In [1] the definition of a connection acting on a cochain complex relative to a differential algebra is given, generalizing the notion of an ordinary covariant derivative:

$$\nabla : \Gamma(E) \to \Omega^1(X) \otimes \Gamma(E)$$

and similar to that of a Quillen superconnection.

In [2] a general concept of actions of differential algebras is given. Here I talk about how the construction in [1] fits into the concept of [2].

Recall the definition of DGA action from [2]:

Definition 1 *Given a DGA* $(A^\bullet, d_A)$ *and a cochain complex* $(E^\bullet, d_E)$ *of* A^0*-modules, an action is witnessed by a DGA extension*

$$\begin{tikzcd} \bigwedge^\bullet E^\bullet & (E^\bullet, A^\bullet) \arrow[l, two heads] & (A^\bullet, d_A) \arrow[l, hook'] \arrow[ll, bend left, "0"] \end{tikzcd} .$$

Motivation: action ∞-groupoids. One way to motivate this is to concentrate on differential graded-*commutative* algebras (DGCA) which are free as graded-commutative algebras (qDGCA). These arise as Chevalley-Eilenberg algebras of L_∞-algebras $\mathfrak{g}$: $(A^\bullet, d_A) = \mathrm{CE}(\mathfrak{g})$. These L_∞-algebras Lie integrate to ∞-groups. An action of them is known to be witnessed by the corresponding action ∞-groupoid, as described in [2]. Its Lie version is the action L_∞-algebroid whose dual CE-algebra is the above $(E^\bullet, A^\bullet)$:

$$(E^\bullet, A^\bullet) = \mathrm{CE}_\rho(\mathfrak{g}, E)\,.$$

Example: ordinary Lie module. The standard example to keep in mind is the ordinary representation

$$\rho : \mathfrak{g} \otimes V \to V$$

of an ordinary Lie algebra $\mathfrak{g}$ on an ordinary vector space V. The relevant qDGCA is $\mathrm{CE}(\mathfrak{g})$ and $E^\bullet$ is V^* regarded as a complex concentrated in degree 0. The action is witnessed by the CE-algebra of the module given by ρ.

$$\wedge^\bullet V^* \leftarrow \mathrm{CE}_\rho(\mathfrak{g}, V) \leftarrow \mathrm{CE}(\mathfrak{g})\,.$$

Notice that with the convention from [2] $\wedge^\bullet V^*$ is the *symmetric* tensor algebra over V^*. For definiteness, here

$$\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet(\underbrace{\mathfrak{g}^*}_{1}), d_{\mathfrak{g}})$$

and

$$\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet(\underbrace{V}_{0} \oplus \underbrace{\mathfrak{g}^*}_{1}), d_{V,\mathfrak{g}})\,.$$

The extended differential $d_{V,\mathfrak{g}}$ is entirely fixed by its restriction to V:

$$d_{V,\mathfrak{g}}|_V : V \to V \otimes \mathfrak{g}^* ,$$

where it is nothing but the dualization of the action morphism $\rho : \mathfrak{g} \otimes V \to V$ with respect to $\mathfrak{g}$.

Lie-integrated, this example comes from the action groupoid sequence

$$V \longrightarrow V//G \longrightarrow G$$

of a Lie group G acting on a vector space V.

Example: flat connections on vector bundles. Another example more directly related to the discussion of connections is the one where $(A^\bullet, d_A) = \Omega^\bullet(X)$ is the deRham-complex of some manifold X, and $(E^\bullet, d_E)$ is concentrated, again, in degree 0, where it is the space of sections $\Gamma(V)$ of some vector bundle $V \to X$. Any extension of the differential of $\Omega^\bullet(X)$ is fixed by its action on $\Gamma(V)$

$$d : \Gamma(V) \to \Gamma(V) \otimes_{\Omega^0(X)} \Omega^1(X) ,$$

hence is a flat connection on V.

Lie integrated, this comes from the integrated Atiyah-sequence of V, where $\Omega^\bullet(X)$ integrates to the fundamental groupoid $\Pi_1(X)$ of X and the action DGA to the Atiyah-groupoid whose objects are the fibers of V and whose morphisms the fiber homomorpohisms.

$$\mathrm{At}(V) \to \Pi(X) .$$

Notice that parallel transport in V is a section of this integrated sequence

$$\Pi(X) \to \mathrm{At}(V) .$$

Block's definition. In def. 6 of [1] Block considers a DGA $(A^\bullet, d_A)$ and a cochain complex $(E^\bullet, d_E)$ equipped with the structure of an A^0-module. He says

Definition 2 *A $\mathbb{Z}$-connection $\mathbb{A}$ is a map linear over the ground field and of degree +1*

$$\mathbb{A} : E^\bullet \otimes_{A^0} A^\bullet \to E^\bullet \otimes_{A^0} A^\bullet$$

satisfying for all $e \in E^\bullet$ and $\omega \in A^\bullet$ the equation

$$\mathbb{A}(e \otimes \omega) = (\mathbb{A}(e)) \otimes \omega + (-1)^{|e|} e \otimes d_A \omega .$$

This connection is flat *if* $\mathbb{A}^2 = 0$.

As he remarks right after the definition, such a map is already fixed by its restriction to $E^\bullet$

$$\mathbb{A} : E^\bullet \to E^\bullet \otimes_{A^0} A^\bullet \,.$$

Indeed, moreover we see that if we consider $E^\bullet \otimes_{A^0} A^\bullet$ to be one term in the GCA

$$\wedge^\bullet_{A^0}(E^\bullet \oplus A^\bullet)$$

then for flat $\mathbb{A}$ the above makes this GCA a DGCA. If we furthermore require that the component of $\mathbb{A}$ which maps

$$\mathbb{A}^0 : E^\bullet \to E^{\bullet+1}$$

coincides with the original differential d_E on $E^\bullet$, then this defines an extension

$$\wedge^\bullet E^\bullet \leftarrow (E^\bullet, A^\bullet) \leftarrow A^\bullet$$

of qDGCAs.

References

[1] Jonathan Block, *Duality and equivalence of module categories in noncommutative geometry I*, [arXiv:math/0509284]

[2] U.S. *On ∞-Lie* [http://www.math.uni-hamburg.de/home/schreiber/action.pdf]