bimodule stuff

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Abstract

This is supposed to be a proof of the canonical weak associativity of the tensor product of bimodules.

Since I need it later, I first recall the definition of the left and right action on a bimodule tensor product.

Consider in the following three bimodules of the form

$$A \xrightarrow{N_1} B \xrightarrow{N_2} C \xrightarrow{N_3} D\,.$$

Definition 1 *The left A-action ℓ and right C-action r on the bimodule tensor product $N_1 \otimes_B N_2$ is defined by the universal arrows filling the following diagram*

$$\begin{tikzcd}
A \otimes N_1 \otimes B \otimes N_2 \arrow[r, "\ell_A"] \arrow[d, shift right, "r_B"'] \arrow[d, shift left, "\ell_B"] & N_1 \otimes B \otimes N_2 \arrow[d, shift right, "r_B"'] \arrow[d, shift left, "\ell_B"] & N_1 \otimes B \otimes N_2 \otimes C \arrow[l, "r_C"'] \arrow[d, shift right, "r_B"'] \arrow[d, shift left, "\ell_B"] \\
A \otimes N_1 \otimes N_2 \arrow[r, "\ell_A"] \arrow[d, "\otimes_B"] & N_1 \otimes N_2 \arrow[d, "\otimes_B"] & N_1 \otimes N_2 \otimes C \arrow[l, "r_C"'] \arrow[d, "\otimes_B"] \\
A \otimes (N_1 \otimes_B N_2) \arrow[r, dashed, "\ell_A"'] & N_1 \otimes_B N_2 & (N_1 \otimes_B N_2) \otimes C \arrow[l, dashed, "\ell_C"]
\end{tikzcd}\ .$$

Here the squares on the top commute due to the bimodule property, i.e. due to the fact that left and right action on a bimodule commute.

This commutativity then implies that the horizontal morphisms in the middle are in fact coequalizers of the respective left and right action.

This in turn implies, by the definition of the bimodule tensor product, the unique existence of the horizontal arrows on the bottom.

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Proposition 1 (weak associativity of bimodule composition) *The bimdule tensor product is* **associative up to canonical isomorphism** *if*

- *the coequalizers defining the bimodule tensor product are preserved by tensoring them with objects (see the proof below for what this means)*
- *and if their coimages exist.*

Proof. We will construct the isomorphism by filling in the bottom of the following diagram

$$
\begin{tikzcd}
 & & N_1 \otimes B \otimes N_2 \otimes C \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & & . \\
 & N_1 \otimes N_2 \otimes C \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, "\otimes_C"] & & \\
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes N_2 \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, "\otimes_C"] & & N_1 \otimes B \otimes (N_2 \otimes_C N_3) \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] & \\
 & (N_1 \otimes_B N_2) \otimes N_3 \arrow[d, "\otimes_C"'] & & N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] & & \\
 & (N_1 \otimes_B N_2) \otimes_C N_3 & \simeq & N_1 \otimes_B (N_2 \otimes_C N_3) & &
\end{tikzcd}
$$

First notice that the squares at the top do commute, due to the compatibility of left and right bimodule actions as well as the functoriality of the monoidal product.

The squares in the middle also do commute, either due to the functoriality of the monoidal product or due to defintion 1 of the left and right action on a bimodule tensor product.

Using the commutativity of the middle squares, we find that the morphism

$$\begin{tikzcd}
N_1 \otimes N_2 \otimes N_3 \arrow[dr, "\otimes_C"] & \\
& N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] \\
& N_1 \otimes_B (N_2 \otimes_C N_3)
\end{tikzcd}$$

coequalizes

$$\begin{tikzcd}
& N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \\
N_1 \otimes N_2 \otimes N_3 &
\end{tikzcd} .$$

By the universal property of the bimodule tensor product, this implies the existence of a unique morphism

$$\begin{tikzcd}
(N_1 \otimes_B N_2) \otimes N_3 \arrow[dr, dashed] & \\
& N_1 \otimes_B (N_2 \otimes_C N_3)
\end{tikzcd} ,$$

if we assume that the diagram characterizing the universal property is not affected by tensoring it with N_3.

The same argument goes through for the other side of the diagram. So that we obtain

$$
\begin{tikzcd}
 & & N_1 \otimes B \otimes N_2 \otimes C \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & \\
 & N_1 \otimes N_2 \otimes C \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, "\otimes_C"] & \\
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes N_2 \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, "\otimes_C"] & & N_1 \otimes B \otimes (N_2 \otimes_C N_3) \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \\
 & (N_1 \otimes_B N_2) \otimes N_3 \arrow[d, "\otimes_C"'] \arrow[drr, dashed] & & N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] \arrow[dll, dashed] & \\
 & (N_1 \otimes_B N_2) \otimes_C N_3 & & N_1 \otimes_B (N_2 \otimes_C N_3) &
\end{tikzcd}
.
$$

If we can now show that

$$
\begin{tikzcd}
(N_1 \otimes_A N_2) \otimes N_3 \arrow[dr, dashed] & \\
 & N_1 \otimes_B (N_2 \otimes_C N_3)
\end{tikzcd}
$$

coequalizes

$$
\begin{tikzcd}
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & \\
 & (N_1 \otimes_B N_2) \otimes N_3
\end{tikzcd}
,
$$

and analogously for the mirror symmetric situation, then the universal morphisms implied by this provide the desired equivalence

$$
\begin{tikzcd}
 & & \mathrm{N}_1 \otimes B \otimes N_2 \otimes C \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & \\
 & N_1 \otimes N_2 \otimes C \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, "\otimes_C"] & \\
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes N_2 \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, "\otimes_C"] & & N_1 \otimes B \otimes (N_2 \otimes_C N_3) \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \\
 & (N_1 \otimes_B N_2) \otimes N_3 \arrow[d, "\otimes_C"'] \arrow[drr, dashed] & & N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] \arrow[dll, dashed] & \\
 & (N_1 \otimes_B N_2) \otimes_C N_3 \arrow[rr, leftrightarrow, dashed] & & N_1 \otimes_B (N_2 \otimes_C N_3) &
\end{tikzcd} .
$$

For that to work, I apparently need to assume that the coimage of the bimodule tensor product exists, i.e. a morphism

$$N_1 \otimes_B N_2 \longrightarrow N_1 \otimes N_2$$

such that

$$
\begin{tikzcd}
N_1 \otimes_B N_2 \arrow[r] \arrow[rr, bend right, "\mathrm{Id}"'] & N_1 \otimes N_2 \arrow[r, "\otimes_B"] & N_1 \otimes_B N_2
\end{tikzcd} .
$$

If that is the case, we can inject the coimage at the top of our diagram

$$
\begin{tikzcd}
 & & (N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[d] & & \\
 & & N_1 \otimes B \otimes N_2 \otimes C \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & \\
 & N_1 \otimes N_2 \otimes C \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes B \otimes N_2 \otimes N_3 \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \arrow[dr, "\otimes_C"] & \\
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & & N_1 \otimes N_2 \otimes N_3 \arrow[dl, "\otimes_B"'] \arrow[dr, "\otimes_C"] & & N_1 \otimes B \otimes (N_2 \otimes_C N_3) \arrow[dl, shift right, "\ell_B"'] \arrow[dl, shift left, "r_B"] \\
 & (N_1 \otimes_B N_2) \otimes N_3 \arrow[d, "\otimes_C"'] \arrow[drr, dashed] & & N_1 \otimes (N_2 \otimes_C N_3) \arrow[d, "\otimes_B"] \arrow[dll, dashed] & \\
 & (N_1 \otimes_B N_2) \otimes_C N_3 & & N_1 \otimes_B (N_2 \otimes_C N_3) &
\end{tikzcd}
$$

and then use the commutativity of the left and the top squares to find that

$$
\begin{tikzcd}
(N_1 \otimes_A N_2) \otimes N_3 \arrow[dr, dashed] & \\
 & N_1 \otimes_B (N_2 \otimes_C N_3)
\end{tikzcd}
$$

indeed coequalizes

$$\begin{tikzcd}
(N_1 \otimes_B N_2) \otimes C \otimes N_3 \arrow[dr, shift left, "r_C"] \arrow[dr, shift right, "\ell_C"'] & \\
& (N_1 \otimes_B N_2) \otimes N_3
\end{tikzcd},$$

Again, the same argument holds for the mirror symmetric situation. □