Lie ∞-algebra connections and their application to String- and Chern-Simons n-Transport Part I: Overview and physical applications

Hisham Sati[*], Urs Schreiber[†] and Jim Stasheff[‡]

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Abstract

We give a generalization of the notion of a Cartan-Ehresmann connection from Lie algebras to L_∞-algebras and use it to study the obstruction theory of lifts through higher String-like extensions of Lie algebras.

It is known that over a D-brane the Kalb-Ramond background field of the string restricts to a 2-bundle with connection (a gerbe) which can be seen as the obstruction to lifting the $PU(H)$-bundle on the D-brane to a $U(H)$-bundle. We discuss how this phenomenon generalizes from the ordinary central extension $U(1) \to U(H) \to PU(H)$ to higher categorical central extensions, like the String-extension $\mathbf{B}U(1) \to \mathrm{String}(G) \to G$. Here the obstruction to the lift is a 3-bundle with connection (a 2-gerbe): the Chern-Simons 3-bundle classified by the first Pontrjagin class. For $G = \mathrm{Spin}(n)$ this obstructs the existence of a String-structure. We discuss how to describe this obstruction problem in terms of Lie n-algebras and their corresponding categorified Cartan-Ehresmann connections. Generalizations even beyond String-extensions are then straightforward. For $G = \mathrm{Spin}(n)$ the next step is "Fivebrane structures" whose existence is obstructed by certain generalized Chern-Simons 7-bundles classified by the second Pontrjagin class.

[*] hisham.sati@yale.edu
[†] schreiber@math.uni-hamburg.de
[‡] jds@math.upenn.edu

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1 Introduction

The study of extended n-dimensional relativistic objects which arise in string theory has shown that these couple to background fields which can be naturally thought of as n-fold categorified generalizations of fiber bundles with connection. There are two popular alternative viewpoints on studying such higher structures geometrically. The first is using the language of gerbes and the second using the language of Cheeger-Simons differential characters or Deligne cohomology.

fundamental object	**background field**
n-particle	n-bundle
$(n-1)$-brane	$(n-1)$-gerbe

Table 1: **The two schools of counting** higher dimensional structures. Here n is in $\mathbb{N} = \{0, 1, 2, \cdots\}$.

The first departure from bundles with connections occurs with the fundamental (super)string which couples to the Neveu-Schwarz (NS) B-field. Locally, the B-field is just an $\mathbb{R}$-valued two-form. However, the study of the path integral, which amounts to 'exponentiation', reveals that the B-field can be thought of as an abelian gerbe with connection whose curving corresponds to the H-field H_3 or as a Cheeger-Simons differential character, whose holonomy [19] can be described [11] in the language of bundle gerbes [35].

The next step up occurs with the M-theory (super)membrane which couples to the C-field [6]. In supergravity, this is viewed locally as an $\mathbb{R}$-valued differential three-form. However, the study of the path integral has shown that this field is quantized in a rather nontrivial way [44]. This makes the C-field not precisely a 2-gerbe or degree 3 Cheeger-Simons differential character but rather a shifted version [16] that can also be modeled using the Hopkins-Singer description of differential characters [26]. Some aspects of the description in terms of Deligne cohomology is given in [14].

From a purely formal point of view, the need of higher connections for the description of higher dimensional branes is not a surprise: n-fold categorified bundles with connection should be precisely those objects that allow us to define a consistent assignment of "phases" to n-dimensional paths in their base space. We address such an assignment as **parallel n-transport**. This is in fact essentially the definition of Cheeger-Simons differential characters [13] as these are consistent assignments of phases to chains. However, abelian bundle gerbes, Deligne cohomology and Cheeger-Simons differential characters all have one major restriction: they only know about assignments of elements in $U(1)$.

While the group of phases that enter the path integral is usually abelian, more general n-transport is important nevertheless. For instance, the latter plays a role at intermediate stages. This is well understood for $n = 2$: over a D-brane the abelian bundle gerbe corresponding to the NS field has the special property that it measures the obstruction to lifting a $PU(H)$-bundle to a $U(H)$-bundle, i.e. lifting a bundle with structure group the infinite projective unitary group on a Hilbert space H to the corresponding unitary group [8] [9]. Hence, while itself an abelian 2-structure, it is crucially related to a nonabelian 1-structure.

That this phenomenon deserves special attention becomes clear when we move up the dimensional ladder: The Green-Schwarz anomaly cancelation [22] in the heterotic string leads to a 3-structure with the special property that, over the target space, it measures the obstruction to lifting an $E_8 \times \mathrm{Spin}(n)$-bundle to a certain nonabelian principal 2-bundle, called a *String 2-bundle*. Such a 3-structure is also known as a Chern-Simons 2-gerbe [12]. By itself this is abelian, but its structure is constrained by certain nonabelian data. Namely this string 2-bundle with connection, from which the Chern-Simons 3-bundle arises, is itself an instance of a structure that yields parallel 2-transport. It can be described neither by abelian bundle gerbes, nor by

Cheeger-Simons differential characters, nor by Deligne cohomology.

In anticipation of such situations, previous works have considered nonabelian gerbes and nonabelian bundle gerbes with connection. However, it turns out that care is needed in order to find the right setup. For instance, the kinds of nonabelian gerbes with connection studied in [10] [1], although very interesting, are not sufficiently general to capture String 2-bundles. Moreover, it is not easy to see how to obtain the parallel 2-transport assignment from these structures. For the application to string physics, it would be much more suitable to have a nonabelian generalization of the notion of a Cheeger-Simons differential character, and thus a structure which, by definition, knows how to assign generalized phases to n-dimensional paths.

The obvious generalization that is needed is that of a parallel transport n-functor. Such a notion was described in [4] [40]: a structure defined by the very fact that it labels n-paths by algebraic objects that allow composition in n different directions, such that this composition is compatible with the gluing of n-paths. One can show that such transport n-functors encompass abelian and nonabelian gerbes with connection as special cases [40]. However, these n-functors are more general. For instance, String 2-bundles with connection are given by parallel transport 2-functors. Ironically, the strength of the latter – namely their knowledge about general phase assignments to higher dimensional paths – is to some degree also a drawback: for many computations, a description *entirely* in terms of differential form data would be more tractable. However, the passage from parallel n-transport to the corresponding differential structure is more or less straightforward: a parallel transport n-functor is essentially a morphism of Lie n-groupoids. As such, it can be sent, by a procedure generalizing the passage from Lie groups to Lie algebras, to a morphism of Lie n-algebroids.

The aim of this paper is to describe two topics: First, to set up a formalism for higher bundles with connections entirely in terms of L_∞-algebras, which may be thought of as a categorification of the theory of Cartan-Ehresmann connections. This is supposed to be the differential version of the theory of parallel transport n-functors, but an exhaustive discussion of the differentiation procedure is not given here. Instead we discuss a couple of examples and then show how the lifting problem has a nice description in this language. To do so, we present a family of L_∞-algebras that govern the gauge structure of p-branes, as above, and discuss the lifting problem for them. By doing so, we characterize Chern-Simons 3-forms as local connection data on 3-bundles with connection which arise as the obstruction to lifts of ordinary bundles to the corresponding String 2-bundles, governed by the String Lie 2-algebra.

The formalism immediately allows the generalization of this situation to higher degrees. Indeed we indicate how certain 7-dimensional generalizations of Chern-Simons 3-bundles obstruct the lift of ordinary bundles to certain 6-bundles governed by the Fivebrane Lie 6-algebra. The latter correspond to what we define as the fivebrane structure, for which the degree seven NS field H_7 plays the role that the degree three dual NS field H_3 plays for the $n = 2$ case.

The paper is organized in such a way that section 2 serves more or less as a self-contained description of the basic ideas and construction, with the rest of the document having all the details and all the proofs.

In this paper we make use of the homotopy algebras usually referred to as L_∞-algebras. These algebras also go by other names such as sh-Lie algebras [31]. In our context we may also call such algebras Lie ∞-algebras which we think of as the abstract concept of an ∞-vector space with an antisymmetric and coherently Jacobi bracket ∞-functor on it, whereas "L_∞-algebra" is concretely a codifferential coalgebra of sorts. In this paper we will nevertheless follow the standard notation of L_∞-algebra.

2 The Setting and Plan

We set up a useful framework for describing higher order bundles with connection entirely in terms of Lie n-algebras, which can be thought of as arising from a categorification of the concept of an Ehresmann connection on a principal bundle. Then we apply this to the study of Chern-Simons n-bundles with connection as obstructions to lifts of principal G-bundles through higher String-like extensions of their structure Lie algebra.

2.1 L_∞-algebras and their String-like central extensions

A Lie group has all the right properties to locally describe the phase change of a charged particle as it traces out a worldline. A Lie n-group is a higher structure with precisely all the right properties to describe locally the phase change of a charged $(n-1)$-brane as it traces out an n-dimensional worldvolume.

2.1.1 L_∞-algebras

Just as ordinary Lie groups have Lie algebras, Lie n-groups have Lie n-algebras. If the Lie n-algebra is what is called *semistrict*, these are [2] precisely L_∞-algebras [31] which have come to play a significant role in cohomological physics. A ("semistrict" and finite dimensional) Lie n-algebra is any of the following three equivalent structures:

- an L_∞-algebra structure on a graded vector space $\mathfrak{g}$ concentrated in the first n degrees $(0, ..., n-1)$;
- a quasi-free differential graded-commutative algebra ("qDGCA": free as a graded-commutative) algebra on the dual of that vector space: this is the Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ of $\mathfrak{g}$;
- an n-category internal to the category of *graded* vector spaces and equipped with a skew-symmetric linear bracket functor which satisfies a Jacobi identity up to higher coherent equivalence.

For every L_∞-algebra $\mathfrak{g}$, we have the following three qDGCAs:

- the **Chevalley-Eilenberg algebra** $\mathrm{CE}(\mathfrak{g})$
- the **Weil algebra** $\mathrm{W}(\mathfrak{g})$
- the algebra of **invariant polynomials** or **basic forms** $\mathrm{inv}(\mathfrak{g})$.

These sit in a sequence

$$\mathrm{CE}(\mathfrak{g}) \twoheadleftarrow \mathrm{W}(\mathfrak{g}) \hookleftarrow \mathrm{inv}(\mathfrak{g}) \;, \tag{1}$$

where all morphisms are morphisms of dg-algebras. This sequence plays the role of the sequence of differential forms on the "universal $\mathfrak{g}$-bundle".

2.1.2 L_∞-algebras from cocycles: String-like extensions

A simple but important source of examples for higher Lie n-algebras comes from the abelian Lie algebra $\mathfrak{u}(1)$ which may be shifted into higher categorical degrees. We write $b^{n-1}\mathfrak{u}(1)$ for the Lie n-algebra which is entirely trivial except in its nth degree, where it looks like $\mathfrak{u}(1)$. Just as $\mathfrak{u}(1)$ corresponds to the Lie group $U(1)$, so $b^{n-1}\mathfrak{u}(1)$ corresponds to the iterated classifying space $B^{n-1}U(1)$, realizable as the topological group given by the Eilenber-MacLane space $K(\mathbb{Z}, n)$. Thus an important source for interesting Lie n-algebras comes from extensions

$$0 \to b^{n-1}\mathfrak{u}(1) \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{3}$$

of an ordinary Lie algebra $\mathfrak{g}$ by such a shifted abelian Lie n-algebra $b^{n-1}\mathfrak{u}(1)$. We find that, for each $(n+1)$-cocycle μ in the Lie algebra cohomology of $\mathfrak{g}$, we do obtain such a central extension, which we describe by

$$0 \to b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0\,. \tag{4}$$

Since, for the case when $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ is the canonical 3-cocycle on a semisimple Lie algebra $\mathfrak{g}$, this $\mathfrak{g}_\mu$ is known ([3] and [25]) to be the Lie 2-algebra of the String 2-group, we call these central extensions *String-like* central extensions. (We also refer to these as Lie n-algebras "of Baez-Crans type" [2].) Moreover, whenever the cocycle μ is related by transgression to an invariant polynomial P on the Lie algebra, we find that $\mathfrak{g}_\mu$ fits into a short *homotopy* exact sequence of Lie $(n+1)$-algebras

$$0 \to \mathfrak{g}_\mu \to \mathrm{cs}_P(\mu) \to \mathrm{ch}_P(\mu) \to 0\,. \tag{5}$$

$$\begin{tikzcd}
 & \text{dg-algebras} & \text{(pointed) topological spaces} & \\
\text{Chevalley-Eilenberg algebra} & \mathrm{CE}(\mathfrak{g}) & G \arrow[d, hook, "i"] & \text{structure group} \\
\text{Weil algebra} & \mathrm{W}(\mathfrak{g}) \arrow[u, two heads, "i^*"] & EG \arrow[d, "p"] & \text{universal } G\text{-bundle} \\
\text{algebra of invariant polynomials} & \mathrm{inv}(\mathfrak{g}) \arrow[u, hook, "p^*"] & BG & \text{classifying space for } G
\end{tikzcd} \tag{2}$$

Figure 1: **The universal G-bundle** and its analog in the world of dg-algebras. See also figure **??**.

Here $\mathrm{cs}_P(\mathfrak{g})$ is a Lie $(n+1)$-algebra governed by the Chern-Simons term corresponding to the transgression element interpolating between μ and P. In a similar fashion $\mathrm{ch}_P(\mathfrak{g})$ knows about the characteristic (Chern) class associated with P.

In summary, from elements of $W(\mathfrak{g})$-cohomology we obtain the String-like extensions of Lie algebras to Lie $2n$-algebras and the associated Chern- and Chern-Simons Lie $(2n-1)$-algebras:

Lie algebra cocycle	μ	Baez-Crans Lie n-algebra	$\mathfrak{g}_\mu$
invariant polynomial	P	Chern Lie n-algebra	$\mathrm{ch}_P(\mathfrak{g})$
transgression element	cs	Chern-Simons Lie n-algebra	$\mathrm{cs}_P(\mathfrak{g})$

2.1.3 L_∞-algebra differential forms

For $\mathfrak{g}$ an ordinary Lie algebra and Y some manifold, one finds that dg-algebra morphisms $\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(Y)$ from the Chevally-Eilenberg algebra of $\mathfrak{g}$ to the DGCA of differential forms on Y are in bijection with $\mathfrak{g}$-valued 1-forms $A \in \Omega^1(Y, \mathfrak{g})$ whose ordinary curvature 2-form

$$F_A = dA + [A \wedge A] \tag{6}$$

vanishes. Without the flatness, the correspondence is with algebra morphisms *not* respecting the differentials. But dg-algebra morphisms $A : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(Y)$ are in bijection with arbitrary $\mathfrak{g}$-valued 1-forms. These are flat precisely if A factors through $\mathrm{CE}(\mathfrak{g})$. This situation is depicted in the following diagram:

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) \arrow[d, "{(A, F_A = 0)}"'] & \mathrm{W}(\mathfrak{g}) \arrow[l, two heads] \arrow[d, "{(A, F_A)}"'] \\
\Omega^\bullet(Y) & \Omega^\bullet(Y)
\end{tikzcd} . \tag{7}$$

This has an obvious generalization for $\mathfrak{g}$ an arbitrary L_∞-algebra. For $\mathfrak{g}$ any L_∞-algebra, we write

$$\Omega^\bullet(Y, \mathfrak{g}) = \mathrm{Hom}_{\mathrm{dg-Alg}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(X)) \tag{8}$$

for the collection of **$\mathfrak{g}$-valued differential forms** and

$$\Omega^\bullet_{\mathrm{flat}}(Y,\mathfrak{g}) = \mathrm{Hom}_{\mathrm{dg-Alg}}(\mathrm{CE}(\mathfrak{g}),\Omega^\bullet(X)) \tag{9}$$

for the collection of **flat $\mathfrak{g}$-valued differential forms**.

2.2 L_∞-algebra Cartan-Ehresmann connections

2.2.1 $\mathfrak{g}$-Bundle descent data

A *descent object* for an ordinary principal G-bundle on X is a surjective submersion $\pi : Y \to X$ together with a functor $g : Y \times_X Y \to \mathbf{B}G$ from the groupoid whose morphisms are pairs of points in the same fiber of Y, to the groupoid $\mathbf{B}G$ which is the one-object groupoid corresponding to the group G. Notice that the groupoid $\mathbf{B}G$ is not itself the classifying space BG of G, but the geometric realization of its nerve, $|\mathbf{B}G|$, is: $|\mathbf{B}G| = BG$.

We may take Y to be the disjoint union of some open subsets $\{U_i\}$ of X that form a good open cover of X. Then g is the familiar concept of a transition function decribing a bundle that has been locally trivialized over the U_i. But one can also use more general surjective submersions. For instance, for $P \to X$ any principal G-bundle, it is sometimes useful to take $Y = P$. In this case one obtains a canonical choice for the cocycle

$$g : Y \times_X Y = P \times_X P \to \mathbf{B}G \tag{10}$$

since P being principal means that

$$P \times_X P \simeq_{\mathrm{diffeo}} P \times G\,. \tag{11}$$

This reflects the fact that every principal bundle canonically trivializes when pulled back to its own total space. The choice $Y = P$ differs from that of a good cover crucially in the following aspect: if the group G is connected, then also the fibers of $Y = P$ are connected. Cocycles over surjective submersions with connected fibers have special properties, which we will make use of: *When the fibers of Y are connected, we may think of the assignment of group elements to pairs of points in one fiber as arising from the parallel transport with respect to a flat vertical 1-form $A_{\mathrm{vert}} \in \Omega^1_{\mathrm{vert}}(Y,\mathfrak{g})$, flat along the fibers.* As we shall see, this can be thought of as the vertical part of a Cartan-Ehresmann connection 1-form. This provides a morphism

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) \tag{12}$$

of differential graded algebras from the Chevalley-Eilenberg algebra of $\mathfrak{g}$ to the vertical differential forms on Y.

Unless otherwise specified, *morphism* will always mean *homomorphism of differential graded algebra.* A_{vert} has an obvious generalization: for $\mathfrak{g}$ any Lie n-algebra, we say that a $\mathfrak{g}$-bundle descent object for a $\mathfrak{g}$-n-bundle on X is a surjective submersion $\pi : Y \to X$ together with a morphism $\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$. Now $A_{\mathrm{vert}} \in \Omega^\bullet_{\mathrm{vert}}(Y,\mathfrak{g})$ encodes a collection of vertical p-forms on Y, each taking values in the degree p-part of $\mathfrak{g}$ and all together satisfying a certain flatness condition, controlled by the nature of the differential on $\mathrm{CE}(\mathfrak{g})$.

2.2.2 Connections on n-bundles: the extension problem

Given a descent object $\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$ as above, a **flat connection** on it is an extension of the morphism A_{vert} to a morphism A_{flat} that factors through differential forms on Y

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \arrow[dl, dotted, "A_{\mathrm{flat}}"] \\ \Omega^\bullet(Y) \arrow[u, "i^*"] & \end{tikzcd} . \tag{13}$$

In general, such an extension does not exist.

A general **connection** on a $\mathfrak{g}$-descent object A_{vert} is a morphism

$$\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) \tag{14}$$

from the Weil algebra of $\mathfrak{g}$ to the differential forms on Y together with a morphism

$$\Omega^\bullet(X) \xleftarrow{\{K_i\}} \mathrm{inv}(\mathfrak{g}) \tag{15}$$

from the invariant polynomials on $\mathfrak{g}$, as in 2.1.1, to the differential forms on X, such that the following two squares commute:

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(Y) \arrow[u, "i^*"] & W(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u] \\ \Omega^\bullet(X) \arrow[u, hook, "\pi^*"] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{K_i\}"] \arrow[u, hook] \end{tikzcd} . \tag{16}$$

Whenever we have such two commuting squares, we say

- $A_{\mathrm{vert}} \in \Omega^\bullet_{\mathrm{vert}}(Y,\mathfrak{g})$ is a $\mathfrak{g}$-bundle **descent object** (playing the role of a **transition function**);
- $A \in \Omega^\bullet(Y,\mathfrak{g})$ is a (Cartan-Ehresmann) **connection** with values in the L_∞-algebra $\mathfrak{g}$ on the total space of the surjective submersion;
- $F_A \in \Omega^{\bullet+1}(Y,\mathfrak{g})$ are the corresponding **curvature** forms;
- and the set $\{K_i \in \Omega^\bullet(X)\}$ are the corresponding **characteristic forms**, whose classes $\{[K_i]\}$ in deRham cohomology

$$\Omega^\bullet(X) \xleftarrow{\{K_i\}} \mathrm{inv}(\mathfrak{g}) \tag{17}$$

$$H^\bullet_{\mathrm{deRham}}(X) \xleftarrow{\{[K_i]\}} H^\bullet(\mathrm{inv}(\mathfrak{g}))$$

are the corresponding **characteristic classes** of the given descent object A_{vert}.

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & & \mathrm{CE}(\mathfrak{g}) \arrow[ll, "A_{\mathrm{vert}}"'] \\
\\
\Omega^\bullet(Y) \arrow[uu, two heads, "i^*"] & & W(\mathfrak{g}) \arrow[ll, "{(A,F_A)}"'] \arrow[uu, two heads] \\
\\
\Omega^\bullet(X) \arrow[uu, hook, "\pi^*"] & & \mathrm{inv}(\mathfrak{g}) \arrow[ll, "\{K_i\}" description] \arrow[uu, hook] \\
H^\bullet_{\mathrm{dR}}(X) & & H^\bullet(\mathrm{inv}(\mathfrak{g})) \arrow[ll, "{\{[K_i]\}}"]
\end{tikzcd}
$$

descent data

first Cartan-Ehresmann condition

connection data

second Cartan-Ehresmann condition

characteristic forms

Chern-Weil homomorphism

Figure 2: **A $\mathfrak{g}$-connection descent object and its interpretation.** For $\mathfrak{g}$-any L_∞-algebra and X a smooth space, a $\mathfrak{g}$-connection on X is an equivalence class of pairs $(Y, (A, F_A))$ consisting of a surjective submersion $\pi : Y \to X$ and dg-algebra morphisms forming the above commuting diagram. The equivalence relation is concordance of such diagrams.

So we realize the curvature of a $\mathfrak{g}$-connection as the *obstruction* to extending a $\mathfrak{g}$-descent object to a *flat* $\mathfrak{g}$-connection.

2.3 Higher String and Chern-Simons n-transport: the lifting problem

Given a $\mathfrak{g}$-descent object

$$
\begin{tikzcd}
 & \mathrm{CE}(\mathfrak{g}) \arrow[dl, "A_{\mathrm{vert}}"] \; , \\
\Omega^\bullet_{\mathrm{vert}}(Y) &
\end{tikzcd}
\tag{18}
$$

and given an extension of $\mathfrak{g}$ by a String-like L_∞-algebra

$$
\begin{tikzcd}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i"'] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \; ,
\end{tikzcd}
\tag{19}
$$

we ask if it is possible to *lift the descent object* through this extension, i.e. to find a dotted arrow in

$$
\begin{tikzcd}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads] \arrow[dr, dotted] & & \mathrm{CE}(\mathfrak{g}) \arrow[ll, hook'] \arrow[dl, "A_{\mathrm{vert}}"] \; . \\
 & & \Omega^\bullet_{\mathrm{vert}}(Y) &
\end{tikzcd}
\tag{20}
$$

In general this is not possible. We seek a straightforward way to compute the obstruction to the existence of the lift. The strategy is to form the *weak* (homotopy) kernel of

$$
\begin{tikzcd}
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i"']
\end{tikzcd}
\tag{21}
$$

which we denote by $\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$ and realize as a mapping cone of qDGCAs.

This comes canonically with a morphism f from $\mathrm{CE}(\mathfrak{g})$ which happens to have a *weak* inverse

$$\begin{tikzcd}
 & & & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[dll] \arrow[dl, bend left, "f^{-1}"] \\
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i"'] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[ur, "f"] &
\end{tikzcd} \quad . \tag{22}$$

Then we see that, while the lift to a $\mathfrak{g}_\mu$-cocycle may not always exist, the lift to a $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$-cocycle does always exist. We form $A_{\mathrm{vert}} \circ f^{-1}$:

$$\begin{tikzcd}
 & & & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[dll] \arrow[dl, "f^{-1}"'] \\
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i"'] \arrow[dr, dotted] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[d, "A_{\mathrm{vert}}"] & \\
 & & \Omega^\bullet_{\mathrm{vert}}(Y) &
\end{tikzcd} \quad . \tag{23}$$

The failure of this lift to be a true lift to $\mathfrak{g}_\mu$ is measured by the component of $A_{\mathrm{vert}} \circ f^{-1}$ on $b^{n-1}\mathfrak{u}(1)[1] \simeq b^n\mathfrak{u}(1)$. Formally this is the composite $A'_{\mathrm{vert}} := A_{\mathrm{vert}} \circ f^{-1} \circ j$ in

$$\begin{tikzcd}
 & & & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[dll] \arrow[dl, "f^{-1}"'] & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l, hook', "j"'] \arrow[ddll, "A'_{\mathrm{vert}}"] \\
\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \mathrm{CE}(\mathfrak{g}_\mu) \arrow[l, two heads, "i"'] \arrow[dr, dotted] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[d, "A_{\mathrm{vert}}"] & & \\
 & & \Omega^\bullet_{\mathrm{vert}}(Y) & &
\end{tikzcd} \quad . \tag{24}$$

The nontriviality of the $b^n\mathfrak{u}(1)$-descent object A'_{vert} is the obstruction to constructing the desired lift.

We thus find the following results, for any $\mathfrak{g}$-cocycle μ which is in transgression with the the invariant polynomial P on $\mathfrak{g}$,

- The characteristic classes (in deRham cohomology) of $\mathfrak{g}_\mu$-bundles are those of the corresponding $\mathfrak{g}$-bundles modulo those coming from the invariant polynomial P.
- The lift of a $\mathfrak{g}$-valued connection to a $\mathfrak{g}_\mu$-valued connection is obstructed by a $b^n\mathfrak{u}(1)$-valued $(n+1)$-connection whose $(n+1)$-form curvature is $P(F_A)$, i.e. the image under the Chern-Weil homomorphism of the invariant polynomial corresponding to μ.
- Accordingly, the $(n+1)$-form connection of the obstructing $b^n\mathfrak{u}(1)$ $(n+1)$-bundle is a Chern-Simons form for this characteristic class.

We call the obstructing $b^n\mathfrak{u}(1)$ $(n+1)$-descent object the corresponding Chern-Simons $(n+1)$-bundle. For the case when $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ is the canonical 3-cocycle on a semisimple Lie algebra $\mathfrak{g}$, this structure (corresponding to a 2-gerbe) has a 3-connection given by the ordinary Chern-Simons 3-form and has a curvature 4-form given by the (image in deRham cohomology of the) first Pontrjagin class of the underlying $\mathfrak{g}$-bundle.

3 Physical applications: String-, Fivebrane- and p-Brane structures

We can now discuss physical applications of the formalism that we have developed. What we describe is a useful way to handle obstructing n-bundles of various kinds that appear in string theory. In particular, we can describe generalizations of string structure in string theory. In the context of p-branes, such generalizations have been suggested based on p-loop spaces [18] [5] [37] and, more generally, on the space of maps $\mathrm{Map}(\mathrm{M},\mathrm{X})$ from the brane worldvolume M to spacetime X [34]. The statements in this section will be established in detail in [45].

From the point of view of supergravity, all branes, called p-branes in that setting, are a priori treated in a unified way. In tracing back to string theory, however, there is a distinction in the form-fields between the Ramond-Ramond (RR) and the Neveu-Schwarz (NS) forms. The former live in generalized cohomology and the latter play two roles: they act as twist fields for the RR fields and they are also connected to the geometry and topology of spacetime. The H-field H_3 plays the role of a twist in K-theory for the RR fields [28] [8] [33]. The twist for the degree seven dual field H_7 is observed in [39] at the rational level.

The ability to define fields and their corresponding partition functions puts constraints on the topology of the underlying spacetime. The most commonly understood example is that of fermions where the ability to define them requires spacetime to be spin, and the ability to describe theories with chiral fermions requires certain restrictions coming from the index theorem. In the context of heterotic string theory, the Green-Schwarz anomaly cancelation leads to the condition that the difference between the Pontrjagin classes of the tangent bundle and that of the gauge bundle be zero. This is called the string structure, which can be thought of as a spin structure on the loop space of spacetime [29] [15]. In M-theory, the ability to define the partition function leads to an anomaly given by the integral seventh-integral Steifel-Whitney class of spacetime [17] whose cancelation requires spacetime to be orientable with respect to generalized cohomology theories beyond K-theory [30] .

In all cases, the corresponding structure is related to the homotopy groups of the orthogonal group: the spin structure amounts to killing the first homotopy group, the string structure and – to some extent– the W_7 condition to killing the third homotopy group. Note that when we say that the n-th homotopy group is killed, we really mean that all homotopy groups up to and including the n-th one are killed. For instance, a String structure requires killing everything up to and including the third, hence everything through the sixth, since there are no homotopy groups in degrees four, five or six.

The Green-Schwarz anomaly cancelation condition for the heterotic string can be translated to the language of n-bundles as follows. We have two bundles, the spin bundle with structure group $G = \mathrm{Spin}(10)$, and the gauge bundle with structure group G' being either $\mathrm{SO}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$. Considering the latter, we have one copy of E_8 on each ten-dimensional boundary component, which can be viewed as an end-of-the-world nine-brane, or M9-brane [27]. The structure of the four-form on the boundary which we write as

$$G_4|_\partial = dH_3 \tag{25}$$

implies that the 3-bundle (2-gerbe) becomes the trivializable lifting 2-gerbe of a $\mathrm{String}(\mathrm{Spin}(10)\times E_8)$ bundle over the M9-brane. As the four-form contains the difference of the Pontrjagin classes of the bundles with structure groups G and G', the corresponding three-form will be a difference of Chern-Simons forms. The bundle aspect of this has been studied in [7] and will be revisited in the current context in [45].

The NS fields play a special role in relation to the homtopy groups of the orthogonal group. The degree three class $[H_3]$ plays the role of a twist for a spin structure. Likewise, the degree seven class plays a role of a twist for a higher structure related to $BO\langle 10\rangle$, the 9-connected cover of BO, which we might call a *Fivebrane*-structure on spacetime. We can talk about such a structure once the spacetime already has a string structure. The obstructions are given in the following table, where A is the connection on the G'

bundle and ω is a connection on the G bundle.

n	2 $= 4 \cdot 0 + 2$	6 $= 4 \cdot 1 + 2$
fundamental object $(n-1)$-brane n-particle	string	5-brane
target space structure	string structure $\mathrm{ch}_2(A) - p_1(\omega) = 0$	fivebrane structure $\mathrm{ch}_4(A) - \frac{1}{48}p_2(\omega) = 0$

Table 2: **Higher dimensional extended objects** and the corresponding topological structures.

In the above we alluded to how the brane structures are related to obstructions to having spacetimes with connected covers of the orthogonal groups as structures. The obstructing classes here may be regarded as classifying the corresponding obstructing n-bundles, after we apply the general formalism that we outlined earlier. The main example of this general mechanism that will be of interest to us here is the case where $\mathfrak{g}$ is an ordinary semisimple Lie algebra. In particular, we consider $\mathfrak{g} = \mathfrak{spin}(n)$.

For $\mathfrak{g} = \mathfrak{spin}(n)$ and μ a $(2n+1)$-cocycle on $\mathfrak{spin}(n)$, we call $\mathfrak{spin}(n)_\mu$ the (skeletal version of the) $(2n-1)$**-brane Lie** $(2n)-algebra$.

Thus, the case of String structure and Fivebrane structure occurring in the fundamental string and NS fivebrane correspond to the cases $n = 1$ and $n = 3$ respectively. Now applying our formalism for $\mathfrak{g} = \mathrm{spin}(n)$, and μ_3, μ_7 the canonical 3- and 7-cocycle, respectively:

- the obstruction to lifting a $\mathfrak{g}$-bundle descent object to a String 2-bundle (a $\mathfrak{g}_{\mu_3}$-bundle descent object) is the first Pontryagin class of the original $\mathfrak{g}$-bundle cocycle;
- the obstruction to lifting a String 2-bundle descent object to a Fivebrane 6-bundle cocycle (a $\mathfrak{g}_{\mu_7}$-bundle descent object) is the second Pontryagin class of the original $\mathfrak{g}$-bundle cocycle.

The cocyles and invariant polynomials corresponding to the two structures are given in the following table

p**-brane**	**cocycle**	**invariant polynomial**	
$p = 1 = 4 \cdot 0 + 1$	$\mu_3 = \langle \cdot, [\cdot, \cdot] \rangle$	$P_1 = \langle \cdot, \cdot \rangle$	first Pontrjagin
$p = 5 = 4 \cdot 1 + 1$	$\mu_7 = \langle \cdot, [\cdot, \cdot], [\cdot, \cdot], [\cdot, \cdot] \rangle$	$P_2 = \langle \cdot, \cdot, \cdot, \cdot \rangle$	second Pontrjagin

Table 3: The Lie algebra cohomology governing NS p-branes.

In case of the fundamental string, the obstruction to lifting the $PU(H)$ bundles to $U(H)$ bundles is measured by a gerbe or a line 2-bundle. In the language of E_8 bundles this corresponds to lifting the loop group LE_8 bundles to the central extension $\hat{L}E_8$ bundles [33]. The obstruction for the case of the String structure is a 2-gerbe and that of a Fivebrane structure is a 6-gerbe. The structures are summarized in the following table

obstruction		G**-bundle**		$\hat{G}$**-bundle**
1-gerbes / line 2-bundles		$PU(H)$-bundles		$U(H)$-bundles
2-gerbes / line 3-bundles	obstruct the lift of	Spin(n)-bundles	to	String(n)-2-bundles
6-gerbes / line 7-bundles		Spin(n)-bundles		FiveBrane(n)-6-bundles

Table 4: **Obstructing line** n**-bundles** appearing in string theory.

A description can also be given in terms of (higher) loop spaces, generalizing the known case where a String structure on a space X can be viewed as a Spin structure on the loop space LX. A fuller discussion of the ideas of this section will be given in [45].

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Hisham Sati
Department of Mathematics
Yale University
10 Hillhouse Avenue
New Haven, CT 06511

Urs Schreiber
Fachbereich Mathematik
Schwerpunkt Algebra und Zahlentheorie
Universität Hamburg
Bundesstraße 55
D–20146 Hamburg

Jim Stasheff
Department of Mathematics
University of Pennsylvania
David Rittenhouse Lab.
209 South 33rd Street
Philadelphia, PA 19104-6395