Bimodules

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Abstract

We collect some facts about the bicategory of bimodules internal to some suitable category. Then we take a closer look at the full sub-bicategories of induced bimodules, of special Frobenius bimodules, and of the intersection of these two.

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1 General Bimodules

Bimodules can be considered internal to any category

$$\mathcal{C}$$

which is

- monoidal
- monoidally cocomplete.

While monoidal cocompleteness (meaning that all colimits exist and are preserved by tensoring with any object) is a convenient assumption, in practice we often just want to require all those colimits to exist that are actually applied.

We denote the tensor product by

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and the tensor unit by

$$\mathbb{1} \in \mathrm{Obj}(\mathcal{C})\,.$$

The bicategory

$$\mathrm{Bim}(\mathcal{C})$$

of **bimodules internal to** $\mathcal{C}$ has objects that are algebras internal to $\mathcal{C}$, morphisms that are bimodules of such algebras, with horizontal composition being the bimodule tensor product, and 2-morphisms which are bimodule homomorphisms.

More generally, bimodules can be considered internal to suitable bicategories. Since $\mathcal{C}$ is monoidal, we may think of it equivalently in terms of its suspension

$$\Sigma\mathcal{C}\,,$$

which is the bicategory with a single object, single Hom-categoy $\mathcal{C}$ and composition functor given by the tensor product.

One way to appreciate the relevance of the notion of $\mathrm{Bim}(\mathcal{C})$ is to notice that any of its full sub-bicategories with n objects is given by the image of a lax functor

$$\mathrm{Codisc}(n) \to \Sigma(\mathcal{C})\,.$$

1.1 Definitions

A **monoid** A in $\mathcal{C}$ is a **monad** in $\Sigma(\mathcal{C})$. Here we shall call such monoids **algebras** (or algebra objects) in $\mathcal{C}$, even though, strictly speaking, this term should refer to monoids in abelian categories only.

For $\mathcal{C} = \mathrm{Vect}$ algebras in $\mathcal{C}$ coincide with the ordinary notion of algebra.

We denote the algebra product, which is a morphism in $\mathcal{C}$, by

$$m : A \otimes A \to A\,.$$

The unit we denote by

$$i : \mathbb{1} \to A\,.$$

A monoid in $\mathcal{C}$ is the same as a $\mathcal{C}$-enriched category with a single object. Accordingly, there is a many-object version of all of the following considerations. In the case that $\mathcal{C}$ is closed, this generalizes the concept of bimodules of algebras to that of profunctors of $\mathcal{C}$-enriched categories.

Definition 1 *Let A and B be algebra objects in $\mathcal{C}$. An A-B* **bimodule** *in $\mathcal{C}$ is an object $N \in \mathrm{Obj}(\mathcal{C})$ together with commuting left and right action morphisms*

$$A \otimes N \xrightarrow{\ell} A$$

and

$$N \otimes B \xrightarrow{r} A$$

satisfying

1. *compatibility with the product*

$$\begin{tikzcd} A \otimes A \otimes N \arrow[r, "m\otimes N"] \arrow[d, "A\otimes\ell"'] & A \otimes N \arrow[d, "\ell"] \\ A \otimes N \arrow[r, "\ell"'] & N \end{tikzcd}$$

$$\begin{tikzcd} N \otimes B \otimes B \arrow[r, "N\otimes m"] \arrow[d, "r\otimes B"'] & N \otimes B \arrow[d, "r"] \\ N \otimes B \arrow[r, "r"'] & N \end{tikzcd}$$

2. *compatibility with the unit*

$$\begin{tikzcd} \mathbb{1} \otimes N \arrow[rr] \arrow[dr, "i_A\otimes N"'] & & N \\ & A \otimes N \arrow[ur, "\ell"'] & \end{tikzcd}$$

$$\begin{tikzcd} N \otimes \mathbb{1} \arrow[rr] \arrow[dr, "N\otimes i_B"'] & & N \\ & A \otimes N \arrow[ur, "r"'] & \end{tikzcd}$$

Definition 2 *Given an A-B bimodule N and a B-C bimodule N', their bimodule tensor product*

$$N \otimes_B N'$$

over B is the A-C bimodule which is

- *as an object of $\mathcal{C}$ the coequalizer of*

$$\begin{tikzcd} N \otimes B \otimes N' \arrow[r, shift left, "r\otimes N'"] \arrow[r, shift right, "N\otimes\ell"'] & N \otimes N' \end{tikzcd} ,$$

  i.e. such that for any other coequalizing morphism $N \otimes N' \xrightarrow{\lambda} Q$ we have a unique morphism

$$\begin{tikzcd} N \otimes B \otimes N' \arrow[r, shift left, "r\otimes N'"] \arrow[r, shift right, "N\otimes\ell"'] & N \otimes N' \arrow[r, "\otimes_B"] \arrow[d, "\lambda"] & N \otimes_B N' \arrow[dl, dashed] \\ & Q & \end{tikzcd} ,$$

- *equipped with the left A and right C action given by the universal arrows at the bottom of this diagram*

$$\begin{tikzcd} A \otimes N_1 \otimes B \otimes N_2 \arrow[r, "\ell_A"] \arrow[d, shift right, "r_B"'] \arrow[d, shift left, "\ell_B"] & N_1 \otimes B \otimes N_2 \arrow[d, shift right, "r_B"'] \arrow[d, shift left, "\ell_B"] & N_1 \otimes B \otimes N_2 \otimes C \arrow[l, "r_C"'] \arrow[d, shift right, "r_B"'] \arrow[d, shift left, "\ell_B"] \\ A \otimes N_1 \otimes N_2 \arrow[r, "\ell_A"] \arrow[d, "\otimes_B"] & N_1 \otimes N_2 \arrow[d, "\otimes_B"] & N_1 \otimes N_2 \otimes C \arrow[l, "r_C"'] \arrow[d, "\otimes_B"] \\ A \otimes (N_1 \otimes_B N_2) \arrow[r, dashed, "\ell_A"'] & N_1 \otimes_B N_2 & (N_1 \otimes_B N_2) \otimes C \arrow[l, dashed, "r_C"] \end{tikzcd} .$$

The above diagram, as similar diagrams to follow, is to be read as a shorthand for *two* different diagrams: one where we pick the left and one where we pick the right morphisms from every pair of parallel arrows.

Here the squares on the top commute due to the bimodule property, i.e. due to the fact that left and right action on a bimodule commute.

This commutativity then implies that the horizontal morphisms in the middle, when postcomposed with $\otimes_B$, in fact coequalize the respective left and right action.

This in turn implies, by the definition of the bimodule tensor product, the unique existence of the horizontal arrows on the bottom.

Remark. Notice that the bimodule tensor product operation

$$N \otimes N' \xrightarrow{\otimes_B} N \otimes_B N'$$

is necessarily an *epimorphism*, as follows directly from the universal property.

Definition 3 *Given two A-B bimodules N and M in $\mathcal{C}$, a* **bimodule homomorphism**

$$N \xrightarrow{\phi} M$$

is a morphism in $\mathcal{C}$ which commutes with the left and right action on the bimodules:

$$\begin{tikzcd}
A \otimes N \arrow[r, "\mathrm{Id}_A \otimes \phi"] \arrow[d, "\ell_N"'] & A \otimes M \arrow[d, "\ell_M"] \\
N \arrow[r, "\phi"'] & M
\end{tikzcd}$$

$$\begin{tikzcd}
N \otimes B \arrow[r, "\phi \otimes \mathrm{Id}_B"] \arrow[d, "r_N"'] & M \otimes B \arrow[d, "r_M"] \\
N \arrow[r, "\phi"'] & M
\end{tikzcd}$$

Definition 4 *The* **bicategory of bimodules internal to** $\mathcal{C}$*, denoted*

$$\mathrm{Bim}(\mathcal{C}),$$

is defined as follows:

1. *objects are all algebras A internal to $\mathcal{C}$*
2. *1-morphisms $A \xrightarrow{N} B$ are all internal A-B bimodules N*
3. *2-morphisms*

$$\begin{tikzcd}
A \arrow[r, bend left, "N"] \arrow[r, bend right, "M"'] \arrow[r, phantom, "\Downarrow\phi"] & B
\end{tikzcd}$$

are all bimodule homomorphisms $N \xrightarrow{\phi} M$.

Horizontal composition in $\mathrm{Bim}(\mathcal{C})$ *is the tensor product of bimodules. Vertical composition is the composition of bimodule homomorphisms.*

Remark.

1. $\mathrm{Bim}(\mathcal{C})$ is really a *weak* 2-category (a bicategory) with nontrivial but canonical associator. This will be described in the following.

2. The tensor unit $\mathbb{1} \in \mathcal{C}$ equipped with the trivial product is always an algebra internal to $\mathcal{C}$. The sub-2-category $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ of $\mathrm{Bim}(\mathcal{C})$ is $\mathcal{C}$ itself:

$$\mathrm{Hom}_{\mathbf{BiMod}(\mathcal{C})}(\mathbb{1}, \mathbb{1}) \simeq \mathcal{C}\,.$$

1.2 Structure Morphisms and Coherence

Proposition 1 *The bimodule tensor product is* **associative up to canonical isomorphism**.

Proof.

First we construct the associator, as an isomorphism in $\mathcal{C}$, then we show that it is in fact a homomorphism of bimodules.

We will construct the isomorphism by filling in the bottom of the following diagram

$$\begin{tikzcd}
 & & \mathrm{N}_1 \otimes B \otimes N_2 \otimes C \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, shift right, "\ell_C"'] \arrow[dr, shift left, "r_C"] & & \\
 & N_1 \otimes N_2 \otimes C \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, shift right, "\ell_C"'] \arrow[dr, shift left, "r_C"] & & N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, "\otimes_C"] & \\
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift right, "\ell_C"'] \arrow[dr, shift left, "r_C"] & & N_1 \otimes N_2 \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, "\otimes_C"] & & N_1 \otimes B \otimes (N_2 \otimes_C N_3) \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \\
 & (N_1 \otimes_B N_2) \otimes N_3 \arrow[d, "\otimes_C"'] & & N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] & \\
 & (N_1 \otimes_B N_2) \otimes_C N_3 & \simeq & N_1 \otimes_B (N_2 \otimes_C N_3) &
\end{tikzcd} \quad .$$

Again, this is shorthand for *two* different diagrams. Either choose the left morphism or the right morphism from every pair of parallel arrows.

First notice that the squares at the top do commute, due to the compatibility of left and right bimodule actions as well as the functoriality of the monoidal product.

The squares in the middle also do commute, either due to the functoriality of the monoidal product or due to defintion 2 of the left and right action on a bimodule tensor product.

Using the commutativity of the middle squares, we find that the morphism

$$\begin{tikzcd}
N_1 \otimes N_2 \otimes N_3 \arrow[dr, "\otimes_C"] & \\
& N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] \\
& N_1 \otimes_B (N_2 \otimes_C N_3)
\end{tikzcd}$$

coequalizes

$$\begin{tikzcd}
& N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \\
N_1 \otimes N_2 \otimes N_3 &
\end{tikzcd} .$$

By the universal property of the bimodule tensor product, this implies the existence of a unique morphism

$$\begin{tikzcd}
(N_1 \otimes_B N_2) \otimes N_3 \arrow[dr, dashed] & \\
& N_1 \otimes_B (N_2 \otimes_C N_3)
\end{tikzcd} .$$

The same argument goes through for the other side of the diagram. So that we obtain

$$
\begin{tikzcd}[column sep=small]
 & & & \mathrm{N}_1 \otimes B \otimes N_2 \otimes C \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & & \\
 & & N_1 \otimes N_2 \otimes C \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, "\otimes_C"] & & \\
 & (N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes N_2 \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, "\otimes_C"] & & N_1 \otimes B \otimes (N_2 \otimes_C N_3) \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] & \\
 & & (N_1 \otimes_B N_2) \otimes N_3 \arrow[d, "\otimes_C"'] \arrow[drr, dashed] & & N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] \arrow[dll, dashed] & & \\
 & & (N_1 \otimes_B N_2) \otimes_C N_3 & & N_1 \otimes_B (N_2 \otimes_C N_3) & &
\end{tikzcd}
.
$$

If we can now show that

$$
\begin{tikzcd}
(N_1 \otimes_A N_2) \otimes N_3 \arrow[dr, dashed] & \\
 & N_1 \otimes_B (N_2 \otimes_C N_3)
\end{tikzcd}
$$

coequalizes

$$
\begin{tikzcd}
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & \\
 & (N_1 \otimes_B N_2) \otimes N_3
\end{tikzcd}
,
$$

and analogously for the mirror symmetric situation, then the universal morphisms implied by this provide the desired equivalence

$$
\begin{tikzcd}[column sep=small]
 & & \mathrm{N}_1 \otimes B \otimes N_2 \otimes C \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & \\
 & N_1 \otimes N_2 \otimes C \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, "\otimes_C"] & \\
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes N_2 \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, "\otimes_C"] & & N_1 \otimes B \otimes (N_2 \otimes_C N_3) \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \\
 & (N_1 \otimes_B N_2) \otimes N_3 \arrow[d, "\otimes_C"'] \arrow[drr, dashed] & & N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] \arrow[dll, dashed] & \\
 & (N_1 \otimes_B N_2) \otimes_C N_3 \arrow[rr, dashed, leftrightarrow] & & N_1 \otimes_B (N_2 \otimes_C N_3) &
\end{tikzcd}
.
$$

(Everything is seen to commute as indicated, and in particular that the constructed associator is an isomorphism, by using again that $\otimes_B$ and $\otimes_C$ are epi.)

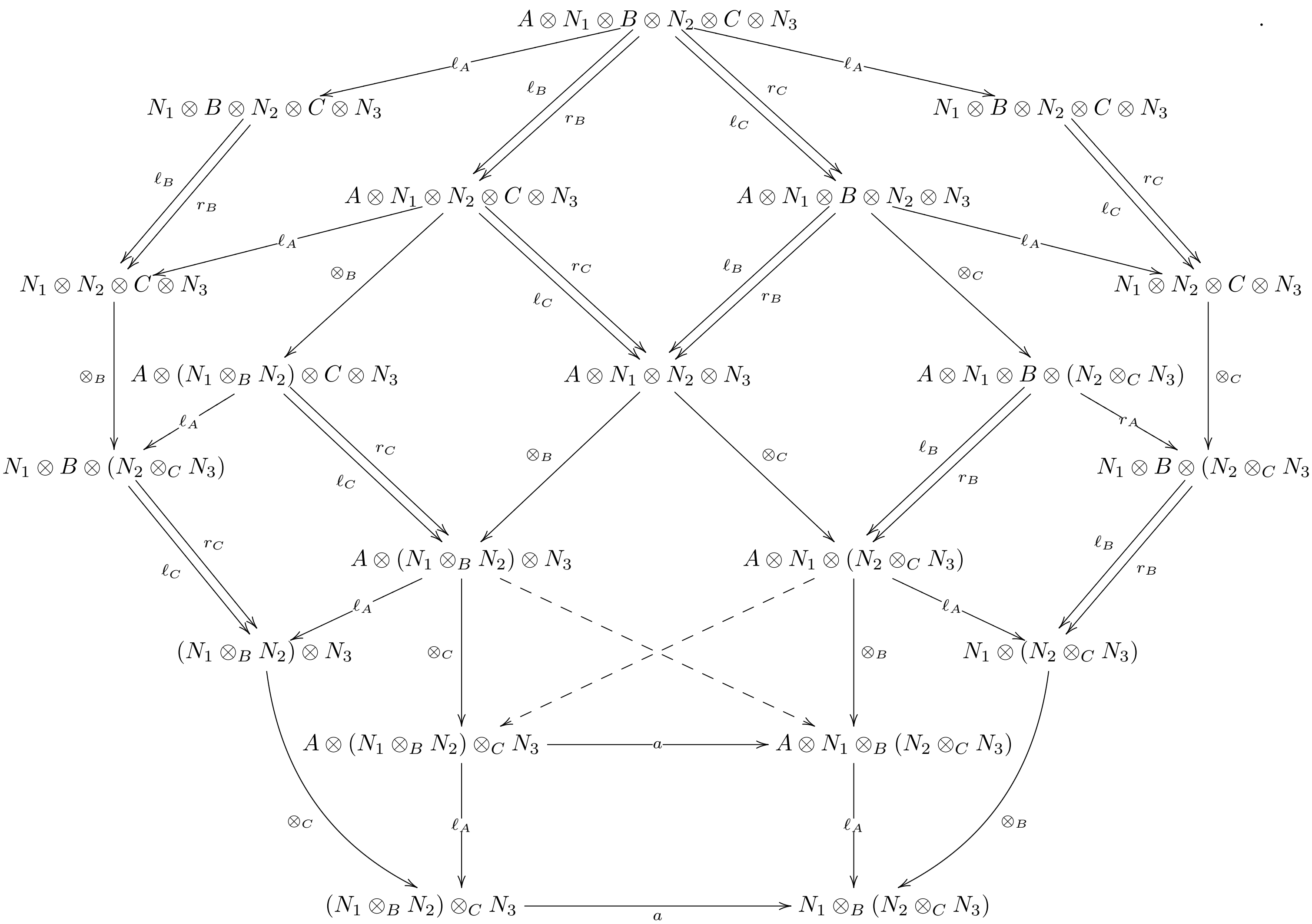
In order to see that, we inject the unit in B at the top of our diagram

$$
\begin{tikzcd}[column sep=small]
 & & N_1 \otimes \mathbb{1} \otimes N_2 \otimes C \otimes N_3 \arrow[d, "i_B"] & & \\
 & & N_1 \otimes B \otimes N_2 \otimes C \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & \\
 & N_1 \otimes N_2 \otimes C \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, "\otimes_C"] & \\
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes N_2 \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, "\otimes_C"] & & N_1 \otimes B \otimes (N_2 \otimes_C N_3) \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \\
 & (N_1 \otimes_B N_2) \otimes N_3 \arrow[d, "\otimes_C"'] \arrow[drr, dashed] & & N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] \arrow[dll, dashed] & \\
 & (N_1 \otimes_B N_2) \otimes_C N_3 & & N_1 \otimes_B (N_2 \otimes_C N_3) &
\end{tikzcd}
,
$$

which makes the entire composite of morphisms on the top left an epimorphism. Then we use the commutativity of the left and the top squares to find the desired coequality.

Again, the same argument holds for the mirror symmetric situation.

Finally, we need to show that the associator constructed this way does respect the bimodule structure. This can be seen by contemplating the diagram obtained by merging the above diagram with that defining the action on a bimodule tensor product from def. 2:



We need to show that the square on the bottom commutes, knowing that all other squares do commute, and knowing that the boundary of the entire diagram does commute.

So we start with the outermost morphisms and use the commutativity of all the squares to pull them inside.

Using the fact that the associator is an isomorphism, this yields the commuting square that we are after, but whiskered with a morphism from one side. But we can see that this morphism is epi and remove it. $\square$

2 Left-Induced Bimodules

Left-induced bimodules are a special kind of bimodules, which, as objects, are tensor products of an algebra with some other object, where the left action is given simply by the product in the algebra, while the right action comes from the product precompoced with a certain morphism.

Left induced bimodules are particularly easy to handle in that their bimodule tensor product amounts essentially just to the composition of these inducing morphisms.

2.1 Definitions

Definition 5 *A* **left-induced bimodule** *in* $\mathrm{Bim}(\mathcal{C})$ *is an A-B bimodule of the form*

$$N = A \otimes V \,,$$

for some object V, with the left A-action given by multiplication

$$A \otimes (A \otimes V) \xrightarrow{\ \ell\ } A \otimes V \quad := \quad A \otimes (A \otimes V) \xrightarrow{\ m \otimes V\ } A \otimes V$$

and with the right B action induced by a morphism

$$V \otimes B \xrightarrow{\ \phi\ } A \otimes V$$

followed by multiplication:

$$\begin{tikzcd}
(A \otimes V) \otimes B \arrow[rr, "r"] \arrow[dr, "A \otimes \phi"'] & & A \otimes V \\
& A \otimes A \otimes V \arrow[ur, "m \otimes V"'] &
\end{tikzcd} .$$

To explicitly indicate the action on the bimodule we sometimes write

$$(N, r, \ell) = \ A \overset{m}{\dashrightarrow} A \otimes V \overset{\phi \circ m}{\dashleftarrow} B \ .$$

The action property of the action of B on such a bimodule implies that the morphism ϕ is required to make the following diagrams commute:

1. compatibility with the product

$$\begin{tikzcd}
V \otimes B \otimes B \arrow[r, "\phi \otimes B"] \arrow[d, "V \otimes m"'] & A \otimes V \otimes B \arrow[r, "A \otimes \phi"] & A \otimes A \otimes V \arrow[d, "m \otimes V"] \\
V \otimes B \arrow[rr, "\phi"'] & & A \otimes V
\end{tikzcd}$$

2. <u>compatibility with the unit</u>

$$\begin{tikzcd}
 & V \arrow[dl, "V\otimes i_B"'] \arrow[dr, "i_A\otimes V"] & \\
V\otimes B \arrow[rr, "\phi"'] & & A\otimes V
\end{tikzcd}$$

Definition 6 *We denote the full sub-2-category of left-induced bimodules by*

$$\mathrm{LFBim}(\mathcal{C}) \subset \mathrm{Bim}(\mathcal{C})\,.$$

2.2 Properties

Proposition 2 *The bimodule tensor product* $A \xrightarrow{N} B \xrightarrow{N'} C$ *of two left-induced bimodules is the left-induced bimodule*

$$N \otimes_B N' = \; A \overset{m}{\dashrightarrow} A\otimes V\otimes V' \overset{\phi'\circ\phi\circ m}{\dashleftarrow} C \;\; .$$

Proof. The morphism

$$\begin{tikzcd}
A\otimes V \;\otimes\; B\otimes V' \arrow[rr, "f"] \arrow[dr, "A\otimes\phi\otimes V'"'] & & A\otimes V\otimes V' \\
 & A\otimes A\otimes V\otimes V' \arrow[ur, "m_A\otimes V\otimes V'"'] &
\end{tikzcd}$$

coequalizes

$$\begin{tikzcd}
(A\otimes V)\otimes B(B\otimes\otimes V') \arrow[r, shift left, "r\otimes(B\otimes V')"] \arrow[r, shift right, "(A\otimes V)\otimes\ell"'] & (A\otimes V)\otimes(B\otimes V')
\end{tikzcd} \; .$$

This follows from the compatibility of ϕ with the product. This f has a left inverse

$$A\otimes V\otimes V' \xrightarrow{\simeq} A\otimes V\otimes \mathbb{1}\otimes V' \xrightarrow{A\otimes V\otimes i\otimes V'} A\otimes V\otimes B\otimes V' \;,$$

as follows from the compatibility of ϕ with the unit. This implies that f is indeed universal. □

Proposition 3 *A* **morphism of left-induced bimodules**

$$\begin{tikzcd}
A \arrow[r, bend left=50, "{(A\otimes V_1,\phi_1)}"{name=U}] \arrow[r, bend right=50, "{(A\otimes V_2,\phi_2)}"'{name=D}] & B
\arrow[from=U, to=D, Rightarrow, "\rho"]
\end{tikzcd}$$

is uniquely specified by a morphism

$$
\begin{tikzcd}
V_1 \arrow[d, "\rho"] \\
A \otimes V_2
\end{tikzcd}
$$

as

$$
\begin{tikzcd}
A \otimes V_1 \arrow[d, "A\otimes\rho"] \\
A \otimes A \otimes V_2 \arrow[d, "m\otimes V_2"] \\
A \otimes V_2
\end{tikzcd}
.
$$

This ρ has to make the diagram

$$
\begin{tikzcd}
V_1 \otimes B \arrow[r, "\phi_1"] \arrow[d, "\rho\otimes B"'] & A \otimes V_1 \arrow[d, "A\otimes\rho"] & \\
A \otimes V_2 \otimes B \arrow[r, "A\otimes\phi_2"] & A \otimes A \otimes V_2 \arrow[dr, "m"] & \\
& & A \otimes V_2
\end{tikzcd}
$$

commute.

Proof. Using the compatibility with the left action, we find that ρ is completely determined already on the image of

$$
\mathbb{1} \otimes V_1 \xrightarrow{i\otimes V} A \otimes V \ ,
$$

and every map ρ on that image uniquely induces a morphism compatible with the left action. The above condition on ρ is then equivalent to the compatibility with the right action. $\square$

Proposition 4 *In terms of these morphisms, the 2-morphism arising as the horizontal product*

$$
\begin{tikzcd}
A \arrow[r, bend left, "{(A\otimes V_1,\phi_1)}"] \arrow[r, bend right, "{(A\otimes V_2,\phi_2)}"'] \arrow[r, phantom, "\Downarrow\rho"] & B \arrow[r, bend left, "{(B\otimes V_1',\phi_1')}"] \arrow[r, bend right, "{(B\otimes V_2',\phi_2')}"'] \arrow[r, phantom, "\Downarrow\rho'"] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "{(A\otimes V_1\otimes V_1',\phi_1'\circ\phi_1)}"] \arrow[r, bend right, "{(A\otimes V_2\otimes V_2',\phi_2'\circ\phi_2)}"'] \arrow[r, phantom, "\Downarrow\rho\cdot\rho'"] & C
\end{tikzcd}
,
$$

in LFBim($\mathcal{C}$) *is given by the morphism* $\rho \cdot \rho'$ *which is defined by*

$$\begin{tikzcd}
V_1 \otimes V_1' \arrow[rr, "\rho\otimes\rho'"] \arrow[d, "\rho\cdot\rho'"] & & A \otimes V_2 \otimes B \otimes V_2' \arrow[d, "A\otimes\phi\otimes V_2'"] \\
A \otimes V_2 \otimes V_2' & & A \otimes A \otimes V_2 \otimes V_2' \arrow[ll, "m\otimes V_2\otimes V_2'"]
\end{tikzcd} .$$

— *proof needs to be spelled out* —

3 Special Frobenius Bimodules

3.1 Definitions

Definition 7 *A* **Frobenius algebra** *in a monoidal category* $\mathcal{C}$ *is an object* $A \in \mathrm{Obj}(\mathcal{C})$ *togther with morphisms*

1. *product*

$$A \otimes A \xrightarrow{m} A$$

2. *unit*

$$\mathbb{1} \xrightarrow{i} A$$

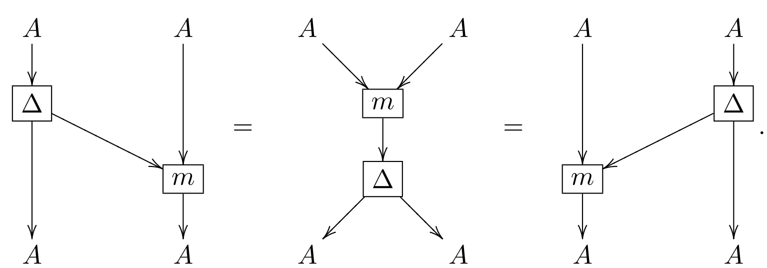
3. *coproduct*

$$A \xrightarrow{\Delta} A \otimes A$$

4. *counit*

$$A \xrightarrow{e} \mathbb{1}$$

such that (m, i) *is an algebra,* (Δ, e) *is a coalgebra and such that product and coproduct satisfy the* **Frobenius property**



Remark. For manipulations of diagrams it is often helpful to think of the Frobenius property as saying that, with A regarded as a bimodule over itself and with $A \otimes A$ regarded as an A-bimodule in the obvious way, the coproduct in A is a bimodule homomorphism form A to $A \otimes A$.

We will be interested in Frobenius algebras with additional properties. The Frobenius algebras of relevance here are

- special (def. 8)
- symmetric (def. **??**) .

Unfortunately, while standard, the terms "special" and "symmetric" are rather unsuggestive of the phenomena they are suppposed to describe.

1. Speciality says that the two "bubble diagrams" in a Frobenius algebra are proportional to identity morphisms.
2. Symmetry of a Frobenius algebra says that the two obvious isomorphisms of A with its dual object $A^\vee$ are equal.

The reader should in particular be warned that symmetry, in this sense, of a Frobenius algebra is not directly related to whether or not that algebra is (braided) *commutative*.

Specialty of a Frobenius algebra is a concept that makes sense when the ambient category is abelian. So we will assume in the following that $\mathcal{C}$ is abelian.

Definition 8 *Let A be a Frobenius algebra object in an abelian tensor category. A is* **special** *precisely if*

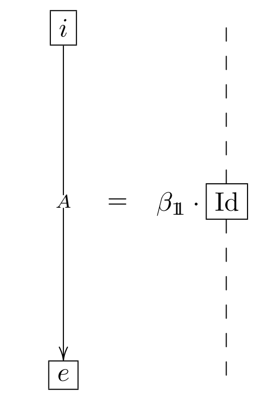
$$\begin{tikzcd} \mathbb{1} \arrow[dd, "\beta_{\mathbb{1}} \cdot \mathrm{Id}"'] \arrow[dr, "i"] & \\ & A \arrow[dl, "e"] \\ \mathbb{1} & \end{tikzcd}$$

and

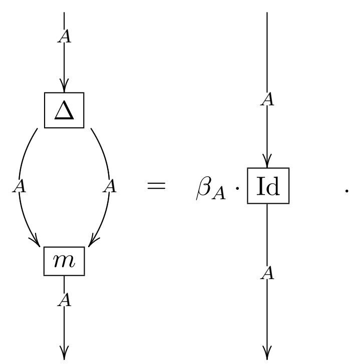
$$\begin{tikzcd} A \arrow[dd, "\beta_A \cdot \mathrm{Id}"'] \arrow[dr, "\Delta"] & \\ & A \otimes A \arrow[dl, "m"] \\ A & \end{tikzcd}$$

for some constants $\beta_{\mathbb{1}}$ and β_A.

In terms of string diagrams in the suspension of $\mathcal{C}$ these two conditions look like



and



3.2 Properties