## §1. Anafunctors

Let $\mathbf{X}$ and $\mathbf{A}$ be categories. An *anafunctor* $F$ *with domain* $\mathbf{X}$ *and codomain* $\mathbf{A}$, in notation $F:\mathbf{X}\xrightarrow{\mathrm{a}}\mathbf{A}$, or just simply $F:\mathbf{X}\longrightarrow\mathbf{A}$, is given by the following data 1.(i),(ii) and conditions 1.(iii)-(v):

**1.(i)** A class $|F|$, with maps $\sigma: |F| \longrightarrow \mathrm{Ob}(\mathbf{X})$ ("source"), $\tau: |F| \longrightarrow \mathrm{Ob}(\mathbf{A})$ ("target"). $|F|$ is the class of *specifications*; $s\in |F|$ "specifies the value $\tau(s)$ at the argument $\sigma(s)$". For $X\in\mathbf{X}$ (that is, $X\in\mathrm{Ob}(\mathbf{X})$), we write $|F|X$ for the class $\{s\in |F| : \sigma(s)=X\}$, and $F_s(X)$ for $\tau(s)$; the notation $F_s(X)$ presumes that $s\in |F|X$.

**1.(ii)** For each $X, Y\in\mathbf{X}$, $x\in |F|X$, $y\in |F|Y$ and $f:X\longrightarrow Y$ ($\in\mathrm{Arr}(\mathbf{X})$), an arrow $F_{x,y}(f):F_x(X)\longrightarrow F_y(Y)$ in $\mathbf{A}$.

**1.(iii)** For every $X\in\mathbf{X}$, $|F|X$ is inhabited.

**1.(iv)** For all $X\in\mathbf{X}$ and $x\in |F|X$, we have $F_{x,x}(1_X) = 1_{F_xX}$.

**1.(v)** Whenever $X, Y, Z\in\mathbf{X}$, $x\in |F|X$, $y\in |F|Y$, $z\in |F|Z$, and

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=2, "\circ"] & & Z \end{tikzcd} \quad \text{, then} \quad \begin{tikzcd} & F_yY \arrow[dr, "F_{y,z}(g)"] & \\ F_xX \arrow[ur, "F_{x,y}(f)"] \arrow[rr, "F_{x,z}(h)"'] \arrow[rr, phantom, shift left=2, "\circ"] & & F_zZ \end{tikzcd}$$

(a circle in a diagram means that the diagram commutes), i.e.,

$F_{x,z}(gf) = F_{y,z}(g)\circ F_{x,y}(f)$ .

With any given $X\in\mathbf{X}$, $A\in\mathbf{A}$, we put $|F|(X, A) \underset{\mathrm{def}}{=} \{x\in |F|X: F_x(X)=A\}$ .

The anafunctor $F:\mathbf{X}\longrightarrow\mathbf{A}$ is *locally small* if all the classes $|F|(X, A)$ ($X\in\mathbf{X}$, $A\in\mathbf{A}$) are sets. It is *weakly small* if the classes $|F|X$ are all small ($X\in\mathbf{X}$); thus, "weakly small" implies "locally small". Finally, $F$ is *small* iff it is weakly small, and the category $\mathbf{X}$ is small. Notice that if $F$ is small, then it is given by a *set* of data, beyond the data for $\mathbf{A}$; in particular, we may consider the *class of* all small anafunctors with a fixed codomain $\mathbf{A}$, an arbitrary (not necessarily small) category.

If $F:\boldsymbol{X} \xrightarrow{\mathrm{a}} \boldsymbol{A}$, and $s \in |F|X$, $t \in |F|X$, then $F_{s,t}(1_X): F_s X \longrightarrow F_t X$ is an isomorphism, with inverse $F_{t,s}(1_X)$. In particular, the *value* of $F$ at $X$, $F_s(X)$, is determined up to isomorphism.

Any (ordinary) functor $F:\boldsymbol{X} \longrightarrow \boldsymbol{A}$ is, essentially, an anafunctor, by putting $|F| = \mathrm{Ob}(\boldsymbol{X})$, $\sigma(X) = X$, $\tau(X) = F(X)$ (thus $|F|X = \{X\}$), with the obvious specification of the rest of the structure.

A more abstract way of defining the concept is as follows. A *discrete* category is one in which all arrows are identities; an *antidiscrete* category is one in which for any pair $(U, V)$ of objects, there is exactly one arrow $U \to V$. A *discrete* (*antidiscrete*) opfibration is one in which every fiber is a discrete (antidiscrete) category. A discrete opfibration is a functor $G:\boldsymbol{S} \longrightarrow \boldsymbol{B}$ such that for any $a:A \to B$ in $\boldsymbol{B}$ and $S \in G^{-1}(A)$, there is exactly one arrow $s:S \to T$ with some $T \in G^{-1}(B)$ such that $G(s) = a$; an antidiscrete opfibration is a functor $G:\boldsymbol{S} \longrightarrow \boldsymbol{B}$ such that for any $a:A \to B$ in $\boldsymbol{B}$, $S \in G^{-1}(A)$ and $T \in G^{-1}(B)$, there is exactly one arrow $s:S \to T$ such that $G(s) = a$. Now,

**1**$^*$. *An anafunctor* $F:\boldsymbol{X} \longrightarrow \boldsymbol{A}$ *may be given by a span*

$$\begin{tikzcd} & {|F|} \arrow[dl, "F_0"'] \arrow[dr, "F_1"] & \\ \boldsymbol{X} & & \boldsymbol{A} \end{tikzcd} \tag{1}$$

*of functors in which* $F_0$ *is an antidiscrete opfibration that is surjective on objects.*

Indeed, with $F:\boldsymbol{X} \to \boldsymbol{A}$ being an anafunctor in the original sense, we let $|F|$ be the category whose object-class is what was $|F|$ above, whose arrows $f:x \to y$ are the same as arrows $f:\sigma(x) \to \sigma(y)$ in $\boldsymbol{X}$, with the obvious composition; $F_0$ is the obvious forgetful functor (clearly an antidiscrete opfibration); $F_1$ maps $s$ to $\tau(s)$ and $f:x \to y$ to $F_{x,y}(f)$. Conversely, if we have an anafunctor in the new sense, we put the object-class of $|F|$ for $|F|$ in the old sense, $\sigma(x) = F_0(x)$, $\tau(x) = F_1(x)$, and for $f:X \to Y$ in $\boldsymbol{X}$, $x, y \in |F|$ with $F_0(x)=X$, $F_0(y)=Y$, we put $F_{x,y}(f) = F_1(\hat{f})$ for the unique $\hat{f}:x \to y$ for which $F_0(\hat{f}) = f$.

**2. Example**. Suppose the category $\boldsymbol{A}$ has binary products; that is, for every $A, B \in \boldsymbol{A}$, there is at least one product diagram

$$\begin{tikzcd} & C \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ A & & B \end{tikzcd} \quad . \tag{1}$$

Then we have the following anafunctor $P:\boldsymbol{A}\times\boldsymbol{A}\longrightarrow\boldsymbol{A}$. $|P|$ consists of all product diagrams of the form (1); for $s$ the diagram in (1), $\sigma(s)=(A, B)$ and $\tau(s)=C$. In the formulation of 1.*, $|P|$ is the category of all product diagrams, where arrows are given as in (2) below. In other words, for $(A, B)\in\boldsymbol{A}\times\boldsymbol{A}$, $|P|((A, B))$ is the class of all product diagrams (1), with the given $A$, $B$, but all other data variable; for $s\in |P|((A, B))$ given by (1), $P_s((A, B)) = C$. For $s\in |P|((A, B))$ given by the data in (1), and $s'\in |P|((A', B'))$, given by data as in (1) but primed, and for $(f, g):(A, B)\longrightarrow(A', B')$ ($\in\mathrm{Arr}(\boldsymbol{A}\times\boldsymbol{A})$), $P_{s, s'}((f, g)):C\longrightarrow C'$ is the unique $h$ making the diagram

$$\begin{tikzcd} & C \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[ddd, "h"] & \\ A \arrow[d, "f"'] & & B \arrow[d, "g"] \\ A' & & B' \\ & C' \arrow[ul, "\pi'_0"] \arrow[ur, "\pi'_1"'] & \end{tikzcd} \tag{2}$$

commute; the universal property of the product consisting of the primed data ensures that $P_{s, s'}((f, g))$ is well-defined. It is fairly clear that the conditions 1.(iii) to (v) are all satisfied.

The above-defined $P$ is the *product-anafunctor* for the category $\boldsymbol{A}$, "replacing" the product-functor $(A, B)\longmapsto A\times B$. Whereas the definition of the latter requires a non-canonical *choice* of a particular product $A\times B$ for each pair $(A, B)$ of objects, and thus, in general, for its definition, the product-functor needs the Axiom of Choice (AC), the product-anafunctor does not involve any non-canonical choice, in particular, it does not need the AC. Of course, it is still to be demonstrated that the product-anafunctor does enough of the job of the product-functor, for it to be a reasonable replacement. At any rate, it will turn out (see below) that if the product-functor exists, then the product-anafunctor is *isomorphic* to it, by an appropriate notion of (natural) isomorphism.

An anafunctor $F:\mathbf{X} \xrightarrow{a} \mathbf{A}$ is *saturated* if it satisfies the following additional condition:

**1.(vi)** (*unique transfer*) Whenever $s \in |F|(X, A)$, and $\mu : A \xrightarrow{\cong} B$ is an isomorphism (in $\mathbf{A}$), then there is a unique $t \in |F|(X, B)$ such that $\mu = F_{s,\,t}(1_X)$.

With $F$ an anafunctor, and continuing with the above notation, if $|F|(X, A)$ is inhabited, then $A$ is a *possible value* of $F$ at the argument $X$. Note that the possible values of $F$ at a given $X$ form a subclass of an isomorphism class of objects in $\mathbf{A}$; if $F$ is saturated, they form a complete isomorphism class.

An anafunctor determines its values at least up to isomorphism; a saturated one determines its values *exactly* up to isomorphism. Among anafunctors, the ordinary functors and the saturated anafunctors represent two extremes; our ultimate goal here is to promote the use of the saturated anafunctors as the ones that stand for the point of view that objects (in this case the values of the anafunctor) should be determined *exactly* up to isomorphism, just as they are when they are determined by a universal property.

**2. Example** (continued). The product anafunctor $P:\mathbf{A}\times\mathbf{A} \longrightarrow \mathbf{A}$ is saturated, as it is immediately seen.

Note that if $F:\mathbf{X} \to \mathbf{A}$ is saturated, $X \in \mathbf{X}$, $s \in |F|(X, A)$, then for any $B \in \mathbf{A}$ we have the bijection

$$\begin{array}{ccc} |F|(X, B) & \xrightarrow{\cong} & \mathrm{Iso}(A, B) \\ t & \longmapsto & F_{s,\,t}(1_X) \end{array} \quad . \tag{2'}$$

This bijection is not canonical; it depends on the choice of $s \in |F|(X, A)$. Nevertheless, it follows that for a saturated anafunctor $F:\mathbf{X} \longrightarrow \mathbf{A}$, if $\mathbf{A}$ is locally small, then so is $F$, and if both $\mathbf{X}$ and $\mathbf{A}$ are small, then so is $F$.

Assume $F:\mathbf{X} \longrightarrow \mathbf{A}$ is a saturated anafunctor. We have a form of "isomorphic transfer" not only for the values but also for the arguments of $F$. More precisely,

**3.** *For* $F:\boldsymbol{X}\longrightarrow\boldsymbol{A}$ *a saturated anafunctor, a pair of isomorphisms* $(f:X\xrightarrow{\cong}Y,\ g:A\xrightarrow{\cong}B)$ *induces a bijection* $|F|(X,A)\xrightarrow{\cong}|F|(Y,B)$*, defined by* $s\mapsto t \iff F_{s,t}(f)=g$.

Let us fix $f:X\xrightarrow{\cong}Y$ and $g:A\xrightarrow{\cong}B$. Let $s\in|F|(X,A)$; I claim that there is a unique $t\in|F|(Y,B)$ such that $F_{s,t}(f)=g$. Once this is shown, for any $t\in|F|(Y,B)$ there is unique $s\in|F|(X,A)$ such that $F_{t,s}(f^{-1})=g^{-1}$, that is, $F_{s,t}(f)=g$, and the definition above indeed gives a bijection $s\mapsto t$.

Let $t_0\in|F|Y$, $s\in|F|(X,A)$, $t\in|F|(Y,B)$, and consider the commutative triangle

$$\begin{tikzcd}
A \arrow[d, "F_{s,t_0}(f)"'] \arrow[dr, "F_{s,t}(f)"] \arrow[dr, phantom, shift right=3, "\circ"] & \\
F_{t_0}Y \arrow[r, "F_{t_0,t}(1_Y)"'] & B
\end{tikzcd}$$

consisting of isomorphisms. It follows that saying that $F_{s,t}(f)=g$ is equivalent to saying that the triangle

$$\begin{tikzcd}
A \arrow[d, "F_{s,t_0}(f)"'] \arrow[dr, "g"] \arrow[dr, phantom, shift right=3, "\circ"] & \\
F_{t_0}Y \arrow[r, "F_{t_0,t}(1_Y)"'] & B
\end{tikzcd}$$

commutes. But by 1.(vi), for any $g:A\xrightarrow{\cong}B$, there is a unique $t$ satisfying this latter condition, that is, $F_{t_0,t}(1_Y)=g\circ(F_{s,t_0}(f))^{-1}$.

With $\boldsymbol{X}^*$ denoting the groupoid of all isomorphisms in $\boldsymbol{X}$, and similarly for $\boldsymbol{A}^*$, we have,

**4.** *With* $F:\boldsymbol{X}\to\boldsymbol{A}$ *a saturated anafunctor, the mapping in* 3. *defines a functor*

$$\begin{array}{l}\mathbf{X}^* \times \mathbf{A}^* \longrightarrow \mathrm{SET}: \\ \qquad \begin{array}{ccc} (X, A) & \longmapsto & |F|(X, A) \\ {\scriptstyle (f,g)}\downarrow & & \downarrow {\scriptstyle s \mapsto t} \\ (Y, B) & \longmapsto & |F|(Y, B) \end{array}\end{array}$$

Equivalently,

**5.** *An anafunctor as in* $1^*.$ *is saturated iff the induced functor* $|F|^* \longrightarrow \mathbf{X}^* \times \mathbf{A}^*$ *is a discrete opfibration.*

A *natural transformation* $h: F \longrightarrow G$ of anafunctors $\mathbf{X} \overset{F}{\underset{G}{\rightrightarrows}} \mathbf{A}$ is given by

**6.(i)** a family $\langle h_{s,\,t}: F_s X \to G_t X \rangle_{X \in \mathbf{X},\, s \in |F| X,\, t \in |G| X}$ of arrows in $\mathbf{A}$ such that

**6.(ii)** (*naturality*) for every $f: X \longrightarrow Y$ in $\mathbf{X}$, and for every $s \in |F| X$, $t \in |G| X$, $u \in |F| Y$, $v \in |G| Y$, the square

$$\begin{array}{ccc} F_s X & \xrightarrow{F_{s,\,u}(f)} & F_u Y \\ {\scriptstyle h_{s,\,t}}\downarrow & & \downarrow {\scriptstyle h_{u,\,v}} \\ G_t X & \xrightarrow[G_{t,\,v}(f)]{} & G_v Y \end{array} \tag{3}$$

commutes .

An equivalent definition is this. Given anafunctors $(\mathbf{X} \xleftarrow{F_0} |F| \xrightarrow{F} \mathbf{A})$, $(\mathbf{X} \xleftarrow{G_0} |G| \xrightarrow{G} \mathbf{A})$ in the style of $1^*.$, a natural transformation from $F$ to $G$ is the same as a natural transformation $h$ in the usual sense as in the following diagram:

$$
\begin{tikzcd}
 & {|\hat{F}|} \arrow[dl, "F_0"'] \arrow[drr, "F"] & & \\
\boldsymbol{X} & {|\hat{F}| \times_{\boldsymbol{X}} |\hat{G}|} \arrow[l] \arrow[u] \arrow[d] \arrow[rr, shift left] \arrow[rr, shift right] \arrow[rr, phantom, "\downarrow h"] & & \boldsymbol{A} \\
 & {|\hat{G}|} \arrow[ul, "G_0"] \arrow[urr, "G"'] & &
\end{tikzcd}
$$

Continuing with the notation of 6., note that if $s, u \in |F|X$, $t, v \in |G|X$, then $h_{u, v}$ is determined by $h_{s, t}$; this is because of the commmutativity of

$$
\begin{tikzcd}
F_s X \arrow[r, "F_{s,u}(1_X)", "\cong"'] \arrow[d, "h_{s,t}"'] & F_u X \arrow[d, "h_{u,v}"] \\
G_t X \arrow[r, "\cong", "G_{t,v}(1_X)"'] & G_v X
\end{tikzcd} \quad . \tag{4}
$$

Suppose we have a family $\langle (s_i \in |F| X_i, t_i \in |G| X_i) \rangle_{i \in I}$ such that for all $X \in \boldsymbol{X}$, $X = X_i$ for some $i \in I$. Suppose we have $\langle h_i : F_{s_i}(X_i) \to G_{t_i}(X_i) \rangle_{i \in I}$ such that the naturality condition (3) holds for these data, that is,

$$
\begin{tikzcd}
F_{s_i} X_i \arrow[r, "F_{s_i, s_j}(f)"] \arrow[d, "h_i"'] \arrow[dr, phantom, "\circ"] & F_{s_j} X_j \arrow[d, "h_j"] \\
G_{s_i} X_i \arrow[r, "G_{t_i, t_j}(f)"'] & G_{t_j} X_j
\end{tikzcd} \tag{3'}
$$

for any $i, j \in I$, and $f : X_i \to X_j$.

**7.** *Under the stated conditions, there is a unique* $h : F \to G$ *such that* $h_{s_i, t_i} = h_i$ *for all* $i \in I$ *;*

the rest of the data for $h$ are determined by appropriate instances of the diagram (4).

For any anafunctor $\boldsymbol{X}\xrightarrow{F}\boldsymbol{A}$, we have the *identity* natural transformation $1_F : F\to F$, defined by $(1_F)_{s,\,t} \overset{=}{\text{def}} F_{s,\,t}(1_X):F_sX\longrightarrow F_tX$. Naturality of $1_F$ is a consequence of 1.(v). As a consequence of 7., $h:F\to F$ is equal to $1_F$ iff $h_{s,\,s}=1_{F_sX}$ for all $X\in\boldsymbol{X}$, $s\in|F|X$.

*Composition* of $k\circ h:F\longrightarrow H$ of $h$, $k$ in $\boldsymbol{X}\begin{smallmatrix}\xrightarrow{F}\\ \downarrow h \quad G \\ \longrightarrow \\ \downarrow k \\ \xrightarrow[H]{}\end{smallmatrix}\boldsymbol{A}$ is defined in the expected manner: for $s\in|F|X$, $u\in|H|X$, $(k\circ h)_{s,\,u}:F_sX\longrightarrow H_uX$ is the composite of $F_sX\xrightarrow{h_{s,\,t}}G_tX\xrightarrow{k_{t,\,u}}H_uX$, with any $t\in|G|X$;

**3.(iii)** $(k\circ h)_{s,\,u} \overset{=}{\text{def}} k_{t,\,u}\circ h_{s,\,t}$;

for one thing, such $t$ exists; for another, with arbitrary $t,\,t'\in|G|X$, the commutative diagram

$$\begin{tikzcd}
F_sX \arrow[r, "1_{F_sX}\,=\,F_{s,\,s}(1_X)"] \arrow[d, "h_{s,\,t}"'] & F_sX \arrow[d, "h_{s,\,t'}"] \\
G_tX \arrow[r, "G_{t,\,t'}(1_X)", "\cong"'] \arrow[d, "k_{t,\,u}"'] & G_{t'}X \arrow[d, "k_{t',\,u}"] \\
H_uX \arrow[r, "1_{H_uX}\,=\,H_{u,\,u}(1_X)"'] & H_uX
\end{tikzcd}$$

shows that $(k\circ h)_{s,\,u}$ is well-defined (independent of the choice of $t$). The naturality (3.(ii)) of $k\circ h$ so defined is seen immediately; and so are the associativity of the composition of natural transformations, and the identity character of the identity natural transformations.

**2. Example** (continued). $Q:\boldsymbol{A}\times\boldsymbol{A}\xrightarrow{a}\boldsymbol{A}$ qualifies as *a* product-anafunctor if, for any $A,\,B\in\boldsymbol{A}$, there is a mapping associating with any $s\in|Q|((A,\,B))$ a product diagram

$$
\begin{tikzcd}
 & Q_s(A,B) \arrow[dl, "\pi_0^s"'] \arrow[dr, "\pi_1^s"] & \\
A & & B
\end{tikzcd}
$$

such that, for any $s \in |Q|((A,B))$, $t \in |Q|((A',B'))$, $a:A \to A'$, $b:B \to B'$, we have that

$$
\begin{tikzcd}
 & Q_s(A,B) \arrow[dl, "\pi_0^s"'] \arrow[dr, "\pi_1^s"] \arrow[ddd, "Q_{st}(a,b)"'] & \\
A \arrow[d, "a"'] & & B \arrow[d, "b"] \\
A' & & B' \\
 & Q_t(A',B') \arrow[ul, "\pi_0^t"] \arrow[ur, "\pi_1^t"'] &
\end{tikzcd}
$$

commutes. Certainly, any product-functor, making a choice of each product, will, as an anafunctor, satisfy the stated condition. But notice that any such $Q$ is isomorphic to $P$: with $s \in |P|((A,B))$ as in (2), and $t \in |Q|((A,B))$, we can put $h_{s,t}:C \to Q_s(A,B)$ to be the unique isomorphism $i$ that makes

$$
\begin{tikzcd}
 & Q_s(A,B) \arrow[dl, "\pi_0^s"'] \arrow[dr, "\pi_1^s"] & \\
A & & B \\
 & C \arrow[ul, "\pi_0"] \arrow[ur, "\pi_1"'] \arrow[uu, "i"'] &
\end{tikzcd}
$$

commute; $h$ so defined is an isomorphism $P \xrightarrow{\cong} Q$ as it is easily seen. In particular, if the product-functor *exists*, it is isomorphic to the product-anafunctor (which always exists).

Given categories $\boldsymbol{A}$, $\boldsymbol{X}$, with $\boldsymbol{X}$ small, $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, $\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$ denote the categories of all *small* anafunctors, respectively *small* saturated anafunctors, $\boldsymbol{X} \to \boldsymbol{A}$; arrows are the natural transformations, with composition as given above; $\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$ is a full subcategory of $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$. When $\boldsymbol{A}$ and $\boldsymbol{X}$ are both small, we might still have anafunctors $\boldsymbol{X} \to \boldsymbol{A}$ that are not small; however, as we said above, all saturated ones are small, and thus belong to $\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$. We should point out that if $\boldsymbol{A}$ has an isomorphism class of objects which is not small (a very common occurrence), and $F:\boldsymbol{X} \to \boldsymbol{A}$ takes a value in such an isomorphism class, then $F$ cannot be saturated and small at the same time; the category $\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$ is of importance mainly when both $\boldsymbol{A}$ and $\boldsymbol{X}$ are small. Let us also point out that, for small $\boldsymbol{X}$ and $\boldsymbol{A}$, $\mathrm{Ana}(\boldsymbol{X}, \boldsymbol{A})$, and even $\mathrm{Sana}(\boldsymbol{X}, \boldsymbol{A})$, cannot be shown to be equivalent to a small category; however, a weak version of the Axiom of Choice will suffice for this last conclusion; see later.

For convenience of expression, we will talk about the metacategories $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$, $\mathrm{SANA}(\boldsymbol{X},\boldsymbol{A})$ of all anafunctors, resp. saturated anafunctors $\boldsymbol{X}\to\boldsymbol{A}$, with natural transformations as arrows. The notations $\mathrm{ANA}_{\mathrm{ls}}(\boldsymbol{X},\boldsymbol{A})$, $\mathrm{SANA}_{\mathrm{ls}}(\boldsymbol{X},\boldsymbol{A})$, $\mathrm{ANA}_{\mathrm{ws}}(\boldsymbol{X},\boldsymbol{A})$, $\mathrm{SANA}_{\mathrm{ws}}(\boldsymbol{X},\boldsymbol{A})$, referring to "locally small", resp. "weakly small" anafunctors, are self-explanatory. The latter are full subcategories of $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$.

Recall the identification of any functor $G:\boldsymbol{X}\longrightarrow\boldsymbol{A}$ with an anafunctor; the latter is obviously weakly small. This identification extends to natural transformations, and we have a fully faithful functor $j=j_{\boldsymbol{X},\boldsymbol{A}}:\mathrm{FUN}(\boldsymbol{X}, \boldsymbol{A})\longrightarrow\mathrm{ANA}_{\mathrm{ws}}(\boldsymbol{X}, \boldsymbol{A})$, to which we will refer as an *inclusion*.

It is easily seen that if $h:F\to G$ is a natural transformation, then $h$ is an isomorphism in $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ iff each component $h_{s,\,t}$ is an isomorphism (in $\boldsymbol{A}$) (for $h^{-1}$ defined by $(h^{-1})_{t,\,s} = (h_{s,\,t})^{-1}$, we get $h^{-1}\circ h = 1_F$, $h\circ h^{-1} = 1_G$ because the $s$, $s$-components of both composites are identities) .

Given anafunctors $\boldsymbol{X}\underset{G}{\overset{F}{\rightrightarrows}}\boldsymbol{A}$, a *renaming transformation* $\bar{h}:F\xrightarrow{\equiv}G$ is a system $\bar{h} = \langle h[X, A]\rangle_{X\in\mathrm{Ob}(\boldsymbol{X}),\, A\in\mathrm{Ob}(\boldsymbol{A})}$ of bijections $h[X, A]=(s\mapsto\bar{s}):|F|(X, A)\xrightarrow{\cong}|G|(X, A)$ preserving the effect of the anafunctors $F$, $G$ on arrows: $F_{s,\,s'}(f) = G_{\bar{s},\,\bar{s}'}(f)$ whenever $f:X\to X'$ is an arrow in $\boldsymbol{X}$, $A\in\boldsymbol{A}$, $s\in|F|(X, A)$, $s'\in|F|(X', A)$. Continuing the above notation,

**7'.** *Every renaming transformation induces a natural isomorphism*

$h:F\xrightarrow{\cong}G$ for which $h_{s,\,\bar{s}} = 1_A$ ( $s\in|F|(X, A)$ ); condition (3') holds because of the assumption on effect on arrows (in general, $h_{s,\,t} = G_{\bar{s},\,t}(1_X)$ ( $s\in|F|X$, $t\in|G|X$ )). We now will see that for saturated anafunctors, natural isomorphisms and renaming transformations are in a bijective correspondence.

Suppose that $F, G \in \mathrm{SANA}(\boldsymbol{X},\boldsymbol{A})$, and $h: F \xrightarrow{\cong} G$. Let us fix $X \in \boldsymbol{X}$. Note that the isomorphism $h$ in particular ensures that any possible values $A, B$ of $F$, resp. $G$, at $X$ are isomorphic; hence, the possible values of $F$ and those of $G$ at $X$ are the same. Let $A$ be any common possible value at $X$. I claim that

**8.** *for any* $s \in |F|(X, A)$, *there is a unique* $t \in |G|(X, A)$ *such that* $h_{s,\,t} = 1_A$.

Indeed, let $s_0 \in |F|(X, A)$, $t_0 \in |G|(X, A)$, and consider, with any $s \in |F|(X, A)$ and $t \in |G|(X, A)$, the following commutative diagram of isomorphisms:

$$\begin{tikzcd} A \arrow[r, "h_{s_0,\, t_0}"] \arrow[d, "F_{s_0,\, s}(1_X)"'] \arrow[dr, phantom, "\circ"] & A \arrow[d, "G_{t_0,\, t}(1_X)"] \\ A \arrow[r, "h_{s,\, t}"'] & A \end{tikzcd}$$

This implies that $h_{s,\,t} = 1_A$ iff

$$\begin{tikzcd} A \arrow[r, "h_{s_0,\, t_0}"] \arrow[d, "F_{s_0,\, s}(1_X)"'] & A \arrow[d, "G_{t_0,\, t}(1_X)"] \\ A \arrow[r, "1_A"'] & A \end{tikzcd}$$

commutes; the last condition determines $G_{t_0,\, t}(1_X)$ in terms of ( $s_0$, $t_0$ and) $s$; by unique transfer (1.(vi)), there is a unique $t$ with this property.

In this argument, we used that $G$ was saturated; using also that $F$ is so, we get

**9.** *For* $h:F\xrightarrow{\cong}G$ *in* $\mathrm{SANA}(\boldsymbol{X},\boldsymbol{A})$ , $X\in\boldsymbol{X}$ , $A\in\boldsymbol{A}$ , *the condition* $h_{s,\,\bar{s}}=1_A$ *for* $s\in|F|(X,A)$ , $\bar{s}\in|G|(X,A)$ *establishes a bijection* $(s\mapsto\bar{s}):|F|(X,A)\xrightarrow{\cong}|G|(X,A)$ *for which* $F_{s,\,t}(f)=G_{\bar{s},\,\bar{t}}(f)$ *holds* for all $f:X\to Y$ , $s\in|F|X$ , $t\in|F|Y$ .

Therefore, by 7'. also, we have

**9'**. *For* $F,\,G\in\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ , *the natural isomorphisms* $h:F\xrightarrow{\cong}G$ *are in a bijective correspondence with renaming transformations* $\bar{h}:F\xrightarrow{\equiv}G$ .

Let us emphasize (in view of the lack of the Axiom of Choice) that a functor $\Phi:\boldsymbol{C}\longrightarrow\boldsymbol{D}$ is an *equivalence of* (*meta*)*categories* if there exist a functor $\Psi:\boldsymbol{D}\longrightarrow\boldsymbol{C}$ and natural isomorphisms $\alpha:1_{\boldsymbol{C}}\xrightarrow{\cong}\Psi\Phi$ , $\beta:1_{\boldsymbol{D}}\xrightarrow{\cong}\Phi\Psi$ . Note that if the functor $\Phi:\boldsymbol{C}\longrightarrow\boldsymbol{D}$ is full and faithful, and there exists a function $\Psi:\mathrm{Ob}(\boldsymbol{D})\longrightarrow\mathrm{Ob}(\boldsymbol{C})$ together with a function $D\mapsto\beta_D$ assigning an isomorphism $\beta_D:D\xrightarrow{\cong}\Phi\Psi D$ to each object $D\in\boldsymbol{D}$ (for which we say that $\Phi$ is *uniformly essentially surjective*), then $\Phi$ is an equivalence; in fact, there is a unique way of making $\Psi$ into a functor $\Psi:\boldsymbol{D}\longrightarrow\boldsymbol{C}$ and defining the isomorphism $\alpha:1_{\boldsymbol{C}}\xrightarrow{\cong}\Psi\Phi$ so that $\langle\beta_D\rangle_D$ becomes an isomorphism $\beta:1_{\boldsymbol{D}}\xrightarrow{\cong}\Phi\Psi$ , and $\alpha\Psi=\Psi\beta$ , $\beta\Phi=\Phi\alpha$ .

**10.** *Let* $\boldsymbol{X}$ , $\boldsymbol{A}$ *be small categories. The inclusion* $i:\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})\longrightarrow\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of categories.*

**Proof.** Let $F\in\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ ; we define $F^{\#}\in\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$ , called the *saturation* of $F$ , as follows. For $X\in\boldsymbol{X}$ , $A\in\boldsymbol{A}$ , we let $S_{X,\,A}$ be the set of all pairs $(s\in|F|X,\,\mu:F_sX\xrightarrow{\cong}A)$ . Let $\sim$ be the relation on $S_{X,\,A}$ defined by

$$(s,\mu)\sim(s',\mu') \;\underset{\text{def}}{\Longleftrightarrow}\; \begin{tikzcd} F_sX \arrow[rr, "F_{s,s'}(1_X)"] \arrow[dr, "\mu"', "\cong"] & & F_{s'}X \arrow[dl, "\mu'", "\cong"'] \\ & A & \end{tikzcd} \quad \text{commutes.}$$

It is immediately seen that $\sim$ is an equivalence relation. We put $|F^{\#}|(X,A)$ to be $S_{X,A}/\sim$, the set of equivalence-classes $[s,\mu]$ of elements $(s,\mu)$ of $S_{X,A}$. Given $a=[s,\mu]\in|F^{\#}|X$, $b=[t,\nu]\in|F^{\#}|Y$ and $f:X\to Y$, $F^{\#}_{a,b}(f)$ is defined so as to make the outside rectangle in the diagram

$$\begin{tikzcd} F^{\#}_aX \arrow[rrr, "F^{\#}_{a,b}(f)"] & & & F^{\#}_bY \\ & F_{s'}X \arrow[ul, "\mu'"] \arrow[r, "F_{s',t'}(f)"] & F_{t'}Y \arrow[ur, "\nu'"] & \\ F_sX \arrow[uu, "\mu", "\cong"'] \arrow[ur, "F_{s,s'}(1_X)"'] \arrow[rrr, "F_{s,t}(f)"'] & & & F_tY \arrow[uu, "\nu"', "\cong"] \arrow[ul, "F_{t,t'}(1_Y)"] \end{tikzcd}$$

commute. The commutativity of the rest of the diagram shows that the definition is independent of the choice of the representatives. It is easy to see that 1.(iv),(v) hold for $F^{\#}$ so defined. To see 1.(vi) for $F^{\#}$, let $a=[s,\mu]\in|F^{\#}|X$, and let $\rho:F^{\#}_aX\xrightarrow{\cong}B$; we want that there is unique $b=[t,\nu]$ with $B=F^{\#}_bX$ and $F^{\#}_{a,b}(1_X)=\rho$; this means that

$$\begin{tikzcd} F_sX \arrow[r, "\mu"] \arrow[d, "F_{s,t}(1_X)"'] & F^{\#}_aX \arrow[d, "\rho"] \\ F_tX \arrow[r, "\nu"'] & B \end{tikzcd} \tag{5}$$

should commute; we can take $b=[s,\rho\circ\mu]$ to satisfy this; clearly, the commutativity of (5) implies that $(t,\nu)\sim(s,\rho\circ\mu)$, which shows the required uniqueness.

We give $\eta_F:F\xrightarrow{\cong}F^{\#}$ ( $F\in\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ ) as an application of 7. We let $I=|F|$, $\langle(s_i,t_i)\rangle_{i\in I}=\langle(s,[s,1_{\tau(s)}])\rangle_{s\in|F|}$, with $X_s=\sigma(s)$, and abbreviating $[s,1_{\tau(s)}]$ as $\bar{s}$, we let, for $s\in|F|X$, $(\eta_F)_s:F_s(X)\longrightarrow F^{\#}_{\bar{s}}(X)$ be the identity

$1_{F_s X}$. It is immediate that $\eta_F$ is a natural transformation (by 7.; (3') now holds), and that it is an isomorphism. This completes the proof of 10.

Let us note the effect of the saturation functor $(\ )^{\#}:\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})\longrightarrow\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$ on arrows. Given $h:F\longrightarrow G$ in $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, $a=[s,\mu]\in F^{\#}X$, $b=[t,\nu]\in G^{\#}X$, $h^{\#}_{a,b}$ is defined so as to make the outside rectangle in

$$\begin{tikzcd}
F^{\#}_a X \arrow[rrr, "h^{\#}_{a,b}"] & & & G^{\#}_b X \\
 & F_{s'}X \arrow[ul, "\mu'"'] \arrow[r, "h_{s',t'}"] & F_{t'}Y \arrow[ur, "\nu'"] & \\
F_s X \arrow[uu, "\mu"] \arrow[uu, phantom, "\cong"', shift right=4] \arrow[ur, "F_{s,s'}(1_X)"'] \arrow[rrr, "h_{s,t}"'] & & & G_t X \arrow[uu, "\nu"'] \arrow[uu, phantom, "\cong", shift left=4] \arrow[ul, "F_{t,t'}(1_Y)"]
\end{tikzcd}$$

commute; the rest of the diagram shows that the definition of $h^{\#}_{a,b}$ is independent of the choice of the representatives; it is easy to see that $h^{\#}$ so defined is a natural transformation $F^{\#}\rightarrow G^{\#}$. Further, it is easily seen that $(\ )^{\#}$ so defined is a functor. The functor $(\ )^{\#}$ is the same as the one obtained from $i$ and $\langle\eta_F\rangle_F$ in the remark before 10., denoted by $\Psi$ there.

Given a weakly small anafunctor $F:\boldsymbol{X}\longrightarrow\boldsymbol{A}$, *using* the Global Axiom of Choice (GAC), the existence of a class-function that picks an element of every inhabited set, we let $(X\in\boldsymbol{X})\mapsto s_X\in|F|X$ be a choice-function, and we consider the functor $F^{!}:\boldsymbol{X}\longrightarrow\boldsymbol{A}$ for which $F^{!}(X)=F_{s_X}(X)$, $F^{!}(f:X\rightarrow Y)\overset{\mathrm{def}}{=}F_{s_X,s_Y}(f):F^{!}X\longrightarrow F^{!}Y$ (it is immediate that $F^{!}$ is a functor). We also have, with any $F$ as above, a natural isomorphism $\alpha:F\longrightarrow jF^{!}$ (with $j$ the inclusion of functors in anafunctors) defined by

$$(\alpha_{s,X}:F_sX\rightarrow F_XX)\overset{\mathrm{def}}{=}(F_{s,s_X}(1_X):F_sX\longrightarrow F_{s_X}X)\,.$$

Making the choices involved simultaneously for all $F\in\mathrm{ANA}_{\mathrm{WS}}(\boldsymbol{X},\boldsymbol{A})$, we obtain, *using* the

GAC , that

**11**$^{(\mathrm{GAC})}$. *The inclusion* $\mathtt{FUN}(\boldsymbol{X},\boldsymbol{A})\longrightarrow\mathtt{ANA}_{\mathtt{WS}}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of metacategories*; *when* $\boldsymbol{X}$ *is small, the inclusion* $\mathtt{Fun}(\boldsymbol{X},\boldsymbol{A})\longrightarrow\mathtt{Ana}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of categories.*

11. is reassuring since it says that we have not strayed from the notion of functor too far.

It should be noted that, without any choice,

**11'.** *Any small anafunctor into* $\mathtt{Set}$ *is isomorphic to a functor; for any small category* $\boldsymbol{X}$ *, the inclusion* $\mathtt{Fun}(\boldsymbol{X},\mathtt{Set})\longrightarrow\mathtt{Ana}(\boldsymbol{X},\mathtt{Set})$ *is an equivalence of categories.*

**Proof.** Let $F:\boldsymbol{X}\longrightarrow\mathtt{Set}$ be a small anafunctor. An *element of* $F$ *at* $X\in\boldsymbol{X}$ is a family $x=\langle x_s\rangle_{s\in|F|X}$ such that $x_s\in F_sX$ , and $(F_{s,t}(1_X))(x_s)=x_t$ for $s,t\in|F|X$ . Clearly, any component $x_s$ of $x$ determines the whole of $x$ , and in fact, any pair $(s\in|F|X,a\in F_sX)$ determines a unique element $x$ at $X$ for which $x_s=a$ ; let us denote $x$ by $[s,a]$ .

Given $F$ , we define the functor $\hat{F}:\boldsymbol{X}\longrightarrow\mathtt{Set}$ as follows. We put $\hat{F}(X)$ equal to the set of all elements of $F$ at $X$ . We define, for $f:X\to Y$ , the function $\hat{F}(f):\hat{F}(X)\to\hat{F}(Y)$ by putting $\hat{F}(f)(x)$ equal to the unique element $y$ at $Y$ for which $y_u=F_{s,u}(f)(x_s)$ for any (equivalently, for some) pair $(s\in|F|X,u\in|F|Y)$ . It is easily seen that $\hat{F}$ is well-defined as a functor $\hat{F}:\boldsymbol{X}\longrightarrow\mathtt{Set}$ by these stipulations. We have the natural isomorphism $\alpha_F:F\xrightarrow{\cong}\hat{F}$ whose components $(\alpha_F)_{s,X}:F_sX\to\hat{F}X$ are given by $(\alpha_F)_{s,X}(a)=[s,a]$ . This completes the proof.

Many concrete categories (categories of algebras, of topological spaces, *etc.*) that have a faithful forgetful functor to $\mathrm{Set}$ share the property of $\mathrm{Set}$ stated in 11'.; I do not see how to make a general-enough statement of this state of affairs.

Another, rather obvious, case of this situation is in the next statement.

**11".** *The anafunctor* $F$ *is isomorphic to an ordinary functor when the domain category of* $F$ *has finitely many objects.*

(By "$S$ is finite", we mean "there are a natural number $n$ and a surjection $\{i:\, i<n\} \twoheadrightarrow S$ ".) Note, however, that we cannot say that the inclusion $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ is an equivalence even when $\boldsymbol{X}$ is $\mathbf{1}$, the terminal category.

We turn to the composition of anafunctors. Let $\boldsymbol{X} \xrightarrow{F} \boldsymbol{A} \xrightarrow{G} \boldsymbol{M}$ be anafunctors. There is a natural composition $G\circ F:\boldsymbol{X}\longrightarrow\boldsymbol{A}$, also written just $GF$, an anafunctor, defined as follows. For $X\in\boldsymbol{X}$, we let $|GF|X$ be the class of all pairs

$$a = (s\in |F|X,\ t\in |G|(F_s X))$$

(in other words,

$$|GF|X \overset{=}{\mathrm{def}} \coprod_{s\in |F|X} |G|(F_s X)\ ) \tag{6}$$

and for $a$ as displayed, $(GF)_a \overset{=}{\mathrm{def}} G_t(F_s(X))$. Note that if also $M\in\boldsymbol{M}$,

$$|GF|(X,M) \cong \coprod_{A\in\boldsymbol{A}} |F|(X,A)\times |G|(A,M)\ .$$

For the action of $GF$ on arrows, with $a$ as above, and with $b = (u\in |F|Y,\ v\in |G|(F_u X))$, and with $f:X\longrightarrow Y$, we put $(GF)_{a,b}(f) \overset{=}{\mathrm{def}} G_{t,v}(F_{s,u}(f)$. It is immediate that $GF$ is a anafunctor.

It is immediate that the composition of weakly small anafunctors is weakly small.

If $F$ and $G$ are given by the spans $(\mathbf{X}\longleftarrow |F| \longrightarrow\mathbf{A})$, $(\mathbf{A}\longleftarrow |G| \longrightarrow\mathbf{M})$, then the composite $GF$ is given by the "composite span" $(\mathbf{X}\longleftarrow |F|\times_{\mathbf{A}}|G| \longrightarrow\mathbf{M})$.

We can extend composition to a functor

$$\mathrm{ANA}(\mathbf{X},\mathbf{A})\times\mathrm{ANA}(\mathbf{A},\mathbf{M}) \xrightarrow{\circ} \mathrm{ANA}(\mathbf{X},\mathbf{M}) \tag{7}$$

in a natural way. With data as in

$$\begin{tikzcd} \mathbf{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\downarrow h"] & \mathbf{A} \arrow[r, bend left, "I"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\downarrow k"] & \mathbf{M} \end{tikzcd} \tag{8}$$

first we define $\circ(h, 1_I)$, denoted $Ih$, by

$$(Ih)_{a,b} = I_{t,v}(h_{s,u});$$

here, $a=(s\in |F|X, t\in |I|(F_sX))$, $b=(u\in |G|X, v\in |I|(F_uX))$;

the naturality of $Ih$ is immediate.

In defining $\circ(1_F, k)$, denoted $kF$, we make use of the fact that, to specify a natural transformation of anafunctors, it suffices to specify "enough" components of it, with the appropriate naturality conditions satisfied (see 7.). Accordingly, let
$a=(s\in |F|X, t\in |I|(F_sX))$, $b=(s\in |F|X, u\in |J|(F_sX))$; we let
$(kF)_{a,b}:(IF)_a(X)\longrightarrow(JF)_b(X)$ be
$(kF)_{a,b} \overset{\mathrm{def}}{=} k_{t,u}:I_tF_s(X)\longrightarrow J_uF_s(X)$; the (needed partial) naturality of $kF$ is immediate.

Next, we need to verify that thus we have defined functors

$$(\ )\circ I:\mathrm{ANA}(\mathbf{X},\mathbf{A})\longrightarrow\mathrm{ANA}(\mathbf{X},\mathbf{M}),$$

$$F\circ(\ ):\mathrm{ANA}(\mathbf{A},\mathbf{M})\longrightarrow\mathrm{ANA}(\mathbf{X},\mathbf{M});$$

we leave the task to the reader.

Finally, we need that

$$\begin{tikzcd} I\circ F \arrow[r, "Ih"] \arrow[d, "kF"'] & I\circ G \arrow[d, "kG"] \\ J\circ F \arrow[r, "Jh"'] & J\circ G \end{tikzcd}$$

commutes. With evaluating $I\circ F$ at $(s\in|F|X, t\in|I|(F_sX))$, $J\circ G$ at $(u\in|G|X, v\in|J|(G_uX))$, $I\circ G$ at $(u\in|G|X, w\in|I|(G_uX))$, and $J\circ F$ at $(s\in|F|X, r\in|J|(F_sX))$, the diagram becomes

$$\begin{tikzcd} I_tF_sX \arrow[r, "I_{t,w}(h_{s,u})"] \arrow[d, "k_{t,r}"'] & I_wG_uX \arrow[d, "k_{w,v}"] \\ J_rF_sX \arrow[r, "J_{r,v}(h_{s,u})"'] & J_vG_uX \end{tikzcd} \quad ,$$

whose commutativity is an instance of the naturality of $k$. By 7. again, this suffices.

It is well-known (Prop. 1, II.3, p. 37 in [CWM]) that what we did above determines uniquely the functor (7).

It is clear that, for $\boldsymbol{X}$ and $\boldsymbol{A}$ small, (7) restricts to a composition-functor

$$\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})\times\mathrm{Ana}(\boldsymbol{A},\boldsymbol{M}) \xrightarrow{\circ} \mathrm{Ana}(\boldsymbol{X},\boldsymbol{M}) \qquad (7')$$

Let us turn to the question of associativity of composition of functors. With anafunctors

$$\boldsymbol{X} \xrightarrow{F} \boldsymbol{A} \xrightarrow{G} \boldsymbol{M} \xrightarrow{H} \boldsymbol{S},$$

we find the associativity isomorphism

$$\alpha = \alpha_{F,G,H} : H(GF) \xrightarrow{\cong} (HG)F$$

given (see 7'.) by the renaming transformation $\bar{\alpha}$ for which

$$\alpha[X, S] : ((s, t), u) \longmapsto (s, (t, u))$$

whenever $X \in \mathbf{X}$, $s \in |F| X$, $t \in |G| (F_s X)$, $u \in |H| (G_t F_s(X))$, $S = H_u G_t F_s(X)$. It is easy to see that $\alpha_{F, G, H}$ is natural in each of $F$, $G$ and $H$, and that the pentagonal associativity coherence diagram ((1.1)(A.C.) in [Be1], pp. 5 and 6) commutes. With the identity functor $1_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$ as an anafunctor, we have the left and right identity isomorphisms

$$\lambda_F : 1_{\mathbf{A}} F \xrightarrow{\cong} F \ , \quad \rho_F : I 1_{\mathbf{A}} \xrightarrow{\cong} I$$

(see (8)) defined by $(\lambda_F)_{((s, X), s), X} = 1_{F_s X}$ ($s \in |F| X$), and similarly for $\rho_F$. Both $\lambda_F$ and $\rho_F$ are natural in $F$, and they satisfy identity coherence ((1.1)(I.C.) *loc.cit.*).

We have the ingredients of a metabicategory (see *loc.cit.*).

**12. Conclusion**. *Categories, anafunctors between them, and natural transformations between the latter form, with the given notions of composition, a meta-bicategory* $\mathtt{ANACAT}$. *The identification of ordinary functors with anafunctors provides an inclusion* $i : \mathtt{CAT} \longrightarrow \mathtt{ANACAT}$ *(* $\mathtt{CAT}$ *is the meta 2-category of categories, functors and natural transformations), which is the identity on objects, and locally fully faithful.*

*We also have the bicategory* $\mathtt{AnaCat}$ *of small categories, small anafunctors between them, and all natural transformations between the latter. The 2-category* $\mathtt{Cat}$ *of small categories has a locally fully faithful inclusion into* $\mathtt{AnaCat}$*, which is an equivalence of bicategories provided the Axiom of Choice holds.*

G. M. Kelly gave us once the healthy advice to use simple terminology in higher dimensional category theory. For instance, "functor" of bicategories should mean "homomorphism of bicategories"; a functor between bicategories cannot reasonably mean anything but a mapping that respects the whole bicategory structure and not just the reduct to the category structure. Similarly, "product" in a bicategory should mean what is usually called "*bi*product". Also, I say "equivalence of bicategories" for "biequivalence". (As a reminder, I note that by an *equivalence of bicategories* $\mathcal{S}$ and $\mathcal{A}$, I mean a pair of functors $\mathcal{S}\underset{G}{\overset{F}{\rightleftarrows}}\mathcal{A}$ such that $GF\simeq 1_{\mathcal{S}}$, $FG\simeq 1_{\mathcal{A}}$, the latter equivalences meant in the metabicategories of endofunctors of $\mathcal{S}$, $\mathcal{A}$, respectively. As usual, we say of a single functor $F:\mathcal{S}\longrightarrow\mathcal{A}$ that it is an equivalence if it can be expanded with further data to form an equivalence). Maybe I am carrying Kelly's advice farther than he intended; I hope no confusion will arise.

Small categories with saturated anafunctors between them also form a bicategory named $\mathtt{SanaCat}$, which is equivalent to $\mathtt{AnaCat}$. This is a consequence of 10., together with the fact that, in the proof of 10., the isomorphisms $\eta_F$ are obtained uniformly from $F$ not just within a given $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, but also uniformly in the variables $\boldsymbol{X}$, $\boldsymbol{A}$.

In some detail, $\mathtt{SanaCat}$ has the following structure. With reference to the saturation-functor

$$(\ )^{\#}=(\ )^{\#}_{\boldsymbol{X},\boldsymbol{M}}:\mathrm{Ana}(\boldsymbol{X},\boldsymbol{M})\longrightarrow\mathrm{Sana}(\boldsymbol{X},\boldsymbol{M})$$

(see 10.), a composition-functor in $\mathtt{SanaCat}$,

$$\circ^{\#}=\circ^{\#}_{\boldsymbol{X},\boldsymbol{A},\boldsymbol{M}}:\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})\times\mathrm{Sana}(\boldsymbol{A},\boldsymbol{M})\longrightarrow\mathrm{Sana}(\boldsymbol{X},\boldsymbol{M})\ ,$$

is defined by $G\circ^{\#}F=(G\circ F)^{\#}$, and correspondingly for natural transformations. The associativity isomorphisms

$$\alpha^{\#}_{F,\,G,\,H}:H\circ^{\#}(G\circ^{\#}F)\longrightarrow(H\circ^{\#}G)\circ^{\#}F$$

are determined so as to make

$$\begin{tikzcd}
H\circ^{\#}(G\circ^{\#}F) \arrow[rr, "\alpha^{\#}_{F,G,H}"] & & (H\circ^{\#}G)\circ^{\#}F \\
H\circ(G\circ^{\#}F) \arrow[u, "\eta"] & & (H\circ^{\#}G)\circ F \arrow[u, "\eta"'] \\
H\circ(G\circ F) \arrow[u, "H\circ\eta"] \arrow[rr, "\alpha_{F,G,H}"'] & & (H\circ G)\circ F \arrow[u, "\eta\circ F"']
\end{tikzcd}$$

commute.

Using 10., we can see that

**12.'** *The inclusion mapping* $\mathtt{SanaCat}\longrightarrow\mathtt{AnaCat}$ *is an equivalence of bicategories.*

It is more natural to make the totality of small categories, with saturated anafunctors between them, an *anabicategory* in which composition is an anafunctor; see §4.

*Terminal object* and *product* in a (meta-)bicategory are defined as expected by universal properties defining the result of the operation up to an equivalence rather than isomorphism. Placing ourselves in a fixed (meta)bicategory, we say that $A\xleftarrow{\pi}C\xrightarrow{\pi'}B$ is a *product diagram* if, for any object $D$, the functor

$$(\pi(-),\pi'(-)):\mathrm{Hom}(D,C)\longrightarrow\mathrm{Hom}(D,A)\times\mathrm{Hom}(D,B)$$
$$D\xrightarrow{f}C\quad\longmapsto\quad(\pi f,\pi' f)$$

is an equivalence of categories. As usual, $A\xleftarrow{\pi}A\times B\xrightarrow{\pi'}B$ denotes, ambiguously, a product diagram on $(A,B)$.

$T$ is a *terminal object* if, for any $A$, $\mathrm{Hom}(A,T)\longrightarrow\mathbf{1}$, with $\mathbf{1}$ the one-object, one-arrow category, is an equivalence of categories.

We say that a bicategory is *Cartesian* if it has a terminal object and binary products.

## 13. AnaCat *and* ANACAT *are Cartesian.*

In fact, the Cartesian structure in ANACAT ( AnaCat ) is computed as in CAT ( Cat ).

The Cartesian closed nature of Cat , the (2-)category of all small categories is a fundamental fact. What prevents AnaCat from being Cartesian closed is that, for $\mathbf{A}$ , $\mathbf{X}$ small categories, $\mathrm{Ana}(\mathbf{X},\mathbf{A})$ is not necessarily equivalent to a small category. In §5, we will see that a weak form of the Axiom of Choice will ensure this, and hence the Cartesian closed nature of AnaCat . Here, we give the relevant facts that hold without further set-theoretical hypotheses.

We first formulate a characterization of anafunctors of the form $F:\mathbf{X}\times\mathbf{M}\longrightarrow\mathbf{A}$ , ("bi-anafunctors") analogous to Prop. II.3.1 in [CWM]. Suppose we have

classes $|F|((X, M))$ $(X\in\mathbf{X}, M\in\mathbf{M})$ ,
objects $F_s(X, M)\in\mathbf{A}$ $(s\in|F|((X, M)))$,
arrows $F_{s,t}(f, M):F_s(X, M)\longrightarrow F_t(Y, M)$ ,
$F_{s,u}(X, g):F_s(X, M)\longrightarrow F_u(X, N)$ $(f:X\rightarrow Y,\ g:M\rightarrow N,\ s\in|F|((X, M)),$
$t\in|F|((Y, M)),\ u\in|F|((X, N)))$

such that

(i) for any $X\in\mathbf{X}$ , the data define an anafunctor $F_X=F(X, -):\mathbf{M}\longrightarrow\mathbf{A}$ ( $|F_X|M=|F|((X, M))$ , etc.), and similarly for $F(-, M):\mathbf{X}\longrightarrow\mathbf{A}$ ;

(ii) for any $f:X\rightarrow Y$ in $\mathbf{X}$ , $g:M\rightarrow N$ in $\mathbf{M}$ , and for all appropriate specifications, the diagram

$$\begin{tikzcd} F_s(X, M) \arrow[r, "F_{s,t}(X, g)"] \arrow[d, "F_{s,u}(f, M)"'] & F_t(X, N) \arrow[d, "F_{t,v}(f, N)"] \\ F_u(Y, M) \arrow[r, "F_{u,v}(Y, g)"'] & F_v(Y, N) \end{tikzcd}$$

commutes. Then we have a unique anafunctor $F:\mathbf{X}\times\mathbf{M}\longrightarrow\mathbf{A}$ having as sections $F(X, -)$ ,

$F(-, M)$ the given data.

I leave the verification to the reader.

Given categories $\boldsymbol{X}$ , $\boldsymbol{A}$ , we consider the metacategory $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ , and the *evaluation* anafunctor

$$e = \mathrm{e}_{\boldsymbol{X},\,\boldsymbol{A}} : \boldsymbol{X}\times\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \boldsymbol{A} \tag{8}$$

determined as follows. For $X\in\boldsymbol{X}$ , $F\in\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ ,

$$|e|((X, F)) \underset{\mathrm{def}}{=} |F|X ; \tag{8'}$$

for $s\in |F|X$ ,

$$e_s(X, F) \underset{\mathrm{def}}{=} F_s(X) ;$$

with also $u\in |F|Y$ , $f:X\to Y$ ,

$$e_{s,\,u}((f, F)) \underset{\mathrm{def}}{=} F_{s,\,u}(f) .$$

With $h:F\to G$ ( $\in \mathrm{Arr}(\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A}))$ ) , $t\in |G|X$ ,

$$e_{s,\,t}((X, h)) \underset{\mathrm{def}}{=} h_{s,\,t} ;$$

the diagram

$$\begin{tikzcd}
e_s(X, F) \arrow[r, "e_{s,\,t}(X,\,h)"] \arrow[d, "e_{s,\,u}(f,\,F)"'] & e_t(X, G) \arrow[d, "e_{t,\,v}(f,\,G)"] \\
e_u(Y, F) \arrow[r, "e_{u,\,v}(Y,\,h)"'] & e_v(Y, G)
\end{tikzcd}$$

is identical to

$$\begin{array}{ccc} F_s(X) & \xrightarrow{h_{s,t}} & G_t(X) \\ {\scriptstyle F_{s,u}(f)}\downarrow & & \downarrow{\scriptstyle G_{t,v}(f)} \\ F_u(Y) & \xrightarrow[h_{u,v}]{} & G_v(Y) \end{array} \qquad ,$$

which commutes by the naturality of $h$. This shows (by the characterization of "bi-anafunctors") that $e$ is an anafunctor.

Whereas $\mathrm{e}_{\boldsymbol{X},\boldsymbol{A}}$ in (8) is a metafunctor, for $\boldsymbol{X}$ small, its restriction

$$\mathrm{e}_{\boldsymbol{X},\boldsymbol{A}} : \boldsymbol{X}\times\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \boldsymbol{A} \tag{9}$$

to $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, the category of small anafunctors $\boldsymbol{X}\to\boldsymbol{A}$, is a functor (denoted by the same symbol as the metafunctor in (8)).

In propositions 14., 15., 16. and 17. below, $\boldsymbol{X}$, $\boldsymbol{Y}$ are small categories, $\boldsymbol{A}$ is an arbitrary category.

**14.** $e = \mathrm{e}_{\boldsymbol{X},\boldsymbol{A}}$ (see (9)) *induces an equivalence of categories*

$$\varphi \underset{\mathrm{def}}{=} e\circ(\boldsymbol{X}\times(-)) : \mathrm{Ana}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) \xrightarrow{\simeq} \mathrm{Ana}(\boldsymbol{X}\times\boldsymbol{Y},\boldsymbol{A}) .$$

**15.** *The inclusion*

$$i : \mathrm{Fun}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) \xrightarrow{\simeq} \mathrm{Ana}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$$

*is an equivalence of categories.*

Note that 15. implies that $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ shares the property of $\mathrm{Set}$ given in 11'.

**16.** *There is an isomorphism*

$$\psi : \mathrm{Fun}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) \xrightarrow{\cong} \mathrm{Ana}(\boldsymbol{X}\times\boldsymbol{Y},\boldsymbol{A})$$

*of categories for which* $\psi\cong i\circ\varphi$, with $i$ and $\varphi$ from 15. and 14.

**Proof of 14.**, **15. and 16.** The functors in these assertions form the diagram

$$\begin{tikzcd} \mathrm{Ana}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) \arrow[rr, "\varphi"] & & \mathrm{Ana}(\boldsymbol{X}\times\boldsymbol{Y},\boldsymbol{A}) \\ & \mathrm{Fun}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) \arrow[ul, "i"] \arrow[ur, "\psi"'] & \end{tikzcd} \tag{9}$$

We'll define $\psi$, show the properties given in 16., and show that $\varphi$ is full and faithful. Since $i$ is full and faithful, both assertions 14. and 15. will follow. We will have that, in (9), all three functors are equivalences of categories, one in fact is an isomorphism.

Given $H\in\mathrm{Ana}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$, $X\in\boldsymbol{X}$, $Y\in\boldsymbol{Y}$, we have

$$|e\circ(\boldsymbol{X}\times H)|((X,Y))=\{((X,a),s): a\in|H|Y,\ s\in|H_aY|X\}$$

(remember that $|e|((X,H_aY))=|H_aY|X$) and

$$(e\circ(\boldsymbol{X}\times H))_{((X,a),s)}(X,Y)\ =\ (H_aY)_sX\ .$$

Let also $K\in\mathrm{Ana}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$. A natural transformation $h:e\circ(\boldsymbol{X}\times H)\longrightarrow e\circ(\boldsymbol{X}\times K)$ has components

$$h_{((X,a),s),((X,b),t)}:(H_aY)_sX\longrightarrow(K_bY)_tX\ .$$

Starting with $h$, we define $j:H\longrightarrow K$ by specifying the natural transformation $j_{a,b}:H_aY\longrightarrow K_bY$ by making $(j_{a,b})_{s,t}:(H_aY)_sX\longrightarrow(K_bY)_tX$ equal to $h_{((X,a),s),((X,b),t)}$. This works, and $j$ is the unique natural transformation $H\to K$ mapped by the functor (9) to $h$; this amounts to the fact that $\varphi$ is fully faithful.

Given the small anafunctor $G:\boldsymbol{X}\times\boldsymbol{Y}\longrightarrow\boldsymbol{A}$, we define $H=\psi^{-1}(G)$, $H:\boldsymbol{Y}\longrightarrow\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ as follows. With $Y\in\boldsymbol{Y}$, $H(Y):\boldsymbol{X}\to\boldsymbol{A}$ is the (obviously small) anafunctor $G(-,Y)$, that is

$$|H(Y)|X=|G|((X,Y))\ ,\ \ (H(Y))_sX=G_s(X,Y)$$

and

$$(H(Y))_{s,\,t}(f) = G_{s,\,t}(f,\, Y) \quad (\, s\in |H(Y)|X\,,\;\; t\in |H(Y)|X'\,,\;\; f{:}X\to X'\,);$$

moreover, for $g{:}Y\to Y'$, $H(g){:}H(Y)\longrightarrow H(Y')$ is the natural transformation for which

$$(H(g))_{s,\,t} = G_{s,\,t}(X,\, g)\ .$$

Conversely, given any functor $H{:}\boldsymbol{Y}\longrightarrow \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, the listed equalities define a unique small $G{:}\boldsymbol{X}\times\boldsymbol{Y}\xrightarrow{\mathrm{a}}\boldsymbol{A}$ ; in other words, $\psi$ is a bijection of the object-classes of the two categories in 16. If $g{:}G\to F$, then $\psi^{-1}(g)=h{:}H\to K$ for $h$ defined by

$$(h_Y)_{s,\,t} = g_{s,\,t} \qquad\qquad (\, s\in |G|(X,\, Y)\,,\;\; t\in |F|(X,\, Y)\,)$$

(here, $G,\, F\in \mathrm{Ana}(\boldsymbol{X}\times\boldsymbol{Y},\boldsymbol{A})$, $H=\psi^{-1}(G)$, $K=\psi^{-1}(F)$ ), and the mapping $g\mapsto h$ is a bijection $\mathrm{Nat}(G,\, F)\xrightarrow{\cong}\mathrm{Nat}(\psi^{-1}G,\, \psi^{-1}F)$ . This defines the isomorphism $\psi$ of 16.

To show the isomorphism $\psi\cong i\circ\varphi$, for a functor $H{:}\boldsymbol{Y}\longrightarrow\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, and $G=\psi(H)$, we exhibit an isomorphism $\alpha_H{:}e\circ(\boldsymbol{X}\times H)\cong G$. Calculating $e\circ(\boldsymbol{X}\times H)$ in this case, we get

$$|e\circ(\boldsymbol{X}\times H)|\,((X,\, Y),\, A) = \{((X,\, Y),\, s){:}s\in |G|\,((X,\, Y),\, A)\}\ .$$

We can define the renaming transformation $\bar{\alpha}{:}e\circ(\boldsymbol{X}\times H)\xrightarrow{\equiv}G$ by defining

$$\begin{array}{llcl} \bar{\alpha}[(X,\, Y),\, A]\ : & |e\circ(\boldsymbol{X}\times H)|\,((X,\, Y),\, A) & \xrightarrow{\cong} & |G|\,((X,\, Y),\, A) \\ & ((X,\, Y),\, s) & \longmapsto & s\ . \end{array}$$

as

The corresponding natural isomorphism $\alpha_H{:}e\circ(\boldsymbol{X}\times H)\xrightarrow{\cong}G$ has

$$(\alpha_H)_{((X,\, Y),\, s),\, s} = 1_{G_s(X,\, Y)}\ . \tag{10}$$

We need to see that $\alpha_H$ is natural in $H\in\mathrm{Fun}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$. Because of (10), naturality means that for $H,\, K\in\mathrm{Fun}(\boldsymbol{Y},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$, $j{:}H\to K$, $h=\varphi(j)$, $\ell=\psi^{-1}(j)$, $s\in |H(Y)|X$, $t\in |K(Y)|X$, we have

$$h_{((X,Y),s),((X,Y),t)} = \ell_{s,t} \quad (\,:(HY)_s X \to (KY)_t X\,).$$

But this equality is true; both sides are equal to $(j_Y)_{s,t}$ .

This completes the proof.

We also arrive at the conclusion mentioned after 15.: if $K:\mathbf{Y}\longrightarrow\mathrm{Ana}(\mathbf{X},\mathbf{A})$ is an anafunctor, we have a functor $H:\mathbf{Y}\longrightarrow\mathrm{Ana}(\mathbf{X},\mathbf{A})$ isomorphic to it; $H$ is obtained from $G=e\circ(\mathbf{X}\times K)$ as above. In particular, the anafunctor $H(Y):\mathbf{X}\to\mathbf{A}$ has

$$|H(Y)|\,X = \{(a,s):\ a\in|K|\,Y,\ s\in|K_a(Y)|\,X\}\ ;$$

the "uncertainty" from $K$ is absorbed into the values of $H$ .

Here is a rather special, but useful, result.

**17.** *When the category* $\mathbf{X}$ *has finitely many objects, the functor*

$$\iota\circ(\ )\ :\ \mathrm{Ana}(\mathbf{Y},\mathrm{Fun}(\mathbf{X},\mathbf{A}))\longrightarrow\mathrm{Ana}(\mathbf{Y},\mathrm{Ana}(\mathbf{X},\mathbf{A}))$$

*induced by the inclusion* $\iota:\mathrm{Fun}(\mathbf{X},\mathbf{A}))\longrightarrow\mathrm{Ana}(\mathbf{X},\mathbf{A})$ *is an equivalence of categories.*

**Proof.** Since $\iota$ is full and faithful, it is immediate that so is $\iota\circ(\ )$ . To show that $\iota\circ(\ )$ is uniformly essentially surjective on objects, it suffices to show that the composite with the equivalence $\varphi$ of (9) ,

$$\varphi\circ(\iota\circ(\ ))\ :\ \mathrm{Ana}(\mathbf{Y},\mathrm{Fun}(\mathbf{X},\mathbf{A}))\longrightarrow\mathrm{Ana}(\mathbf{X}\times\mathbf{Y},\mathbf{A})\ ,$$

is so. Let $G:\mathbf{X}\times\mathbf{Y}\xrightarrow{a}\mathbf{A}$ . Define $F:\mathbf{Y}\xrightarrow{a}\mathrm{Fun}(\mathbf{X},\mathbf{A})$ as follows. Put $|F|\,Y \overset{\text{def}}{=} \prod_{X\in|\mathbf{X}|}|G|\,((X,Y))$ ; for $a\in|F|\,Y$ , $F_a(Y)(X) \overset{\text{def}}{=} G_{a(X)}(X,Y)$ ; for $x:X\to X'$ , $F_a(Y)(x) \overset{\text{def}}{=} G_{a(X),a(X')}(x,Y)$ (note that $a(X)\in|G|\,((X,Y))$ , $a(X')\in|G|\,((X',Y))$ ); for $y:Y\to Y'$ , $a\in|F|\,Y$ , $a'\in|F|\,Y'$ , the components of the natural transformation $F_{a,a'}(f):F_a(Y)\to F_{a'}(Y')$ are defined as

$$(F_{a,a'}(f))_X \overset{\text{def}}{=} G_{a(X),a'(X)}(X,y) : G_{a(X)}(X,Y) \longrightarrow G_{a'(X)}(X,Y') \,.$$

It is easy to check that $F$ is an anafunctor; the only point where the finiteness of $|\boldsymbol{X}|$ is used is the inhabitedness of the set $|F|\,Y = \prod_{X\in|\boldsymbol{X}|} |G|\,((X,Y))$ ; as a finite product of inhabited sets, it is inhabited.

We need to exhibit a natural isomorphism $h:(\varphi\circ(\iota\circ()))(F) \xrightarrow{\cong} G$ . But $(\varphi\circ(\iota\circ()))(F) = e\circ(\boldsymbol{X}\times\iota(F))$ has

$$|e\circ(\boldsymbol{X}\times\iota(F))|\,((X,Y)) = \{((X,a),X) : a\in|F|\,Y\}$$

and

$$(e\circ(\boldsymbol{X}\times\iota(F)))_{((X,a),X)}(X,Y) \;=\; F_a(Y)(X)\;(=G_{a(X)}(X,Y)\,)\,.$$

Thus, we may define $h$ by

$$h_{((X,a),X),a(X)} = 1_{G_{a(X)}(X,Y)}\;;$$

7. ensures that $h$ is well-defined.

When in 16., we put $\boldsymbol{X} = \mathbf{1}$ , we note the isomorphism $\mathbf{1}\times\boldsymbol{Y} \cong \boldsymbol{Y}$ , and we write $\boldsymbol{A}^+$ for $\mathrm{Ana}(\mathbf{1},\boldsymbol{A})$ (we may call $\boldsymbol{A}^+$ the category of *small anaobjects* of $\boldsymbol{A}$ ), we obtain the isomorphism $\mathrm{Ana}(\boldsymbol{Y},\boldsymbol{A}) \cong \mathrm{Fun}(\boldsymbol{Y},\boldsymbol{A}^+)$ of categories. In other words, (small) anafunctors $\boldsymbol{Y}\to\boldsymbol{A}$ may be identified with ordinary functors from the same domain $\boldsymbol{Y}$ into the category $\boldsymbol{A}^+$ of (small) anaobjects of the codomain $\boldsymbol{A}$ , and this identification extends to natural transformations. This shows that the notion of anafunctor and that of natural transformation of anafunctors can be reduced to the case when the domain category is $\mathbf{1}$ . This fact was suggested by the Referee.

When in 14., we put both $\boldsymbol{X}$ and $\boldsymbol{Y}$ equal to $\mathbf{1}$ , we obtain the equivalence $\boldsymbol{A}^{++} \simeq \boldsymbol{A}^+$ . In fact, writing $\mu_{\boldsymbol{A}}:\boldsymbol{A}^{++} \xrightarrow{\simeq} \boldsymbol{A}^+$ for a (the) quasi-inverse of the equivalence $\varphi:\boldsymbol{A}^+ \xrightarrow{\simeq} \boldsymbol{A}^{++}$ given in 14., and $\eta_{\boldsymbol{A}}:\boldsymbol{A}\longrightarrow\boldsymbol{A}^+$ for the inclusion functor $\boldsymbol{A} \cong \mathrm{Fun}(\mathbf{1},\boldsymbol{A}) \longrightarrow \mathrm{Ana}(\mathbf{1},\boldsymbol{A})$ , we have an idempotent monad $(()^+,\mu,\eta)$ on the bicategory $\mathrm{AnaCat}$ (both "idempotent" and "monad" understood in the suitable bicategorical sense); this fact will be explored in [M/P]. Further, in [M/P], it will be shown that $\boldsymbol{A}^+$ is a *stack-completion* of $\boldsymbol{A}$ ; the full explanation of this fact requires putting anafunctors into the context of indexed category theory.

As I mentioned in the Introduction, the construction of the category $\boldsymbol{A}^+$ is also given in [J/S], where $\boldsymbol{A}^+$ is named the category of *cliques* of $\boldsymbol{A}$; see Chapter 1, §1 of [J/S]. The general properties of cliques and $\boldsymbol{A}^+$ are not developed in [J/S]; $\boldsymbol{A}^+$ is used in [J/S] for purposes different from those of this paper.

Written out explicitly, $\boldsymbol{A}^+$ is the following category. An object $A$ of $\boldsymbol{A}^+$ (a clique, or a small anaobject of $\boldsymbol{A}$) is given by an inhabited set $|A|$, an $|A|$-indexed family $\langle A_s \rangle_{s\in |A|}$ of objects $A_s$ of $\boldsymbol{A}$, and an assignment of an isomorphism $A_{s,t}:A_s \xrightarrow{\cong} A_t$ to each pair $(s,t)$ of elements of $S$ such that $A_{s,s}=1_{A_s}$ and $A_{t,u}\circ A_{s,t}=A_{s,u}$ whenever $s,t,u\in S$. A morphism $h:A\longrightarrow A'$ is a family

$$h = \langle h_{s,s'}:A_s \longrightarrow A'_{s'}\rangle_{s\in |A|,\, s'\in |A'|}$$

such that

$$\begin{tikzcd} A_s \arrow[r, "h_{s,s'}"] \arrow[d, "A_{s,t}"'] \arrow[dr, phantom, "\circ"] & A_{s'} \arrow[d, "A_{s',t'}"] \\ A_t \arrow[r, "h_{t,t'}"'] & A_{t'} \end{tikzcd}$$

for all appropriate values of the parameters.

By a (not necessarily small) anaobject of $\boldsymbol{A}$, we mean a (not necessarily small) anafunctor $\mathbf{1}\longrightarrow\boldsymbol{A}$; we will use (in the next section) a similar notation in relation to anaobjects in general as we did above for small anaobjects; for a general anaobject $A$, $|A|$ may be a proper class.

## §2. Adjoint anafunctors

Anafunctors provide solutions without introducing non-canonical choices to existence problems when data are given by universal properties. The best example for this is the existence of an adjoint anafunctor when the "local existence criterion" is satisfied.

Given the anafunctors $\mathbf{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{A}$, we say that $F$ is a *left-adjoint to* $G$ ($F \dashv G$) if we have, for any $X \in \mathbf{X}$, $A \in \mathbf{A}$, $s \in |F|X$, $v \in |G|A$ a bijection $\varphi_{s,v}$, mapping $f$ to $g$ as in

$$\frac{F_s X \xrightarrow{f} A}{X \xrightarrow[g]{} G_v A} \tag{1}$$

between $\mathbf{A}(F_s X, A)$ and $\mathbf{X}(X, G_v A)$, which is natural in $X$ and $A$ in the expected sense: for any $Y \in \mathbf{X}$, $t \in |F|Y$ and $h: X \to Y$ in addition to the above data, in

$$\frac{F_s X \xrightarrow{F_{st}h} F_t Y \xrightarrow{f} A}{X \xrightarrow[h]{} Y \longrightarrow G_v A},$$

we have $\varphi_{s,v}(f \circ F_{st}h) = \varphi_{t,v}(f) \circ h$, and similarly for data in $\mathbf{A}$.

We leave it to the reader to check that this is the same as the standard internal definition in the metabicategory $\mathrm{ANACAT}$: the existence of $\eta: 1_{\mathbf{X}} \longrightarrow GF$ and $\varepsilon: FG \longrightarrow 1_{\mathbf{A}}$ such that

$$\begin{tikzcd} F(GF) \arrow[rr, "\alpha", "\cong"'] & & (FG)F \\ & F \arrow[ul, "F\eta"] \arrow[ur, "\varepsilon G"'] \arrow[u, phantom, "\circ"] & \end{tikzcd} \qquad \begin{tikzcd} (GF)G \arrow[rr, "\alpha", "\cong"'] & & G(FG) \\ & G \arrow[ul, "\eta G"] \arrow[ur, "G\varepsilon"'] \arrow[u, phantom, "\circ"] & \end{tikzcd}$$

where the $\alpha$'s are the appropriate associativity isomorphisms. In particular, if $F \dashv G$, and $F' \cong F$, $G' \cong G$, then $F' \dashv G'$; and if $F \dashv G$, $F' \dashv G$, then $F' \cong F$.

Let $\mathbf{X} \xleftarrow{G} \mathbf{A}$ be an anafunctor (in particular, $G$ may be an ordinary functor), and $X \in \mathbf{X}$. We say that the triple $(B \in \mathbf{A}, u \in |G|B, \eta: X \to G_u B)$ is *good* for $X$ if it has the universal property that for any $(A\in\mathbf{A},\ v\in|G|A,\ g:X\to G_vA)$ there is a unique $f:B\to A$ with $g=G_{u,v}(f)\circ\eta$. $\mathbf{X}\xleftarrow{G}\mathbf{A}$ satisfies the condition *of local existence of a left adjoint* if for every $X\in\mathbf{X}$, there is at least one good triple for $X$.

**1.** *Assume that the anafunctor* $\mathbf{X}\xleftarrow{G}\mathbf{A}$ *satisfies the condition of local existence of a left adjoint. Then there is a* (*canonical*) *anafunctor* $F:\mathbf{X}\to\mathbf{A}$ *which is left adjoint to* $G$.

**Proof.** We define $F:\mathbf{X}\longrightarrow\mathbf{A}$ as follows. For any $X\in\mathbf{X}$, $|F|X$ is the class of all good triples for $X$. If $s=(B,u,\eta)\in|F|X$, $F_s(X)\ \overset{=}{\text{def}}\ B$. If also $t=(C,v,\theta)\in|F|Y$, $g:X\to Y$, then $F_{s,t}(g)$ is the unique $f:B\to C$ such that

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "g"'] \arrow[dr, phantom, "\circ"] & G_uB \arrow[d, "G_{u,v}(f)"] \\ Y \arrow[r, "\theta"'] & G_vC \end{tikzcd}\quad .$$

The bijection $\varphi_{s,v}$ (see (1)) is as follows. If $s=(B,u,\eta)\in|F|X$ and $v\in|G|A$, for $f:F_sX\to A$, the corresponding $g:X\to G_vA$ is $g=G_{u,v}(f)\circ\eta$. The remaining details are similar to the ones in the basic theory of adjoint functors (see [CWM]).

When $G$ is a functor, $F$ constructed above is a saturated anafunctor. Indeed, given $s=(B,B,\eta)\in|F|X$ and $\mu:B\xrightarrow{\cong}C$, the condition for $t=(C,C,\theta)\in|F|X$ to satisfy $F_{s,t}(1_X)=\mu$ is that the diagram

$$\begin{tikzcd} & GB \arrow[d, "G\mu"] \\ X \arrow[ur, "\eta"] \arrow[r, "\theta"'] & GC \end{tikzcd}$$

commutes, which determines $\theta$.

Let us also note that if $\mathbf{X}$, $\mathbf{A}$, $G$ are all small, then so is $F$.

The example in 1.2. is, of course, a special case of 1., which is the main source of naturally occurring anafunctors.

Another special case of 1. says that any functor, or even anafunctor, which is fully faithful and essentially surjective has a quasi-inverse *anafunctor*; thus it is an equivalence (without the axiom of choice) in the sense of the metabicategory $\mathrm{ANACAT}$ . We call an anafunctor which is an equivalence in the sense of $\mathrm{ANACAT}$ an *anaequivalence* of categories. $F:\boldsymbol{X}\xrightarrow{a}\boldsymbol{A}$ is *fully faithful* if for every $X\in\boldsymbol{X}$ and $Y\in\boldsymbol{X}$ , for some (equivalently, for all) $s\in|F|X$ , $t\in|F|Y$ , the mapping $F_{s,t}:\boldsymbol{X}(X,Y)\longrightarrow\boldsymbol{A}(F_sX,F_tY)$ is a bijection. The same $F$ is *essentially surjective* if for all $A\in\boldsymbol{A}$ , there is $X\in\boldsymbol{X}$ and $s\in|F|X$ such that $A\cong F_sX$ . We have

**2.** *Any fully faithful and essentially surjective* (*ana*)*functor is an anaequivalence of categories.*

By 1.11".,

**2'.** *The inclusion* $\boldsymbol{A}\longrightarrow\boldsymbol{A}^+$ *(* $=\mathrm{Ana}(\mathbf{1},\boldsymbol{A})$ *) is an anaequivalence.*

Completeness properties of functor-categories depend, in the usual treatment, on non-canonical choices. Assume $\boldsymbol{I}$ ,$\boldsymbol{X}$ and $\boldsymbol{A}$ are categories, and $\boldsymbol{A}$ has $\boldsymbol{I}$-indexed limits. Then the proof that the functor category $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{A})$ has $\boldsymbol{I}$-indexed limits proceeds by picking particular limits in $\boldsymbol{A}$ of the $\boldsymbol{I}$-indexed diagrams in $\boldsymbol{A}$ obtained by evaluating the given $\boldsymbol{I}$-indexed diagram in $\mathrm{Fun}(\boldsymbol{X},\boldsymbol{A})$ .

For the case when the category $\boldsymbol{I}$ has finitely many objects, we can avoid the choices. In fact, in this case the metacategory $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ of anafunctors is *better* than the base category $\boldsymbol{A}$ ; it has *specified* limits (given as a function with arguments the $\boldsymbol{I}$-diagrams in $\boldsymbol{A}$ ) even if $\boldsymbol{A}$ is not assumed to have specified limits. We will have results concerning arbitrary small limit types $\boldsymbol{I}$ ; see propositions 6. and 7. below, and also the last section of the paper.

**3.** *Suppose that the small category* $\boldsymbol{I}$ *has finitely many objects, and the category* $\boldsymbol{A}$ *has* $\boldsymbol{I}$*-indexed limits. Then* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *has specified* $\boldsymbol{I}$*-indexed limits.*

**Proof.** For simplicity of notation, we show why $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ has specified binary products if $\boldsymbol{A}$ has binary products; the general case is only notationally different (but also see 4. below). Given $F, G\in\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$, we define $F\times G\in\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ as follows. We put

$$|F\times G|\,X=\{(s\in|F|\,X,\ t\in|G|\,X,\ F_sX\xleftarrow{\ \pi\ }A\xrightarrow{\ \pi'\ }G_tX):\ (\pi,\pi')\text{ is a product in }\boldsymbol{A}\}\ .$$

For $a\in|F\times G|\,X$ as displayed, $(F\times G)_aX=A$. If also $a'\in|F\times G|\,X'$ with similar ingredients, and $f{:}X\longrightarrow X'$, $(F\times G)_{a,a'}(f)$ is the arrow $g$ in the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] \arrow[ddd, "g"] & \\
F_sX \arrow[d, "F_{s,s'}(f)"'] & & G_tX \arrow[d, "F_{t,t'}(f)"] \\
F_{s'}X' & & G_{t'}X' \\
 & A' \arrow[ul, "\pi"] \arrow[ur, "\pi'"'] &
\end{tikzcd}$$

.

I leave it to the reader to define the projections $F\xleftarrow{\ \pi\ }F\times G\xrightarrow{\ \pi'\ }G$, and to check the universal property of the product.

We have the following variant of 3.

**4.** *Suppose that* $\boldsymbol{X}$, $\boldsymbol{A}$ *and* $\boldsymbol{I}$ *are small categories, and* $\boldsymbol{I}$ *has finitely many objects. Assume that* $\boldsymbol{A}$ *has* $\boldsymbol{I}$*-indexed limits. Then* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *has specified* $\boldsymbol{I}$*-indexed limits.*

**Proof.** By 1., we have $\mathrm{Lim}{:}\boldsymbol{A}^{\boldsymbol{I}}\ (=\mathrm{Fun}(\boldsymbol{I},\boldsymbol{A}))\longrightarrow\boldsymbol{A}$, an anafunctor right adjoint to $\Delta{:}\boldsymbol{A}\to\boldsymbol{A}^{\boldsymbol{I}}$. Since $\boldsymbol{A}$ is small, $\mathrm{Lim}$ is (can be taken to be) small; thus, the adjunction $\Delta\dashv\mathrm{Lim}$ lives in the bicategory $\mathrm{AnaCat}$. As any bicategory, $\mathrm{AnaCat}$ has a representable functor to $\mathrm{ANACAT}$, represented by any object of it:

$$\mathrm{Ana}(\boldsymbol{X},-) = \mathrm{AnaCat}(\boldsymbol{X},-) : \mathrm{AnaCat} \longrightarrow \mathrm{ANACAT}$$

(as explained before, we mean a homomorphism of bicategories when we talk about a functor of bicategories). As any functor of bicategories, $\mathrm{Ana}(\boldsymbol{X},-)$ preserves any adjunction in its domain. Thus, we have the adjunction

$$\mathrm{Ana}(\boldsymbol{X},\mathrm{Fun}(\boldsymbol{I},\boldsymbol{A})) \underset{\Delta^*}{\overset{\mathrm{Lim}^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}) , \qquad (2)$$

where $\mathrm{Lim}^*$ , $\Delta^*$ are the *functors* $\mathrm{Ana}(\boldsymbol{X},\mathrm{Lim})$ , $\mathrm{Ana}(\boldsymbol{X},\Delta)$ , resp. We have the equivalences

$$\begin{aligned}\mathrm{Ana}(\boldsymbol{X},\mathrm{Fun}(\boldsymbol{I},\boldsymbol{A})) &\underset{17}{\simeq} \mathrm{Ana}(\boldsymbol{X},\mathrm{Ana}(\boldsymbol{I},\boldsymbol{A})) \\ &\underset{14}{\simeq} \mathrm{Ana}(\boldsymbol{I},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) \underset{15}{\simeq} \mathrm{Fun}(\boldsymbol{I},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) .\end{aligned}$$

Composing them with (2), we get

$$\mathrm{Fun}(\boldsymbol{I},\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) \underset{\hat{\Delta}}{\overset{\widehat{\mathrm{Lim}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}) .$$

Going through the above equivalences, one can check that $\hat{\Delta}$ is isomorphic to $\Delta:\boldsymbol{B}\longrightarrow\boldsymbol{B}^{\boldsymbol{I}}$ for $\boldsymbol{B} = \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ . Thus, up to isomorphism, $\widehat{\mathrm{Lim}}$ is the desired limit-functor.

The conclusion of 4. holds, in particular, for $\boldsymbol{A}^+ = \mathrm{Ana}(\mathbf{1},\boldsymbol{A})$ .

Of course, the similar result for colimits is a consequence, by passing to the opposite category. But also for other finitary categorical operations defined by universal properties, we have similar conclusions, at least for $\boldsymbol{A}^+$ . E.g.,

**5.** *Suppose that the small category* $\boldsymbol{A}$ *is Cartesian closed. Then* $\boldsymbol{A}^{+}$ *, the category of small anaobjects of* $\boldsymbol{A}$ (a category anaequivalent to $\boldsymbol{A}$ ; see 2'.), *is also Cartesian closed, and in fact has specified finite products and exponentials.*

**Proof.** An *exponential diagram on* a pair $(X, Y)$ of objects in $\boldsymbol{A}$ is a diagram of the form

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[rr, "e"] & & Y \\ X & & W & \end{tikzcd} \tag{3}$$

such that $(p, q)$ is a product, and $e$ satisfies the usual universal property of the evaluation morphism of an exponential (think of

$$\begin{tikzcd} & X{\times}Y^X \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] \arrow[rr, "e"] & & Y \\ X & & Y^X & \end{tikzcd} \quad ;$$

the definition is that for any

$$\begin{tikzcd} & Z' \arrow[dl, "p'"'] \arrow[dr, "q'"] \arrow[rr, "e'"] & & Y \\ X & & W' & \end{tikzcd}$$

such that $(p', q')$ is a product, there is a unique commutative diagram of the form

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[rr, "e"] & & Y \\ X & & W & \\ X \arrow[u, "1_X"] & & W' \arrow[u, "f"'] & \\ & Z' \arrow[ul, "p'"] \arrow[ur, "q'"'] \arrow[uuu] \arrow[rr, "e'"'] & & Y \arrow[uuu, "1_Y"'] \end{tikzcd} \quad . \tag{4}$$

Of course, a category with finite products is Cartesian closed iff there exists an exponential diagram on any pair of objects.)

If $\Delta$ abbreviates (3), we indicate the components of $\Delta$ by putting the subscript $\Delta$ to the corresponding symbol in (3); e.g., $W_\Delta$ for the object $W$ in (3), *etc*.

Let $A$, $B$ be anaobjects of $\boldsymbol{A}$. Define the anaobject $B^A$ as follows. Let $|B^A|$ be the set of all $(s, u, \Delta)$ such that $s\in|A|$, $u\in|B|$, and $\Delta$ is an exponential diagram on $(A_s, B_u)$. For $a=(s, u, \Delta)\in|B^A|$, let $(B^A)_a \overset{=}{\mathrm{def}} W_\Delta$. Here and below, $a=(s, u, \Delta)\in|B^A|$ and $a'=(s', u', \Delta')\in|B^A|$. $(B^A)_{a,a'}:W_\Delta\longrightarrow W_{\Delta'}$ is defined to be the arrow $g$ in the unique commutative diagram

$$\begin{tikzcd}
 & Z_\Delta \arrow[rr, "e"] \arrow[dl, "p_\Delta"'] \arrow[dr, "q_\Delta"] \arrow[ddd, "h"] & & B_u \arrow[ddd, "B_{u,u'}", "\cong"'] \\
A_s \arrow[d, "A_{s,s'}"', "\cong"] & & W_\Delta \arrow[d, "g"] & \\
A_{s'} & & W_{\Delta'} & \\
 & Z_{\Delta'} \arrow[ul, "p_{\Delta'}"] \arrow[ur, "q_{\Delta'}"'] \arrow[rr, "e'"'] & & B_{u'}
\end{tikzcd} ; \qquad (5)$$

the reasons why the latter uniquely exists are the universal property of $\Delta'$, and the fact that $A_{s,s'}$, $B_{u,u'}$ are isomorphisms.

The exponential diagram

$$\begin{tikzcd}
 & A\times B^A \arrow[r, "e"] \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & B \\
A & & B^A
\end{tikzcd}$$

on $(A, B)$ is given as follows. $|A\times B^A| \overset{=}{\mathrm{def}} |B^A|$; $(A\times B^A)_a = Z_\Delta$; $(A\times B^A)_{a,a'}$ is the arrow $h$ in (5). For $t\in|A|$, $\pi_{a,t}:Z\longrightarrow A_{s'}$ is $A_{s,t}\circ p_\Delta$; $\pi'$ is similar. For $v\in|B|$, $e_{a,v}:Z\longrightarrow B_v$ is $B_{u,v}\circ e_\Delta$.

The verification of the needed properties of these data is omitted.

**6.** *Let* $\boldsymbol{X}$ *be a small category,* $\boldsymbol{A}$ *a category having all small limits. Then every small diagram in* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *has a limit in* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *; that is, with* $\varphi:\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})\to\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *the inclusion, for any small* $\boldsymbol{I}$ *and* $\Gamma:\boldsymbol{I}\to\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *,* $\lim(\varphi\circ\Gamma)$ *exists in* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *. Moreover, there is a class-function assigning, to any small diagram* $\Gamma$ *in* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *, a limit-cone in* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *on* $\varphi\circ\Gamma$ *. If* $\boldsymbol{A}$ *is locally small, the limit-objects in the assigned limit-cones are locally small anafunctors.*

**Proof.** Let $\Gamma = (\langle F_I\rangle_{I\in\boldsymbol{I}}, \langle f_i:F_I\to F_J\rangle_{(i:I\to J)\in\boldsymbol{I}})$ be a small diagram in $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ . We define $L = \lim\Gamma\in\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ as follows.

Fix $X\in\boldsymbol{X}$ , to define $|L|X$ . We let $\boldsymbol{I}|X$ be the category whose objects are pairs $(I, s)$ with $I\in\boldsymbol{I}$ and $s\in|F_I|X$ , and whose arrows $(I, s)\to(J, t)$ are $(s, t, i)$ with $i:I\to J$ (that is, an arrow $(I, s)\to(J, t)$ is just an arrow $I\to J$ , with the information on the domain $(I, s)$ and the codomain $(J, t)$ attached; we will write $i:(I, s)\to(J, t)$ instead of $(s, t, i):(I, s)\to(J, t)$ ). By the hypotheses, $\boldsymbol{I}|X$ is a small category. Consider the diagram $\Gamma|X:\boldsymbol{I}|X\to\boldsymbol{A}$ that assigns the object $F_{I,\, s}X\equiv(F_I)_s(X)$ to $(I, s)$ , and the arrow $f_{i,\, s,\, t}\equiv(f_i)_{s,\, t}:F_{I,\, s}X\to F_{J,\, t}X$ to $i:(I, s)\to(J, t)$ . We define $|L|X$ to be the class of all limit-cones on $\Gamma|X$ in $\boldsymbol{A}$ ; for $\pi\in|L|X$ , $\pi=\langle\pi_{I,\, s}:[\pi]\longrightarrow F_{I,\, s}X\rangle_{(I,\, s)\in\Gamma|X}$ , we put $L_\pi(X) = [\pi]$ .

Let $g:X\to Y$ be an arrow, $\pi\in|L|X$ , $\rho\in|L|Y$ , to define $h\equiv L_{\pi,\,\rho}(g):L_\pi(X)\to L_\rho(X)$ . $h$ is given uniquely by the condition that

$$\begin{array}{ccc} L_\pi X & \xrightarrow{\pi_{I,\, s}} & F_{I,\, s}X \\ {\scriptstyle h}\downarrow & \circ & \downarrow{\scriptstyle F_{I,\, s,\, t}g} \\ L_\rho Y & \xrightarrow[\pi_{I,\, t}]{} & F_{I,\, t}Y \end{array}$$

commutes for all $I\in\boldsymbol{I}$ , $s\in|F|X$ , $t\in|F|Y$ . Indeed, first of all, the diagram

$$
\begin{tikzcd}
 & L_{\pi}X \arrow[dl, "\pi_{I,s}"'] \arrow[dr, "\pi_{I,s'}"] \arrow[d, phantom, "\circ"] & \\
F_{I,s}X \arrow[rr, "f_{1_I,s,s'}", "F_{I,s,s'}1_X"'] \arrow[dr, "F_{I,s,t}g"'] & {} \arrow[d, phantom, "\circ"] & F_{I,s'}X \arrow[dl, "F_{I,s',t}g"] \\
 & F_{I,t}Y &
\end{tikzcd}
$$

shows that the arrow $k_{I,t} \overset{\text{def}}{=} F_{I,s,t}g \circ \pi_{I,s} : L_{\pi}X \to F_{I,t}Y$ does not depend on $s$ (the upper commutativity is by $\pi$ being a cone, the lower by the functoriality of $F_I$ ; the equality $f_{1_I,s,s'} = (1_{F_I})_{s,s'} = F_{s,s'}(1_X)$ holds by the compatibility of the diagram $\Gamma$ , and the definition of $1_{F_I}$ .) Next, the diagram

$$
\begin{tikzcd}
 & L_{\pi}X \arrow[dl, "\pi_{I,s}"'] \arrow[dr, "\pi_{J,u}"] \arrow[d, phantom, "\circ"] & \\
F_{I,s}X \arrow[rr, "f_{i,s,u}"] \arrow[d, "F_{I,s,t}"'] & {} \arrow[d, phantom, "\circ"] & F_{J,u}X \arrow[d, "F_{J,u,v}"] \\
F_{I,t}Y \arrow[rr, "f_{i,t,v}"'] & {} & F_{J,v}Y
\end{tikzcd}
$$

shows that $\langle k_{I,t} \rangle_{(I,t) \in \mathrm{Ob}(\boldsymbol{I}|Y)}$ is a cone on the diagram $\Gamma|Y$ . Since $\langle \rho_{I,t} : L_{\rho}Y \to F_{I,t}Y \rangle_{I,t}$ is a limit cone, there is a unique $h : L_{\pi}X \to L_{\rho}Y$ such that $h \circ \rho_{I,t} = k_{I,t}$ for all $I$ and $t$ , which is our assertion on $h$ .

Having defined $L_{\pi,\rho}(g)$ , I leave it to the reader to check that $L$ so defined is indeed an anafunctor. We have $\lambda_I : L \to F_I$ for which $\lambda_{I,\pi,s} = \pi_{I,s}$ , for all appropriate values of the parameters; moreover, $\langle \lambda_I \rangle_I$ is a limit cone on the diagram $\varphi \circ \Gamma$ ; the verification is omitted.

Note that, in this proof, in order to build the required $\boldsymbol{I}$-type limit, we use a whole class of other limit-types, to construct limits in $\boldsymbol{A}$ . However, when each $F_I$ is in particular a functor, than each $\boldsymbol{I}|X$ is isomorphic to $\boldsymbol{I}$ ; this shows that we have

**7.** *Assuming that* $\boldsymbol{A}$ *has* $\boldsymbol{I}$*-type limits, then* $\boldsymbol{I}$*-type diagrams of functors* $\boldsymbol{X}\to\boldsymbol{A}$ *have specified limits in* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ .

The last observation is due to the Referee.