute the obstruction to lifting this G-bundle to a String 2-bundle, i.e. to lift the $\mathfrak{g}$-connection descent object to a $\mathfrak{g}_\mu$-connection descent object, for

$$0 \to b\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0 \tag{254}$$

the ordinary String extension from definition 12.

By the above discussion in 4.3, the obstruction is the (class of the) $b^2\mathfrak{u}(1)$-connection descent object $(Y, (H_{(3)}, G_{(4)}))$ whose connection and curvature are given by the composite

$$\begin{tikzcd}
 & & & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[dl] \arrow[dddlll, bend left, "{(H_{(3)},G_{(4)})}"] \\
 & & (\mathrm{W}(b\mathfrak{u}(1)) \to \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))) \arrow[dl, "\simeq"] & \\
 & \mathrm{W}(\mathfrak{g}) \arrow[dl, "{(A,F_A)}"'] & & \\
 \Omega^\bullet(Y) & & &
\end{tikzcd} , \tag{255}$$

where, as discussed above, we are making use of the isomorphism $\mathrm{W}(\mathfrak{g}_\mu) \simeq \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))$ from proposition 12.

The crucial aspect of this composite is the isomorphism

$$\mathrm{W}(\mathfrak{g}) \xleftarrow[\simeq]{f^{-1}} (\mathrm{W}(b\mathfrak{u}(1)) \to \mathrm{CE}_P(\mathfrak{g})) \tag{256}$$

from proposition 26. This is where the obstruction data is picked up. The important formula governing this is equation 194, which describes how the shifted elements coming from $\mathrm{W}(b\mathfrak{u}(1))$ in the mapping cone $(\mathrm{W}(b\mathfrak{u}(1)) \to \mathrm{CE}_P(\mathfrak{g}))$ are mapped to $\mathrm{W}(\mathfrak{g})$.

Recall that $\mathrm{W}(b^2\mathfrak{u}(1)) = \mathrm{F}(\mathbb{R}[3])$ is generated from elements (h, dh) of degree 3 and 4, respectively, that $\mathrm{W}(b\mathfrak{u}(1)) = \mathrm{F}(\mathbb{R}[2])$ is generated from elements (c, dc) of degree 2 and 3, respectively, and that $\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))$ is generated from $\mathfrak{g}^* \oplus \mathfrak{g}^*[1]$ together with elements b and c of degree 2 and 3, respectively, with

$$d_{\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))} b = c - \mathrm{cs} \tag{257}$$

and

$$d_{\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))} c = P \,, \tag{258}$$

where $\mathrm{cs} \in \wedge^3(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$ is the transgression element interpolating between the cocycle $\mu = \langle \cdot, [\cdot, \cdot] \rangle \in \wedge^3(\mathfrak{g}^*)$ and the invariant polynomial $P = \langle \cdot, \cdot \rangle \in \wedge^2(\mathfrak{g}^*[1])$. So the map f^{-1} acts as

$$f^{-1} : \sigma b \mapsto -(d_{\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))} b)|_{\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])} = +\mathrm{cs} \tag{259}$$

and

$$f^{-1} : \sigma c \mapsto -(d_{\mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))} c)|_{\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])} = -P \,. \tag{260}$$

Therefore the above composite $(H_{(3)}, G_{(4)})$ maps the generators (h, dh) of $\mathrm{W}(b^2\mathfrak{u}(1))$ as

$$\begin{tikzcd}
 & & & h \arrow[dl, mapsto] \arrow[dddlll, mapsto, bend left, "{(H_{(3)},G_{(4)})}"] \\
 & & \sigma b \arrow[dl, mapsto, "\simeq"'] & \\
 & \mathrm{cs} \arrow[dl, mapsto, "{(A,F_A)}"'] & & \\
 \mathrm{CS}_P(F_A) & & &
\end{tikzcd} \tag{261}$$

and

$$\begin{tikzcd}
 & & & dh \arrow[dl, mapsto] \arrow[dddlll, mapsto, bend left, "{(H_{(3)},G_{(4)})}"] \\
 & & \text{-}\sigma c \arrow[dl, mapsto, "\simeq"'] & \\
 & P \arrow[dl, mapsto, "{(A,F_A)}"'] & & \\
 P(F_A) & & &
\end{tikzcd} . \tag{262}$$

Notice the signs here, as discussed around equation 214.

So the connection 3-form of the Chern-Simons 3-bundle given by our obstructing $b^2\mathfrak{u}(1)$-connection descent object is the Chern-Simons form

$$H_{(3)} = -\mathrm{CS}(A, F_A) = -\langle A \wedge dA \rangle - \frac{1}{3}\langle A \wedge [A \wedge A] \rangle \in \Omega^3(Y) \tag{263}$$

of the original Ehresmann connection 1-form A, and its 4-form curvature is therefore the corresponding 4-form

$$G_{(4)} = -P(F_A) = \langle F_A \wedge F_A \rangle \in \Omega^4(Y)\,. \tag{264}$$

This descends down to X, where it constitutes the characteristic form which classifies the obstruction. Indeed, noticing that $\mathrm{inv}(b^2\mathfrak{u}(1)) = \wedge^\bullet(\mathbb{R}[4])$, we see that (this works the same for all line n-bundles, i.e., for all $b^{n-1}\mathfrak{u}(1)$-connection descent objects) the characteristic forms of the obstructing Chern-Simons 3-bundle

$$\begin{tikzcd}
 & & & \mathrm{inv}(b^2\mathfrak{u}(1)) \arrow[dl] \arrow[dddlll, bend left, "\{G_{(4)}\}"] \\
 & & \mathrm{inv}(b\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[dl, "\simeq"'] & \\
 & \mathrm{inv}(\mathfrak{g}) \arrow[dl, "\{K_i\}"'] & & \\
 \Omega^\bullet(X) & & &
\end{tikzcd} \tag{265}$$

consist only and precisely of this curvature 4-form: the second Chern-form of the original G-bundle P.

5 L_∞-algebra parallel transport

We close by indicating briefly how our notion of $\mathfrak{g}$-connections give rise to a notion of parallel transport.

5.1 Parallel transport

Given an $(n-1)$-brane ("n-particle") whose n-dimensional worldvolume is modeled on the smooth parameter space Σ (for instance $\Sigma = S^1$ for the closed string) and which propagates on a target space X in that its configurations are given by maps

$$\phi : \Sigma \to X$$

hence by dg-algebra morphisms

$$\Omega^\bullet(\Sigma) \xleftarrow{\phi^*} \Omega^\bullet(X)$$

we can couple it to a $\mathfrak{g}$-descent connection object $(Y, (A, F_A))$ over X pulled back to Σ if Y is such that for every map

$$\phi : \Sigma \to X \tag{266}$$

the pulled back surjective submersion has a global section

$$\begin{tikzcd} & \phi^* Y \arrow[d, "\pi"] \\ \Sigma \arrow[ur, "\hat{\phi}"] \arrow[r, "\mathrm{Id}"'] & \Sigma \end{tikzcd} \;. \tag{267}$$

Definition 23 (parallel transport) *Given a $\mathfrak{g}$-descent object $(Y, (A, F_A))$ on a target space X and a parameter space Σ such that for all maps $\phi : \Sigma \to X$ the pullback $\phi^* Y$ has a global section, we obtain a map*

$$\mathrm{tra}_{(A)} : \mathrm{Hom}_{\mathrm{dgca}}(\Omega^\bullet(X), \Omega^\bullet(\Sigma)) \to \mathrm{Hom}_{\mathrm{dgca}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(\Sigma)) \tag{268}$$

by precomposition with

$$\Omega^\bullet(Y) \xleftarrow{(A, F_A)} \mathrm{W}(\mathfrak{g}) \;. \tag{269}$$

This is essentially the parallel transport of the $\mathfrak{g}$-connection object $(Y, (A, F_A))$. In the physics literature this parallel transport is known as the **gauge coupling** part in the action functional. A full discussion is beyond the scope of this article, but for the special case that our L_∞-algebra is $(n-1)$-fold shifted $\mathfrak{u}(1)$, $\mathfrak{g} = b^{n-1}\mathfrak{u}(1)$, the elements in

$$\mathrm{Hom}_{\mathrm{dgca}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(\Sigma)) = \Omega^\bullet(\Sigma, b^{n-1}\mathfrak{u}(1)) \simeq \Omega^n(\Sigma)$$

are in bijection with n-forms on Σ. Therefore they can be integrated over Σ. Then the functional

$$\int_\Sigma \mathrm{tra}_A : \mathrm{Hom}_{\mathrm{dgca}}(\Omega^\bullet(Y), \Omega^\bullet(\Sigma)) \to \mathbb{R}$$

is the full parallel transport of A.

Proposition 31 *The map $\mathrm{tra}_{(A)}$ is indeed well defined, in that it depends at most on the homotopy class of the choice of global section $\hat{\phi}$ of ϕ.*

Proof. Let $\hat{\phi}_1$ and $\hat{\phi}_2$ be two global section of $\phi^* Y$. Let $\hat{\phi} : \Sigma \times I \to \phi^* Y$ be a homotopy between them, i.e. such that $\hat{\phi}|_0 = \hat{\phi}_1$ and $\hat{\phi}|_1 = \hat{\phi}_2$. Then the difference in the parallel transport using $\hat{\phi}_1$ and $\hat{\phi}_2$ is the integral of the pullback of the curvature form of the $\mathfrak{g}$-descent object over $\Sigma \times I$. But that vanishes, due to the commutativity of

$$\begin{tikzcd}
 & \Omega^\bullet(\phi^* Y) \arrow[dl, "\hat{\phi}^*"'] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \\
\Omega^\bullet(\Sigma \times I) & \Omega^\bullet(\Sigma) \arrow[l, hook', "\phi^*"'] \arrow[u, hook] & \mathrm{inv}(b^{n-1}\mathfrak{u}(1)) = b^n\mathfrak{u}(1) \arrow[l, "K"'] \arrow[u, hook] \arrow[ll, bend left, "0"]
\end{tikzcd}$$

The composite of the morphisms on the top boundary of this diagram send the single degree $(n+1)$-generator of $\mathrm{inv}(b^{n-1}\mathfrak{u}(1)) = b^n\mathfrak{u}(1)$ to the curvature form of the $\mathfrak{g}$-connection descent object pulled back to Σ.

It is equal to the composite of the horizontal morphisms along the bottom boundary. These vanish, as there is no nontrivial $(n+1)$-form on the n-dimensional Σ. $\Box$

5.1.1 Examples.

Proposition 32 *For G simply connected, the parallel transport coming from the Chern-Simons 3-bundle discussed in 4.3.1 for $\mathfrak{g} = \mathrm{Lie}(G)$ reproduces the familiar Chern-Simons action functional*

$$\int_\Sigma \left(\langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \right)$$

over 3-dimensional Σ.

Proof. Recall from 4.3.1 that we csan build the connection descent object for the Chern-Simons connection on the surjective submersion Y coming from the total space P of the underlying G-bundle $P \to X$. Then $\phi^* Y = \phi^* P$ is simply the pullback of that G-bundle to Σ. For G simply connected, BG is 3-connected and hence any G-bundle on Σ is trivializable. Therefore the required lift $\hat{\phi}$ exists and we can construct the above diagram. By equation (263) one sees that the integral which gives the parallel transport is indeed precisely the Chern-Simons action functional. $\Box$

Higher Chern-Simons n-bundles, coming from obstructions to fivebrane lifts or higher lifts, similarly induce higher dimensional generalizations of the Chern-Simons action functional.

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References

[1] P. Aschieri and B. Jurco, *Gerbes, M5-brane anomalies and E_8 gauge theory* J. High Energy Phys. **0410** (2004) 068, [`arXiv:hep-th/0409200v1`].

[2] J. Baez and A. Crans, *Higher-dimensional algebra VI: Lie 2-algebras*, Theory and Applications of Categories **12** (2004) 492-528, [`arXiv:math/0307263v5`].

[3] J. Baez, A. Crans, U. Schreiber, D. Stevenson, *From Loop Groups to 2-Groups*, Homology, Homotopy and Applications, Vol. 9 (2007), No. 2, pp.101-135, [`arXiv:math/0504123v2`]

[4] J. Baez and U. Schreiber, *Higher gauge theory*, in Contemporary Mathematics, 431, *Categories in Algebra, Geometry and Mathematical Physics*, [`arXiv:math/0511710`]

[5] E. Bergshoeff, R. Percacci, E. Sezgin, K. S. Stelle, and P. K. Townsend, *$U(1)$-extended gauge algebras in p-loop space*, Nucl. Phys. **B 398** (1993) 343.

[6] E. Bergshoeff, E. Sezgin, and P. K. Townsend, *Properties of the eleven-dimensional super membrane theory*, Annals Phys.**185** (1988) 330.

[7] L. Bonora, P. Cotta-Ramusino, M. Rinaldi, and J. Stasheff, *The evaluation map in field theory, sigma-models and strings I*, Commun. Math. Phys. **112** (1987) 237.

[8] P. Bouwknegt and V. Mathai, *D-branes, B-fields and twisted K-theory*, J. High Energy Phys. **03** (2000) 007, [`arXiv:hep-th/0002023`].

[9] P. Bouwknegt, A. Carey, V. Mathai, M. Murray and D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, Commun. Math. Phys. **228** (2002) 17, [`arXiv:hep-th/0106194`].

[10] L. Breen and W. Messing, *Differential geometry of gerbes*, Adv. Math. **198** (2005), no. 2, 732–846, [`arXiv:math/0106083v3`].

[11] A. Carey, S. Johnson, and M. Murray, *Holonomy on D-branes*, J. Geom. Phys. **52** (2004), no. 2, 186–216, [`arXiv:hep-th/0204199v3`].

[12] A. Carey, S. Johnson, M. Murray, D. Stevenson, and B.-L. Wang, *Bundle gerbes for Chern-Simons and Wess-Zumino-Witten theories* , Commun. Math. Phys. 259 (2005) 577, [`arXiv:math/0410013`].

[13] J. Cheeger and J. Simons, *Differential characters and geometric invariants*, in Geometry and Topology, 50–80, Lecture Notes in Math., 1167, Springer, Berlin, 1985.

[14] A. Clingher, *Heterotic string data and theta functions*, Adv. Theor. Math. Phys. **9** (2005) 173, [`arXiv:math/0110320v2`].

[15] R. Coquereaux and K. Pilch, *String structures on loop bundles*, Commun. Math. Phys. **120** (1989) 353.

[16] E. Diaconescu, D. Freed, and G. Moore, *The M theory three form and E_8 gauge theory*, [`arXiv:hep-th/0312069`].

[17] E. Diaconescu, G. Moore and E. Witten, *E_8 gauge theory, and a derivation of K-theory from M-theory* Adv. Theor. Math. Phys. **6** (2003) 1031, [`arXiv:hep-th/0005090`].

[18] J. A. Dixon, M. J. Duff, and E. Sezgin, *The coupling of Yang-Mills to extended objects*, Phys. Lett. **B 279** (1992) 265.

[19] D. S. Freed and E. Witten, *Anomalies in string theory with D-Branes*, Asian J. Math. **3** (1999) 819, [`arXiv:hep-th/9907189`].

[20] F. Girelli and H. Pfeiffer, *Higher gauge theory – differential versus integral formulation*, J. Math. Phys. **45** (2004) 3949-3971, [`arXiv:hep-th/0309173v2`].

[21] F. Girelli, H. Pfeiffer, and E. Popescu, *Topological higher gauge theory - from BF to BFCG theory*, [`arXiv:0708.3051v1`].

[22] M. B. Green and J. H. Schwarz, *Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory*, Phys. Lett. **B149** (1984) 117.

[23] W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology. Vol. II: Lie groups, principal bundles, and characteristic classes*, Academic Press, New York-London, 1973.

[24] S. Halperin and J.-C. Thomas, *Rational equivalence of fibrations with fibre G/K*, Canad. J. Math. **34** (1982), no. 1, 31–43.

[25] A. Henriques, *Integrating L_∞ algebras*, [`arXiv:math/0603563v2`].

[26] M.J. Hopkins, I.M. Singer, *Quadratic functions in geometry, topology, and M-theory*, [`arXiv:math.AT/0211216`].

[27] P. Horava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, Nucl.Phys. **B475** (1996) 94, [`hep-th/9603142`].

[28] A. Kapustin, *D-branes in a topologically nontrivial B-field*, Adv. Theor. Math. Phys. **4** (2000) 127, [`arXiv:hep-th/9909089`].

[29] T. P. Killingback, *World-sheet anomalies and loop geometry*, Nucl. Phys. B 288 (1987) 578.

[30] I. Kriz and H. Sati, *M Theory, type IIA superstrings, and elliptic cohomology*, Adv. Theor. Math. Phys. **8** (2004) 345, [`arXiv:hep-th/0404013`].

[31] T. Lada and J. Stasheff, *Introduction to sh Lie algebras for physicists*, Int. J. Theor. Phys. **32** (1993) 1087, [`arXiv:hep-th/9209099`].

[32] T. Lada, M. Markl, *Strongly homotopy Lie algebras*, [`arXiv:hep-th/9406095v1`]

[33] V. Mathai and H. Sati, *Some relations between twisted K-theory and E_8 gauge theory*, J. High Energy Phys. **0403** (2004) 016, [`arXiv:hep-th/0312033v4`].

[34] J. Mickelsson and R. Percacci, *Global aspects of p-branes*, J. Geom. and Phys. **15** (1995) 369-380.

[35] M. Murray, *Bundle gerbes*, J. London Math. Soc. **(2) 54** (1996), no. 2, 403–416.

[36] M. Murray and D. Stevenson, *Higgs fields, bundle gerbes and string structures*, Commun. Math. Phys. **243** (2003) 541-555, [`arXiv:math/0106179v1`].

[37] R. Percacci and E. Sezgin, *Symmetries of p-branes*, Int. J. Theor. Phys. **A 8** (1993) 5367.

[38] D. Roberts and U. Schreiber, *The inner automorphism 3-group of a strict 2-group*, [`arXiv:0708.1741`].

[39] H. Sati, *On Higher twists in string theory*, [`arXiv:hep-th/0701232v2`].

[40] U. Schreiber and K. Waldorf, *Parallel transport and functors*, [`arXiv:0705.0452v1`].

[41] U. Schreiber and K. Waldorf, *The geometry of smooth 2-functors*, to appear.

[42] U. Schreiber and K. Waldorf, *Parallel transport and 2-functors*, to appear.

[43] St. Stolz and P. Teichner, *What is an elliptic object?* in Topology, geometry and quantum field theory, London Math. Soc. LNS 308, Cambridge University Press 2004, 247-343.

[44] E. Witten, *On Flux quantization in M-Theory and the effective action*, J. Geom. Phys. **22** (1997) 1-13, [`arXiv:hep-th/9609122`].

[45] *Dual string theories and fivebrane structures*, work in progress.

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