On the Canonical Quantization of the 1-Particle

Urs Schreiber

August 15, 2007

What you are seeing here

The following is currently nothing but a few sketchy remarks supposed to illustrate a discussion given elsewhere in some blog entry on *n*-functorial quantum theory.
As time permits, it might evolve into something self-contained eventually. Or it might not.

Plan

1. The charged n-Particle.
2. Sections and Quantum States of the n-Particle.
3. Pull-Push Propagation of States of the n-Particle.
4. Example: The charged 1-particle on a lattice.

What on Earth is a "charged n-particle"?

See elsewhere.

Associated Sections

Definition

Given a parallel transport n-functor $\mathrm{tra} : \mathrm{tar} \to D$, its ("associated") sections are transformations

$$\begin{tikzcd}
\mathrm{tar} \arrow[rr, "E"] \arrow[dr, "\mathrm{tra}"'] \arrow[rr, bend right=40, "\delta\mathrm{tra}"] \arrow[rr, phantom, bend right=20, "\Downarrow e"] & & n\mathrm{Cat} \\
& D \arrow[ur, "TD"'] &
\end{tikzcd}$$

Associated Sections

Definition

A morphism of sections $(E, e) \longrightarrow (E', e')$ is a transformation $f : E \longrightarrow E'$ together with an n-equivalence

$$
\begin{tikzcd}
\mathrm{tar} \arrow[rr, "E"] \arrow[dr, "\mathrm{tra}"'] \arrow[rr, bend right=60, "\delta\mathrm{tra}"] & {} \arrow[d, phantom, "\Downarrow e"] & n\mathrm{Cat} \\
& D \arrow[ur, "TD"'] &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
& {} \arrow[d, phantom, "\Downarrow f"] & \\
\mathrm{tar} \arrow[rr, "E'"] \arrow[rr, bend left=60, "E"] \arrow[dr, "\mathrm{tra}"'] \arrow[rr, bend right=60, "\delta\mathrm{tra}"] & {} \arrow[d, phantom, "\Downarrow e'"] & n\mathrm{Cat} \\
& D \arrow[ur, "TD"'] &
\end{tikzcd}
$$

The space of histories

Definition

For conf the n-category of configurations of the n-particle, we say that an n-category of one-step histories is a span

$$\begin{tikzcd} & \mathrm{hist} \arrow[dl, "\mathrm{out}"'] \arrow[dr, "\mathrm{in}"] & \\ \mathrm{conf} & & \mathrm{conf} \end{tikzcd}$$

together with a transformation

$$\begin{tikzcd} & \mathrm{hist} \times \mathrm{par} \arrow[dl, "\mathrm{out}"'] \arrow[dr, "\mathrm{in}"] & \\ \mathrm{conf} \times \mathrm{par} \arrow[dr, "\mathrm{ev}"'] & \Leftarrow & \mathrm{conf} \times \mathrm{par} \arrow[dl, "\mathrm{ev}"] \\ & \mathrm{tar} & \end{tikzcd} \cdot$$

Pasting a section to the space of histories

Definition

The pullback of a section through the space of histories is the pasting diagram

$$
\begin{tikzcd}
 & \mathrm{hist}\times\mathrm{par} \arrow[dl, "\mathrm{out}"'] \arrow[dr, "\mathrm{in}"] & \\
\mathrm{conf}\times\mathrm{par} \arrow[dr, "\mathrm{ev}"'] \arrow[rr, phantom, "\Leftarrow"] & & \mathrm{conf}\times\mathrm{par} \arrow[dl, "\mathrm{ev}"] \arrow[dddl, bend left, "E"] \\
 & \mathrm{tar} \arrow[d, "\mathrm{tra}"] & \arrow[dl, phantom, "\overset{e}{\Leftarrow}"] \\
 & D \arrow[d, "\mathrm{TD}"] & \\
 & n\mathrm{Cat} &
\end{tikzcd}
$$

Pasting a section to the space of histories

Definition

The propagation of a section with respect to a given space of histories is the pushforward of the previous pullback along out.

The charged 1-particle on a lattice

The setup

The 1-particle on a lattice and charged under a trivial G-bundle is modeled by taking $\mathrm{tar} = \mathbb{Z}//\mathbb{Z}$ and $\mathrm{tra} : \mathrm{tar} \to \Sigma G$.
We take $\mathrm{hist} \subset \mathrm{Hom}(a \to b, \mathrm{tar})$ to be that subcategory whose objects come only from generators of tar. This expresses the fact that this are histories of "one time step".

The charged 1-particle on a lattice

Remark

Notice that $\delta\mathrm{tra}$ takes values in $T_\bullet \Sigma G = \mathrm{INN}(G) = G//G$.
In the combinatoric picture of linear algebra following the *Tale of Groupoidification* this corresponds to the regular representation space of G.
Therefore we will – probably – eventually, following the *Tale*, split idempotents and project out of that regular rep an irrep.

The charged 1-particle on a lattice

Remark

Consider, for simplicity, that we have a section (E, e) of tra where E factors through $0\mathrm{Cat}$. Then this is

- over each object x of target space a G-phased set $E(x) \to G \hookrightarrow G//G$.
- over each morphism of target space the covariant derivative with respect to tar of these G-phased sets.

$$
\begin{tikzcd}
E(x) \arrow[rr] \arrow[d, "e(x)"] & & E(y) \arrow[d, "e(y)"] \arrow[dll, Rightarrow, "e(\gamma)"'] \\
G//G \arrow[rr, "\mathrm{tra}(x\to y)^*"] & & G//G
\end{tikzcd}
.
$$

More generally we get G-phased groupoids.

The charged 1-particle on a lattice

Push-forward of 1-sections of the 1-particle corresponds to taking the coproduct of these G-phased sets.

The charged 1-particle on a lattice

If we can assume that functor underlying a transformation e coming from a section is *free* (I am not sure yet how this can be nicely implemented), then propagating a section along the space of one-step histories corresponds, unless I am mixed up, to the ordinary (euclidean, latticized in our example) quantum propagation.

While more needs to be done, clearly, it seems we find evidence that

- The measures needed when quantizing may arise canonically from colimits by expressing the path integral as a pull-push operation of states of an *n*-particle propagating on an *n*-categorical target space.
- Three mysteries are apparently solved in one stroke by this method: the funny shift in categorical dimension, the appearance of phases and the need to pass from principal to associated transport all come from the categorical differential $\delta\mathrm{tra}$ of the classical transport *n*-functor tra.