morphisms of anafunctors

Schreiber*

December 11, 2006

In his article on anafunctors, M. Makkai presents almost everything in terms of two equivalent definitions. The one exception is the composition of morphisms of anafunctors, which is not presented in the otherwise more elegant definition in terms of spans.

Here I want to recall the definition of anafunctors in terms of spans and write down the composition of their morphisms in that form.

In fact, Toby Bartels does exactly that, even internally, in his thesis. Just for my own benefit, I want to see the relevant structure stripped off the complexity introduced by writing down everything internalized.

Definition 1 (Makkai) *Given two categories A and B, an* **anafunctor**

$$F : A \to B$$

is a span

$$\begin{tikzcd} {|F|} \arrow[r, "F_1"] \arrow[d, "F_0"'] & B \\ A & \end{tikzcd}$$

such that F_0 is surjective on objects and on morphisms and such that every morphism in A has at most one lift with given source and target.

Example 1

Let

$$A = \mathcal{P}_1(X)$$

be paths in a smooth space X, let

$$Y \to X$$

be a surjective submersion, let $Y^\bullet$ be the associated groupoid and

$$\mathcal{P}_1(Y^\bullet)$$

*E-mail: urs.schreiber at math.uni-hamburg.de

be the category of paths in $Y^\bullet$. Then the canonical projection

$$p : \mathcal{P}_1(Y^\bullet) \to \mathcal{P}_1(X)$$

is surjective on objects and morphisms and every path in X has a unique lift for given lift of its endpoints.

A smooth functor

$$(\mathrm{tra}_Y, g) : \mathcal{P}_1(Y^\bullet) \to \Sigma(G)$$

for G any Lie group is precisely the cocycle data of a locally trivialized G-bundle with connection on X.

Definition 2 (Makkai) *A* **morphism of anafunctors**

$$\begin{tikzcd} A \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow h"] & & B \end{tikzcd}$$

is a natural transformation

$$\begin{tikzcd} & {|F|} \arrow[dr, "F_1"] & \\ {|F| \times_A |G|} \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow h_1"] & & B \\ & {|G|} \arrow[ur, "G_1"'] & \end{tikzcd}$$

.

The next definition is supposed to be equivalent to what Makkai defines in other terms.

Definition 3 *The composition*

$$\begin{tikzcd} A \arrow[rr, bend left=50, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=50, "H"'] \arrow[rr, phantom, shift left=3, "\Downarrow h"] \arrow[rr, phantom, shift right=3, "\Downarrow k"] & & B \end{tikzcd}$$

of morphisms of anafunctors is the morphism given by the natural transforma-

tion

$$\begin{tikzcd}
 & & & {|F|} \arrow[dd, phantom, "\Downarrow h_1"] \arrow[ddr, "F_1"] & \\
 & & {|F| \times_A |G|} \arrow[ur] \arrow[dr] & & \\
{|F| \times_A |H|} \arrow[r, "t"] & {|F| \times_A |G| \times_A |H|} \arrow[ur] \arrow[dr] & & {|G|} \arrow[r, "G_1"] \arrow[dd, phantom, "\Downarrow k_1"] & B \\
 & & {|G| \times_A |H|} \arrow[ur] \arrow[dr] & & \\
 & & & {|H|} \arrow[uur, "H_1"'] &
\end{tikzcd},$$

where t *is*

$$t:\ |F| \times_A |H| \xrightarrow{\simeq} |F| \times_A A \times_A |H| \xrightarrow{\mathrm{Id}\times s\times \mathrm{Id}} |F| \times_A |G| \times_A |H|$$

for any *lift* $s : A \to |G|$.

The crucial point which makes this work is that a s always exists and, crucially, that the above natural transformation is completely independent of the choice of s.

To see this notice first that all choices of s are isomorphic. Then use the rules for horizontal composition of natural transformations to see that

$$\begin{tikzcd}
 & & & {|F|} \arrow[dd, phantom, "\Downarrow h_1"] \arrow[ddr, "F_1"] & \\
 & & {|F| \times_A |G|} \arrow[ur] \arrow[dr] & & \\
{|F| \times_A |H|} \arrow[r, bend left, "t"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow\sim"] & {|F| \times_A |G| \times_A |H|} \arrow[ur] \arrow[dr] & & {|G|} \arrow[r, "G_1"] \arrow[dd, phantom, "\Downarrow k_1"] & B \\
 & & {|G| \times_A |H|} \arrow[ur] \arrow[dr] & & \\
 & & & {|H|} \arrow[uur, "H_1"'] &
\end{tikzcd}$$

still equals the expression in the above definition, because everything involving G is projected out by the 1-morphisms bounding this diagram.