In [1] the definition of a connection acting on a cochain complex relative to a differential algebra is given, generalizing the notion of an ordinary covariant derivative:

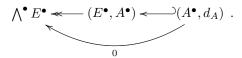
$$\nabla: \Gamma(E) \to \Omega^1(X) \otimes \Gamma(E)$$

and similar to that of a Quillen superconnection.

In [2] a general concept of actions of differential algebras is given. Here I talk about how the construction in [1] fits into the concept of [2].

Recall the definition of DGA action from [2]:

**Definition 1** Given a DGA  $(A^{\bullet}, d_A)$  and a cochain complex  $(E^{\bullet}, d_E)$  of  $A^0$ -modules, an action is witnessed by a DGA extension



Motivation: action  $\infty$ -groupoids. One way to motivate this is to concentrate on differential graded-commutative algebras (DGCA) which are free as graded-commutative algebras (qDGCA). These arise as Chevalley-Eilenberg algebras of  $L_{\infty}$ -algebras  $\mathfrak{g}$ :  $(A^{\bullet}, d_A) = \operatorname{CE}(\mathfrak{g})$ . These  $L_{\infty}$ -algebras Lie integrate to  $\infty$ -groups. An action of them is known to be witnessed by the corresponding action  $\infty$ -groupoid, as described in [2]. Its Lie version is the action  $L_{\infty}$ -algebraid whose dual CE-algebra is the above  $(E^{\bullet}, A^{\bullet})$ :

$$(E^{\bullet}, A^{\bullet}) = \operatorname{CE}_{\rho}(\mathfrak{g}, E).$$

**Example: ordinary Lie module.** The standard example to keep in mind is the ordinary representation

$$\rho:\mathfrak{g}\otimes V\to V$$

of an ordinary Lie algebra  $\mathfrak{g}$  on an ordinary vector space V. The relevant qDGCA is  $CE(\mathfrak{g})$  and  $E^{\bullet}$  is  $V^*$  regarded as a complex concentrated in degree 0. The action is witnessed by the CE-algebra of the module given by  $\rho$ .

$$\wedge^{\bullet} V^* \leftarrow \mathrm{CE}_{\rho}(\mathfrak{g}, V) \leftarrow \mathrm{CE}(\mathfrak{g}).$$

Notice that with the convention from [2]  $\wedge^{\bullet} V^*$  is the *symmetric* tensor algebra over  $V^*$ . For definiteness, here

$$\operatorname{CE}(\mathfrak{g}) = (\wedge^{\bullet}(\underbrace{\mathfrak{g}^{*}}_{1}), d_{\mathfrak{g}})$$

and

$$\operatorname{CE}(\mathfrak{g}) = (\wedge^{\bullet}(\underbrace{V}_{0} \oplus \underbrace{\mathfrak{g}^{*}}_{1}), d_{V,\mathfrak{g}}).$$

The extended differential  $d_{V,\mathfrak{g}}$  is entirely fixed by its restriction to V:

$$d_{V,\mathfrak{g}}|_V: V \to V \otimes \mathfrak{g}^*$$
,

where it is nothing but the dualization of the action morphism  $\rho : \mathfrak{g} \otimes V \to V$ with respect to  $\mathfrak{g}$ .

Lie-integrated, this example comes from the action groupoid sequence

$$V \longrightarrow V//G \longrightarrow G$$

of a Lie group G acting on a vector space V.

**Example: flat connections on vector bundles.** Another example more directly related to the discussion of connections is the one where  $(A^{\bullet}, d_A) = \Omega^{\bullet}(X)$  is the deRham-complex of some manifold X, and  $(E^{\bullet}, d_E)$  is concentrated, again, in degree 0, where it is the space of sections  $\Gamma(V)$  of some vector bundle  $V \to X$ . Any extension of the differential of  $\Omega^{\bullet}(X)$  is fixed by its action on  $\Gamma(V)$ 

$$d: \Gamma(V) \to \Gamma(V) \otimes_{\Omega^0(X)} \Omega^1(X),$$

hence is a flat connection on V.

Lie integrated, this comes from the integrated Atiyah-sequence of V, where  $\Omega^{\bullet}(X)$  integrates to the fundamental groupoid  $\Pi_1(X)$  of X and the action DGA to the Atiyah-groupoid whose objects are the fibers of V and whose morphisms the fiber homomorphisms.

$$\operatorname{At}(V) \to \Pi(X)$$
.

Notice that parallel transport in V is a section of this integrated sequence

$$\Pi(X) \to \operatorname{At}(V) \,.$$

**Block's definition.** In def. 6 of [1] Block considers a DGA  $(A^{\bullet}, d_A)$  and a cochain complex  $(E^{\bullet}, d_E)$  equipped with the structure of an  $A^0$ -module. He says

**Definition 2** A  $\mathbb{Z}$ -connection  $\mathbb{A}$  is a map linear over the ground field and of degree +1

$$\mathbb{A}: E^{\bullet} \otimes_{A^0} A^{\bullet} \to E^{\bullet} \otimes_{A^0} A^{\bullet}$$

satisfying for all  $e \in E^{\bullet}$  and  $\omega \in A^{\bullet}$  the equation

$$\mathbb{A}(e \otimes \omega) = (\mathbb{A}(e)) \otimes \omega + (-1)^{|e|} e \otimes d_A \omega.$$

This connection is flat if  $\mathbb{A}^2 = 0$ .

As he remarks right after the definition, such a map is already fixed by its restriction to  $E^\bullet$ 

$$\mathbb{A}: E^{\bullet} \to E^{\bullet} \otimes_{A^0} A^{\bullet}$$

Indeed, moreover we see that if we consider  $E^{\bullet} \otimes_{A^0} A^{\bullet}$  to be one term in the GCA

 $\wedge_{A^0}^{\bullet}(E^{\bullet} \oplus A^{\bullet})$ 

then for flat  $\mathbbm{A}$  the above makes this GCA a DGCA. If we furthermore require that the component of  $\mathbbm{A}$  which maps

$$\mathbb{A}^0: E^{\bullet} \to E^{\bullet+1}$$

coincides with the original differential  $d_E$  on  $E^{\bullet}$ , then this defines an extension

$$\wedge^{\bullet} E^{\bullet} \leftarrow (E^{\bullet}, A^{\bullet}) \leftarrow A^{\bullet}$$

of qDGCAs.

## References

- Jonathan Block, Duality and equivalence of module categories in noncommutative geometry I, [arXiv:math/0509284]
- [2] U.S. On ∞-Lie [http://www.math.uni-hamburg.de/home/schreiber/action.pdf]