On Transport Theory

Urs Schreiber[*]

May 1, 2006

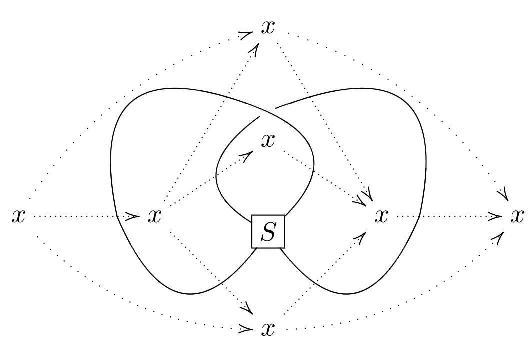
Abstract

n-transports are an n-functors describing

- parallel trasport in n-bundles
- propagation in n-dimensional QFT.

We describe basic notions of n-transport theory, such as trivialization, transition and trace and discuss examples.
This text is a synthesis of the material contained in [1, 2, 3, 4, 5, 6].

Contents



[*]E-mail: urs.schreiber at math.uni-hamburg.de

Parallel transport in a vector bundle as well a propagation in quantum mechanics is a functor from "paths" to vector spaces. As a generalization of this, n-dimensional QFT has been proposed to be described by functors from n-cobordisms to vector spaces. But locality requirements suggest that this should be resolved in n-functors on n-paths. Indeeed, 2-functors on 2-paths have been shown to describe connections in 2-bundles ($\sim$ gerbes) (BaezSchreiber:2005). I claim that, in an analogous way, there is **2-vector transport** which describes 2-dimensonal field theories like the state sum model for 2D TFT introduced by Fukuma-Hosono-Kawai (FHK), as well as the "internal state sum model" for 2D CFT discussed by Fuchs-Runkel-Schweigert (FRS).

This suggests a general theory of **n-transport** which describes both parallel transport in n-bundles as well as n-dimensional quantum field theory. Here I try to give an overview of my (unfinished) work [1, 2, 3, 4, 5, 6] concerning this issue.

1 Transport

In our context, an 'n-transport' is nothing but an n-functor. We shall address n-functors as **n-transport** whenever we want to think of them as realizing an n-categorical analogue of parallel transport in a fiber bundle.

This implies that, usually, the domain of an n-transport is a 'geometric' n-category. We shall address an n-category as a **geometric n-category** whenever we want to think of its p-morphisms as p-dimensional spaces of some sort.

Hence, for our purposes, n-transport is an n-functor

$$\mathrm{tra} : \mathcal{P} \to T$$

from a geometric domain $\mathcal{P}$ to some target n-category T.

Example 1

Examples for transport functors come from parallel transport in bundles, as well as from functorial descriptions of quantum field theory (QFT).

- <u>parallel transport</u>
  - A **vector bundle** $E \xrightarrow{p} M$ with connection ∇ is given by its parallel transport
    $$\mathcal{P}_1(M) \to \mathbf{Vect}\,,$$
    which is a 1-transport 1-functor from the groupoid $\mathcal{P}_1(M)$ of thin homotopy classes of paths in M to the category **Vect** of (finite dimensional) vector spaces (over some field).
  - A **principal G-bundle** $E \xrightarrow{p} M$ with connection ∇ is given by its parallel transport
    $$\mathcal{P}_1(M) \to G\mathbf{Tor}\,,$$

which is a 1-transport 1-functor with values in the category $G\mathbf{Tor}$ of (left, say) G-torsors.

– An **abelian bundle gerbe**

$$\begin{tikzcd} L \arrow[d] & \\ Y^{[2]} \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[d] \\ & M \end{tikzcd}$$

with connection and curving is a trivialization of a **line-2-bundle** wtih 2-connection, which is a 2-transport

$$\mathcal{P}_2(M) \to \Sigma(\mathbf{Vect}) \,,$$

where $\Sigma(\mathbf{Vect})$ is the suspension of the monoidal 1-category **Vect** to a 2-category with a single object.

– A **nonabelian** $\mathrm{Aut}(H)$**-bundle-gerbe** with connection and curving is a trivialization of a 2-transport

$$\mathcal{P}_2(M) \to \Sigma(\mathbf{BiTor}(H)) \,,$$

where $\Sigma(\mathbf{BiTor}(H))$ is the suspension of the category of H-bitorsors.

– A **principal** G_2**-2-bundle** with 2-connection over a 1-space M is a 2-transport

$$\mathcal{P}(M) \to G_2\mathbf{Tor}\,,$$

where $G_2\mathbf{Tor}$ is the 2-category of (left, say) G_2-torsors.

– Parallel transport in a **2-vector bundle** is a 2-transport

$$\mathcal{P}_2(M) \to {}_{\mathcal{C}}\mathbf{Mod}$$

with ${}_{\mathcal{C}}\mathbf{Mod}$ the 2-category of module categories of a tensor category $\mathcal{C}$.

• <u>QFT</u>

The concept of n-transport is intended to capture functorial constructions in quantum field theory. Commonly, 1-functors whose domain is an n-cobordism category are addressed as n**-dimensional quantum field theories**. n-transport is supposed to refine this description.

– Propagation in **quantum mechanics** is a 1-transport

$$\mathbf{1Cob} \to \mathbf{Hilb}\,.$$

– Evaluation of **Feynman diagrams** is a 1-transport

$$\mathbf{FGrph} \to \mathbf{Rep}(G)$$

from Feynman graphs **FGrph** to representations of the symmetry group.

– **Segal's formulation** of 2D QFT is a 1-transport

$$\mathbf{2Cob} \to \mathbf{Hilb}\,,$$

where **2Cob** is the 1-category of 2-dimensional cobordisms.

– **Stolz&Teichner's refinement** of Segal's description is a 2-transport

$$\mathcal{P}_2 \to \mathbf{BiMod}_{\mathrm{vN}}$$

with values in bimodules of vonNeumann algebras.

– Propagation in **categorified quantum mechanics** is a 2-transport

$$\mathcal{P}_2 \to {}_{\mathcal{C}}\mathbf{Mod}$$

with ${}_{\mathcal{C}}\mathbf{Mod}$ the 2-category of module categories of a tensor category $\mathcal{C}$.

The vague notions 'n-transport' and 'geometric n-category' do not affect the content of our constructions (which could be carried out with arbitrary n-functors on arbitrary domains), but do affect the choice of our constructions. Regarding an n-functor as an n-transport implies that we want to apply certain 'geometric' operations to that functor, notably that we may want to

- **locally trivalize**

it (express it in terms of "**local data**"), make

- **transitions**

between local trivializations and

- **take a trace**

of (trivialized) transport.

In order to indicate the context in which we think of certain n-categories and n-functors below, we will use the following symbols.

$\mathcal{P}$	a geometric n-category
T	a codomain of an n-transport
$\mathrm{tra} : \mathcal{P} \to T$	an n-transport n-functor
$T' \xrightarrow{i} T$	an injection of n-transport codomains
$\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P}$	a surjection of n-transport domains

1.1 Trivialization

Given any n-transport functor, it is often desireable to study its global and its local properties seperately. If the functor is locally trivializable in some suitable sense, we may express its global behaviour by gluing of local data.

Local Trivialization. Our notion of local trivialization of n-transport is a generalization and refinement of similar constructions in 1- and 2-bundles.

Definition 1 *Given a transport* $\mathrm{tra} : \mathcal{P} \to T$ *as well as a morphism*

$$T' \xrightarrow{i} T$$

of codomains, we say that tra *is* **trivial** *with respect to* i*, or* i**-trivial** *iff there exists* $\mathrm{tra}^i : \mathcal{P} \to T'$ *such that*

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{tra}^i"'] \arrow[dr, phantom, "//"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

We say that tra *is* i**-trivializable** *iff there is a* **trivialization** t

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{tra}^i"'] \arrow[dr, phantom, "\overset{t}{\swarrow}\sim"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

Finally, given a morphism

$$\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P} ,$$

we say that tra *is* p**-locally** i**-trivializable** *iff there is* t *such that*

$$\begin{tikzcd} \mathcal{P}_{\mathcal{U}} \arrow[r, "p"] \arrow[d, "\mathrm{tra}^i"'] \arrow[dr, phantom, "\overset{t}{\swarrow}\sim"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

Example 2 (injections along which to i-trivialize)

- The suspension

$$\Sigma(M(n\times n,\mathbb{C})) = \left\{ \bullet \xrightarrow{A} \bullet \;\middle|\; A \in M(n\times n,\mathbb{C}) \right\}$$

of the monoid $M(n\times n,\mathbb{C})$ of complex $n\times n$ matrices sits inside the category of complex vector spaces

$$\begin{array}{ccc} \Sigma(M(n\times n,\mathbb{C})) & \xrightarrow{i} & \mathbf{Vect}_{\mathbb{C}} \\ \bullet & & \mathbb{C}^n \\ \downarrow A & \mapsto & \downarrow A \\ \bullet & & \mathbb{C}^n \end{array} .$$

Local trivialization of a transport functor $\mathcal{P} \to \mathbf{Vect}_{\mathbb{C}}$ with respect to this i evidently coincides with the ordinary notion of **local trivialization of a vector bundle** with connection.

- The suspension

$$\Sigma(G) = \left\{ \bullet \xrightarrow{g} \bullet \;\middle|\; g \in G \right\}$$

of the group G sits inside the category of G-torsors

$$\begin{array}{ccc} \Sigma(G) & \xrightarrow{i} & G\mathbf{Tor} \\ \bullet & & G \\ \downarrow g & \mapsto & \downarrow r(-,g) \\ \bullet & & G \end{array} .$$

(Here r is the right action of G on itself.) Local trivialization of a transport functor $\mathcal{P} \to G\mathbf{Tor}$ with respect to this i evidently coincides with the ordinary notion of **local trivialization of a principal G-bundle** with connection.

- The suspension

$$\Sigma(G_2) = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow (g,h)"] & \bullet \end{tikzcd} \;\middle|\; (g,h) \in \mathrm{Mor}(G_2) \right\}$$

of the 2-group G_2 sits inside the 2-category of G_2-torsors

$$\begin{array}{ccc} \Sigma(G_2) & \xrightarrow{i} & G_2\mathbf{Tor} \\ \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "{\Downarrow (g,h)}"] & \bullet \end{tikzcd} & \mapsto & \begin{tikzcd} G_2 \arrow[r, bend left, "{r(-,g)}"] \arrow[r, bend right, "{r(-,g')}"'] \arrow[r, phantom, "{\Downarrow r(-,(g,h))}"] & G_2 \end{tikzcd} \end{array} .$$

(Here r is the right action of G_2 on itself.) Local trivialization of a transport 2-functor $\mathcal{P} \to G_2\mathbf{Tor}$ with respect to this i coincides with the ordinary notion of **local trivialization of a principal G_2-2-bundle** with connection.

- The double suspension

$$\Sigma(\Sigma(\mathbb{C})) = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "\bullet"] \arrow[r, bend right, "\bullet"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \end{tikzcd} \;\middle|\; c \in \mathbb{C} \right\}$$

of the monoid of complex numbers sits inside the suspension

$$\Sigma(\mathbf{Vect}) = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "V"] \arrow[r, bend right, "W"'] \arrow[r, phantom, "\Downarrow R"] & \bullet \end{tikzcd} \;\middle|\; R \in \mathrm{Mor}_{\mathbf{Vect}}(V, W) \right\}$$

of **Vect**

$$\begin{array}{ccc} \Sigma(\Sigma(\mathbb{C})) & \xrightarrow{i} & \Sigma(\mathbf{Vect}) \\ \begin{tikzcd} \bullet \arrow[r, bend left, "\bullet"] \arrow[r, bend right, "\bullet"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \end{tikzcd} & \mapsto & \begin{tikzcd} \bullet \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \end{tikzcd} \end{array} .$$

Local trivialization of a transport 2-functor $\mathcal{P} \to \Sigma(\mathbf{Vect})$ with respect to this i coincides with the process of obtaining an **abelian bundle gerbe from a line-2-bundle by pre-trivialization**.

- The suspension

$$\Sigma(\mathrm{Aut}(H)) = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "{\Downarrow (g,h)}"] & \bullet \end{tikzcd} \;\middle|\; (g,h) \in \mathrm{Mor}(\mathrm{Aut}(H)) \right\}$$

of the 2-group $\mathrm{Aut}(H)$ sits inside the suspension of the category of H-bitorsors

$$
\begin{array}{ccc}
\Sigma(\mathrm{Aut}(H)) & \xrightarrow{\sim} & \Sigma(\mathbf{BiTor}(H)) \\
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow (g,h)"] & \bullet
\end{tikzcd}
& \mapsto &
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "H_g"] \arrow[r, bend right=50, "H_{g'}"'] \arrow[r, phantom, "\Downarrow h"] & \bullet
\end{tikzcd}
\end{array}
.
$$

Local trivialization of a transport 2-functor $\mathcal{P} \to \Sigma(\mathbf{BiTor}(H))$ with respect to this i coincides with the process of obtaining a **nonabelian bundle gerbe by pre-trivialization**.

- Let $\mathcal{C}$ be a modular tensor category. The chain of injections

$$
\begin{array}{ccccc}
\Sigma(\mathcal{C}) & \longrightarrow & \mathbf{BiMod}(\mathcal{C}) & \xrightarrow{\sim} & {}_{\mathcal{C}}\mathbf{Mod} \\
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "V_1"] \arrow[r, bend right=50, "V_2"'] \arrow[r, phantom, "\Downarrow \phi"] & \bullet
\end{tikzcd}
& \mapsto &
\begin{tikzcd}
\mathbb{1} \arrow[r, bend left=50, "V_1"] \arrow[r, bend right=50, "V_2"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathbb{1}
\end{tikzcd}
& \mapsto &
\begin{tikzcd}
\mathbf{Mod}_{\mathbb{1}} \arrow[r, bend left=50, "{r(-,V_1)}"] \arrow[r, bend right=50, "{r(-,V_2)}"'] \arrow[r, phantom, "\Downarrow r(-{,}\phi)"] & \mathbf{Mod}_{\mathbb{1}}
\end{tikzcd}
\end{array}
.
$$

governs the derivation of **FRS formalism** from locally trivialized 2-transport.

(That the second inclusion is in fact an equivalence goes back to a theorem by Ostrik.)

- The chain of injections

$$
\begin{array}{ccccc}
\Sigma(\mathbf{Vect}) & \longrightarrow & \mathbf{KVVect} & \xrightarrow{\sim} & {}_{\mathbf{Vect}}\mathbf{Mod} \\
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "V_1"] \arrow[r, bend right=50, "V_2"'] \arrow[r, phantom, "\Downarrow \phi"] & \bullet
\end{tikzcd}
& \mapsto &
\begin{tikzcd}
1 \arrow[r, bend left=50, "{[V_1]}"] \arrow[r, bend right=50, "{[V_2]}"'] \arrow[r, phantom, "\Downarrow {[\phi]}"] & 1
\end{tikzcd}
& \mapsto &
\begin{tikzcd}
\mathbf{Mod}_{\mathbb{C}} \arrow[r, bend left=50, "{r(-,V_1)}"] \arrow[r, bend right=50, "{r(-,V_2)}"'] \arrow[r, phantom, "\Downarrow r(-{,}\phi)"] & \mathbf{Mod}_{\mathbb{C}}
\end{tikzcd}
\end{array}
.
$$

governs the derivation of the **FHK state sum model** from locally trivialized 2-transport.

(That the second inclusion is in fact an equivalence is due to a theorem by Yetter.)

Proper Local Trivialization. An i-trvialization is a pullback cone of

$$\begin{tikzcd} & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

It need not, however, in general be the pullback itself (the universal pullback cone), which might not even exist. Rather, we are interested in those p-local trivializations which admit a **splitting** $\begin{tikzcd} & \mathcal{P} \arrow[dl, "s"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\swArrow\sim"] & \\ \mathcal{P}_{\mathcal{U}} \arrow[rr, "p"'] & & \mathcal{P} \end{tikzcd}$ of $\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P}$.

Definition 2 *We call the transport* $\mathrm{tra} : \mathcal{P} \to T$ **properly** *p-locally i-trivializable if a 2-morphism*

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "s"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\swArrow\sim"] & \\ \mathcal{P}_{\mathcal{U}} \arrow[rr, "p"] \arrow[dd, "\mathrm{tra}^i"'] & & \mathcal{P} \arrow[dd, "\mathrm{tra}"] \\ & \swArrow\sim & \\ T' \arrow[rr, "i"'] & & T \end{tikzcd}$$

exists.

Hence a properly p-locally i-trivializable n-transport factors (weakly) through an i-trivial transport. A major aspect of the study of n-transport is the determination of proper local trivializations. Proper local trivializations provide what is often called the **local data** of parallel transport.

Example 3 (parallel transport in vector bundles)

Let $E \xrightarrow{\pi} M$ be a smooth rank-n $\mathbb{C}$-vector bundle with connection ∇. Denote by T_E the transport groupoid of E and by $\mathcal{P}(M)$ the groupoid of thin homotopy classes of paths in M. The connection gives rise to the smooth parallel transport functor

$$\mathrm{tra}_\nabla : \mathcal{P}_1(M) \to T_E \,,$$

which acts as

$$\mathrm{tra}\left(\begin{tikzcd} x \arrow[r, bend left, "\gamma"] & y \end{tikzcd} \right) = \begin{tikzcd} E_x \arrow[r, bend left, "\mathrm{tra}_\nabla(\gamma)"] & E_y \end{tikzcd} \,,$$

where $E_x \equiv \pi^{-1}(x)$. Using the forgetful functor

$$T_E \to \mathbf{Vect}$$

which forgets the smooth structure on T_E, we obtain a transport

$$\mathcal{P}_1(M) \xrightarrow{\mathrm{tra}} T_E \longrightarrow \mathbf{Vect}$$

as in example 1.

Consider the suspension $\Sigma(M(n \times n, \mathbb{C}))$ from example 2.

Picking any $x \in M$ together with a basis $E_x \underset{\sim}{\xrightarrow{\mathcal{A}}} \mathbb{C}^n$ induces an injection

$$\begin{array}{ccc} \Sigma(M(n \times n, \mathbb{C})) & \xrightarrow{i} & T_E \\ \bullet \xrightarrow{A} \bullet & \mapsto & E_x \xrightarrow{\mathcal{A}} \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n \xrightarrow{\mathcal{A}^{-1}} E_x \end{array} .$$

Obviously, E is trivial in the ordinary sense iff tra is i-trivial.

Now let $\mathcal{U} = \bigsqcup_i U_i$ be a good covering of M by open contractible sets. Let $\check{\mathrm{C}}(\mathcal{U})$ be the Čech-groupoid of $\mathcal{U}$ and let $\mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right)$ be the groupoid of paths in $\check{\mathrm{C}}(\mathcal{U})$ [6]. A typical morphism in $\mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right)$ looks like

$$\begin{tikzcd} (x,i) \arrow[r, bend left, "(\gamma_1, i)"] & (y,i) \arrow[r] & (y,j) \arrow[r, bend left, "(\gamma_2, j)"] & (z,j) \end{tikzcd} .$$

This is sent

- by the canonical surjection

$$\mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right) \xrightarrow{p} \mathcal{P}_1(M)$$

to

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] & y \arrow[r, "\mathrm{Id}"] & y \arrow[r, bend left, "\gamma_2"] & z \end{tikzcd} ,$$

- by the pulled back transport

$$\begin{tikzcd} \mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right) \arrow[r, "p"] & \mathcal{P}_1(M) \arrow[d, "\mathrm{tra}_\nabla"] \\ & T_E \end{tikzcd}$$

to

$$\begin{tikzcd} E_x \arrow[r, bend left, "\mathrm{tra}_\nabla(\gamma_1)"] & E_y \arrow[r, "\mathrm{Id}"] & E_y \arrow[r, bend left, "\mathrm{tra}_\nabla(\gamma_2)"] & E_y \end{tikzcd} ,$$

- by an i-trivial transport

$$\begin{tikzcd} \mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right) \arrow[d, "\mathrm{tra}^i"'] & \\ \Sigma(M(n\times n,\mathbb{C})) \arrow[r, "i"'] & T_E \end{tikzcd}$$

to

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, bend left, "\mathrm{tra}^i_i(\gamma_1)"] & \mathbb{C}^n \arrow[r, "g_{ij}(x)"] & \mathbb{C}^n \arrow[r, bend left, "\mathrm{tra}^i_j(\gamma_2)"] & \mathbb{C}^n \end{tikzcd} .$$

A p-local i-trivialization t of tra

$$\begin{tikzcd} \mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right) \arrow[r, "p"] \arrow[d, "\mathrm{tra}^i"'] \arrow[dr, phantom, "\overset{t}{\swarrow}\sim"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ \Sigma(M(n\times n,\mathbb{C})) \arrow[r, "i"'] & T_E \end{tikzcd}$$

is hence given by naturality squares of the form

$$\begin{tikzcd} (x,i) \arrow[r, bend left, "(\gamma_1,i)"] & (y,i) \arrow[r] & (y,j) \arrow[r, bend left, "(\gamma_2,j)"] & (z,j) \end{tikzcd}$$

$$\begin{tikzcd} E_x \arrow[r, "\mathrm{tra}(\gamma_1)"] \arrow[d, "t_i(x)"] & E_y \arrow[r, "\mathrm{Id}"] \arrow[d, "t_i(y)"] & E_y \arrow[r, "\mathrm{tra}(\gamma_2)"] \arrow[d, "t_j(y)"] & E_z \arrow[d, "t_j(z)"] \\ \mathbb{C}^n \arrow[r, "\mathrm{tra}^i_i(\gamma_1)"'] & \mathbb{C}^n \arrow[r, "g_{ij}(y)"'] & \mathbb{C}^n \arrow[r, "\mathrm{tra}^i_j(\gamma_2)"'] & \mathbb{C}^n \end{tikzcd} .$$

This encodes local trivialization of E in the ordinary sense. The cocylce relation follows from functoriality.

This trivialization is in fact proper. We obtain a splitting

$$
\begin{tikzcd}
 & \mathcal{P}_1(M) \arrow[dl, "s"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "//"] & \\
\mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right) \arrow[rr, "p"'] & & \mathcal{P}_1(M)
\end{tikzcd}
$$

by choosing for each $x \in M$ a lift $(x,i) \in \mathcal{U}$. In terms of this choice $\mathcal{P}_1(M) \xrightarrow{s} \mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right)$ acts by decomposing each path $\gamma \in \mathrm{Mor}(\mathcal{P})$ into open intervals with a smooth lift and inserting a transition morphism $(x,i) \longrightarrow (x,j)$ at each jump.

Had we chosen a trivilization with $\mathcal{P}_1(\mathcal{U})$ instead of $\mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right)$ there would not have been any splitting at all.

In the presence of this splitting we have an isomorphism

$$
\begin{tikzcd}
 & \mathcal{P}_1(M) \arrow[dl, "s"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "//"] & \\
\mathcal{P}_1\left(\check{\mathrm{C}}(\mathcal{U})\right) \arrow[rr, "p"] \arrow[d, "\mathrm{tra}^i"'] \arrow[drr, phantom, "\swarrow\sim"] & & \mathcal{P}_1(M) \arrow[d, "\mathrm{tra}"] \\
\Sigma(M(n \times n, \mathbb{C})) \arrow[rr, "i"'] & & \mathbf{Vect}
\end{tikzcd}
$$

which gives rise to naturality squares of the following kind.

Let $x \xrightarrow{\gamma} x$ be a closed path in M and let there be a good covering together with a choice of splitting such that x is lifted to (x,i) while the lift of $\gamma(1-\epsilon)$ goes to (x,j) for $\epsilon \to 0$. Then

$$
s\left(
\begin{tikzcd}
x \arrow[r, bend left, "\gamma"] & x
\end{tikzcd}
\right) =
\begin{tikzcd}
(x,i) \arrow[r, bend left, "(\gamma,i)"] & (x,j) \arrow[r] & (x,i)
\end{tikzcd}
$$

and the naturality squares are

$$
\begin{tikzcd}
(x,i) \arrow[r, bend left, "{(\gamma,i)}"] & (x,j) \arrow[r] & (x,i)
\end{tikzcd}
$$

$$
\begin{tikzcd}
V_x \arrow[r, "\mathrm{tra}(\gamma)"] \arrow[d, "t_i(x)"'] & V_x \arrow[r, "\mathrm{Id}"] \arrow[d, "t_j(x)"] & V_x \arrow[d, "t_i(x)"] \\
\mathbb{C}^n \arrow[r, "\mathrm{tra}_i^i(\gamma)"'] & \mathbb{C}^n \arrow[r, "g_{ji}(x)"'] & \mathbb{C}^n
\end{tikzcd}
.
$$

Note that the image of the path γ appearing at the bottom of this diagram is the one we would trace over.

Example 4 (parallel transport in 2-bundles)

The discussion is closely analogous to the previous example, with everything lifted from paths to surfaces. Let $\mathcal{P}_2\left(\check{\mathrm{C}}_2\left(\mathcal{U}\right)\right)$ the 2-category of 2-paths in the Čech 2-category of a good covering $\mathcal{U} \to M$ [6].

A typical 2-morphism in $\mathcal{P}_2\left(\check{\mathrm{C}}_2\left(\mathcal{U}\right)\right)$ looks like

$$\begin{tikzcd}
 & & (y,k) \arrow[dr] \arrow[d, phantom, "\Downarrow"] & & \\
(x,i) \arrow[r, "(\gamma_1',i)"'] \arrow[r, bend left=70, "(\gamma_1,i)"] \arrow[r, phantom, shift left=2, "\Downarrow(S_1,i)"] & (y,i) \arrow[ur] \arrow[rr] & & (y,j) \arrow[r, "(\gamma_2',j)"'] \arrow[r, bend left=70, "(\gamma_2,j)"] \arrow[r, phantom, shift left=2, "\Downarrow(S_2,j)"] & (z,j)
\end{tikzcd}.$$

This is sent

- by the canonical surjection

$$\mathcal{P}_2\left(\check{\mathrm{C}}_2\left(\mathcal{U}\right)\right) \xrightarrow{p} \mathcal{P}_2\left(M\right)$$

  to

$$\begin{tikzcd}
 & & y \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow\mathrm{Id}"] & & \\
x \arrow[r, "\gamma_1'"'] \arrow[r, bend left=70, "\gamma_1"] \arrow[r, phantom, shift left=2, "\Downarrow S_1"] & y \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}" description] & & y \arrow[r, "\gamma_2'"'] \arrow[r, bend left=70, "\gamma_2"] \arrow[r, phantom, shift left=2, "\Downarrow S_2"] & z
\end{tikzcd}$$

- by the pulled back transport

$$\begin{tikzcd}
\mathcal{P}_2\left(\check{\mathrm{C}}_2\left(\mathcal{U}\right)\right) \arrow[r, "p"] & \mathcal{P}_2\left(M\right) \arrow[d, "\mathrm{tra}"] \\
 & T_E
\end{tikzcd}$$

  to

$$\begin{tikzcd}
 & & E_y \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow\mathrm{Id}"] & & \\
E_x \arrow[r, "\mathrm{tra}(\gamma_1')"'] \arrow[r, bend left=70, "\mathrm{tra}(\gamma_1)"] \arrow[r, phantom, shift left=2, "\Downarrow\mathrm{tra}(S_1)"] & E_y \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}" description] & & E_y \arrow[r, "\mathrm{tra}(\gamma_2')"'] \arrow[r, bend left=70, "\mathrm{tra}(\gamma_2)"] \arrow[r, phantom, shift left=2, "\Downarrow\mathrm{tra}(S_2)"] & E_z
\end{tikzcd}$$

- by an i-trivial transport

$$\begin{tikzcd}
\mathcal{P}_2\left(\check{\mathrm{C}}_2\left(\mathcal{U}\right)\right) \arrow[d, "\mathrm{tra}^i"'] & \\
\Sigma\left(M\left(n \times n, \mathbb{C}\right)\right) \arrow[r, "i"'] & T_E
\end{tikzcd}$$

to

$$
\begin{tikzcd}
 & & \bullet \arrow[dr, "g_{kj}(y)"] & & \\
\bullet \arrow[r, bend left=60, "\mathrm{tra}^i_i(\gamma_1)"] \arrow[r, "\mathrm{tra}^i_i(\gamma_1')"'] \arrow[r, phantom, shift left=2, "\Downarrow \mathrm{tra}^i_i(S_1)"] & \bullet \arrow[ur, "g_{ik}(y)"] \arrow[rr, "g_{ij}(y)"'] \arrow[rr, phantom, shift left=3, "\Downarrow f_{ikj}(y)"] & & \bullet \arrow[r, bend left=60, "\mathrm{tra}^i_j(\gamma_2)"] \arrow[r, "\mathrm{tra}^i_j(\gamma_2')"'] \arrow[r, phantom, shift left=2, "\Downarrow \mathrm{tra}^i_j(S_2)"] & \bullet
\end{tikzcd}
$$

A p-local i-trivialization t of tra

$$
\begin{tikzcd}
\mathcal{P}_2\left(\check{\mathrm{C}}_2(\mathcal{U})\right) \arrow[r, "p"] \arrow[d, "\mathrm{tra}^i"'] \arrow[dr, phantom, "\overset{t}{\swarrow} \sim"] & \mathcal{P}_2 \arrow[d, "\mathrm{tra}"] \\
\Sigma(M(n \times n, \mathbb{C})) \arrow[r, "i"'] & T_E
\end{tikzcd}
$$

is hence given by naturality tin cans of the form depicted in figure 1. This encodes the transition relations discussed in BaezSchreiber:2005.

$$
\begin{tikzcd}
 & & (y,k) \arrow[dr] \arrow[d, phantom, "\Downarrow"] & & \\
(x,i) \arrow[r, "(\gamma_1',i)"'] \arrow[r, bend left=60, "(\gamma_1,i)"] \arrow[r, phantom, shift left=3ex, "\Downarrow (S_1,i)"] & (y,i) \arrow[ur] \arrow[rr] & & (y,j) \arrow[r, "(\gamma_2',j)"'] \arrow[r, bend left=60, "(\gamma_2,j)"] \arrow[r, phantom, shift left=3ex, "\Downarrow (S_2,j)"] & (z,j)
\end{tikzcd}
$$

.

$$
\begin{tikzcd}
 & & E_y \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & & \\
E_x \arrow[r, "\mathrm{tra}(\gamma_1')"'] \arrow[r, bend left=60, "\mathrm{tra}(\gamma_1)"] \arrow[r, phantom, shift left=3ex, "\Downarrow \mathrm{tra}(S_1)"] \arrow[d, "t_i(x)"'] \arrow[dr, phantom, "\swarrow t_i(\gamma_1')"] & E_y \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}"] \arrow[d, "t_i(y)"'] \arrow[drr, phantom, "\swarrow \tilde{\phi}_{ij}(y)"] & & E_y \arrow[r, "\mathrm{tra}(\gamma_2')"'] \arrow[r, bend left=60, "\mathrm{tra}(\gamma_2)"] \arrow[r, phantom, shift left=3ex, "\Downarrow \mathrm{tra}(S_2)"] \arrow[d, "t_j(x)"'] \arrow[dr, phantom, "\swarrow t_j(\gamma_2')"] & E_z \arrow[d, "t_j(z)"] \\
\bullet \arrow[r, "\mathrm{tra}_i^i(\gamma_1)"'] & \bullet \arrow[rr, "g_{ij}(y)"'] & & \bullet \arrow[r, "\mathrm{tra}_j^i(\gamma_2)"'] & \bullet
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & & E_y \arrow[dr, "\mathrm{Id}"] \arrow[d, "t_k(y)"] & & \\
E_x \arrow[r, bend left=60, "\mathrm{tra}(\gamma_1)"] \arrow[d, "t_i(x)"'] \arrow[dr, phantom, "\swarrow t_i(\gamma_1)"] & E_y \arrow[ur, "\mathrm{Id}"] \arrow[d, "t_i(y)"'] \arrow[r, phantom, "\swarrow \tilde{\phi}_{ik}(y)"] & \bullet \arrow[dr, "g_{kj}(y)"] \arrow[r, phantom, "\swarrow \tilde{\phi}_{kj}(y)"] & E_y \arrow[r, bend left=60, "\mathrm{tra}(\gamma_2)"] \arrow[d, "t_j(x)"'] \arrow[dr, phantom, "\swarrow t_j(\gamma_2)"] & E_z \arrow[d, "t_j(z)"] \\
\bullet \arrow[r, "\mathrm{tra}_i(\gamma_1')"'] \arrow[r, bend left=60, "\mathrm{tra}_i(\gamma_1)"] \arrow[r, phantom, shift left=3ex, "\Downarrow \mathrm{tra}_i(S_1)"] & \bullet \arrow[ur, "g_{ik}(y)"] \arrow[rr, "g_{ij}(y)"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow f_{ikj}(y)"] & & \bullet \arrow[r, "\mathrm{tra}_j(\gamma_2')"'] \arrow[r, bend left=60, "\mathrm{tra}_j(\gamma_2)"] \arrow[r, phantom, shift left=3ex, "\Downarrow \mathrm{tra}_j(S_2)"] & \bullet
\end{tikzcd}
$$

Figure 1: **Tin can equation** expressing the existence of a local trivialization of 2-transport in a 2-bundle, as discussed in example 4.

Example 5 (trivialization on covering space)

Consider 2-transport $\mathrm{tra} : \mathcal{P}_2(M) \to T$ with $M = T^2$ the torus. In order to trivialize this, realize the torus as a $\mathbb{Z} \times \mathbb{Z}$ orbifold

$$\begin{tikzcd} \mathbb{R}^2 \arrow[d, "p"] \\ T^2 \end{tikzcd}$$

and consider the pullback

$$\begin{tikzcd} \mathcal{P}_2\left(\mathbb{R}^2/\mathbb{Z}\times\mathbb{Z}\right) \arrow[r, "p"] & \mathcal{P}_2\left(T^2\right) \arrow[d, "\mathrm{tra}"] \\ & T \end{tikzcd} ,$$

where $\mathcal{P}_2\left(\mathbb{R}^2/\mathbb{Z}\times\mathbb{Z}\right)$ is the 2-category of 2-paths in the 2-groupoid representing the orbifold [6]. This pullback is split with the lift of the full torus

$$\begin{tikzcd} & x \arrow[dr, "B"] & \\ x \arrow[ur, "A"] \arrow[dr, "B"'] \arrow[rr, phantom, "\Downarrow S"] & & x \\ & x \arrow[ur, "B"'] & \end{tikzcd}$$

under

$$\begin{tikzcd} & \mathcal{P}_2\left(T^2\right) \arrow[dl, "s"'] \arrow[dr, "\mathrm{Id}"] & \\ \mathcal{P}_2\left(\mathbb{R}^2/\mathbb{Z}\times\mathbb{Z}\right) \arrow[rr, "p"] & & \mathcal{P}_2\left(T^2\right) \end{tikzcd}$$

being given by the 2-morphism

$$
\begin{tikzcd}
 & & & (x,k) \arrow[dr] \arrow[d, phantom, "\Downarrow"] & & & \\
 & & (x,l) \arrow[ur] \arrow[rr] \arrow[dr] \arrow[dd, phantom, "\Downarrow"] & & (x,i) \arrow[ddrr] \arrow[dd, phantom, "\Downarrow"] & & \\
 & & & (x,j) \arrow[ur] \arrow[u, phantom, "\Downarrow"] \arrow[ddrr, "{(B,j)}"] \arrow[dddd, phantom, "\Downarrow{(S,i)}"] & & & \\
(x,k) \arrow[uurr] \arrow[dr] & & & & & & (x,l) \\
 & (x,i) \arrow[uurr, "{(A,i)}"] \arrow[ddrr, "{(B,i)}"'] & & & & (x,l) \arrow[ur] & \\
 & & & & & & \\
 & & & (x,k) \arrow[uurr, "{(A,k)}"'] & & &
\end{tikzcd}
\quad .
$$

Trivial transport of the transition boundary of this 2-morphism yields the local gluing data. More generally, for orbifolds this yields the “twisted sector phases” [5].

Example 6 (transport in KV 2-vector bundles)

(Warning: This example is rather sketchy. Handle with care.)

Assume we have a transport $\mathrm{tra} : \mathcal{P}(M) \to \mathbf{KVVect}$, where $\mathcal{P}(M)$ is the 2-gropupoid of thin-homotopy classes of 2-paths in some smooth space M and where **KVVect** is the 2-category of Kapranov-Voevodsky 2-vector spaces.

Let

$$\mathrm{tra}_{1,2} : \mathcal{P}(M) \to \Sigma(\mathbf{Vect})$$

be two line-2-bundles [3]. tra shall be expressible in terms of these as follows

$$\mathrm{tra}\left(\begin{tikzcd} x \arrow[rr, bend left, "\gamma"] \arrow[rr, bend right, "\gamma'"'] & \Downarrow S & y \end{tikzcd} \right) \equiv \begin{tikzcd}[column sep=small] & \begin{bmatrix} \mathrm{tra}_1(\gamma) & 0 \\ 0 & \mathrm{tra}_2(\gamma) \end{bmatrix} \arrow[dr] & \\ 2 \arrow[ur, no head] \arrow[dr, no head] & \begin{array}{c} \| \\ \begin{bmatrix} \mathrm{tra}_1(S) & 0 \\ 0 & \mathrm{tra}_2(S) \end{bmatrix} \\ \Downarrow \end{array} & 2 \\ & \begin{bmatrix} \mathrm{tra}_1(\gamma') & 0 \\ 0 & \mathrm{tra}_2(\gamma') \end{bmatrix} \arrow[ur] & \end{tikzcd} .$$

We might think of this as a "batch" of two "uncoupled" line-2-bundles on top of each other. Assume now furthermore that both these line-2-bundles have trivializing gerbe modules, i.e. that they are trivializable with trivializations possibly given by morphisms in higher-rank vector bundles. Assume furthermore that these gerbe modules form special ambidextrous adjunctions in the 2-category of gerbes (see §1.2 for more on this).

Under these conditions tra may be trivialized with respect to

$$\Sigma(\mathbf{Vect}) \longrightarrow \mathbf{KVVect} .$$

In order to do so, we need special ambidextrous adjunctions

$$n \xrightarrow{t} 1 \ \in \mathrm{Mor}_1(\mathbf{KVVect}) .$$

These are given by a tuple of n vector spaces $(t_i)_{i=1}^n$. Let

$$1 \xrightarrow{\bar{t}} n$$

be given by the tuple of dual vector spaces $(\bar{t}_i)_{i=1}^n = ((t_i)^*)_{i=1}^n$. Let 2-morphisms

$$\begin{tikzcd} & 1 \arrow[dr, bend left, "\bar{t}"] & \\ n \arrow[ur, bend left, "t"] \arrow[rr, bend right, "I_n"'] & \Downarrow & n \end{tikzcd}$$

$$
\begin{tikzcd}
 & n \arrow[dr, bend left, "t"] & \\
1 \arrow[ur, bend left, "\bar{t}"] \arrow[rr, bend right, "I_1"'] \arrow[rr, phantom, "\Downarrow"] & & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
n \arrow[rr, bend left, "I_n"] \arrow[dr, bend right, "t"'] \arrow[rr, phantom, "\Downarrow"] & & n \\
 & 1 \arrow[ur, bend right, "\bar{t}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
1 \arrow[rr, bend left, "I_1"] \arrow[dr, bend right, "\bar{t}"'] \arrow[rr, phantom, "\Downarrow"] & & 1 \\
 & n \arrow[ur, bend right, "t"'] &
\end{tikzcd}
$$

be given by componentwise idenity and evaluation morphisms in the obvious way. (See below for an example.)

The Frobenius algebra induced by this adjunction is the algebra of the direct sum of endomorphisms

$$
\bigoplus_{i=1}^{n} \mathrm{End}\,(t_i)\ .
$$

By Wedderburn's theorem [**?**] every semisimple algebra is isomorphic to a direct sum of matrix algebras, hence to an algebra of the above kind. Note that there are different Frobenius structures on these algebras. Compare example 4.8 in LaudaPfeiffer:2006.

It follows that [...] (compare claim 1, p. 41)

For $n = 2$, as in the above mentioned setup, these morphisms look as follows

- $$
2 \xrightarrow{t} 1 \;=\; 2 \xrightarrow{\left[\begin{array}{cc} V & W \end{array}\right]} 1\ , \qquad 1 \xrightarrow{\bar{t}} 2 \;=\; 1 \xrightarrow{\left[\begin{array}{c} V^* \\ W^* \end{array}\right]} 2
$$

- $$
2 \xrightarrow{t} 1 \xrightarrow{\bar{t}} 2 \;=\; 2 \xrightarrow{\left[\begin{array}{cc} V^*\otimes V & V^*\otimes W \\ W^*\otimes V & W^*\otimes W \end{array}\right]} 2
$$

$$1 \xrightarrow{\bar{t}} 2 \xrightarrow{t} 1 \;=\; 1 \xrightarrow{\left[\; V \otimes V^* \oplus W \otimes W^* \;\right]} 1$$

$$\begin{tikzcd} & 1 \arrow[dr, bend left, "\bar{t}"] & \\ 2 \arrow[ur, bend left, "t"] \arrow[rr, bend right, "I_2"'] \arrow[rr, phantom, "\Downarrow"] & & 2 \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=huge, row sep=large] 2 \arrow[r, bend left=40, "\begin{bmatrix} V^* \otimes V & V^* \otimes W \\ W^* \otimes V & W^* \otimes W \end{bmatrix}"] \arrow[r, bend right=40, "\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}"'] \arrow[r, phantom, "\begin{array}{c} \| \\ \begin{bmatrix} e_V & 0 \\ 0 & e_W \end{bmatrix} \\ \Downarrow \end{array}"] & 2 \end{tikzcd}$$

-

$$\begin{tikzcd} & 2 \arrow[dr, bend left, "t"] & \\ 1 \arrow[ur, bend left, "\bar{t}"] \arrow[rr, bend right, "I_1"'] \arrow[rr, phantom, "\Downarrow"] & & 1 \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=huge, row sep=large] 1 \arrow[r, bend left=40, "\left[\; V \otimes V^* \oplus W \otimes W^* \;\right]"] \arrow[r, bend right=40, "\left[\; K \;\right]"'] \arrow[r, phantom, "\begin{array}{c} \| \\ \left[\; e_V \oplus e_W \;\right] \\ \Downarrow \end{array}"] & 1 \end{tikzcd}$$

Pullback. We have seen that local trivialization of transport is related to a pullback cone. On the other hand, what one would want to call *pullback of transport*

$$\begin{tikzcd} \mathcal{P}' \arrow[r, "p"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ & T \end{tikzcd}$$

is just composition of morphisms. There is not (to my knowledge) any sensible universal property that would complete this diagram to a square.

Note that this composition by itself already induces ordinary pullback of the bundles induced by the transport functor, since $E'_x = (\mathrm{tra} \circ p)(x) = \mathrm{tra}(p(x)) = E_{p(x)}$.

In certain situations, however, we may want to demand that pulled back transport factors as

$$\begin{tikzcd} \mathcal{P}' \arrow[r, "p"] \arrow[d, "\mathrm{tra}'"'] \arrow[dr, phantom, "\overset{\sim}{\swarrow}"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} ,$$

for specified T'. For instance if $T = \mathrm{Trans}(E)$ is the transport n-groupoid of an n-bundle $E \to M$ and $\mathrm{tra} : \mathcal{P}_n(M) \to \mathrm{Trans}(E)$ is a smooth transport on smooth n-paths in M, and if $M' \xrightarrow{f} M$ is a smooth map, then we may want to factor[1]

$$\begin{tikzcd} \mathcal{P}_n(M') \arrow[r, "f"] \arrow[d, "f^*\mathrm{tra}"'] \arrow[dr, phantom, "\overset{\sim}{\swarrow}"] & \mathcal{P}_n(M) \arrow[d, "\mathrm{tra}"] \\ \mathrm{Trans}(f^*E) \arrow[r] & \mathrm{Trans}(E) \end{tikzcd} .$$

[1] I am indebted to Konrad Waldorf for discussion of this point.

1.2 Transition

Trivialization allows to relate transport with codomain T to transport with some codomain T'. Under suitable conditions we may forget about T alltogether and perform transitions entirely within T'.

Definition 3 *Given a p-local i-trivialization*

$$\begin{tikzcd}
\mathcal{P}_{\mathcal{U}} \arrow[r, "p"] \arrow[d, "\mathrm{tra}_{\mathcal{U}}"'] \arrow[dr, phantom, "\overset{\sim}{\swarrow}\, t"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\
T' \arrow[r, "i"'] & T
\end{tikzcd}$$

we call a choice of 2-morphism

$$\begin{tikzcd}
 & & T' \arrow[dr, "i"] \arrow[dd, phantom, "\Downarrow g"] & \\
\mathcal{P}_{\mathcal{U}}^{[2]} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_{\mathcal{U}} \arrow[ur, "\mathrm{tra}_{\mathcal{U}}"] \arrow[dr, "\mathrm{tra}_{\mathcal{U}}"'] & & T \\
 & & T' \arrow[ur, "i"'] &
\end{tikzcd}$$

together with a choice of isomorphism

$$\begin{tikzcd}
 & & T' \arrow[dr, "i"] \arrow[dd, phantom, "\Downarrow g"] & \\
\mathcal{P}_{\mathcal{U}}^{[2]} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_{\mathcal{U}} \arrow[ur, "\mathrm{tra}_{\mathcal{U}}"] \arrow[dr, "\mathrm{tra}_{\mathcal{U}}"'] & & T \\
 & & T' \arrow[ur, "i"'] &
\end{tikzcd}
\overset{\phi}{\simeq}
\begin{tikzcd}
 & & & T' \arrow[drr, "i"] \arrow[d, phantom, "\Downarrow \bar{t}"] & & \\
\mathcal{P}_{\mathcal{U}}^{[2]} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_{\mathcal{U}} \arrow[urr, "\mathrm{tra}_{\mathcal{U}}"] \arrow[drr, "\mathrm{tra}_{\mathcal{U}}"'] \arrow[rr, "p"] & & \mathcal{P} \arrow[rr, "\mathrm{tra}"] \arrow[d, phantom, "\Downarrow t"] & & T \\
 & & & T' \arrow[urr, "i"'] & &
\end{tikzcd}$$

a choice of p**-local** i**-transition***.*

Let us write

$$A \xrightarrow{p^*\mathrm{tra}} C \quad \equiv \quad A \xrightarrow{p} B \xrightarrow{\mathrm{tra}} C \ .$$

Then a choice of transition is a choice of a 2-morphisms of the following form

$$\begin{tikzcd}
 & {\begin{array}{c} p_1^* p^* \mathrm{tra} \\ = p_2^* p^* \mathrm{tra} \end{array}} \arrow[dr, "p_2^* t"] \arrow[d, phantom, "\sim\Downarrow \phi"] & \\
p_1^* \mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_1^* \bar{t}"] \arrow[rr, "g"'] & {} & p_2^* \mathrm{tra}_{\mathcal{U}}
\end{tikzcd} .$$

Definition 4 *Given a transition, we can construct*

$$
\begin{tikzcd}[column sep=small]
 & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[dr, "p_{23}^*g"] & \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_{12}^*g"] \arrow[rr, "p_{13}^*g"'] \arrow[rr, phantom, shift left=8, "f\Uparrow"] & & p_3^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
\equiv
\begin{tikzcd}[column sep=small]
 & & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[drr, "p_{23}^*g"] \arrow[d, bend left, "p_2^*\bar{t}"] & & \\
 & & p^*\mathrm{tra} \arrow[u, bend left, "p_2^*t"] \arrow[u, phantom, "\Uparrow"] \arrow[drr, "p_3^*t"'] \arrow[ull, phantom, "p_{12}^*\bar{\phi}\,\Swarrow"] \arrow[urr, phantom, "\Nearrow\, p_{23}^*\bar{\phi}"] & & \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[uurr, "p_{12}^*g"] \arrow[urr, "p_1^*t"] \arrow[rrrr, "p_{13}^*g"'] \arrow[rrrr, phantom, shift left=3, "p_{13}^*\phi\Uparrow"] & & & & p_3^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[dr, "p_{23}^*g"] & \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_{12}^*g"] \arrow[rr, "p_{13}^*g"'] \arrow[rr, phantom, shift left=8, "\tilde{f}\Downarrow"] & & p_3^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
\equiv
\begin{tikzcd}[column sep=small]
 & & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[drr, "p_{23}^*g"] \arrow[d, bend left, "p_2^*\bar{t}"] & & \\
 & & p^*\mathrm{tra} \arrow[u, bend left, "p_2^*t"] \arrow[u, phantom, "\Downarrow"] \arrow[drr, "p_3^*t"'] \arrow[ull, phantom, "p_{12}^*\phi\,\Searrow"] \arrow[urr, phantom, "\Swarrow\, p_{23}^*\phi"] & & \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[uurr, "p_{12}^*g"] \arrow[urr, "p_1^*t"] \arrow[rrrr, "p_{13}^*g"'] \arrow[rrrr, phantom, shift left=3, "p_{13}^*\bar{\phi}\Downarrow"] & & & & p_3^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
.
$$

Proposition 1

The 2-morphisms f and $\tilde{f}$ have the following properties.

1. For $(n = 1)$-transport f and $\tilde{f}$ are identity morphisms.
2. For $(n = 2)$-transport and $t, \bar{t}$ a special ambidextrous adjunction $\tilde{f}$ is associative, f is coassociative and together they satisfy the Frobenius property.

Proof. Follows from standard properties of adjunctions. See [2] for more details. □

Example 7

1. A choice of ($\Sigma(G) \xrightarrow{i} G\mathbf{Tor}$)-transition in a principal G-bundle with connection is a choice of Čech 1-cocycles.
2. A choice of ($\Sigma(G_2) \xrightarrow{i} G_2\mathbf{Tor}$)-transition in a principal G_2-2-bundle with connection is a choice of Čech 2-cocycles.
3. A choice of ($\Sigma(\Sigma(\mathbb{C})) \xrightarrow{i} \Sigma(\mathbf{Vect})$)-transition in a line-2-bundle with connection is an **abelian bundle gerbe** with connection and curving.

4. A choice of $(\Sigma(\mathrm{Aut}(H)) \xrightarrow{i} \Sigma(\mathbf{BiTor}(H)))$-transition is a (fake flat) **nonabelian bundle gerbe** with connection and curving.

Morphisms of Trivializations. There are several ways along which to motivate the notion of a morphism between choices of local trivializations. One is to regard a choice of local trivialization including a choice of transition

$$\begin{tikzcd}
& {\substack{p_1^* p^* \mathrm{tra} \\ = p_2^* p^* \mathrm{tra}}} \arrow[dr, "p_2^* t"] \arrow[d, phantom, "\Downarrow\phi"] & \\
p_1^* \mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_1^* \bar{t}"] \arrow[rr, "g"'] & {} & p_2^* \mathrm{tra}_{\mathcal{U}}
\end{tikzcd}$$

as the 2-functorial image of an abstract triangle

$$\begin{tikzcd}
& \cdot \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
\cdot \arrow[ur] \arrow[rr] & {} & \cdot
\end{tikzcd},$$

and then to define morphisms between these images following the definition of morphisms of 2-functors.

Definition 5 *Consider the n-category $T^{\mathcal{P}}$ of n-transport functors with domain $\mathcal{P}$ and codomain T. Fix $T' \xrightarrow{i} T$ and $\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P}$. The* **2-category of p-local i-trivializations** *of transport in $T^{\mathcal{P}}$ is the 2-category defined as follows:*

1. *objects are p-local i-trivializations together with i-transitions $\mathcal{G} = (\mathrm{tra}_{\mathcal{U}}, t, \phi)$*

2. *a morphism $\mathcal{G} \xrightarrow{\epsilon} \mathcal{G}'$ is a morphism*

$$\mathrm{tra} \xrightarrow{f} \mathrm{tra}'$$

together with a map

$$\epsilon : \{t, \bar{t}, g\} \longrightarrow \mathrm{Mor}_2(\mathbf{L2B}(\mathcal{U}))$$

given by

$$t \quad \mapsto \quad \begin{tikzcd}
p^* \mathrm{tra} \arrow[r, "t"] \arrow[d, "p^* f"'] \arrow[dr, phantom, "\swarrow \epsilon_t"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h"] \\
p^* \mathrm{tra}' \arrow[r, "t'"'] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}$$

$$
\begin{tikzcd}
\bar{t} \arrow[r, phantom, "\mapsto"] & \mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow \epsilon_{\bar{t}}"] & p^*\mathrm{tra} \arrow[d, "p^*f"] \\
 & \mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"'] & p^*\mathrm{tra}'
\end{tikzcd}
$$

$$
\begin{tikzcd}
g \arrow[r, phantom, "\mapsto"] & p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p_1^*h"'] \arrow[dr, phantom, "\swarrow \epsilon_{g}"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h"] \\
 & p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "g'"'] & p_2^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\tag{1}
$$

such that all relevant tin can equations hold:

(a) <u>*tin can based on the transition modification*</u>

$$
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, "g"] \arrow[d, "p_1^*h"'] \arrow[drr, phantom, "\swarrow \epsilon_{g}"] & & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "g'"] \arrow[dr, "p_1^*\bar{t}'"'] & \Downarrow \phi' & p_2^*\mathrm{tra}'_{\mathcal{U}} \\
 & p_1^*p^*\mathrm{tra} \arrow[ur, "p_2^*t'"'] &
\end{tikzcd}
=
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_1^*\bar{t}"] \arrow[rr, bend left=50, "g"] \arrow[d, "p_1^*h"'] \arrow[dr, phantom, "\swarrow p_1^*\epsilon_{\bar{t}}"] & p_1^*p^*\mathrm{tra} \arrow[u, phantom, "\Downarrow \phi"] \arrow[r, "p_2^*t"] \arrow[d, "p_1^*p^*f"] \arrow[dr, phantom, "\swarrow p_2^*\epsilon_{t}"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "p_1^*\bar{t}'"] & p_1^*p^*\mathrm{tra}' \arrow[r, "p_2^*t'"] & p_2^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\tag{2}
$$

(b) *tin can based on the unit on $t \circ \bar{t}$*

$$
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[rr, "\mathrm{Id}"] \arrow[d, "h"'] \arrow[drr, phantom, "\swArrow\,\mathrm{Id}"] & & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "\mathrm{Id}"] \arrow[dr, "\bar{t}'"'] & \Downarrow & \mathrm{tra}'_{\mathcal{U}} \\
& p^*\mathrm{tra}' \arrow[ur, "t'"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h"'] \arrow[rr, bend left=60, "\mathrm{Id}"] \arrow[dr, phantom, "\swArrow\,\epsilon_{\bar{t}}"] & p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f"] \arrow[dr, phantom, "\swArrow\,\epsilon_t"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"] & p^*\mathrm{tra}' \arrow[r, "t'"] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\tag{3}
$$

(c) *tin can based on the unit on $\bar{t} \circ t$*

$$
\begin{tikzcd}
p^*\mathrm{tra} \arrow[rr, "\mathrm{Id}"] \arrow[d, "p^*f"'] \arrow[drr, phantom, "\swArrow\,\mathrm{Id}"] & & p^*\mathrm{tra} \arrow[d, "p^*f"] \\
p^*\mathrm{tra}' \arrow[rr, "\mathrm{Id}"] \arrow[dr, "t'"'] & \Downarrow & p^*\mathrm{tra}' \\
& \mathrm{tra}'_{\mathcal{U}} \arrow[ur, "t'"'] &
\end{tikzcd}
=
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f"'] \arrow[rr, bend left=60, "\mathrm{Id}"] \arrow[dr, phantom, "\swArrow\,\epsilon_t"] & \mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h"] \arrow[dr, phantom, "\swArrow\,\epsilon_{\bar{t}}"] & p^*\mathrm{tra} \arrow[d, "p^*f"] \\
p^*\mathrm{tra}' \arrow[r, "t'"] & \mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"] & p^*\mathrm{tra}'
\end{tikzcd}
. \tag{4}
$$

Note that this implies in particular the following tin can equation:

$$
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, "p_{13}^*g"] \arrow[d, "p_1^*h"'] \arrow[drr, phantom, "\swarrow p_{13}^*\epsilon_g"] & & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_3^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "p_{13}^*g'"] \arrow[dr, "p_{12}^*g'"'] \arrow[rr, phantom, bend right=20, "\Downarrow f'"] & & p_3^*\mathrm{tra}'_{\mathcal{U}} \\
& p_2^*\mathrm{tra}' \arrow[ur, "p_{23}^*g'"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_{12}^*g"] \arrow[rr, bend left=50, "p_{13}^*g"] \arrow[d, "p_1^*h"'] \arrow[dr, phantom, "\swarrow p_{12}^*\epsilon_g"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_{23}^*g"] \arrow[d, "p_2^*h"] \arrow[dr, phantom, "\swarrow p_{23}^*\epsilon_g"] \arrow[u, phantom, "\Downarrow f"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_3^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "p_{12}^*g'"] & p_2\mathrm{tra}'_{\mathcal{U}} \arrow[r, "p_{23}^*g'"] & p_3^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\tag{5}
$$

3. *a 2-morphism between 1-morphisms between local pre-trivializations*

$$
\begin{tikzcd}
\mathcal{G} \arrow[r, bend left=50, "\epsilon_1"] \arrow[r, bend right=50, "\epsilon_2"'] \arrow[r, phantom, "\Downarrow E"] & \mathcal{G}'
\end{tikzcd}
$$

is a "modification of the above pseudonatural transformations" in the sense that it is a map

$$E : \{h, f\} \longrightarrow \mathrm{Mor}_2(\mathbf{L2B}(\mathcal{U}))$$

given by

$$
h \quad \mapsto \quad
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, bend left=50, "h_1"] \arrow[r, bend right=50, "h_2"'] \arrow[r, phantom, "\Downarrow E_h"] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

and

$$
f \quad \mapsto \quad
\begin{tikzcd}
\mathrm{tra} \arrow[r, bend left=50, "f_1"] \arrow[r, bend right=50, "f_2"'] \arrow[r, phantom, "\Downarrow E_f"] & \mathrm{tra}'
\end{tikzcd}
$$

such that the modification tin can equations

$$
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f_1"] \arrow[d, bend right=60, "p^*f_2"', ""{name=L}] \arrow[dr, phantom, "\swArrow \epsilon_{t1}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h_1"] \\
p^*\mathrm{tra}' \arrow[r, "t'"'] & \mathrm{tra}'_{\mathcal{U}}
\arrow[from=L, to=1-1, phantom, "\Leftarrow p^*E_f"]
\end{tikzcd}
=
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f_2"] \arrow[dr, phantom, "\swArrow \epsilon_{t2}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h_2"] \arrow[d, bend left=60, "h_1", ""'{name=R}] \\
p^*\mathrm{tra}' \arrow[r, "t"'] & \mathrm{tra}'_{\mathcal{U}}
\arrow[from=R, to=1-2, phantom, "\Leftarrow E_h"]
\end{tikzcd}
\tag{6}
$$

and

$$
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h_1"] \arrow[d, bend right=60, "h_2"', ""{name=L}] \arrow[dr, phantom, "\swArrow \epsilon_{\bar{t}1}"] & p^*\mathrm{tra} \arrow[d, "p^*f_1"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"'] & p^*\mathrm{tra}'
\arrow[from=L, to=1-1, phantom, "\Leftarrow E_h"]
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h_2"] \arrow[dr, phantom, "\swArrow \epsilon_{\bar{t}2}"] & p^*\mathrm{tra} \arrow[d, "p^*f_2"] \arrow[d, bend left=60, "p^*f_1", ""'{name=R}] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}"'] & p^*\mathrm{tra}'
\arrow[from=R, to=1-2, phantom, "\Leftarrow p^*E_f"]
\end{tikzcd}
\tag{7}
$$

hold.

It is straightforward to slightly generalize this definition, for instance such as to allow morphisms between transport trivialized with respect to different p.

Example 8

For trivializations with respect to $\left(\Sigma(\Sigma(\mathbb{C})) \xrightarrow{i} \Sigma(\mathbf{Vect}) \right)$ a morphism of i-transitions is what is called a **stable isomorphism of bundle gerbes** [3] if all 2-morphisms take values in 1-dimensional vector spaces.

Such a morphism with target the trivial transition is called a trivialization of a bundle gerbe.

A morphism of the same sort but now with the 2-morphisms in the tin can equation allowed to take values in all of $\Sigma(\mathbf{Vect})$ is called a **bundle gerbe module**.

Similar statement hold for transitions with respect to $\left(\Sigma(\mathrm{Aut}(H)) \xrightarrow{i} \Sigma(\mathbf{BiTor}(H)) \right)$ and their relation to stable isomorphisms for and modules of nonabelian bundle gerbes.

Trivialization of Transition. For $(n \geq 2)$-transport a transition

$$p_1^*\mathrm{tra}_{\mathcal{U}} \xrightarrow{g} p_2^*\mathrm{tra}_{\mathcal{U}}$$

is a natural transformation internal to **Cat** and hence itself an $(n-1)$-transport. Therefore there is a notion of local trivialization of g itself, and so on. An n-transport admits up to n-layers of local trivializations.

Example 9

Trivializing the transition of example 8 amounts to trivializing the bundle involved in a bundle gerbe. This yields Čech cocycles representing the bundle gerbe.

1.3 Trace

Of particular interest is n-transport over n-paths of **nontrivial topology**, those which are not isomorphic to an n-disk. Describing transport $\mathrm{tra} : \mathcal{P} \to T$ over such n-paths in terms of n-morphisms of a geometric n-category requires certain structure at least on the codomain T, possibly also on the domain $\mathcal{P}$.

The structure needed on T is the existence of **partial traces** which implement the gluing of n-paths along $(n-1)$-paths. This gluing may, or may not, be already present in $\mathcal{P}$.

In Segal's description of n-dimensional QFT in terms of 1-functors on 1-categories of n-cobordisms this is not a seperate issue, since the cobordisms may have arbitrary topology. The n-categorical refinement which we are considering here, however, requires a framework which allows to construct topologically nontrivial n-cobordisms by gluing topologically trivial n-morphisms.

Dimension $n = 2$. Let $\mathcal{P}$ be some geometric 2-category. Assume that $\mathcal{P}$ has the following special properties

1. Every 1-morphism $x \xrightarrow{\gamma} y$ is part of an ambidextrous adjunction.
2. All the monoidal 1-categories $\mathrm{Hom}_{\mathcal{P}}(x, x)$ are braided.

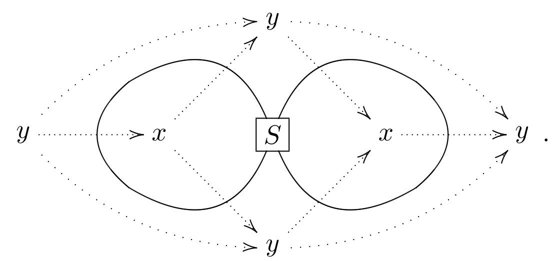
Sphere. Consider a 2-morphism

$$
\begin{tikzcd}
& y \arrow[dr, "B"] & \\
x \arrow[ur, "A"] \arrow[dr, "A"'] \arrow[rr, phantom, "\Downarrow S"] & & x \\
& y \arrow[ur, "B"'] &
\end{tikzcd}
$$

in $\mathcal{P}$. Glue the two copies of A and the two copies of B by composing with unit and counit of the respective adjunctions.

$$
\begin{tikzcd}
& & y \arrow[dr, "B"] \arrow[drr, bend left, "\mathrm{Id}"] & & \\
y \arrow[r, "\bar{A}"] \arrow[urr, bend left, "\mathrm{Id}"] \arrow[drr, bend right, "\mathrm{Id}"'] & x \arrow[ur, "A"] \arrow[dr, "A"'] \arrow[rr, phantom, "\Downarrow S"] \arrow[u, phantom, "\Searrow"] \arrow[d, phantom, "\Swarrow"] & & x \arrow[r, "\bar{B}"] \arrow[u, phantom, "\Swarrow"] \arrow[d, phantom, "\Searrow"] & y \\
& & y \arrow[ur, "B"'] \arrow[urr, bend right, "\mathrm{Id}"'] & &
\end{tikzcd}
$$

The Poincaré-dual string diagram is



Torus. Consider a 2-morphism

$$\begin{tikzcd}
& x \arrow[dr, "B"] & \\
x \arrow[ur, "A"] \arrow[dr, "B"'] \arrow[rr, phantom, "\Downarrow S"] & & x \\
& x \arrow[ur, "A"'] &
\end{tikzcd}$$

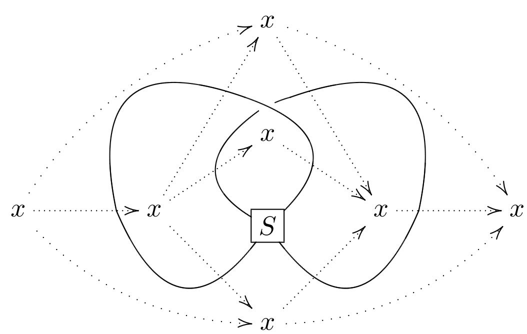
in $\mathcal{P}$. In order to be able to glue A with A and B with B, first move them on the same side by composing with a braiding

$$\begin{tikzcd}
& x \arrow[ddr, "A"] \arrow[d, phantom, "\Downarrow"] & \\
& x \arrow[dr, "B"] & \\
x \arrow[uur, "B"] \arrow[ur, "A"] \arrow[dr, "B"'] \arrow[rr, phantom, "\Downarrow S"] & & x \\
& x \arrow[ur, "A"'] &
\end{tikzcd} .$$

Then glue by composing with unit and counit of the respective adjunctions.

$$\begin{tikzcd}
& & x \arrow[ddrr, bend left, "\mathrm{Id}"] \arrow[ddr, "A"] \arrow[d, phantom, "\Downarrow"] & & \\
& & x \arrow[dr, "B"] & & \\
x \arrow[uurr, bend left, "\mathrm{Id}"] \arrow[r, "\bar{B}"] \arrow[drr, bend right, "\mathrm{Id}"'] & x \arrow[uur, "B"] \arrow[ur, "A"] \arrow[dr, "B"'] \arrow[rr, phantom, "\Downarrow S"] & & x \arrow[r, "\bar{A}"] & x \\
& & x \arrow[ur, "A"'] \arrow[urr, bend right, "\mathrm{Id}"'] & &
\end{tikzcd}$$

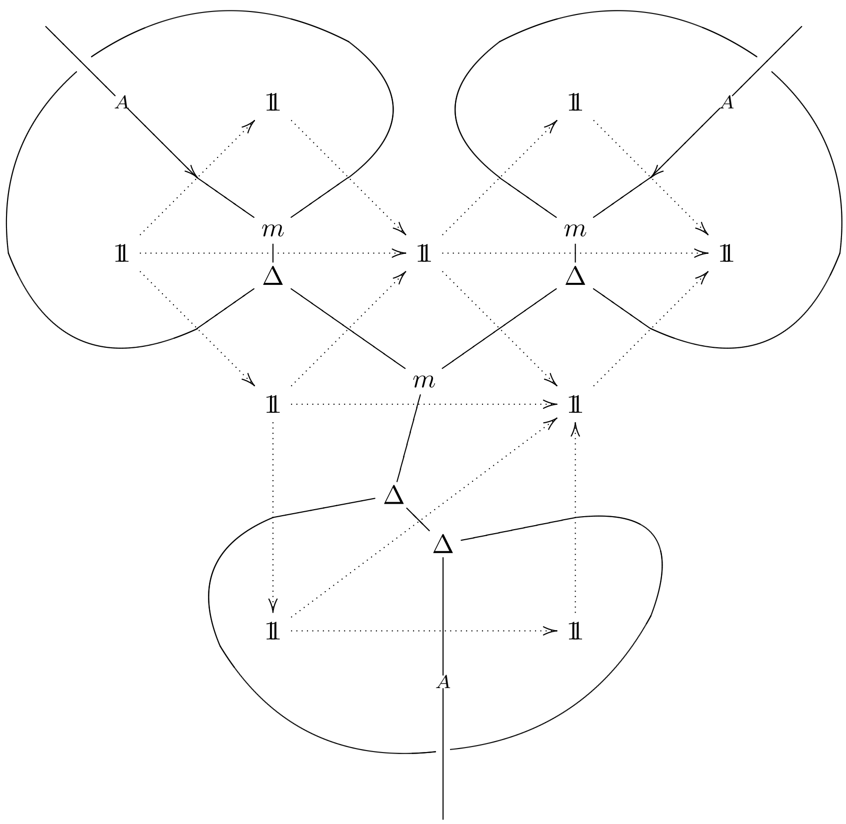
The Poincaré-dual string diagram is



Trinion (Pair of Pants). Consider the pair of pants

$$
\begin{tikzcd}
 & x \arrow[dr, "A"] & & z \arrow[dr, "\gamma_2"] & \\
x \arrow[ur, "\gamma_1"] \arrow[dr, "A"'] \arrow[rr, phantom, "\Downarrow"] & & w \arrow[ur, "B"] \arrow[dr] \arrow[rr, phantom, "\Downarrow"] \arrow[d, phantom, "\Downarrow"] & & z \\
 & w \arrow[ur] \arrow[rr] \arrow[d, "C"'] \arrow[drr, phantom, "\Downarrow"] & {} & w \arrow[ur, "B"'] & \\
 & z \arrow[rr] & & z \arrow[u, "\bar{C}"'] &
\end{tikzcd}
$$

With the structure described above we cannot do the required braiding in order to contract identitfied boundaries. But we may consider the image under some 2-transport of this 2-morphism in a braided tensor category (possibly obtained by first locally trivializing) and then braid and trace in that image. For instance, for a trivialization as in [1] this yields



.

2 Results

Those examples given in §1 which are not by themselves obvious or well-known, follow from a couple of results which are outlined in the following.

2.1 Trivialization

Proper Local Trivialization. Much of the theory of transport revolves around the question how a given transport looks like in term of "local data". In our language this amounts to the question on which $\mathcal{P}_{\mathcal{U}}$ a given transport may be *properly* trivialized.

Our main result concerning trivialization of 2-transport is [1, 2]

Proposition 2 *Let* $\mathrm{tra} : \mathcal{P}_2(M) \to T$ *be 2-transport on 2-paths in* M *and let* $T' \xrightarrow{i} T$ *be given. The transport* tra

- *admits a (* $\mathcal{P}_2\left(\check{C}_2(M)\right) \xrightarrow{p} \mathcal{P}_2(M)$ *)-local* i*-trivialization*
- *if there is a good covering* $\mathcal{U} = \bigsqcup_i U_i$ *of* M *such that all* $\mathrm{tra}|_{U_i}$ *are* i*-trivializable with the trivialization fitting into a special ambidextrous adjunction.*

This trivialization is proper (def. 2).

Here $\mathcal{P}_2\left(\check{\mathrm{C}}_2(M)\right)$ is the 2-category of 2-paths in the Čech 2-category induced by the good covering [6]. Note that a ($\mathcal{P}_2\left(\check{\mathrm{C}}_2(M)\right) \xrightarrow{p} \mathcal{P}_2(M)$)-local trivialization implies a ($\mathcal{P}_2(\mathcal{U}) \xrightarrow{p} \mathcal{P}_2(M)$)-local trivialization. But the latter is proper is and only if the good covering contains a patch which covers all of M.

This proposition is based on two results which say that

1. if trivialization of $\mathrm{tra}|_{U_i}$ is a special ambidextrous adjunction, then $\mathrm{tra}|_{U_i}$ may be expressed entirely in terms of trivial transport and trivialization data (prop 3).
2. The trivialization data glues over double intersection $U_i \cap U_j$ to transition data (§1.2).

Proposition 3 *If two (transport) 2-functors are related by a special ambidextrous adjunction*

$$\begin{tikzcd}
\mathrm{tra}^i \arrow[rr, bend left, "\bar{t}"] \arrow[rr, phantom, shift left=2, "\overset{i}{\Leftrightarrow}"] \arrow[rr, phantom, shift right=2, "\underset{e}{\Leftrightarrow}"] & & \mathrm{tra} \arrow[ll, bend left, "t"]
\end{tikzcd}$$

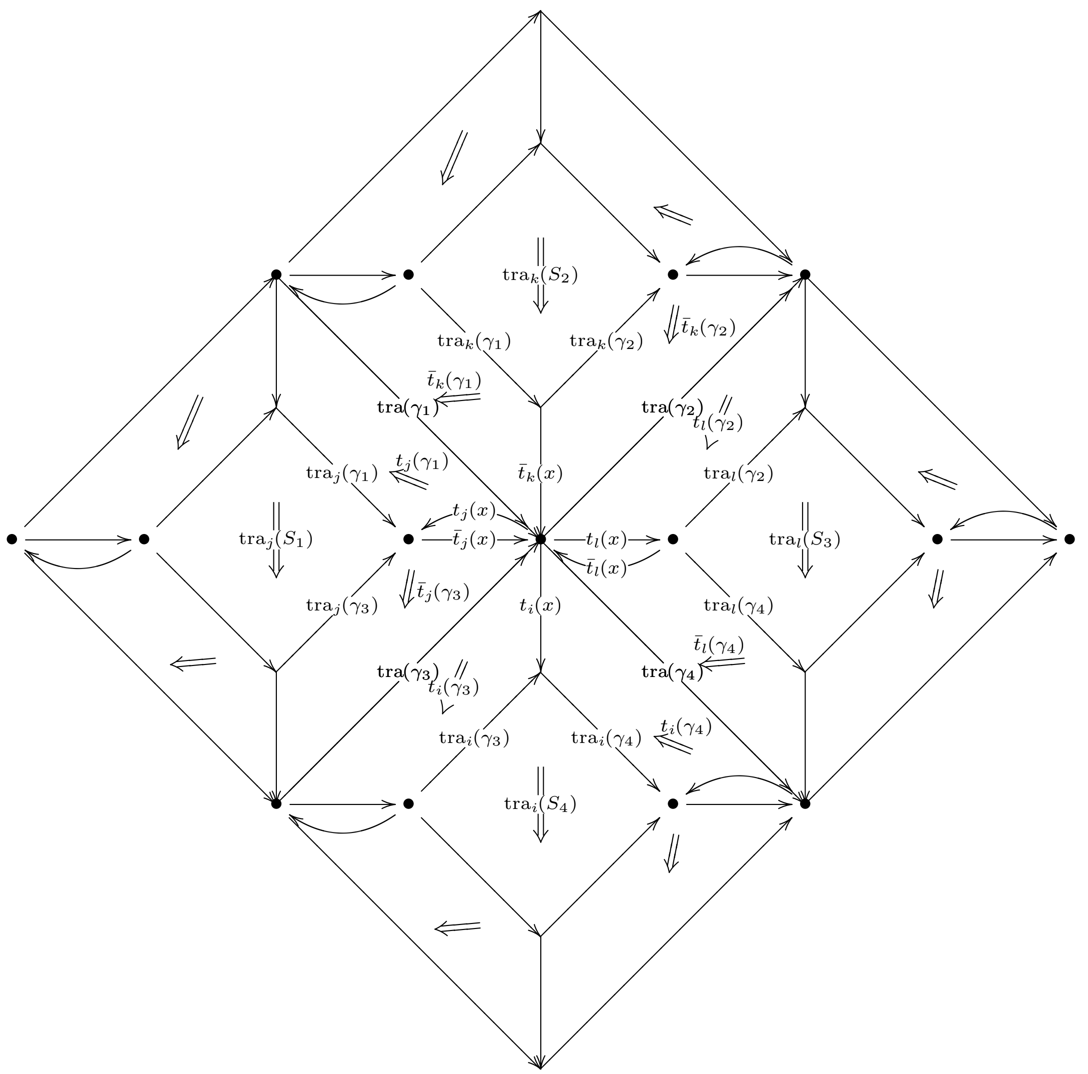
then

$$\mathrm{tra}\left(\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_1"'] \arrow[r, phantom, "\Downarrow S"] & y \end{tikzcd}\right) = \begin{tikzcd} & \mathrm{tra}(x) \arrow[r, "\mathrm{tra}(\gamma_1)"] \arrow[d, "t(x)"] \arrow[dd, bend right=70, "A_y"'] \arrow[dr, phantom, "t(\gamma_1)\,\Swarrow"] & \mathrm{tra}(y) \arrow[d, "t(y)"] \arrow[dd, bend left=70, "A_y"] & \\ {} \arrow[r, phantom, "\overset{\tilde{e}}{\Leftarrow}"] & \mathrm{tra}^i(x) \arrow[r, bend left, "\mathrm{tra}^i(\gamma_1)"] \arrow[r, bend right, "\mathrm{tra}^i(\gamma_2)"'] \arrow[r, phantom, "\Downarrow \mathrm{tra}^i(S)"] \arrow[d, "\bar{t}(x)"] & \mathrm{tra}^i(y) \arrow[d, "\bar{t}(y)"] \arrow[r, phantom, "\overset{j}{\Leftarrow}"] & {} \\ & \mathrm{tra}(x) \arrow[r, "\mathrm{tra}(\gamma_2)"'] \arrow[ur, phantom, "\bar{t}(\gamma_2)\,\Swarrow"] & \mathrm{tra}(y) & \end{tikzcd} .$$

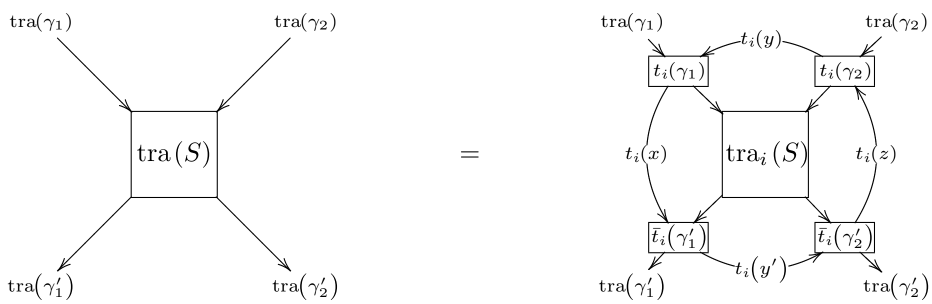
We shall find it convenient to write this as

$$\begin{tikzcd} & \bullet \arrow[dr, "\mathrm{tra}(\gamma_2)"] \arrow[dd, phantom, "\Downarrow \mathrm{tra}(S)"] & \\ \bullet \arrow[ur, "\mathrm{tra}(\gamma_1)"] \arrow[dr, "\mathrm{tra}(\gamma_1')"'] & & \bullet \\ & \bullet \arrow[ur, "\mathrm{tra}(\gamma_2')"'] & \end{tikzcd} = \frac{1}{D} \begin{tikzcd} & & & \bullet \arrow[d, "t_i(y)"] \arrow[ddrrr, "\mathrm{tra}(\gamma_2)"] & & & \\ & & & \bullet \arrow[drr, "\mathrm{tra}_i(\gamma_2)"] \arrow[dd, phantom, "\Downarrow \mathrm{tra}_i(S)"] & & & \\ \bullet \arrow[uurrr, "\mathrm{tra}(\gamma_1)"] \arrow[ddrrr, "\mathrm{tra}(\gamma_1')"'] \arrow[r, "t_i(x)"] \arrow[r, phantom, "\Leftarrow e_i(x)"'] & \bullet \arrow[l, bend left, "\bar{t}_i(x)"] \arrow[urr, "\mathrm{tra}_i(\gamma_1)"] \arrow[drr, "\mathrm{tra}_i(\gamma_1')"'] \arrow[u, phantom, "t_i(\gamma_1)\,\Swarrow"] \arrow[d, phantom, "\bar{t}_i(\gamma_1)\,\Leftarrow"] & & & & \bullet \arrow[r, "\bar{t}_i(z)"'] \arrow[r, phantom, "\Leftarrow \iota_i(z)"] \arrow[u, phantom, "\Swarrow\, t_i(\gamma_2)"] \arrow[d, phantom, "\Downarrow \bar{t}_i(\gamma_2)"] & \bullet \arrow[l, bend right, "t_i(z)"'] \\ & & & \bullet \arrow[urr, "\mathrm{tra}_i(\gamma_2')"'] \arrow[d, "\bar{t}_i(y')"] & & & \\ & & & \bullet \arrow[uurrr, "\mathrm{tra}(\gamma_2')"'] & & & \end{tikzcd} .$$

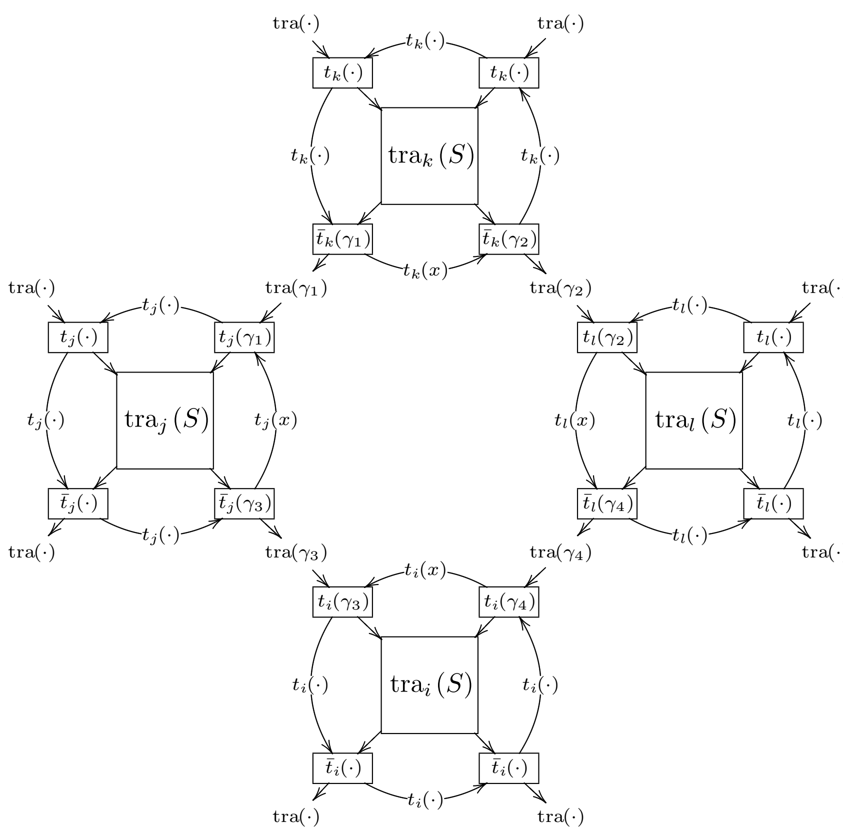
In order to show that attaching such diagrams over double intersections one obtains transition data consider a generic point where up to four patches intersect



In order to simplify this it is convenient to pass to Poincaré-dual string diagram notation

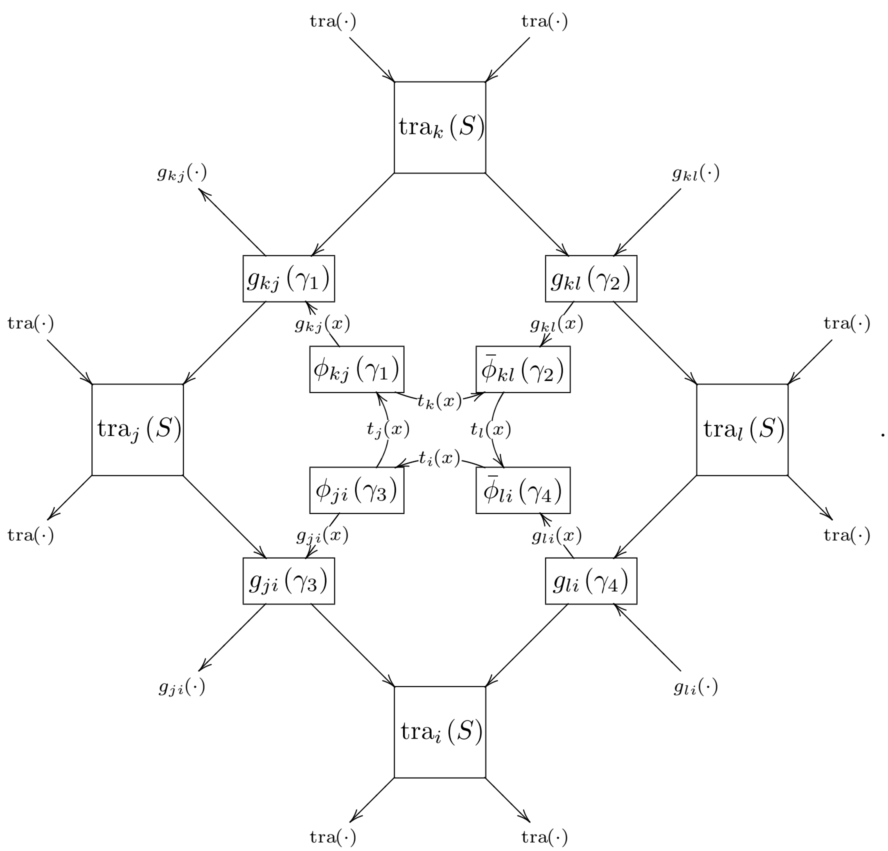


in terms of which the above reads



Now using the definition of transition one gets

Proposition 4 *The above equals*



Passing back to the globular version of this diagram one manifestly sees how this defines a 2-transport on 2-paths in the Čech-2-category of the good covering.

(While this is “obvious” it should eventually be turned into a more formal discussion.)

2.2 Transition

One motivation for the abstract definition of n-transport is to realize several known structures, such as

- – abelian bundle gerbes with connection and curving
  – nonabelian bundle gerbes with connection and curving
- – Fukuma-Hosono-Kawai description of 2D TFT
  – Fuchs-Runkel-Schweigert description of 2D CFT

as **trivialization and transition** of certain 2-transport.

The first two of these items have easy answers.

Proposition 5 *Abelian bundle gerbes* $L \longrightarrow Y^{[2]} \longrightarrow M$ *with connection and curving are in bijection with transitions in* ($\mathcal{P}_2(Y) \xrightarrow{p} \mathcal{P}_2(M)$)*-locally* ($\Sigma(\Sigma(\mathbb{C})) \xrightarrow{i} \Sigma(\mathbf{Vect}_{\mathbb{C}})$) *trivialized 2-transport.*

This is the content of [3]. If properly set up, one has the stronger statement that the 2-category of bundle gerbes with connection and curving is equivalent to that of transitions of $\Sigma(\mathbf{Vect})$-transport.

Proposition 6 *Nonabelian bundle gerbes* $L \longrightarrow Y^{[2]} \longrightarrow M$ *with connection and curving are in bijection with transitions in* ($\mathcal{P}_2(Y) \xrightarrow{p} \mathcal{P}_2(M)$)*-locally* ($\Sigma(\mathrm{Aut}(H)) \xrightarrow{i} \Sigma(\mathbf{BiTor}(H))$) *trivialized 2-transport.*

Part of this is the content of [4]. A full proof is pretty much analogous to that for abelian bundle gerbes but still needs to be written down.

In order to make progress with the third item on the above list it is necessary to have a relation between morphisms of transport and morphisms of trivializations of transport. That the former embed into the latter, as one would hope, is the content of the following propositions.

Morphisms of Transitions.

Proposition 7 *Let* tra *and* tra' *be transport 2-functors with local pre-trivializations* $\mathcal{G}$ *and* $\mathcal{G}'$*, respectively. For every morphism*

$$\mathrm{tra} \xrightarrow{f} \mathrm{tra}'$$

there is (at least) one morphism

$$\mathcal{G} \xrightarrow{\epsilon(f)} \mathcal{G}'$$

in the 2-category of pre-trivializations.

Proof. The proof is given in [3]. $\square$

Corollary 1 *Let* tra *be a transport 2-functor with two p-local i-trivializations* $\mathcal{G}$ *and* $\mathcal{G}'$*. There is (at least) one morphism*

$$\mathcal{G} \xrightarrow{\epsilon(\mathcal{G},\mathcal{G}')} \mathcal{G}' \ .$$

Proof. Set $f = \mathrm{Id}$ in the above proposition. $\square$

Proposition 8 *Let* tra *and* tra′ *be transport 2-functors with p-local i-trivializations* $\mathcal{G}$ *and* $\mathcal{G}'$*, respectively. For every 2-morphisms of transport 2-functors*

$$\begin{tikzcd} \mathrm{tra} \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[rr, phantom, "\Downarrow\mathcal{A}"] & & \mathrm{tra}' \end{tikzcd}$$

there is (at least) one 2-morphism

$$\begin{tikzcd} \mathcal{G} \arrow[rr, bend left, "\epsilon(f_1)"] \arrow[rr, bend right, "\epsilon(f_2)"'] \arrow[rr, phantom, "\Downarrow E(\mathcal{A})"] & & \mathcal{G}' \end{tikzcd}$$

of local pre-trivializations.

Proof. The proof can be found in [3]. $\square$

State Sum Models from Transition of 2-Transport. Using this, we make the following (still somewhat vague) claims

Claim 1 *Let* tra *be a 2-transport in a Kapranov-Voevodsky 2-vector bundle which comes from a* **matrix of line-2-bundles** *with connection (see example 6, p. 19) that can be locally trivialized on all of* M *by means of gerbe modules. Then the local data of this transport are those of Fukuma-Hosono-Kawai.*

This is essentially the content of [2].

Claim 2 *Let* $\mathrm{tra} : \mathcal{P} \to {}_{\mathcal{C}}\mathbf{Mod}$ *be a transport with values in module categories of a modular tensor category. Locally trivializing this with respect to*

$$\Sigma(\mathcal{C}) \longrightarrow \mathbf{BiMod}(\mathcal{C}) \longrightarrow {}_{\mathcal{C}}\mathbf{Mod}$$

yields local data as given by Fuchs-Runkel-Schweigert.

Aspects of this claim have been demonstrated in [1]. More work has to be done. Here we shall content ourselves with sketching one example.

Example 10 (FRS disk diagram with one insertion)

Let the worldsheet Σ be a disk

$$\begin{tikzcd}
x \arrow[r, bend left=60, "\gamma_1"] \arrow[r, bend right=60, "\gamma_2"'] \arrow[r, phantom, "\Downarrow\Sigma"] & y
\end{tikzcd}$$

and let the transport 2-functor $\mathrm{tra} : \mathcal{P}_2 \to \mathbf{BiMod}(\mathcal{C})$ be such that

$$\mathrm{tra}\left(\begin{tikzcd}
x \arrow[r, bend left=60, "\gamma_1"] \arrow[r, bend right=60, "\gamma_2"'] \arrow[r, phantom, "\Downarrow\Sigma"] & y
\end{tikzcd}\right) = \begin{tikzcd}
A \arrow[r, bend left=50, "A\otimes^+ U"] \arrow[r, bend right=50, "A\otimes^- V"'] \arrow[r, phantom, "\Downarrow\rho"] & A
\end{tikzcd}$$

for A some algebra, $A \otimes^+ U$, $A \otimes^- V$ left-free A-bimodules induced by some objects U and V with right action induced by left braiding ($\otimes^+$) and right braiding ($\otimes^-$), respectively.

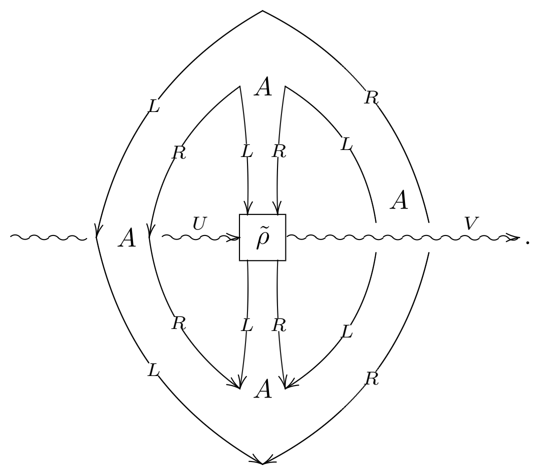
Attaching "trivial boundary conditions" (this is explained in [1]) and trivializing with respect to

$$\Sigma(\mathcal{C}) \longrightarrow \mathbf{BiMod}(\mathcal{C})$$

yields the corresponding trivialized 2-morphism

$$\begin{tikzcd}
& \mathbb{1} \arrow[r, "U"] \arrow[d, "L"'] \arrow[dddddd, bend right=80, "\mathbb{1}"'] \arrow[dr, phantom, "\swarrow_{+}"] & \mathbb{1} \arrow[d, "L"] \arrow[dddddd, bend left=80, "\mathbb{1}"] & \\
& A \arrow[r, "A\otimes^+ U"] \arrow[d, "R"] \arrow[dd, bend right=50, "A"'] \arrow[dr, phantom, "\swarrow_{\mathrm{Id}}"] & A \arrow[d, "R"'] \arrow[dd, bend left=50, "A"] & \\
& \mathbb{1} \arrow[r, "U"] \arrow[d, "L"] \arrow[l, phantom, "\overset{\tilde{e}}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow_{+}"] & \mathbb{1} \arrow[d, "L"'] \arrow[r, phantom, "\overset{j}{\Leftarrow}"] & \\
\mathbb{1} \arrow[r, phantom, "\Leftarrow_{e}"] & A \arrow[r, bend left, "A\otimes^+ U"] \arrow[r, bend right, "A\otimes^- V"'] \arrow[r, phantom, "\Downarrow\rho"] \arrow[d, "R"] \arrow[dd, bend right=50, "A"'] \arrow[dr, phantom, "\swarrow_{\mathrm{Id}}"] & A \arrow[d, "R"'] \arrow[dd, bend left=50, "A"] \arrow[r, phantom, "\Leftarrow_{\tilde{i}}"] & \mathbb{1} \\
& \mathbb{1} \arrow[r, "V"] \arrow[d, "L"] \arrow[l, phantom, "\overset{\tilde{e}}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow_{-}"] & \mathbb{1} \arrow[d, "L"'] \arrow[r, phantom, "\overset{j}{\Leftarrow}"] & \\
& A \arrow[r, "A\otimes^- V"] \arrow[d, "R"'] \arrow[dr, phantom, "\swarrow_{\mathrm{Id}}"] & A \arrow[d, "R"] & \\
& \mathbb{1} \arrow[r, "V"'] & \mathbb{1} &
\end{tikzcd}$$

living in $\mathcal{C}$. The Poincaré-dual string diagram of this globular diagram is



This is the diagram that describes 1-point disk correlators in FRS formalism.

2.3 Trace

[To be written. The main point here is to show that tracing 2-transport correctly captures the prescription for how to evaluate non-disk-shaped surfaces in gerbe holonomy, FHK and FRS.]

This text is based on the following notes. Please see the list of references in these for a collection of relevant literature.

References

[1] U. S., FRS Formalism from 2-Transport, private notes, available at http://www.math.uni-hamburg.de/home/schreiber/FRSfrom2Transport.pdf

[2] ____, 2D TFT from 2-Transport, http://www.math.uni-hamburg.de/home/schreiber/2TransportToTFT.pdf

[3] ____, Line-2-Bundles and Bundle Gerbes, http://www.math.uni-hamburg.de/home/schreiber/Line2Bundle.pdf

[4] ____, Nonabelian Bundle Gerbes from 2-Transport, Part I: Synthetic Bibundles, http://www.math.uni-hamburg.de/home/schreiber/NABGfrom2Trans.pdf

[5] ____, Equivariant 2-Bundles and 2-Transport

[6] ____, Paths in Categories, http://www.math.uni-hamburg.de/home/schreiber/CatPath.pdf