on anafunctors and transitions

Schreiber*

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Anafunctors and Transitions. We recall the defintion of an anafunctor and of the transition data of a functor. Then we want to show that both are equivalent. The connection is made by the **universal transition**. We use this to propose a notion of higher anafunctors.

Definition 1 (Makkai) *Given two categories A and Q, an* **anafunctor**

$$\mathbf{F} : A \to Q$$

is a span

$$\begin{tikzcd} {|F|} \arrow[r, "F_1"] \arrow[d, "F_0"'] & Q \\ A & \end{tikzcd}$$

such that F_0 is surjective on objects and on morphisms and such that every morphism in A has at most one lift with given source and target.

I would like to reformulate this slightly.

Definition 2 *For A any category, a* **cover** *of A is a morphism*

$$p : K \to A$$

such that the image of p generates *A.*

Example 1

Let $A = \mathcal{P}_1(X)$ be the category of paths in a space X. Let $U \to X$ be an ordinary cover at the level of objects and let $K = P_1(U)$ be the category of paths in the cover. The obvious projection $p : \mathcal{P}_1(U) \to \mathcal{P}_1(X)$ hits all paths that remain within one patch of the cover. Under composition, these generate all paths in X.

*E-mail: urs.schreiber at math.uni-hamburg.de

We write $K^{[n]}$ for the n-fold strict pullback of K along itself. For instance $K^{[2]}$ is the universal category making

$$\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] & K \arrow[d, "p"] \\
K \arrow[r, "p"'] & A
\end{tikzcd}$$

commute.

In our example, $K^{[2]} = \mathcal{P}_1(U^{[2]})$ is the category of paths in double intersections of the given cover of X.

Neither of the $K^{[n]} \to A$ is, in general, epi. But we can throw in *gluing morphisms* into $K^{[2]}$ such that we do get a surjection in a universal way by forming a certain weak pushout.

Definition 3 *Given a cover $K \to A$, denote by $K^\bullet$ the object sitting in a diagram*

$$\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{v}{\swarrow}\sim"] & K \arrow[d, "i"] \\
K \arrow[r, "i"'] & K^\bullet
\end{tikzcd}$$

satisfying

$$\begin{tikzcd}
& p_2^* i \arrow[dr, "p_{23}^* v"] \arrow[d, phantom, "\mathrm{Id}\Downarrow"] & \\
p_1^* i \arrow[ur, "p_{12}^* v"] \arrow[rr, "p_{13}^* v"'] & {} & p_3^* i
\end{tikzcd}$$

on $K^{[3]}$, that is (strictly) universal *in the sense that for any other*

$$\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[d] \\
K \arrow[r] & Q
\end{tikzcd}$$

satisfying a triangle law we have

$$
\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[ddr] & & K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{v}{\swarrow}\sim"] & K^{[2]} \arrow[d, "i"] \arrow[ddr] & \\
K \arrow[drr] & & = & K^{[2]} \arrow[r, "i"] \arrow[drr] & K^{\bullet} \arrow[dr, dashed] & \\
& & Q & & & Q
\end{tikzcd}
$$

for a unique morphism

$$K^{\bullet} \dashrightarrow Q$$

Proposition 1 *$K^{\bullet}$ is given in terms of generators and relations as follows. The generators are the morphism of K together with new morphisms – the* **gluing morphisms** *–*

$$p_1(x) \longrightarrow p_2(x)$$

and their inverses, for all $x \in \mathrm{Obj}(K^{[2]})$. The relations are

$$
\begin{tikzcd}
p_1(x) \arrow[r, "p_1(\gamma)"] \arrow[d] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & p_1(y) \arrow[d] \\
p_2(x) \arrow[r, "p_2(\gamma)"'] & p_2(y)
\end{tikzcd}
$$

for all $\gamma \in \mathrm{Mor}(K^{[2]})$ and

$$
\begin{tikzcd}
& p_2(x) \arrow[dr] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \\
p_1(x) \arrow[ur] \arrow[rr] & \, & p_3(x)
\end{tikzcd}
$$

for all $x \in \mathrm{Obj}(K^{[3]})$.

Example 2

In terms of the previous example, the gluing morphism would form precisely the groupoid

$$U^{[2]} \rightrightarrows U$$

of the ordinary cover of X. In other words, there is then a unique gluing morphism

$$(x,i) \longrightarrow (x,j)$$

for every $(x,i,j) \in U^{[2]}$. If we denote by

$$(\gamma,i): \ (x,i) \longrightarrow (y,i)$$

any path in U_i, then the first kind of relation says that

$$\begin{tikzcd}
(x,i) \arrow[r, "(\gamma,i)"] \arrow[d] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & (y,i) \arrow[d] \\
(x,j) \arrow[r, "(\gamma,j)"'] & (y,j)
\end{tikzcd}$$

for every path

$$(\gamma,i,j): \ (x,i,j) \longrightarrow (y,i,j)$$

in $U^{[2]}$.

Notice the following:

for $Q = \Sigma(G)$ a category with a single object and a Lie group G worth of morphisms, a smooth functor $\mathcal{P}_1(U) \to \Sigma(G)$, for $\mathcal{P}_1(U)$ the groupoid of thin homotopy classes of paths in U, is precisely a **trivial G-bundle with connection** on U.

Moreover, a smoth functor $U^{[2]} \to \Sigma(G)$ is precisely a **G-cocycle** relative to U. Or in other words: the transition function of a G-principal bundle locally trivialized with respect to U.

Finally, a smooth functor $K^\bullet \to \Sigma(G)$ is both of that, together with the compatibility condition, induced by the respect for the rectangular relation relation above, which makes the cocycle a **differential G-cocycle**, hence the transition data of a locally trivialized G-bundle with connection.

Proposition 2 *For any*

$$\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "g \swarrow \sim"] & K \arrow[d, "F"] \\
K \arrow[r, "F"'] & Q
\end{tikzcd}$$

satisfying a triangle law, the morphism

$$K^\bullet \dashrightarrow Q$$

is the functor

$$K^\bullet \xrightarrow{(F,g)} Q$$

that acts as F on the generators from K and assigns $g(x)$ to the gluing morphism at x.

Proposition 3 *The obvious epimorphism*

$$p: \; K^\bullet \longrightarrow A$$

has the property that it has unique lifts with given source and target.

Proof. The relations mentioned above precisely ensure that any two lifts with given source and target are equal. □

It follows that

Proposition 4 *From a given cover and a functor F with transition g on that cover, we get an anafunctor*

$$(F,g): \; A \longrightarrow Q$$

given by the span

$$\begin{tikzcd} K^\bullet \arrow[r, "(F{,}g)"] \arrow[d, "p"'] & Q \; . \\ A & \end{tikzcd}$$

Also the converse is true:

Proposition 5 *For every anafunctor*

$$\mathbf{F} : A \to Q$$

there is a cover $K \to A$ and a functor $F : K \to Q$ with transition g such that $\mathbf{F}$ *is the corresponding anafunctor according to the above proposition.*

Proof. Identitfy all nontrivial morphisms in $|\mathbf{F}|$ that get sent to identity morphisms in A with gluing morphisms.

Then take K to be the minimal sub-category of $|\mathbf{F}|$ such that K together with the gluing morphisms generate all of $|\mathbf{F}|$. This implies that the image of $F : K \to A$ generates all of A.

The two relations to be satisfied by the generators follow directly from the fact that F_0 has unique lifts with given source and target.

Finally, identify F with the restriction of $\mathbf{F}$ to K and g with the restriction of $\mathbf{F}$ to the gluing morphisms. □

2-Anafunctors. I would like to use the above equivalence between anafunctors and transition data in order to formulate higher anafunctors. The reason is that a good notion of higher versions of transitions is relatively obvious and has proven its value in applications.

I'll work with strict 2-categories, pseudonatural transformation between them and modifications between these.

There are obvious higher versions of the definitions in the previous paragraph:

Definition 4 *For A any 2-category, a* **cover** *of A is a morphism*

$$p : K \to A$$

such that the image of p generates *A.*

Definition 5 *Given a 2-category A and a cover $K \to A$, denote by $K^\bullet$ the object sitting in a diagram*

$$\begin{tikzcd} K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{v}{\swarrow}\sim"] & K \arrow[d, "i"] \\ K \arrow[r, "i"'] & K^\bullet \end{tikzcd}$$

together with a morphism

$$\begin{tikzcd} & p_2^* i \arrow[dr, "p_{23}^* v"] \arrow[d, phantom, "w\Downarrow"] & \\ p_1^* i \arrow[ur, "p_{12}^* v"] \arrow[rr, "p_{13}^* v"'] & {} & p_3^* i \end{tikzcd}$$

on $K^{[3]}$ that satisfies a tetrahedron law on $K^{[4]}$ and that is (strictly) universal *in the sense that for any other*

$$\begin{tikzcd} K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[d] \\ K \arrow[r] & Q \end{tikzcd}$$

satisfying a tetrahedron law we have

$$
\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{g}{\swarrow}\sim"] & K \arrow[ddr] & \\
K \arrow[drr] & & \\
& & Q
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
K^{[2]} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{v}{\swarrow}\sim"] & K^{[2]} \arrow[d, "i"] \arrow[ddr] & \\
K^{[2]} \arrow[r, "i"] \arrow[drr] & K^{\bullet} \arrow[dr, dashed] & \\
& & Q
\end{tikzcd}
$$

for a unique morphism

$$
K^{\bullet} \dashrightarrow Q \ .
$$

This morphism should be addressed as a **2-anafunctor**:

Definition 6 *A* **2-anafunctor**

$$
\mathbf{F} : A \to Q
$$

between 2-categories A and Q is a cover $K \to A$ of A together with a 2-functor $F : K \to Q$ and its transition data such that

$$
\mathbf{F} : K^{\bullet} \to Q
$$

is the universal morphism obtained from this transition data as above.