Global curvature 2-transport

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Abstract

Every principal 1-transport, tra, gives rise to its curvature 2-transport curv. I discuss how this may be regarded as a principal 2-transport which is trivialized by tra in that tra is canonically identified with the component functor of the trivializing transformation

 $\operatorname{tra}: I_G \xrightarrow{\sim} \operatorname{curv}.$

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1 Introduction

A principal G-transport functor is a smoothly locally tivializable functor

$$\operatorname{tra}: \mathcal{P}_1(X) \to G\operatorname{Tor}$$
.

Let Bitor be the 2-category whose objects are groups, whose morphisms are group bitorsors and whose 2-morphisms are homomorphisms of these. Smoothly locally trivializable 2-functors with values in this

$$\operatorname{tra}_2: \mathcal{P}_2(X) \to \operatorname{Bitor}$$

play the role of principal 2-transport. Now, let

$$I_G: \mathcal{P}_2(X) \to \text{Bitor}$$

be the 2-functor which sends everything to the identity on the group G. Then there is a 2-functor

$$\operatorname{curv}: \mathcal{P}_2(X) \to \operatorname{Bitor}$$

which is trivialized by tra in that tra is the component of a pseudonatural transformation

 $I_G \xrightarrow{\sim} \operatorname{curv}$.

2 Principal 1-Transport and its curvature 2-transport

2.1 A word on torsors and bitorsors

Let BiTor be the 2-category whose objects are groups, whose morphisms $G \xrightarrow{T} G'$ are spaces with a left G and a right G'-action and whose 2-morphisms are maps between these, commuting with both the left and the right action.

If we restrict to bitorsors over a single group, $\operatorname{BiTor}(G)$, this yields a 1-object 2-category hence a monoidal 1-category. This is equivalent to the strict 2-group $\operatorname{AUT}(G)$,

$$\operatorname{BiTor}(G) \simeq \operatorname{AUT}(G)$$

Under this equivalence, every G-bitorsor is identitied with one of the form

 G_{μ} .

This denotes the bitorsor which is G itself as an object, with the obvious left action of G on itself and with the right action of G on itself twisted by a group automorphism $\mu \in \text{Aut}(G)$.

Let P_x be any ordinary right *G*-torsor, i.e. a right *G*-space which is isomorphic to *G* as a *G*-space. Write $\operatorname{Aut}(P_x)$ for the group of automorphisms of this space, which commute with this right *G*-action. This group is in fact isomorphic to *G*

$$\operatorname{Aut}(P_x) \simeq G$$

but not canonically so. There is precisely one such isomorphism for every choice of element in P_x .

$$\operatorname{Hom}_{\operatorname{Tor}(G)}(G, P_x) = P_x$$

Here $t \in P_x$ is identified with the map $G \to P_x$ which sends

$$t: 1 \mapsto t$$
.

Conceiving P_x as $\operatorname{Hom}_{\operatorname{Tor}(G)}(G, P_x)$ makes the bi-action on P_x again manifest: torsor automorphisms of P_x act on $\operatorname{Hom}_{\operatorname{Tor}}(G, P_x)$ from one side, while torsor automorphisms of G (to be distinguished from group automorphisms of G!) act from the other side.

Notice how it looks like the group G acts on this from the left, while it is really an action from the right: the composite

$$G \xrightarrow{g_1} G \xrightarrow{g_2} G \xrightarrow{t} P_x$$

sends

$$1 \mapsto (g_1 = 1 \cdot g_1) \mapsto 1 \cdot g_2 g_1 \mapsto t \cdot g_2 g_1.$$

We use this to define, for every right G-torsor P_x a left G-torsor

$$P_x^* := \operatorname{Hom}(P_x, G)$$

This now is automatically a left G-, right $\operatorname{Aut}(P_x)$ -torsor. We should call it the dual to P_x .

By definition we have

$$P_x \times_G P_x^* = \operatorname{Aut}(P_x).$$

Moreover, since the torsor automorphisms of a group itself are canonically isomorphic to that group

$$\operatorname{Hom}_{\operatorname{Tor}(G)}(G,G) = G$$

we get

$$P_x^* \times_{\operatorname{Aut}(P_x)} P_x = \operatorname{Hom}_{\operatorname{Tor}(G)}(G, G)$$

Therefore P_x and P_x^* form an equivalence

$$G \simeq \operatorname{Aut}(P_x)$$

in $\operatorname{Bitor}(G)$.

2.2 1-Transport trivializing its curvature 2-transport

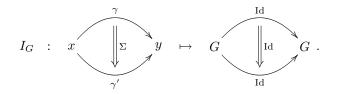
It is of interest to understand which 2-functors

$$\operatorname{curv}: \mathcal{P}_2(X) \to \operatorname{Bitor}$$

are equivalent to the trivial 2-functor (here "trivial" is a sloppy shorthand for "factors through a point")

$$I_G: \mathcal{P}_2(X) \to \text{Bitor}$$

which are such that



Instead of trying to characterize the situation in full generality, I will for the moment just highlight the special case of interest as far as curvature of ordinary ("untwisted") 1-transport goes.

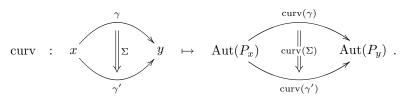
So let's look at the case where there is a principal G-bundle P over X such that curv takes values only in the associated bundle of groups

$$\operatorname{Ad}P := P \times_G G$$

whose fibers are the fiber automorphism groups of ${\cal P}$

$$(\operatorname{Ad} P)_x := \operatorname{Aut}(P_x)$$

This means that



Moreover, assume for the time being that the bitorsors $\operatorname{curv}(\gamma)$ for each path γ are all of the form G_{μ} , i.e. as objects they are just the group $\operatorname{Aut}(P_x)$

$$\operatorname{curv}(\gamma) := \operatorname{Aut}(P_x)_{\mu(\gamma)}$$

with the right $\operatorname{Aut}(P_y)$ -action coming from an group isomorphism

$$\mu(\gamma) : \operatorname{Aut}(P_y) \to \operatorname{Aut}(P_x)$$

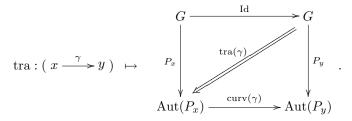
If curv is of this form, then it canonically trivializes. The pseudonatural transformation

$$\operatorname{tra}: I_G \xrightarrow{\sim} \operatorname{curv}$$

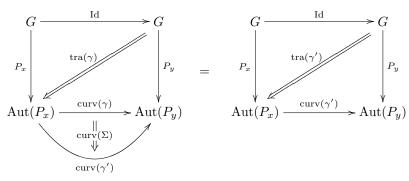
is given by the 1-functorial component map

tra :
$$\mathcal{P}_1(X) \to$$
Squares(Bitor)

which acts as



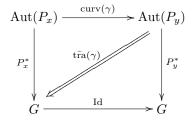
The naturality condition to be satisfied by this is the diagrammatic incarnation of Stokes' theorem



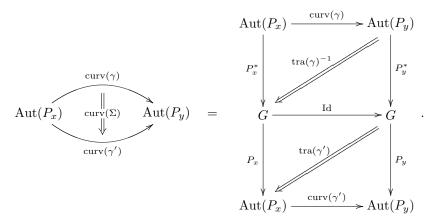
saying that the value of curv over a surface equals the value of tra over the boundary: if we write

tra : curv
$$\rightarrow I_G$$

for the inverse transformation with components



then the above naturality condition gets the explicit Stokes-like form:



Now, due to our assumption that all bitors ors $\operatorname{curv}(\gamma)$ are induced, one easily sees that the bitors or morphisms encoded by tra

$$\operatorname{tra}(\gamma): P_y \to P_x \times_{\operatorname{Aut}(P_x)} \operatorname{curv}(\gamma) \simeq P_x \times_{\operatorname{Aut}(P_x)} \operatorname{Aut}(P_x)_{\mu(\gamma)}$$

simply act as a principal G-transport between the fibers of P, and that

$$\mu(\gamma) = \operatorname{Ad}_{\operatorname{tra}(\gamma)}$$
.

This means that curv indeed canonically factors through $\mathrm{INN}_{P\times_G P}(\mathrm{Ad} P)$

$$\operatorname{curv}: \mathcal{P}_2(X) \to \operatorname{INN}_{P \times_G P}(\operatorname{Ad} P) \hookrightarrow \operatorname{Biton}$$

as the description of curvature as an obstruction for homotopy invariance, described elsewhere, requires.

