The Canonical 2-Representation

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Abstract

Every finite strict 2-group has a canonical 2-representation on Vect-module categories. This easily generalizes to strict Lie 2-groups and possibly to Fréchet Lie 2-groups.

Contents

1 Introduction

The following two observations are almost tautological, and yet prove quite useful in the context of higher group theory.

- We have canonical inclusions of 2-categories

$$\mathrm{Intertwin} \hookrightarrow \mathrm{Bimod} \hookrightarrow {}_{\mathrm{Vect}}\mathrm{Mod} =: 2\mathrm{Vect}\,.$$

- For every strict 2-group $G_{(2)}$ there is a canonical 2-representation

$$\rho : \Sigma G_{(2)} \to 2\mathrm{Vect}$$

  that factors through this inclusion.

In its simplest version this statement holds for finite 2-groups and finite dimensional vector spaces. Then Bimod is the weak 2-category whose objects are finite-dimensional algebras, whose morphisms are bimodules for these and whose 2-morphisms are bimodule homomorphisms. Intertwin is the sub-2-category of bimodules coming from algebra homomorphisms. Finally $\Sigma G_{(2)}$ is our notation for $G_{(2)}$ regarded as a 1-object 2-groupoid.

All this is discussed in the following. It is easy to generalize the construction to finite dimensional Lie 2-groups and representations on finite-dimensional vector spaces.

A main motivation for considering these 2-representations is, however, the existence of a Fréchet Lie 2-group

$$\mathrm{String}_k(G)$$

for every simple, simply connected compact Lie group G, the realization of whose nerve is [2] the topological String group [4].

Generalizing the above 2-representation ρ to this infinite-dimensional case would allow to conceive String bundles as ρ-associated 2-bundles.

2 2-Groups and Crossed Modules

Before discussing 2-representations, we recall the required basics concerning strict 2-groups and crossed modules from [1]. In particular, we explicitly fix one identification of the two and exhibit the relevant identities.

Definition 1. A **crossed module of groups** is a diagram

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

in Grp such that

$$\begin{tikzcd} H \arrow[rr, "\mathrm{Ad}"] \arrow[dr, "t"'] & & \mathrm{Aut}(H) \\ & G \arrow[ur, "\alpha"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} G \times H \arrow[r, "\mathrm{Id}\times t"] \arrow[d, "\alpha"'] & G \times G \arrow[d, "\mathrm{Ad}"] \\ H \arrow[r, "t"] & G \end{tikzcd} .$$

Definition 2. A **strict 2-group** $G_{(2)}$ is any of the following equivalent entities

- a group object in Cat

- a category object in Grp
- a strict 2-groupoid with a single object

As for groups, we shall write $G_{(2)}$ when we think of $G_{(2)}$ as a monoidal category, and $\Sigma G_{(2)}$ when we think of it as a 1-object 2-groupoid.

Proposition 3. *Crossed modules of groups and strict 2-groups are equivalent.*

We now spell out this identification in detail. It is unique only up to a few conventional choices.

The same is in principle already true for the identification of 1-groups with categories, which is unique only up to reversal of all arrows.

To start with, we take all principal actions to be from the *right*.

So for G any group, GTor denotes the category of right-principal G-spaces. This implies that if we want the canonical inclusion

$$i_G : \Sigma G \to G\mathrm{Tor}$$

to be covariant, we need to take composition in ΣG to work like

$$g_2 \circ g_1 = g_2 g_1 \,,$$

where on the left the composition is that of morphisms in ΣG, while on the right it is the product in G. Notice that this implies that diagrammatically we have

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \;=\; \bullet \xrightarrow{g_2 g_1} \bullet \,.$$

If G comes to us as a group of maps, we accordingly take the group product to be given by $g_2 g_1 := g_2 \circ g_1$.

When we then pass to strict 2-groups $G_{(2)}$ coming from crossed modules $(t : H \to G)$ of groups, and want to label 2-morphisms in $\Sigma G_{(2)}$ with elements in H and G, we have one more convention to fix.

Let $G_{(2)}$ be a (strict) 2-group which we may alternatively think of a crossed module $t\colon H \to G$. To recover $G_{(2)}$ from the crossed module $t\colon H \to G$ we set

$$\begin{aligned} \mathrm{Ob}(G_{(2)}) &= G \\ \mathrm{Mor}(G_{(2)}) &= G \ltimes H \,. \end{aligned}$$

A 2-morphism in $\Sigma G_{(2)}$ will be denoted by

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & \bullet \end{tikzcd}$$

for $g, g' \in G$ and $h \in H$, where g' will turn out to be fixed by $(g,h) \in G \ltimes H$. The semi-direct product structure on $G \ltimes H$, the source, target and composition homomorphisms are defined as follows.

We shall agree that

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} := \begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} . \tag{1}$$

One could also use the opposite conventions, but this one ensures that the representations of 2-groups on bimodules, to be presented below, work out nicely.

From the requirement that $t : H \to G$ be a homomorphism, it follows that

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h')"'] \arrow[r, phantom, "\Downarrow h'"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h'h)"'] \arrow[r, phantom, "\Downarrow h'h"] & \bullet \end{tikzcd} .$$

Together with the convention above this means that the source-target matching condition then reads

$$t(h)g = g' . \tag{2}$$

The exchange law then implies that

$$\begin{tikzcd} \bullet \arrow[r, bend left=70, "\mathrm{Id}"] \arrow[r, "t(h)" description] \arrow[r, bend right=70] \arrow[r, phantom, bend left=35, "\Downarrow h"] \arrow[r, phantom, bend right=35, "\Downarrow h'"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h'h)"'] \arrow[r, phantom, "\Downarrow h'h"] & \bullet \end{tikzcd} \tag{3}$$

Finally, since in the crossed module we have $t(\alpha(g)(h)) = gt(h)g^{-1}$ we find that inner automorphisms in the 2-group have to be labeled like this:

$$\begin{tikzcd} \bullet \arrow[r, "g^{-1}"] & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(\alpha(g)(h))"'] \arrow[r, phantom, "\Downarrow \alpha(g)(h)"] & \bullet \end{tikzcd} . \tag{4}$$

3 Bimod

Definition 4. Inside the 2-category Bimod of all bimodules, we have the strict sub-2-category Intertwin $\subset$ Bimod whose objects are algebras, whose morphisms are algebra homomorphisms

$$A \xrightarrow{f} B$$

and whose 2-morphisms

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow u"] & & B \end{tikzcd}$$

are elements $u \in B$ that intertwine the morphisms f and g in that for all $a \in A$ we have

$$uf(a) = g(a)u\,.$$

Horizontal composition is given by the obvious composition of morphisms together with the relations

$$\begin{tikzcd} A \arrow[r, "q"] & B \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow u"] & & C \end{tikzcd} := \begin{tikzcd} A \arrow[rr, bend left, "f \circ q"] \arrow[rr, bend right, "g \circ q"'] \arrow[rr, phantom, "\Downarrow u"] & & C \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow u"] & & B \arrow[r, "q"] & C \end{tikzcd} := \begin{tikzcd} A \arrow[rr, bend left, "q \circ f"] \arrow[rr, bend right, "g \circ q"'] \arrow[rr, phantom, "\Downarrow q(u)"] & & C \end{tikzcd} \qquad (5)$$

The inclusion

$$\mathrm{Intertwin} \hookrightarrow \mathrm{Bimod}$$

sends a homomorphism $f : A \to B$ to the bimodule B_f, which, as an object, is B, with the obvious right B action and with the left A-action induced by f.

Moreover, it sends an intertwiner u to the bimodule homomorphism

$$h_u : B_f \to B_g$$

which acts as

$$h_u : b \mapsto ub\,.$$

The intertwiner condition is precisely the property that guarantees that this is a homomorphisms of bimdodules.

Notice that when the intertwiner is invertible, we may equivalently read this condition as

$$g = \mathrm{Ad}_u \circ f\,. \qquad (6)$$

4 Representations of 2-groups

A representation of a group G on a vector space V can be encoded in the data of a functor $\rho : \Sigma G \to \mathrm{Vect}$, where we denote by ΣG the category with a unique object $\bullet$ whose group of automorphisms is the group G. This functor sends the unique object $\bullet$ of G to the vector space V, and sends a morphism to the corresponding automorphism of V.

Proposition 5. *For $G_{(2)}$ any strict 2-group coming from the crossed module $(H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H))$ the assignment*

$$\begin{tikzcd} \rho : \bullet \arrow[rr, bend left, "g"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\Downarrow h"] & & \bullet \end{tikzcd} \mapsto \begin{tikzcd} \mathbb{C}[H] \arrow[rr, bend left, "\alpha(g)"] \arrow[rr, bend right, "\alpha(g')"'] \arrow[rr, phantom, "\Downarrow h"] & & \mathbb{C}[H] \end{tikzcd}$$

is a strict 2-functor

$$\rho : \Sigma G_{(2)} \to \mathrm{Intertwin} \hookrightarrow \mathrm{Bimod}\,.$$

Proof. Clearly composition of 1-morphisms is respected. Vertical composition of 2-morphisms is respected according to (3). Source-target matching follows from (2) and (6). Horizontal composition of 1-morphisms with 2-morphisms is respected due to (1, 4) and (5). This already implies strict 2-functoriality. $\square$

Definition 6. For $G_{(2)}$ any strict 2-group, we call the 2-functor

$$\rho : \Sigma G_{(2)} \to \mathrm{Intertwin}$$

when regarded as taking values in 2-vector spaces

$$\Sigma G_{(2)} \xrightarrow{\rho} \mathrm{Intertwin} \hookrightarrow \mathrm{Bimod} \hookrightarrow {}_{\mathrm{Vect}}\mathrm{Mod}$$

the **canonical 2-representation** of $G_{(2)}$.

In many application we are intersted in slightly less canonical, but very similar 2-representations:

Corollary 7. *For $G_{(2)}$ any strict 2-group coming from the crossed module*

$$(H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H))$$

and for

$$\rho_0 : \Sigma(H) \to \mathrm{Vect}$$

any ordinary faithful representation of H, with image algebra

$$\langle \mathrm{im}(\rho_0) \rangle$$

the assignment

$$\rho : \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} \langle \mathrm{im}(\rho_0)\rangle \arrow[r, bend left, "\alpha(g)"] \arrow[r, bend right, "\alpha(g')"'] \arrow[r, phantom, "\Downarrow h"] & \langle \mathrm{im}(\rho_0)\rangle \end{tikzcd}$$

is a strict 2-functor

$$\rho : \Sigma G_{(2)} \to \mathrm{Intertwin}\,.$$

5 The canonical 2-Representation of $\Sigma U(1)$

A simple example is the canonical 2-representation of the strict 2-group $G_{(2)} = \Sigma U(1)$, which is that coming from the crossed module

$$U(1) \xrightarrow{t} 1$$

and induced by the standard rep of $\Sigma U(1)$ on $\mathbb{C}$, according to corollary 7.

This $\rho : \Sigma U(1) \to \mathrm{Bimod}$ is given by the assignment

$$\rho : \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow \cdot c"] & \mathbb{C} \end{tikzcd}$$

for all $c \in U(1)$.

URS: The following might be strictly true only for finite dimensional vector spaces.

Proposition 8. *Every 2-representation* $\tilde{\rho} : \Sigma U(1) \to \mathrm{Bimod}$ *which is* equivalent *to this* ρ *is of the form*

$$\rho : \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} K(V) \arrow[r, bend left, "K(V)"] \arrow[r, bend right, "K(V)"'] \arrow[r, phantom, "\Downarrow \cdot c"] & K(V) \end{tikzcd}\,,$$

where $K(V)$ *is the algebra of* finite rank opertors

$$K(V) \simeq V \otimes V^*$$

of any vector space V*.*

Proof. Every equivalence of these 2-functors is in particular an equivalence, in Bimod, of the algebra coming with the 2-representation. Equivalence of objects in Bimod is Morita equivalence of algebras. The Morita class of $\mathbb{C}$ in Vect is that of all algebras of finite rank operators, i.e. of all algebras that are isomorphic to $V \otimes V^*$.

The bimodules inducing the respective equivalence are

$$\mathbb{C} \xrightarrow{V^*} K(V)$$

and

$$K(V) \xrightarrow{V} \mathbb{C}\ .$$

Here

$$V \otimes_{\mathbb{C}} V^* \simeq K(V)$$

by definition of $K(V)$, and one checks that

$$V^* \otimes_{K(V)} V \simeq \mathbb{C}\,.$$

Let $\rho : \Sigma G_{(2)} \to \mathrm{Bimod}$ be the above 2-rep and let $\tilde{\rho}$ be another one. An equivalence of the given 2-functors involves transformations

$$\begin{tikzcd} \tilde{\rho} \arrow[r, bend left, "t"] & \rho \arrow[l, bend left, "\bar{t}"] \end{tikzcd}$$

whose composites are related by an invertible modification to the identity. On objects this implies the above Morita equivalences. Hence these components of t and $\bar{t}$ must look like

$$\begin{tikzcd} K(V) \arrow[r, "K(V)"] \arrow[d, "V"'] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & K(V) \arrow[d, "V"] \\ \mathbb{C} \arrow[r, "\mathbb{C}"] & \mathbb{C} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[d, "V^*"'] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & \mathbb{C} \arrow[d, "V^*"] \\ K(V) \arrow[r, "K(V)"] & K(V) \end{tikzcd}\ .$$

URS: Here I am behaving as if we are in the world $2\mathrm{Cat}_{\mathrm{Gray}}$ of strict 2-cats, strict 2-functors, peudonatural transformations and modifications. I am thinking that we can assume to have strictified Bimod.

The corresponding modifications are then nothing but the above isomorphisms $V \otimes_{\mathbb{C}} V^* \simeq K(V)$ and $V^* \otimes_{K(V)} V \simeq \mathbb{C}$. Using the naturality square for t and $\bar{t}$, this implies the claim. $\square$

6 The canonical 2-Representation of $\mathrm{String}_k(G)$

Proposition 9 ([2]). *For G any simply connected compact simple Lie group and $k \in \mathbb{Z}$ any level, there is a crossed module of Fréchet-Lie groups*

$$\hat{\Omega}_k G \xrightarrow{t} PG \xrightarrow{\widehat{\mathrm{Ad}}} \mathrm{Aut}(\hat{\Omega}_k G)\ .$$

Here $\hat{\Omega}_k G$ is the Kac-Moody central extension of the loop group ΩG at level k, t forgets the central extension and injects loops into all based paths, and $\widehat{\mathrm{Ad}}$ is a lift of the the obvious adjoint action of based paths on loops to the central extension.

The corresponding strict Fréchet Lie 2-group we here call

$$\mathrm{String}_k(G)\,.$$

We want to find representations ρ_0 of $\hat{\Omega}_k G$ such that we obtain a 2-representation of this 2-group according to corollary 7.

Let us briefly review what [3] and [4] say about highest weight representations of $\hat{\Omega}_k G$ and their automorphism groups.

Highest weight reps of $\hat{\Omega}_k G$ and its automorphisms. Let

$$\rho : \Omega G \to PU(H)$$

be a projective unitary representation of ΩG on some Hilbert space H. We may pull this back along the short exact sequence

$$1 \longrightarrow U(1) \longrightarrow U(H) \longrightarrow PU(H) \longrightarrow 1$$

to obtain

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] \arrow[d, "\mathrm{Id}"] & \hat{\Omega}_k G \arrow[r] \arrow[d, "\hat{\rho}"] & \Omega G \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & U(1) \arrow[r] & U(H) \arrow[r] & PU(H) \arrow[r] & 1
\end{tikzcd}\ .$$

Think of loops as maps on S^1 and let $\hat{\Omega}_k G|_I \subset \hat{\Omega}_k G$ be the subgroup of loops with support on the upper half circle $I \subset S^1$.

Then we get a von Neumann algebra as the double commutant (in $B(H)$)

$$A_\rho := (\hat{\rho}(\hat{\Omega}_k G|_I))''$$

of the image of this subgroup under our representation $\hat{\rho}$.

If we want to use this algebra in corollary 7 we need to check that the path group PG still acts by algebra automorphisms on A_ρ.

The way to do this is explained in [4]: identify PG with the subgroup of maps on S^1 supported on I and mapping one boundary of I to the identity. Any such map γ may be extended to a map $\hat{\gamma} : S^1 \to G$ on all of S^1, which then may be sent to $PU(H)$ by ρ. The adjoint action of this lifted curve

$$\mathrm{Ad}_{\hat{\rho}(\hat{\gamma})} \in \mathrm{Aut}(A_\rho)$$

is certainly independent of the chosen lift, hence indeed defines an action

$$\widehat{\mathrm{Ad}} : PG \to \mathrm{Aut}(A_\rho)\,.$$

This way, we get, first for level $k = 0$:

Proposition 10. *The assignment*

$$\begin{tikzcd} \rho : \bullet \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \mapsto \begin{tikzcd} A_\rho \arrow[r, bend left, "\widehat{\mathrm{Ad}}(\hat{\gamma})"] \arrow[r, bend right, "\widehat{\mathrm{Ad}}(\hat{\gamma}')"'] \arrow[r, phantom, "\Downarrow \hat{\rho}(h)"] & A_\rho \end{tikzcd}$$

is a strict 2-functor

$$\rho : \Sigma\mathrm{String}_0(G) \to \mathrm{Intertwin}\,.$$

Proof. The only nontrivial thing to check is the compatibility with the action of PG on $\hat{\Omega}_k G$. But

$$\hat{\rho}(\alpha(\gamma)(h)) = \widehat{\mathrm{Ad}}(\hat{\gamma})(\hat{\rho}(h))$$

holds because on both sides the action is simply by conjugation. □

In order to conceive this as a 2-representation on bimodules, we need to find an analog of the inclusion

$$\mathrm{Intertwin} \hookrightarrow \mathrm{Bimod}$$

in the world of von Neumann algebras. As discussed in [4], the right notion is the 2-category

$$\mathrm{Bimod}_{\mathrm{vN}}$$

whose objects are von Neumann algebras, and whose morphisms are Hilbert spaces with a von Neumann bimodules structure. The subtlety introduced by this is that the ordinary algebraic tensor product of bimodules now needs to be followed by a Hilbert space completion. The right way to do this is known as *Connes fusion* of bimodules.

But indeed, Connes fusion $\otimes^{\mathrm{Connes}}$ does respect the composition of twists in that

$$(A_\rho)_f \otimes^{\mathrm{Connes}}_{A_\rho} (A_\rho)_g \simeq (A_\rho)_{g\circ f} \,.$$

This way we do get an inclusion

$$\mathrm{Intertwin}_{\mathrm{vN}} \hookrightarrow \mathrm{Bimod}_{\mathrm{vN}}$$

as before. So the highest weight representation of $\hat{\Omega}_0 G$ provides us with a 2-representation of the String 2-group at level 0

$$\mathrm{String}_0 G \xrightarrow{\rho} \mathrm{Intertwin}_{\mathrm{vN}} \hookrightarrow \mathrm{Bimod}_{\mathrm{vN}} \,.$$

To see if this goes through for nontrivial level one needs to check if the relation

$$\hat{\rho}(\alpha(\gamma)(h)) = \widehat{\mathrm{Ad}}(\hat{\gamma})(\hat{\rho}(h))$$

still holds in that case, i.e. if the lift of the adjoint action of paths on loops to the action on centrally extended loops is the same on both sides.

References

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