Fiber integration and Cheeger-Simons characters

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Abstract

Notes taken in a talk by **Christian Baer** at the Göttingen Topology Seminar Nov 25. 2008. The notes should fairly literally reproduce what was on the board and what was said. But all mistakes are mine.

The following is joint work with **Christian Becker**.

Contents

1 Differential characters **1**

2 Geometric chains **4**

3 Fiber integration **5**

1 Differential characters

fix a connected smooth manifold X, which may have infinite dimension
question: Hilbert or Frechet
answer: Frechet (but could be Hilbert or Banach)
usual notation for smooth singular chains
boundaries

$$B_k(X,\mathbb{Z}) \subset Z_k(X,\mathbb{Z}) \subset C_k(X,\mathbb{Z})$$

question: smooth in the simplest sense?
answer: does not matter in this context
smooth k forms: $\Omega^k(X)$
in there is subset of closed k-forms with integral periods

$$\Omega^k_0(X) \subset \Omega^k(X)$$

(closed when integrated over integral cycle)
Definition: differential characters (Cheeger-Simons, 1985)
for $k \geq 1$

$$\hat{H}^k(X,U(1)) = \{h \in \mathrm{Hom}(Z_{k-1}(X;\mathbb{Z}),U(1)),U(1)|\exists\omega \in \Omega^k(X) : h(\partial c) = \exp(2\pi i \int_c \omega)\}$$

$\hat{H}^0(X;U(1)) := \mathbb{Z}$
original definition except for index shift
elements in these abelian groups (under pointwise multiplication) are called differential characters

Examples.

- $k=1$ $\mathrm{Hom}(Z_0, U(1)) = \mathrm{Map}(X, U(1))$ (picture of path in X cobounding two 0-chains = points) so there must be a 1-form ω which integrated over this path gives the difference of the values at the two points.

  so this implies that the maps to $U(1)$ must be smooth (since the 1-form is)

  $\Rightarrow h \in C^\infty(X, U(1))$

  conversely, if we have such a smooth h, then we can put $\omega := \frac{1}{2\pi i} d\log h$

  so:
  $$\hat{H}^1(X, U(1)) = C^\infty(X, U(1))$$

- $k=2$ let $E \to X$ be a hermitian smooth line bundle, with metric connection ∇, how can we cook up the homomorphism from 1-cycles to $U(1)$

  we can look at the holonomy of the bundle around the loop z a 1-cycle then
  $$\mathrm{hol}(z) =: h(z)$$

  so $\omega =$ curvature 2-form of ∇

  con versely etc. pp.

  so
  $$\hat{H}^2(X, U(1)) \simeq \{\text{isom. classes of } U(1)\text{-bundles over } X \text{ with compatible connection}\}$$

  question: equality or iso?

  answer: well...

- $k=3$ Hitchin's gerbes with connctive structure

  question: which was first, Hitchin of Cheeger-Simons

  answer: Deligne cohomology even older, translated by Hitchin into differential geometry, but younger than Cheeger-Simons

- assume

  $k \geq \dim(X) + 2$ then
  $$\hat{H}^k(X, U(1)) = 0$$

  one limiting case:
  $$k = \underbrace{\dim(X)}_{=:n} + 1$$

  in this case our Cheeger-Simons homomorphism h descends to homology and we have
  $$\hat{H}^{n+1}(X, U(1)) = H^n(X, U(1))$$

First Remark. Recall that ω is unique, closed and whas integral periods: we will call it the <u>curvature</u>
$$\omega =: \mathrm{curv}(h)$$

this gives us an exact sequence
$$0 \to H^{k-1}(X, U(1)) \to \hat{H}^k(X, U(1)) \stackrel{\text{cup}}{\to} \Omega^k_0(X) \to 0$$

so $H^{k-1}(X, U(1))$ are the <u>flat</u> characters
reason: if curvature vanishes then homomorphism vanishes on boundaries so descends to homology

Second Remark. given $h \in \hat{H}^k(X, U(1))$ choose lift/extension

$$\tilde{h} \in C^{k-1}(X, \mathbb{R})$$

then look at the map

$$\mu : C_k(X, \mathbb{Z}) \to \mathbb{R}$$

$$\mu(x) = \int_c \omega - \delta\tilde{h}(c)$$

so this means that

$$\mu : C_k(X, \mathbb{Z}) \to \mathbb{Z}$$

this way we get a cohomology class

$$[\mu] = c(h) \in H^k(X, \mathbb{Z})$$

called the characteristic class

in degree 2 the Chern class, in degree 3 the Dixmier-Douady class.

we have another exact sequence:

$$0 \to \Omega^{k-1}(X)/\Omega_0^{k-1}(X) \to \hat{H}^k(X, U(1)) \xrightarrow{c} H^k(X, \mathbb{Z}) \to 0$$

Third remark: Curvature and characteristic class are compatible in that curvature represents char. class in real cohomology:

$$[\mathrm{curv}(h)]_{\mathrm{dR}} = r(c(h))$$

$$r : H^\bullet(X, \mathbb{Z}) \to H^\bullet(X, \mathbb{R})$$

Aim of talk: construction of fiber integration map

Situation:

we have a fiber bundle

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \pi \\ & & B \end{array}$$

with F compact and oriented

question: somebody did that already, right?

answer: there is a paper by Hopkins and Singer using a different construction, this however is so non-geometric that one cannot see naturality (they don't claim that) this is not useful for infinite-dimensional fibers

there is another construction by Ljungman; he constructed fiber integration in Deligne cohomology, which is the a complicated combinatorial issue, he uses simplicial forms as bookkeeping device

we want something different from this combinatorial apprroach, we want something geometric (which is the main motivation of this work presneted here)

idea:

given

$$h \in \hat{H}^{k+f}(E)$$

idea: $(\pi_* h)(z)$ "=" $h(\pi^{-1}(z))$

of course preimage of singular cycle is not singular cycle, so this does not work naively

we want to make this work now, how can this be done?

2 Geometric chains

it would be nice to represent singular homology by nicer spaces

Def. $y \in C_k(X, \mathbb{Z})$

is called <u>skinny</u>

$$\Leftrightarrow \int \omega = 0 \forall \omega \in \Omega^k)(X)$$

question: thin?

answer: okay, <u>thin</u>

Remark. Thin chains form a submodule of $C_k(X, \mathbb{Z})$ preserved under pullbacks.

crucial Definition.

$$c, c' \in C_k(X, \mathbb{Z})$$
$$c \simeq c' \Leftrightarrow \exists \text{ thin } \eta \in C_{k+1}(X, \mathbb{Z}) : c - c' = \partial y$$

example: $\dim(X) = n$, compact, without boundary and oriented,

c, c' two representatives of fundamental classsince ; there are no non-zero $n+1$-forms so $c \simeq c'$

second part of definition

$$c \sim c' \exists \text{ thin } y \in C_{k+1}(X, \mathbb{Z}) c - c' - \partial y \text{ is thin}$$

example X with boundary, c, c' repr. of fundamental class $\Rightarrow c \sim c'$

remark

1) if $h \in \hat{H}^k(X, U(1))$

h descends to homomorphism

$$Z_{k-1}(X, \mathbb{Z}) / \simeq \rightarrow U(1)$$

direct consequence of definition

implies in particular that evaluation of diff. character on fundamental class does not depend on chosen representative of that class, since any two are strongly thin equivalent

2) ∂ descends to

$$C_{k+1}(X, \mathbb{Z})/_{\sim} \rightarrow B_k(X, mathbbZ)/_{\simeq}$$

3) we have a natural map from

$$C_k(X, \mathbb{Z})/_{\simeq} \rightarrow C_k(X, \mathbb{Z})/_{\sim}$$

4) integration of forms descends to $\sim$-classes

$$c \sim c' \Rightarrow \int_c \omega - \int_{c'} \omega = \int_{\partial y} \omega = \int_y d\omega = 0$$

<u>**geometric chains**</u>

wishlist for geometric model for singular homology:

suppose we have a functor from "Manifolds" with smooth maps to "complexes of abelian ssemigroups with involution"

$$X \mapsto (\mathcal{C}_\bullet(X), \partial, ^-)$$

almost example showing what we are aiming for:

$$\mathcal{C}_k(X) = \{ \text{ diffeo classes of smooth maps } M \rightarrow C \text{ where } M \text{ is compact, } \dim(M) = k \text{, oriented } \}$$

question: can M have boundary
answer: yes and
∂ is geometric boundary
and the involution $^{-}$ is orientation reversal
now homomorphism

$$\phi_n : \mathcal{C}_n(X) \to C_n(X,\mathbb{Z})/_{\sim}$$

$$\psi_n : \mathcal{Z}_n(X) \to Z_n(X,\mathbb{Z})/_{\simeq}$$

such that

$$\begin{array}{ccccccc}
\mathcal{C}_{n+1} & \xrightarrow{\partial} & \mathcal{B}_n(X) & \longrightarrow & \mathcal{Z}_n(X) & \longrightarrow & \mathcal{C}_n \\
\downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \phi_n} \\
C_{n+1}(X,\mathbb{Z}) & \xrightarrow{\partial} & B_n(X)/_{\simeq} & \longrightarrow & Z_n(X,\mathbb{Z})/_{\simeq} & \longrightarrow & C_n(X,\mathbb{Z})
\end{array}$$

axiom A

$$^{-} : \mathcal{Z}_n \to \mathcal{Z}_n$$

induces identity on $\mathcal{H}_n$
also

$$\psi_n : \mathcal{Z}_n(X) \to Z_n(X,\mathbb{Z})/_{\simeq}$$

induces iso

$$\mathcal{H}_n(X) = \mathcal{Z}_n(X)/\mathcal{B}_n(X) \stackrel{\psi_n}{\to} (Z_n/\simeq)/(B_n/\simeq) = H_n(X,\mathbb{Z})$$

axiom B
for biber bundles as above $F \to E \to X$

$$\iota_E^* : \mathcal{C}_n(X) \to \mathcal{C}_{n+f}(E)$$

such that
first pulling back and then taking bouindary is same as doing it other way round:
1)

$$\partial \circ \iota_E^* = \iota_E^* \circ \partial$$

2) it should be compatible with fiber integration of forms and of singular homology classes
3) if F has a boundary then similar boundary condition as before
[** speaker hurries up , details are missing here **]

3 Fiber integration

define now as follows
a map from differential characters on total space to diff characters on basis

$$\pi_* : \hat{H}^{k+f}(E,U(1)) \to \hat{H}^k(X,U(1))$$

degree shift by dimensuion of fiber

$$(\pi_* h)(z) = h(\psi_k(\iota_E^* \zeta)) \cdot \exp(2\pi i \int_a \int_F \mathrm{curv}(h))$$

where:

$\zeta \in \mathcal{Z}_k(X)$ is a lift through ψ_n of z to the geometric chains

and

$$\psi_k(\zeta) = z + \partial a$$

where $\partial a \in C_{k+1}$ there are choices involved,

why is all this important?

because it appears in the notion of <u>transgression</u>

$\mathcal{L}(X) = C^\infty(S^1, X)$

loop space

$$\text{ev} : S^1 \times \mathcal{L}(X) \to X$$

$$\hat{H}^k(X) \xrightarrow{\text{ev}^*} \hat{H}^k(S^1 \times \mathcal{L}(X)) \xrightarrow{\pi_*} \hat{H}^{k-1}(\mathcal{L}(X))$$

in the exmaple of compact maps to X we almost have geometric chains:

just replace compact manifolds with boundary by Kreck's regular c-stratifolds

then maps into X do form a model for geometric chains and we do get a realization of the above formalism, so we get a notion of fiber integration which is natural etc.