AQFT from n-functorial FQFT and Applications

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Abstract

A relation between n-functorial extended quantum field theory (essentially: parallel n-transport) and algebraic quantum field theory. And some examples.
Talk at Hausdorff Center for Math, Bonn. Based on [14].

Contents

Acknowledgement. The discussion of the relation to the Hopf spin chain models of [10] and the emergence of subfactors from Ocneanu's asymptotic inclusion in sections 3.5 and 3.6 owes a lot to very helpful conversation with Pasquale Zito and his thesis [17]. All remaining mistakes and imperfections are of course mine.

1 Motivation

- The construction [3] solves the topological aspects of full 2-dimensional rational CFT.
- observation ([4]): this prescription is descent data for some transport 2-functor [16] (compare with my previous talk on differential nonabelian cohomology [13]).
- which 2-functor? locally it should reproduce the "chiral data" of the CFT: this is known to be encoded either in vertex operator algebra or in *local conformal nets of observable algebras* (the relation between vertex operator algebras and local conformal nets is described in [6])
- observation here: such local nets indeed can be obtained from transport 2-functors in a way that mimics the passage from the Schrödinger picture to the Heisenberg picture in quantum mechanics (= 1-dimensional QFT)

- so our aim is to complete the last line in the analogy

full 2dCFT	=	VOA/conformal net	+	sewing constraints
transport 2-functor	=	local 2-functorial transport	+	descent data

2 FQFT and AQFT

There exist two approaches to axiomatization of QFT:

- FQFT: (extended) cobordism representations (Atiyah-Segal, Baez-Dolan, Hopkins-Lurie and others)
- AQFT: local nets of algebras (of observables) (Haag-Kastler and their school)

Reconsider ordinary QM = 1dQFT

- Schrödinger picture: the propagator of states is a functor from paths in the worldline to Vect
- Heisenberg picture: this functor yields a local net of algebras after sending each space of states to its algebra of (bounded) endomorphism

Generalization to higher dimensional QFT:

- n-extended QFT (Freed: "n-tiered" QFT) assign data in all codimensions
- n-extended topological QFT: n-functor on abstract n-dimensional cobordisms
- here: n-extended pseudo-Riemannian (Lorentzian, really) QFT: n-dimensional cobordisms *embedded* in an ambient pseudo-Riemannian spacetime.

2.1 AQFT

So let $X = \mathbb{R}^2$ thought of as equipped with the standard Minkowski metric on $\mathbb{R}^2$.

By a causal subset of X we shall mean as usual the interior of the intersection of the future of one point with the past of another.

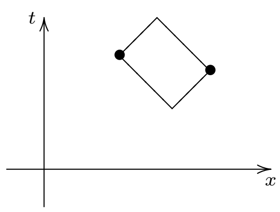
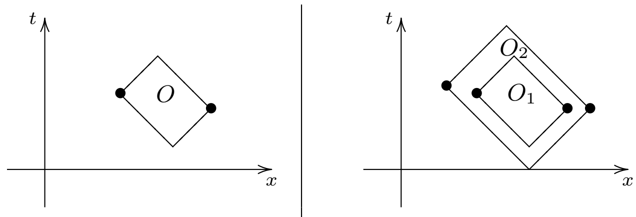


Figure 1: A "causal subset" of 2-dimensional Minkwoski space is the interior of a rectangle all whose sides are lightlike. Such subsets are entirely fixed in particular by their left and right corners.

Definition 1 *We denote by $S(X)$ the category whose objects are open causal subsets $V \subset X$ of X and whose morphisms are inclusions $V \subset V'$.*

In order to concentrate just on the properties crucial for our argument, we shall now talk about nets of local *monoids* (sets equipped with an associative and unital product).



an object O in $S(\mathbb{R}^2)$ | a morphism $O_1 \to O_2$ in $S(\mathbb{R}^2)$

Figure 2: The category $S(\mathbb{R}^2)$ of causal subsets of 2-dimensional Minkowski space. Objects are causal subsets, morphisms are inclusions of these.

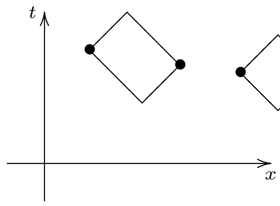


Figure 3: Two spacelike separated causal subsets of $\mathbb{R}^2$.

Definition 2 *Two objects O_1, O_2 in $S(X)$ are called spacelike separated if all pairs of points $(x_1, x_2) \in O_1 \times O_2$ are spacelike separated.*

Definition 3 *A functor*

$$\mathcal{A} : S(\mathbb{R}^2) \to \text{Monoids}\,,$$

is a **net** *of monoids on 2-dimensional Minkwoski if it sends all morphisms in $S(\mathbb{R}^2)$ to injections (monomorphisms) of monoids. This is a* **net of local monoids** *if for all spacelike separated $O_1, O_2 \subset O$ the corresponding algebras commute with each other in O, i.e.*

$$[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$$

as an identity in $\mathcal{A}(O)$. The net $\mathcal{A}$ is said to satisfy the **time slice axiom** *if for any region O, any Cauchy surface in O and any collection of causal subset $\{O'_i \subset O\}$ covering the Cauchy surface we have*

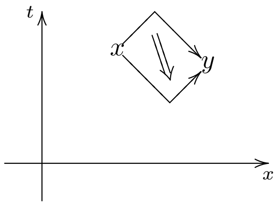
$$\cup_i \mathcal{A}(O_i) = \mathcal{A}(O)\,,$$

where the union is taken in $\mathcal{A}(O)$.

2.2 FQFT

Instead of regarding causal subsets as a category under inclusion of subsets, we can think of them as living in a 2-category under *composition* (gluing).

Definition 4 *Let $P_2(\mathbb{R}^2)$ be the 2-category whose objects are the points of $\mathbb{R}^2$, whose morphisms are piecewise* *lightlike right-moving paths in $\mathbb{R}^2$ and whose 2-morphisms are* generated *from the closure of causal bigons*



regarded as 2-morphisms as indicated, under gluing along pieces of joint boundary. Composition is by gluing along pieces of joint boundary, in the obvious way.

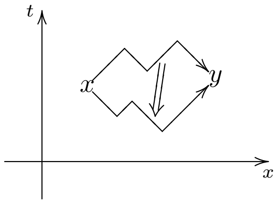


Figure 4: A typical 2-morphism in $P_2(\mathbb{R}^2)$

Remark. The restriction that 1-morphism have to go “right” and 2-morphisms “downwards” simplifies the discussion a bit but is otherwise of no real relevance. Various generalizations of $P_2(\mathbb{R}^2)$ can be considered without changing the substance of the following arguments.

Just as with local nets, there are many variations of definitions of extended quantum field theories on 2-dimensional Minkowski space which one could consider. We choose to take the following simple definition. (Compare with the notion of parallel surface transport [1, 15, 16]).

Definition 5 *For any 2-groupoid C, an extended FQFT on 2-dimensional Minkowski space is a 2-functor*

$$Z : P_2(\mathbb{R}^2) \to C .$$

We write $\mathbf{FQFT}(\mathbb{R}^2, C) := 2\mathrm{Funct}(P_2(\mathbb{R}^2), C)$ for the 2-functor 2-category and $\mathbf{FQFT}_{\mathrm{isos}}(\mathbb{R}^2, C)$ for the maximal strict 2-groupoid inside it.

In concrete application C will usually be a 2-category of 2-vector spaces (which in general is not strict), as for instance those whose objects are (von Neumann) algebras, whose morphisms are bimodules over these, and whose 2-morphisms are bimodule homomorphisms [**?**]. We will see such an example in section **??** based on some constructions summarized in appendix **??**.

2.3 The relation

We define a map from FQFTs in the sense of definition 5 to AQFTs in the sense of definition 3 and demonstrate, theorem 1, that it indeed sends 2-functors to local nets of monoids satisfying the time slice axiom. Then we observe, theorem **??**, that this construction extends to a 2-functor from FQFTs to AQFTs on $\mathbb{R}^2$.

Definition 6 *Given any extended 2-dimensional FQFT, i.e. a 2-functor*

$$Z : P_2(\mathbb{R}^2) \to C$$

we define a functor

$$\mathcal{A}_Z : S(\mathbb{R}^2) \to \mathrm{Monoids}\,.$$

On objects it it acts as

$$\mathcal{A}_Z : \left(\begin{tikzcd} & \cdot \arrow[dr, dash] & \\ x \arrow[ur, dash] \arrow[dr, dash, "\gamma"'] & & y \\ & \cdot \arrow[ur, dash] & \end{tikzcd}\right) \mapsto \mathrm{End}_C\left(Z\left(\begin{tikzcd} x \arrow[dr, dash, "\gamma"'] & & y \\ & \cdot \arrow[ur] & \end{tikzcd}\right)\right),$$

where on the right we form the monoid of 2-endomorphism a in C on the 1-morphism $Z(x \xrightarrow{\gamma} y)$ in C that is the past boundary of $O_{x,y}$,

$$\begin{tikzcd} Z(x) \arrow[rr, bend left, "Z(x \xrightarrow{\gamma} y)", ""{name=U, below}] \arrow[rr, bend right, "Z(x \xrightarrow{\gamma} y)"', ""{name=D, above}] & & Z(y) \arrow[from=U, to=D, Rightarrow, "a"] \end{tikzcd}\,.$$

On morphisms $\mathcal{A}_Z$ is defined to act as follows.

For any inclusion $O_{x',y'} \subset O_{x,y} \in S(\mathbb{R}^2)$

$$\begin{tikzcd} & & & 1 \arrow[drrr, dash] & & & \\ x \arrow[urrr, dash] \arrow[ddr, dash] & & & 2 \arrow[dr, dash] & & & y \\ & & x' \arrow[ur, dash] \arrow[dr, dash] & & y' & & \\ & 3 \arrow[drr, dash] & & 4 \arrow[ur, dash] & & 5 \arrow[uur, dash] & \\ & & & 6 \arrow[urr, dash] & & & \end{tikzcd}$$

(the numbers here and in the following are just labels for various points in order to help us navigate these diagrams) we form the pasting diagram

$$\begin{tikzcd} & & & 1 \arrow[drrr] & & & \\ x \arrow[urrr] \arrow[ddr] & & & 2 \arrow[dr] & & & y \\ & & x' \arrow[ur] \arrow[dr] & & y' \arrow[dr] & & \\ & 3 \arrow[ur] \arrow[drr] & & 4 \arrow[ur] \arrow[d, Rightarrow, "f"] & & 5 \arrow[uur] & \\ & & & 6 \arrow[urr] & & & \end{tikzcd}$$

in $P_2(\mathbb{R}^2)$. Here the obvious projections along light-like directions (for instance from x' onto $x \to 6$ yielding 3) is used. It is at this point that the light-cone structure crucially enters the construction.

Let f' be the 2-morphism obtained by whiskering (= horizontal composition with identity 2-morphisms) the indicated 2-morphism f with the 1-morphisms $x \to 3$ and $5 \to y$.

$$f' := \begin{tikzcd} x \arrow[ddr] & & & & & & y \\ & & x' \arrow[dr] & & y' \arrow[dr] & & \\ & 3 \arrow[ur] \arrow[drr] & & 4 \arrow[ur] \arrow[d, Rightarrow, "f"] & & 5 \arrow[uur] & \\ & & & 6 \arrow[urr] & & & \end{tikzcd} .$$

For any $a \in \mathrm{End}_C Z(x', 4, y')$,

$$\begin{tikzcd} Z(x') \arrow[rr, bend left, "Z(x'\to 4\to y')"] \arrow[rr, bend right, "Z(x'\to 4\to y')"'] \arrow[rr, phantom, "\Downarrow a"] & & Z(y') \end{tikzcd} ,$$

let a' be the corresponding re-whiskering by $Z(x,3,x')$ from the left and by $Z(y',5,y)$ from the right:

$$\begin{tikzcd} Z(x) \arrow[rr, bend left, "Z(x\to 3\to x'\to 4\to y'\to 5\to y)"] \arrow[rr, bend right, "Z(x\to 3\to x'\to 4\to y'\to 5\to y)"'] \arrow[rr, phantom, "\Downarrow a'"] & & Z(y) \end{tikzcd} := \begin{tikzcd} Z(x) \arrow[r, "Z(x\to 3\to x')"] & Z(x') \arrow[r, bend left, "Z(x'\to 4\to y')"] \arrow[r, bend right, "Z(x'\to 4\to y')"'] \arrow[r, phantom, "\Downarrow a"] & Z(y') \arrow[r, "Z(y'\to 5\to y)"] & Z(y) \end{tikzcd} ,$$

Then we obtain an injection

$$\mathrm{End}_C(Z(x',4,y')) \hookrightarrow \mathrm{End}_C(Z(x,3,6,5,y))$$

by setting

$$a \mapsto Z(f') \circ a' \circ Z(f')^{-1} ,$$

i.e.

$$\begin{tikzcd} Z(x') \arrow[rr, bend left, "Z(x'\to 4\to y')"] \arrow[rr, bend right, "Z(x'\to 4\to y')"'] \arrow[rr, phantom, "\Downarrow a"] & & Z(y') \end{tikzcd} \mapsto \begin{tikzcd} Z(x) \arrow[r, "Z(x\to 3\to x')"] \arrow[rrr, bend left=50, "Z(x\to 3\to 6\to 5\to y)"] \arrow[rrr, bend right=50, "Z(x\to 3\to 6\to 5\to y)"'] & Z(x') \arrow[r, bend left, "Z(x'\to 4\to y')"] \arrow[r, bend right, "Z(x'\to 4\to y')"'] \arrow[r, phantom, "\Downarrow a"] \arrow[r, phantom, shift left=6ex, "\Downarrow Z(f')^{-1}"] \arrow[r, phantom, shift right=6ex, "\Downarrow Z(f')"] & Z(y') \arrow[r, "Z(y'\to 5\to y)"] & Z(y) \end{tikzcd} .$$

Now we come to our main point.

Theorem 1 *The functor $\mathcal{A}_Z$ is a net of local monoids satisfying the time slice axiom.*

Proof. We need to demonstrate three things

1. that the above assignment is functorial;

2. that the above assignment satisfies the locality axiom;

3. that the above assignment satisfies the time slice axiom.

The third property is immediate from the construction. The first two properties turn out to be a direct consequence of 2-functoriality of Z and the exchange law in 2-categories.

To see functoriality, consider a chain of inclusions

$$\begin{tikzcd}
O_{x'',y''} \arrow[rr, hook] \arrow[dr, hook] & & O_{x,y} \\
& O_{x',y'} \arrow[ur, hook] &
\end{tikzcd}$$

in $S(\mathbb{R}^2)$ and the corresponding pasting diagram

$$\begin{tikzcd}
x \arrow[ddr] & & & & & & 1 \arrow[drrrr] & & & & & & y \\
& & x' \arrow[urrrr] \arrow[ddrr] \arrow[dddr, Rightarrow, "f_l"] & & & & 2 \arrow[dr] & & & & y' \arrow[dr] \arrow[dddl, Rightarrow, "f_r"'] & & \\
& 3 \arrow[ur] \arrow[ddrr] & & & & x'' \arrow[ur] \arrow[dr] & & y'' \arrow[dr] & & & & 4 \arrow[uur] & \\
& & & & 5 \arrow[ur] \arrow[drr] & & 6 \arrow[ur] \arrow[d, Rightarrow, "f'"] & & 7 \arrow[uurr] \arrow[dr] & & & & \\
& & & 8 \arrow[ur] \arrow[drrr] & & & 9 \arrow[urr] \arrow[d, Rightarrow, "f_c"] & & & 10 \arrow[uurr] & & & \\
& & & & & & 11 \arrow[urrr] & & & & & &
\end{tikzcd}$$

in $P_2(\mathbb{R}^2)$. The composite inclusion

$$\mathrm{End}_C(Z(x'' \to 6 \to y'')) \hookrightarrow \mathrm{End}_C(Z(x' \to 5 \to 9 \to 7 \to y')) \hookrightarrow \mathrm{End}_C(Z(x \to 3 \to 8 \to 11 \to 10 \to 4 \to y))$$

sends $\begin{tikzcd} Z(x'') \arrow[rr, bend left, "Z(x''\to6\to y'')"] \arrow[rr, bend right, "Z(x''\to6\to y'')"'] \arrow[rr, phantom, "\Downarrow a"] & & Z(y'') \end{tikzcd}$ to

$$\begin{tikzcd}[column sep=large]
Z(x) \arrow[r, "Z(x\to3)"] & Z(3) \arrow[r, "Z(3\to x'\to5)" description] \arrow[r, bend left=50, "Z(3\to8\to5)"] \arrow[r, bend right=50, "Z(3\to8\to5)"'] \arrow[r, phantom, shift left=3ex, "\Downarrow Z(f_l)^{-1}"] \arrow[r, phantom, shift right=3ex, "\Downarrow Z(f_l)"] \arrow[rrrrr, bend left=70, "Z(3\to8\to11\to10\to4)"] \arrow[rrrrr, bend right=70, "Z(3\to8\to11\to10\to4)"'] & Z(5) \arrow[r, "Z(5\to x'')"] \arrow[rrrr, bend left=50, "Z(5\to9\to7)"] \arrow[rrrr, bend right=50, "Z(5\to9\to7)"'] & Z(x'') \arrow[r, bend left, "Z(x''\to6\to y'')"] \arrow[r, bend right, "Z(x''\to6\to y'')"'] \arrow[r, phantom, "\Downarrow a"] \arrow[r, phantom, shift left=8ex, "\Downarrow Z(f')^{-1}"] \arrow[r, phantom, shift right=8ex, "\Downarrow Z(f')"] \arrow[r, phantom, shift left=16ex, "\Downarrow Z(3\to8)\cdot Z(f_c)^{-1}\cdot Z(10\to4)"] \arrow[r, phantom, shift right=16ex, "\Downarrow Z(3\to8)\cdot Z(f_c)\cdot Z(10\to4)"] & Z(y'') \arrow[r, "Z(y''\to7)"] & Z(7) \arrow[r, "Z(7\to y'\to4)" description] \arrow[r, bend left=50, "Z(7\to10\to4)"] \arrow[r, bend right=50, "Z(7\to10\to4)"'] \arrow[r, phantom, shift left=3ex, "\Downarrow Z(f_r)^{-1}"] \arrow[r, phantom, shift right=3ex, "\Downarrow Z(f_r)"] & Z(4) \arrow[r, "Z(4\to y)"] & Z(y)
\end{tikzcd} .$$

The contributions from f_l and f_r manifestly cancel and we are left with the pasting diagram for the direct inclusion

$$\mathrm{End}_C(Z(x'' \to 6 \to y'')) \hookrightarrow \mathrm{End}_C(Z(x \to 3 \to 8 \to 11 \to 10 \to 4 \to y))\,.$$

This shows that

$$\begin{tikzcd}
\mathcal{A}_Z(O'') \arrow[rr, hook] \arrow[dr, hook] & & \mathcal{A}_Z(O) \\
& \mathcal{A}_Z(O') \arrow[ur, hook] &
\end{tikzcd}$$

commutes, as desired.

To see locality, let $O_{x,y}$ and $O_{x',y'}$ be two spacelike separated causal subsets inside $O_{(3,5')}$. The relevant pasting diagram in $P_2(\mathbb{R}^2)$ is of the form

$$\begin{tikzcd}
 & & 7 \arrow[dr] & & & & 7' \arrow[dr] & & \\
 & x \arrow[ur] \arrow[dr] & & y \arrow[dr] & & x' \arrow[ur] \arrow[dr] & & y' \arrow[dr] & \\
3 \arrow[ur] \arrow[drr] & & 8 \arrow[ur] \arrow[d, Rightarrow, "f_1"] & & 5 \arrow[ur] \arrow[drr] \arrow[dd, Rightarrow, "f_0"] & & 8' \arrow[ur] \arrow[d, Rightarrow, "f_2"] & & 5' \\
 & & 9 \arrow[urr] \arrow[drr] & & & & 9' \arrow[urr] & & \\
 & & & & 10 \arrow[urr] & & & &
\end{tikzcd}.$$

(We are displaying a very symmetric configuration only for notational convenience. The argument does not depend on that symmetry but just on the fact that $O_{x,y}$ does not intersect the past of $O_{x',y'}$ and vice versa.)

Now given any two endomorphisms $\begin{tikzcd} Z(x) \arrow[rr, bend left, "Z(x\to 8\to y)"] \arrow[rr, bend right, "Z(x\to 8\to y)"'] \arrow[rr, phantom, "\Downarrow a"] & & Z(y) \end{tikzcd}^a$ and $\begin{tikzcd} Z(x') \arrow[rr, bend left, "Z(x'\to 8'\to y')"] \arrow[rr, bend right, "Z(x'\to 8'\to y')"'] \arrow[rr, phantom, "\Downarrow a'"] & & Z(y') \end{tikzcd}$ we can either first include a in $\mathrm{End}_C(Z(3\to 9\to 10\to 9'\to 5'))$ and then a', or the other way around. Either way, the total endomorphism in $\mathrm{End}_C(Z(3\to 9\to 10\to 9'\to 5'))$ is

$$\begin{tikzcd}
 & & Z(9) \arrow[rrrr, bend left=10, "Z(9\to 10\to 9')"] \arrow[drr, "Z(9\to 5)"] \arrow[d, phantom, "\Downarrow Z(f_l)^{-1}"] & & & & Z(9') \arrow[drr, "Z(9'\to 5')"] \arrow[d, phantom, "\Downarrow Z(f_r)^{-1}"] & & \\
Z(3) \arrow[urr, bend left, "Z(3\to 9)"] \arrow[drr, bend right, "Z(3\to 9)"'] \arrow[r] & Z(x) \arrow[rr, bend left, "Z(x\to 8\to y)"] \arrow[rr, bend right, "Z(x\to 8\to y)"'] \arrow[rr, phantom, "\Downarrow a"] & {} & Z(y) \arrow[r] & Z(5) \arrow[r] \arrow[urr, "Z(5\to 9')"] \arrow[drr, "Z(5\to 9')"'] \arrow[u, phantom, "\Downarrow Z(f_0)^{-1}"] \arrow[d, phantom, "\Downarrow Z(f_0)"] & Z(x') \arrow[rr, bend left, "Z(x'\to 8'\to y')"] \arrow[rr, bend right, "Z(x'\to 8'\to y')"'] \arrow[rr, phantom, "\Downarrow a'"] & {} & Z(y') \arrow[r] & Z(5') \\
 & & Z(9) \arrow[rrrr, bend right=10, "Z(9\to 10\to 9')"'] \arrow[urr, "Z(9\to 5)"'] \arrow[u, phantom, "\Downarrow Z(f_l)"] & & & & Z(9') \arrow[urr, "Z(9'\to 5')"'] \arrow[u, phantom, "\Downarrow Z(f_r)"] & &
\end{tikzcd}.$$

This means that the inclusions of a and a' in $\mathrm{End}_C(Z(3\to 9\to 10\to 9'\to 5'))$ commute. $\square$

Theorem 2 *Every G-equivariant structure, definition* **??**, *on the FQFT $Z : P_2(\mathbb{R}^2)\to C$ induces a G-equivariant structure, definition* **??**, *on the AQFT $\mathcal{A}_Z$ obtained from it according to definition 6.*

3 Examples

The most interesting examples of local nets are those where each local algebra is a von Neumann algebra type III factor. I am not yet sure about the best way to get these directly from FQFT along the lines described above. The following lists a couple of simpler applications (some more general in their ways), that should nevertheless help to illustrate the general principle.

As the examples towards the end indicate, a practical way to get fully-fledge nets of von Neumann factors should be to start with a lattice model and then pass to its continuum limit local net. Remarkably, essentially no literature on this question (local nets of factors from continuum limits of lattice models) seems to exist to date.

3.1 AQFT from 2-vector parallel surface transport

Using our theorem, we get large classes of examples of local nets from 2-transport with values in 2-vector spaces from smooth parallel transport 2-functors [15].

$$P_2(\mathbb{R}^2) \xrightarrow{Z_{(A,B)}} \mathbf{B}G_{(2)} \xrightarrow{\rho} \mathrm{Bimod} = 2\mathrm{Vect}_{\mathrm{w/basis}} \hookrightarrow 2\mathrm{Vect} = \mathrm{Vect} - \mathrm{Mod}$$

$$\begin{tikzcd} & y \arrow[dr] \arrow[dd, Rightarrow, "\Sigma"] & \\ x \arrow[ur] \arrow[dr] & & z \\ & y' \arrow[ur] & \end{tikzcd} \mapsto \begin{tikzcd} & \bullet \arrow[dr, "g_{(y,z)}"] \arrow[dd, Rightarrow, "h_\Sigma"] & \\ \bullet \arrow[ur, "g_{(x,y)}"] \arrow[dr, "g_{(x,y')}"'] & & \bullet \\ & \bullet \arrow[ur, "g_{(y',z)}"'] & \end{tikzcd} \mapsto \begin{tikzcd} & A \arrow[dr, "N_{g_{(y,z)}}"] \arrow[dd, Rightarrow, "-\cdot h_\Sigma"] & \\ A \arrow[ur, "N_{g_{(x,y)}}"] \arrow[dr, "N_{g_{(x,y')}}"'] & & A \\ & A \arrow[ur, "N_{g_{(y',z)}}"'] & \end{tikzcd} \mapsto \begin{tikzcd} & \mathrm{Mod}_A \arrow[dr, "-\otimes_A N_{g_{(y,z)}}"] \arrow[dd, Rightarrow, "-\cdot h_\Sigma"] & \\ \mathrm{Mod}_A \arrow[ur, "-\otimes_A N_{g_{(x,y)}}"] \arrow[dr, "-\otimes_A N_{g_{(x,y')}}"'] & & \mathrm{Mod}_A \\ & \mathrm{Mod}_A \arrow[ur, "-\otimes_A N_{g_{(y',z)}}"'] & \end{tikzcd}$$

Figure 5: 2-Vector transport coming from a 2-connection $(A, B) \in \Omega^\bullet(\mathbb{R}^2, (\mathfrak{h} \to \mathfrak{g}))$ with values in the strict Lie 2-algebra $(\mathfrak{h} \to \mathfrak{g})$ and the canonical representation ρ of the corresponding strict Lie 2-group $G_{(2)}$ on 2-vector spaces. The 2-FQFT obtained this way assigns algebras to points, bimodules to paths and bimodule homomorphisms to surfaces. The corresponding local net $A_{Z_{(A,B)}}$ assigns algebras of bimodule endomorphisms.

Let $G_2 := (H \to G)$ be a strict Lie 2-group. According to [15] for each set of differential form data with value in the Lie 2-algebra of G_2 we get a smooth 2-functor

$$\mathrm{tra} : \mathcal{P}_2(X) \to \mathbf{B}G_2 \,.$$

Choosing any 2-representation ρ of G_2 on 2-vector spaces (see also [16]) we get a 2-functor

$$\rho_* \mathrm{tra} : \mathcal{P}_2(X) \to 2\mathrm{Vect} \,.$$

We can simply restrict this to paths in $P_2(X)$ to get an FQFT 2-functor

$$Z := \; P_2(X) \hookrightarrow \mathcal{P}_2(X) \xrightarrow{\mathrm{tra}} \mathbf{B}G \xrightarrow{\rho} 2\mathrm{Vect} \;.$$

3.2 Lattice models

All our definitions and constructions make sense for $S(\mathbb{R}^2)$ and $P_2(\mathbb{R}^2)$ replaced by their restrictions $S(\mathbb{Z}^2)$ and $P_2(\mathbb{Z}^2)$ along that embedding $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ which makes addition of $(1, 0)$ a lightlike translation. This allows to see a class of important examples without the need to worry about weak 2-categories and issues in functional analysis.

Let

$$C := \mathbf{B}\mathrm{Vect} = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "V"] \arrow[r, bend right, "W"'] \arrow[r, phantom, "\Downarrow\phi"] & \bullet \end{tikzcd} \;|\!\left(V \xrightarrow{\phi} W \right) \in \mathrm{Vect} \right\}$$

be the strict 2-category obtained from the strict monoidal category of finite-dimensional vector spaces: it has a single object, its 1-morphisms are finite dimensional vector spaces with composition of morphisms being the tensor product of vector spaces, and 2-morphisms are linear maps $V \xrightarrow{\phi} W$ between vector spaces.

Pick a fixed finite dimensional vector space V and consider the two 2-FQFT 2-functors

$$Z_{\|} : P_2(\mathbb{Z}^2) \to \mathbf{B}\mathrm{Vect}$$

and

$$Z_{\times} : P_2(\mathbb{Z}^2) \to \mathbf{B}\mathrm{Vect}$$

which assign V to every elementary 1-morphism in $P_2(\mathbb{Z}^2)$ and which assign to every elementary square the linear map

$$Z_{\|}\left(\begin{tikzcd} & y \arrow[dr] \arrow[dd, Rightarrow] & \\ x \arrow[ur] \arrow[dr] & & z \\ & y' \arrow[ur] & \end{tikzcd} \right) := \begin{tikzcd} & \bullet \arrow[dr, "V"] \arrow[dd, phantom, "\Downarrow\mathrm{Id}\quad\Downarrow\mathrm{Id}"] & \\ \bullet \arrow[ur, "V"] \arrow[dr, "V"'] & & \bullet \\ & \bullet \arrow[ur, "V"'] & \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left, "V\otimes V"] \arrow[r, bend right, "V\otimes V"'] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & \bullet \end{tikzcd}$$

and

$$Z_{\times}\left(\begin{tikzcd} & y \arrow[dr] \arrow[dd, Rightarrow] & \\ x \arrow[ur] \arrow[dr] & & z \\ & y' \arrow[ur] & \end{tikzcd} \right) := \begin{tikzcd} & \bullet \arrow[dr, "V"] \arrow[dd, phantom, "\mathrm{Id}\ \mathrm{Id}"] & \\ \bullet \arrow[ur, "V"] \arrow[dr, "V"'] & & \bullet \\ & \bullet \arrow[ur, "V"'] & \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left, "V\otimes V"] \arrow[r, bend right, "V\otimes V"'] \arrow[r, phantom, "\Downarrow\theta_{V,V}"] & \bullet \end{tikzcd},$$

respectively, where $V \otimes W \xrightarrow{\theta_{V,W}} W \otimes V$ denotes the canonical symmetric braiding isomorphism in Vect.

The monoids assigned by the corresponding local nets $\mathcal{A}_{Z_{\|}}$ and $\mathcal{A}_{Z_{\times}}$ are algebras of the form $\mathrm{End}(V^{\otimes n})$, where n is the total number of elementary edges in the respective boundary of a region.

Given the inclusion of regions $O_{a,b} \subset O_{x,x'}$

$$\begin{tikzcd} & & & c \arrow[dr] \arrow[d, Rightarrow, "O_{a,b}"] & & & \\ x \arrow[dr] & & a \arrow[ur] \arrow[dr] & \, & b \arrow[dr] & & x' \\ & y \arrow[ur] \arrow[dr] & & d \arrow[ur] \arrow[dd, Rightarrow, "f"] & & y' \arrow[ur] & \\ & & z \arrow[dr] & & z' \arrow[ur] & & \\ & & & w \arrow[ur] & & & \end{tikzcd}$$

we get, according to definition 6, inclusions

$$A_{Z_{\|}}, A_{Z_{\times}} : \mathrm{End}(V^{\otimes 2}) \hookrightarrow \mathrm{End}(V^{\otimes 6})$$

of endomorphism algebras given by

$$A_{Z_\parallel} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A & B & 0 & 0 \\ 0 & 0 & C & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \; ; \; A_{Z_\times} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & B & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & C & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ,$$

where each entry in these matrices is an endomorphism of V.

The locality of the net $\mathcal{A}_{Z_\parallel}$ is manifest. The algebras assigned to two elementary regions clearly commute if and only if the two regions are spacelike separated. For $\mathcal{A}_{Z_\times}$ the algebras of course also commute if the regions are spacelike separated, but here they also commute if the two regions are *timelike* separated. Only if two elementary regions are lightlike separated do the inclusions of algebras due to $\mathcal{A}_{Z_\times}$ not commute.

There are various variations of this example. In particular for $Z_\times$ one would want to consider the case where two different vector spaces V_l and V_r and two nontrivial automorphisms $U_l : V_l \to V_l$ and $U_r : V_R \to V_r$ are assigned to elementary causal subsets as follows:

$$Z_\times \left(\begin{tikzcd} & y \arrow[dr] \arrow[dd, Rightarrow] & \\ x \arrow[ur] \arrow[dr] & & z \\ & y' \arrow[ur] & \end{tikzcd} \right) := \begin{tikzcd} & \bullet \arrow[dr, "V_r"] & \\ \bullet \arrow[ur, "V_l"] \arrow[dr, "V_r"'] \arrow[rr, phantom, "U_l\ U_r"] & & \bullet \\ & \bullet \arrow[ur, "V_l"'] & \end{tikzcd} = \begin{tikzcd} \bullet \arrow[rr, bend left, "V_l \otimes V_r"] \arrow[rr, bend right, "V_r \otimes V_l"'] \arrow[rr, phantom, "\Downarrow \theta_{V_l, V_r} \circ U_l \otimes U_r"] & & \bullet \end{tikzcd} ,$$

Denote by

$$c : \mathrm{End}(V_r) \otimes \mathrm{End}(V_l) \hookrightarrow \mathrm{End}(V_r \otimes V_l)$$

the canonical inclusion of algebras and by

$$c^* \mathcal{A}_{Z_\times} \hookrightarrow \mathcal{A}_{Z_\times}$$

the local sub-net of $\mathcal{A}_{Z_\times}$ obtained by restricting along c everywhere. Then $c^* \mathcal{A}_{Z_\times}$ is what is called a *chiral* AQFT. Its structure is encoded entirely in the two independent projections onto two orthogonal lightlike curves.

$$c^* \mathcal{A}_{Z_\times} : \begin{tikzcd} & y \arrow[dr] \arrow[dd, Rightarrow] & \\ x \arrow[ur] \arrow[dr] & & z \\ & y' \arrow[ur] & \end{tikzcd} \mapsto \mathcal{A}_l \left(\begin{tikzcd} & z \\ y' \arrow[ur] & \end{tikzcd} \right) \otimes \mathcal{A}_r \left(\begin{tikzcd} x \arrow[dr] & \\ & y' \end{tikzcd} \right) = \mathrm{End}(V_l) \otimes \mathrm{End}(V_r) \,.$$

Restricting attention to just one of these and then "compactifying" that to a circle leads to the models [5, **?**] of 2-dimensional (conformal) field theories as local nets on the circle.

This important example is further expanded on in section 3.3.

3.3 Boundary FQFT and boundary AQFT

By taking endomorphisms this defines a net of algebras on the boundary, which entirely encodes the chiral part $c^* \mathcal{A}_{Z^<_\times}$ of $\mathcal{A}_{Z^<_\times}$. This way we arrive at the picture of boundary AQFT given in [8]. Further details should be discussed elsewhere.

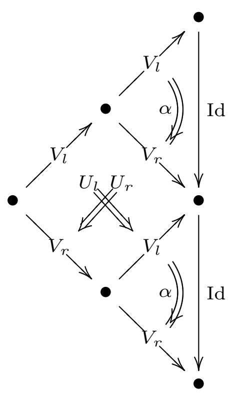


Figure 6: The image under the boundary FQFT 2-functor $Z_\times^<$ of a spacelike wedge on the left Minkowski half plane.

3.4 2-C^*-category codomains

In most applications to physics one wants the algebras in a local net to be C^*-algebras. A natural type of 2-category in which endomorphism algebras of 1-morphisms are C^*-algebras is that of 2-C^*-categories: categories enriched in C^*-categories.

Definition 7 *A* C^*-category *(or* C^*-algebroid*: the many-object version of a C^*-algebra) is a category C enriched in complex Banach spaces (meaning that for all objects ρ, σ, τ of C we have that $C(\rho, \sigma)$ is a complex Banach space and that composition*

$$\circ_{\rho,\sigma,\tau} : C(\rho, \sigma) \times C(\sigma, \tau) \to C(\rho, \tau)$$

is a morphism of complex Banach spaces) which is equipped with an involutive antilinear functor

$$(\cdot)^* : C \to C^{\mathrm{op}}$$

that satisfies the C^*-condition

$$\forall \rho, \sigma \in \mathrm{Obj}(C) : \forall S \in C(a, b) : \begin{cases} S^* \circ S \text{ is positive in } C(\rho, \rho) \\ \|S^* \circ S\| = \|S\|^2 \end{cases} ,$$

where $\|\cdot\| : C(\rho, \sigma) \to \mathbb{C}$ is the Banach norm.

A C^*-algebra A is precisely the endomorphism algebra of an object ρ in a C^*-category, $A = C(\rho, \rho)$. We write $\mathbf{B}A$ for the one object C^*-category whose single endomorphism algebra is A.

C^*-categories form a strict monoidal 2-category $(C^*\mathrm{Cat}, \times)$ whose morphisms are Banach space functors (continuous on each Hom-space). Therefore one can enrich in C^*-categories themselves:

Definition 8 *A (strict)* 2-C^*-category *is a category enriched in* $C^*\mathrm{Cat}$.

A discussion of aspects of 2-C^*-categories can be found in [17].

The canonical example of a strict 2-C^*-category is $\mathrm{Ampli}_{C^*} \subset \mathrm{Bimod}_{C^*}$, the 2-category whose objects are unital C^*-algebras, whose morphisms are amplimorphisms between these and whose 2-morphisms are intertwiners between those. Bimod_{C^*} is very similar, but is not strict. See [?] and section 2 of [17].

So we have

Observation 1 *For $Z : P_2(X) \to \mathcal{C}$ a transport 2-functor with values in a 2-C^*-category $\mathcal{C}$, the corresponding local net A_Z is a net of C^*-algebras.*

3.5 Hopf spin chain models

Recall the description of lattice models with boundary from section 3.3. Consider the extreme case where there is a left and right boundary which are separated only by a single lattice spacing:

$$
\begin{tikzcd}
a \arrow[dr, "\rho"] \arrow[dd, "\mathrm{Id}"'] & \\
\arrow[r, phantom, "\Downarrow"] & b \arrow[dd, "\mathrm{Id}"] \\
a \arrow[ur, "\rho"'] \arrow[dr, "\rho"] \arrow[dd, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow"] & \\
\arrow[r, phantom, "\Downarrow"] & b \\
a \arrow[ur, "\rho"'] &
\end{tikzcd}
,
$$

where for simplicity we are concentrating on the case that Z sends each edge to one and the same morphism $\rho : a \to b$ in $\mathcal{C}$.

Physically, we can think of this as a lattice model for an open string stretching from an brane of type a to a brane of type b. It's a crude lattice model, consisting of a single "string bit".

Consider another such strip, labeled by another morphism $\bar{\rho} : b \to a$

$$
\begin{tikzcd}
& a \arrow[dd, "\mathrm{Id}"] \\
b \arrow[ur, "\bar{\rho}"] \arrow[dr, "\bar{\rho}"'] \arrow[dd, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow"] & \\
\arrow[r, phantom, "\Downarrow"] & a \arrow[dd, "\mathrm{Id}"] \\
b \arrow[ur, "\bar{\rho}"] \arrow[dr, "\bar{\rho}"'] \arrow[r, phantom, "\Downarrow"] & \\
& {}
\end{tikzcd}
.
$$

As the notation suggests, we want to think of $\bar{\rho}$ to be *conjugate* to ρ, meaning that ρ and $\bar{\rho}$ form an ambidextrous adjunction between a and b such that the unit of the left-handed adjunction is the *-adjoint of the counit of the right-handed adjunction, and vice versa. (see p. 8 of [17]).

Then it makes sense to think of this as a lattice model for an open string, or rather a "string bit", as before, but now with that string taken to stretch from the b-type brane to the a-type brane.

In any case, we can now consider lattice models built from the above building blocks by gluing the above strip-wise 2-functors horizontally:

$$
\begin{tikzcd}
a \arrow[dr, "\rho"] \arrow[dd, "\mathrm{Id}"'] & & a \arrow[dd, "\mathrm{Id}"] \\
& b \arrow[ur, "\bar\rho"] \arrow[dr, "\bar\rho"] & \\
a \arrow[ur, "\rho"] \arrow[dr, "\rho"] \arrow[dd, "\mathrm{Id}"'] & & a \arrow[dd, "\mathrm{Id}"] \\
& b \arrow[ur, "\bar\rho"] \arrow[dr, "\bar\rho"] & \\
a \arrow[ur, "\rho"] & & a
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
a \arrow[dr, "\rho"] \arrow[dd, "\mathrm{Id}"'] & & a \arrow[dr, "\rho"] & \\
& b \arrow[ur, "\bar\rho"] \arrow[dr, "\bar\rho"] & & b \arrow[dd, "\mathrm{Id}"] \\
a \arrow[ur, "\rho"] \arrow[dr, "\rho"] \arrow[dd, "\mathrm{Id}"'] & & a \arrow[ur, "\rho"] \arrow[dr, "\rho"] & \\
& b \arrow[ur, "\bar\rho"] \arrow[dr, "\bar\rho"] & & b \\
a \arrow[ur, "\rho"] & & a \arrow[ur, "\rho"] &
\end{tikzcd}
\quad , \cdots
$$

The algebras assigned by the corresponding net A_Z to the elementary causal bigon $O_{\rho,\bar\rho}$ and $O_{\bar\rho,\rho}$ are

$$A_Z(O_{\rho,\bar\rho}) = \mathrm{End}_{\mathcal{C}}(\bar\rho \circ \rho)$$

and

$$A_Z(O_{\bar\rho,\rho}) = \mathrm{End}_{\mathcal{C}}(\rho \circ \bar\rho)\,.$$

If $\mathcal{C}$ is a 2-C^*-category, these are C^*-Hopf algebras H and $\hat{H}$ which are duals of each other [9, 17]. Due to the fact that the 2-morphisms in the above diagrams do not mix ρ and $\bar\rho$, we can understand the nature of the net A_Z obtained from the above 2-functor Z already by concentrating on the endomorphism algebras assigned to a horizontal zig-zag

$$
\begin{tikzcd}
a \arrow[dr, "\rho"] & & a \arrow[dr, "\rho"] & & a \arrow[dr, "\rho"] & \\
& b \arrow[ur, "\bar\rho"] & & b \arrow[ur, "\bar\rho"] & & b
\end{tikzcd}
\,.
$$

$$| \text{———} | \text{———} | \text{———} | \text{———} | \text{———} |$$

If we to restrict evaluating the net A_Z on zig-zags of even length, this gives rise to a net on the latticized real axis with the property that algebras $A_Z(I_1)$ and $A_Z(I_2)$ commute if the intervals I_1 and I_2 are not just disjoint but differ by at least one lattice spacing.

Precisely these kind of 1-dimensional nets are considered in [10], where they are addressed as *Hopf spin chain models*.

3.6 Subfactors and asymptotic inclusion

We can interpret the analysis of the direct limit algebras of the above nets A_Z given in [10] in terms of Ocneanu' notion of asymptotic inclusion [11] and its relation to subfactors. (Thanks to Pasqual Zito for pointing this out.)

Consider first a lattice of the above sort unbounded (only) to the right. The direct limit algebra

$$A := \mathrm{colim}\, A_Z$$

of the chain of inclusions of finite algebras

$$\mathrm{End}_{\mathcal{C}}(\bar\rho\circ\rho) \hookrightarrow \mathrm{End}_{\mathcal{C}}(\rho\circ\bar\rho\circ\rho) \hookrightarrow \mathrm{End}_{\mathcal{C}}(\bar\rho\circ\rho\circ\bar\rho\circ\rho) \hookrightarrow \cdots$$

naturally carries a trace, which we can assume to be normalized. Completing with respect to the norm $\|a\| := \mathrm{tr}(a^*\circ a)$ yields an algebra $\bar A$ which is a type II vonNeumann algebra factor.

We can shift everything one lattice spacing to the right and consider the poset of algebras

$$\mathrm{Id}_\rho \cdot \mathrm{End}_{\mathcal{C}}(\bar\rho) \hookrightarrow \mathrm{Id}_\rho \cdot \mathrm{End}_{\mathcal{C}}(\rho\circ\bar\rho) \hookrightarrow \mathrm{Id}_\rho \cdot \mathrm{End}_{\mathcal{C}}(\bar\rho\circ\rho\circ\bar\rho) \hookrightarrow \cdots\,,$$

where $\cdot$ denotes the horizontal composition in our 2-C^*-category $\mathcal{C}$. The completion of the direct limit of this chain of inclusions is a type II factor $\bar B$ which has a canonical inclusion into $\bar A$

$$\bar B \hookrightarrow \bar A\,.$$

This inclusion of subfactors obtained from a pair of conjugate morphisms $\rho, \bar\rho$ in a 2-C^*-category is Ocneanu' *asymptotic inclusion* [11, 2]. Following the discussion on p. 10 of [5] one can understand this in the context of [7] and read $\bar A$ and $\bar A^o$ as two chiral open string algebras and K as the corresponding closed string algebra.

If the 2-C^*-category $\mathcal{C}$ that we started with is $\mathcal{C} = \mathrm{Bimod}_{C^*}$ and the original morphism $\rho : a \to b$ in $\mathcal{C}$ itself an inclusion of subfactors, then this is recovered by the above constrction.

$\mathcal{A}$ and $\mathcal{B}$ and their inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ encode a QFT on the right half plane. From the above setup we can analogously obtain a subfacvtor $\bar B^o \hookrightarrow \bar A^o$ for the left half plane.

Moreover, the completion of the direct limit algebra over all endomorphism algebras of zig-zags that are allowed to extend finitely to the right and the left yields a factor $\bar K$ which has a canonical inclusion of the factor $\bar A^o \otimes \bar A$

$$\bar A^o \otimes \bar A \hookrightarrow K\,.$$

The Hopf algebra

$$\mathcal{D}(H) = \mathrm{End}_{\mathrm{Bimod}_{C^*}}(\bar A^o \otimes A \overset{\sigma}{\hookrightarrow} K \xrightarrow{\bar\sigma} \bar A^o \otimes A)$$

induced from this inclusion is the Drinfeld double Hopf algebra of the original Hopf algebra $H = \mathrm{End}_{\mathcal{C}}(a \overset{\rho}{to}$ $b \xrightarrow{\bar\rho} b)$. This can be understood as providing the closed string sector corresponding to the above open string scenarios.

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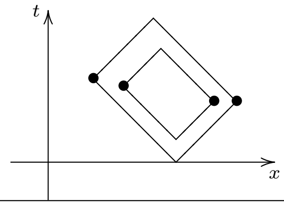
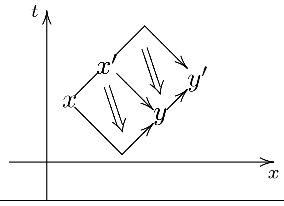
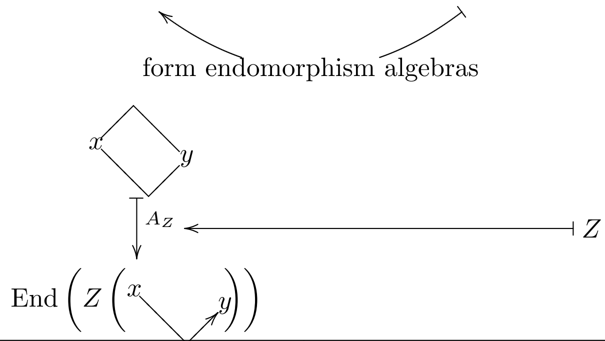
names	algebraic QFT (also: axiomatic QFT, local QFT)	functorial QFT
abbreviations	**AQFT**	**FQFT**
idea	assign algebras (of observables) to patches, compatible with inclusion	(time evolution) operators composition (gluing)
axioms due to	Haag, Kastler	Atiyah, Segal
aspect of QFT	Heisenberg picture	Schrödinger picture
formal structure	co-presheaf	transport n-functor
cartoon of domain structure		
relation		
main existing general theorems	spin-statistics theorem, PCT theorem	results about topological invariants
main existing nontrivial examples	chiral 2-d CFT	topological QFTs full rational 2-d CFT

Table 1: **The two approaches** to the axiomatization of quantum field theory together with their interpretation and relation as discussed here. The rectangular diagrams are explained in section 2. The construction of the AQFT $\mathcal{A}_Z$ from the extended FQFT Z is our main point, described in section **??**.