Global curvature 2-transport

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Abstract

Every principal 1-transport, tra, gives rise to its curvature 2-transport curv. I discuss how this may be regarded as a principal 2-transport which is trivialized by tra in that tra is canonically identified with the component functor of the trivializing transformation

$$\mathrm{tra} : I_G \stackrel{\sim}{\to} \mathrm{curv}\,.$$

Contents

1 Introduction

A principal G-transport functor is a smoothly locally tivializable functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}\,.$$

Let Bitor be the 2-category whose objects are groups, whose morphisms are group bitorsors and whose 2-morphisms are homomorphisms of these.

Smoothly locally trivializable 2-functors with values in this

$$\mathrm{tra}_2 : \mathcal{P}_2(X) \to \mathrm{Bitor}$$

play the role of principal 2-transport.

Now, let

$$I_G : \mathcal{P}_2(X) \to \mathrm{Bitor}$$

be the 2-functor which sends everything to the identity on the group G.

Then there is a 2-functor

$$\mathrm{curv} : \mathcal{P}_2(X) \to \mathrm{Bitor}$$

which is trivialized by tra in that tra is the component of a pseudonatural transformation

$$I_G \xrightarrow{\sim} \mathrm{curv}\,.$$

2 Principal 1-Transport and its curvature 2-transport

2.1 A word on torsors and bitorsors

Let BiTor be the 2-category whose objects are groups, whose morphisms $G \xrightarrow{T} G'$ are spaces with a left G and a right G'-action and whose 2-morphisms are maps between these, commuting with both the left and the right action.

If we restrict to bitorsors over a single group, $\mathrm{BiTor}(G)$, this yields a 1-object 2-category hence a monoidal 1-category. This is equivalent to the strict 2-group $\mathrm{AUT}(G)$,

$$\mathrm{BiTor}(G) \simeq \mathrm{AUT}(G)\,.$$

Under this equivalence, every G-bitorsor is identitfied with one of the form

$$G_\mu\,.$$

This denotes the bitorsor which is G itself as an object, with the obvious left action of G on itself and with the right action of G on itself twisted by a group automorphism $\mu \in \mathrm{Aut}(G)$.

Let P_x be any ordinary right G-torsor, i.e. a right G-space which is isomorphic to G as a G-space. Write $\mathrm{Aut}(P_x)$ for the group of automorphisms of this space, which commute with this right G-action. This group is in fact isomorphic to G

$$\mathrm{Aut}(P_x) \simeq G\,,$$

but not canonically so. There is precisely one such isomorphism for every choice of element in P_x.

$$\mathrm{Hom}_{\mathrm{Tor}(G)}(G, P_x) = P_x\,.$$

Here $t \in P_x$ is identified with the map $G \to P_x$ which sends

$$t : 1 \mapsto t\,.$$

Conceiving P_x as $\mathrm{Hom}_{\mathrm{Tor}(G)}(G, P_x)$ makes the bi-action on P_x again manifest: torsor automorphisms of P_x act on $\mathrm{Hom}_{\mathrm{Tor}}(G, P_x)$ from one side, while torsor automorphisms of G (to be distinguished from group automorphisms of G!) act from the other side.

Notice how it looks like the group G acts on this from the left, while it is really an action from the right: the composite

$$G \xrightarrow{g_1} G \xrightarrow{g_2} G \xrightarrow{t} P_x$$

sends

$$1 \mapsto (g_1 = 1 \cdot g_1) \mapsto 1 \cdot g_2 g_1 \mapsto t \cdot g_2 g_1\,.$$

We use this to define, for every right G-torsor P_x a left G-torsor

$$P_x^* := \mathrm{Hom}(P_x, G)\,.$$

This now is automatically a left G-, right $\mathrm{Aut}(P_x)$-torsor. We should call it the dual to P_x.

By definition we have

$$P_x \times_G P_x^* = \mathrm{Aut}(P_x)\,.$$

Moreover, since the torsor automorphisms of a group itself are canonically isomorphic to that group

$$\mathrm{Hom}_{\mathrm{Tor}(G)}(G, G) = G$$

we get

$$P_x^* \times_{\mathrm{Aut}(P_x)} P_x = \mathrm{Hom}_{\mathrm{Tor}(G)}(G, G)\,.$$

Therefore P_x and P_x^* form an equivalence

$$G \simeq \mathrm{Aut}(P_x)$$

in $\mathrm{Bitor}(G)$.

2.2 1-Transport trivializing its curvature 2-transport

It is of interest to understand which 2-functors

$$\mathrm{curv} : \mathcal{P}_2(X) \to \mathrm{Bitor}$$

are equivalent to the trivial 2-functor (here "trivial" is a sloppy shorthand for "factors through a point")

$$I_G : \mathcal{P}_2(X) \to \mathrm{Bitor}$$

which are such that

$$I_G \;:\; \begin{tikzcd} x \arrow[r, bend left, "\gamma"{name=U}] \arrow[r, bend right, "\gamma'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\Sigma"] & y \end{tikzcd} \;\mapsto\; \begin{tikzcd} G \arrow[r, bend left, "\mathrm{Id}"{name=U}] \arrow[r, bend right, "\mathrm{Id}"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\mathrm{Id}"] & G \end{tikzcd}\,.$$

Instead of trying to characterize the situation in full generality, I will for the moment just highlight the special case of interest as far as curvature of ordinary ("untwisted") 1-transport goes.

So let's look at the case where there is a principal G-bundle P over X such that curv takes values only in the associated bundle of groups

$$\mathrm{Ad}P := P \times_G G$$

whose fibers are the fiber automorphism groups of P

$$(\mathrm{Ad}P)_x := \mathrm{Aut}(P_x)\,.$$

This means that

$$\begin{tikzcd} \mathrm{curv} : & x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow\Sigma"] & y & \mapsto & \mathrm{Aut}(P_x) \arrow[r, bend left, "\mathrm{curv}(\gamma)"] \arrow[r, bend right, "\mathrm{curv}(\gamma')"'] \arrow[r, phantom, "\Downarrow \mathrm{curv}(\Sigma)"] & \mathrm{Aut}(P_y) \end{tikzcd}\,.$$

Moreover, assume for the time being that the bitorsors $\mathrm{curv}(\gamma)$ for each path γ are all of the form G_μ, i.e. as objects they are just the group $\mathrm{Aut}(P_x)$

$$\mathrm{curv}(\gamma) := \mathrm{Aut}(P_x)_{\mu(\gamma)}$$

with the right $\mathrm{Aut}(P_y)$-action coming from an group isomorphism

$$\mu(\gamma) : \mathrm{Aut}(P_y) \to \mathrm{Aut}(P_x)\,.$$

If curv is of this form, then it canonically trivializes. The pseudonatural transformation

$$\mathrm{tra} : I_G \stackrel{\sim}{\to} \mathrm{curv}$$

is given by the 1-functorial component map

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Squares}(\mathrm{Bitor})$$

which acts as

$$\mathrm{tra} : (\, x \xrightarrow{\gamma} y \,) \quad \mapsto \quad \begin{tikzcd} G \arrow[r, "\mathrm{Id}"] \arrow[d, "P_x"'] & G \arrow[d, "P_y"] \arrow[dl, Rightarrow, "\mathrm{tra}(\gamma)"'] \\ \mathrm{Aut}(P_x) \arrow[r, "\mathrm{curv}(\gamma)"] & \mathrm{Aut}(P_y) \end{tikzcd}\,.$$

The naturality condition to be satisfied by this is the diagrammatic incarnation of Stokes' theorem

$$\begin{tikzcd} G \arrow[r, "\mathrm{Id}"] \arrow[d, "P_x"'] & G \arrow[d, "P_y"] \arrow[dl, Rightarrow, "\mathrm{tra}(\gamma)"'] \\ \mathrm{Aut}(P_x) \arrow[r, "\mathrm{curv}(\gamma)"] \arrow[r, bend right=60, "\mathrm{curv}(\gamma')"'] \arrow[r, phantom, bend right=30, "\Downarrow \mathrm{curv}(\Sigma)"] & \mathrm{Aut}(P_y) \end{tikzcd} \quad = \quad \begin{tikzcd} G \arrow[r, "\mathrm{Id}"] \arrow[d, "P_x"'] & G \arrow[d, "P_y"] \arrow[dl, Rightarrow, "\mathrm{tra}(\gamma')"'] \\ \mathrm{Aut}(P_x) \arrow[r, "\mathrm{curv}(\gamma')"] & \mathrm{Aut}(P_y) \end{tikzcd}$$

saying that the value of curv over a surface equals the value of tra over the boundary: if we write

$$\bar{\mathrm{tra}} : \mathrm{curv} \to I_G$$

for the inverse transformation with components

$$\begin{tikzcd}
\mathrm{Aut}(P_x) \arrow[r, "\mathrm{curv}(\gamma)"] \arrow[d, "P_x^*"'] & \mathrm{Aut}(P_y) \arrow[d, "P_y^*"] \arrow[dl, Rightarrow, "\bar{\mathrm{tra}}(\gamma)"'] \\
G \arrow[r, "\mathrm{Id}"] & G
\end{tikzcd}$$

then the above naturality condition gets the explicit Stokes-like form:

$$\begin{tikzcd}
\mathrm{Aut}(P_x) \arrow[r, bend left, "\mathrm{curv}(\gamma)", ""{name=U, below}] \arrow[r, bend right, "\mathrm{curv}(\gamma')"', ""{name=D, above}] & \mathrm{Aut}(P_y) \arrow[from=U, to=D, Rightarrow, "\mathrm{curv}(\Sigma)"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{Aut}(P_x) \arrow[r, "\mathrm{curv}(\gamma)"] \arrow[d, "P_x^*"'] & \mathrm{Aut}(P_y) \arrow[d, "P_y^*"] \arrow[dl, Rightarrow, "\mathrm{tra}(\gamma)^{-1}"'] \\
G \arrow[r, "\mathrm{Id}"] \arrow[d, "P_x"'] & G \arrow[d, "P_y"] \arrow[dl, Rightarrow, "\mathrm{tra}(\gamma')"'] \\
\mathrm{Aut}(P_x) \arrow[r, "\mathrm{curv}(\gamma')"] & \mathrm{Aut}(P_y)
\end{tikzcd} .$$

Now, due to our assumption that all bitorsors $\mathrm{curv}(\gamma)$ are induced, one easily sees that the bitorsor morphisms encoded by tra

$$\mathrm{tra}(\gamma) : P_y \to P_x \times_{\mathrm{Aut}(P_x)} \mathrm{curv}(\gamma) \simeq P_x \times_{\mathrm{Aut}(P_x)} \mathrm{Aut}(P_x)_{\mu(\gamma)}$$

simply act as a principal G-transport between the fibers of P, and that

$$\mu(\gamma) = \mathrm{Ad}_{\mathrm{tra}(\gamma)} .$$

This means that curv indeed canonically factors through $\mathrm{INN}_{P\times_G P}(\mathrm{Ad}P)$

$$\mathrm{curv} : \mathcal{P}_2(X) \to \mathrm{INN}_{P\times_G P}(\mathrm{Ad}P) \hookrightarrow \mathrm{Bitor}$$

as the description of curvature as an obstruction for homotopy invariance, described elsewhere, requires.

$$\mathrm{curv} \quad : \quad
\begin{tikzcd}
x \arrow[r, bend left, "\gamma", ""{name=U, below}] \arrow[r, bend right, "\gamma'"', ""{name=D, above}] & y \arrow[from=U, to=D, Rightarrow, "\Sigma"]
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\mathrm{Aut}(P_x) \arrow[r, bend left, "\mathrm{Aut}(P_x)_{\mathrm{Ad}(\mathrm{tra}(\gamma))}", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Aut}(P_x)_{\mathrm{Ad}(\mathrm{tra}(\gamma'))}"', ""{name=D, above}] & \mathrm{Aut}(P_y) \arrow[from=U, to=D, Rightarrow, "\mathrm{curv}(\Sigma)"]
\end{tikzcd}$$