Transport, Trivialization, Transition

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Abstract

Parallel transport across surfaces is conceived 2-functorially. The 2-categories of local trivialization and transition of general 2-transport are defined. The chain of injections between them is established.

Parallel transport in a fiber bundle can be thought of as a functor from a category of paths in base space to a category of fibers. Accordingly, we here want to think of n-functors on n-paths as encoding parallel transport in higher order structures, like $(n-1)$-gerbes. One of our aims is to show that this is justified.

In our context, an 'n-transport' is nothing but an n-functor. We shall address n-functors as **n-transport** whenever we want to think of them as realizing an n-categorical analogue of parallel transport in a fiber bundle.

This implies that, usually, the domain of an n-transport is a 'geometric' n-category. We shall address an n-category as a **geometric n-category** whenever we want to think of its p-morphisms as p-dimensional spaces of some sort.

Hence, for our purposes, n-transport is an n-functor

$$\mathrm{tra} : \mathcal{P} \to T$$

from a geometric domain $\mathcal{P}$ to some target n-category T.

The point of addressing some n-functors as n-transport is that this suggests that we are interested in performing certain operations on them, notably, that we are interested in

- **local trivialization**
- **transition**

of our n-functors.

This is described in the following subsections. It turns out that various well-known and seemingly independend concepts are all special cases of locally trivialized n-transport.

In order to indicate the context in which we think of certain n-categories and n-functors below, we will use the following symbols.

$\mathcal{P}$	a geometric n-category
T	a codomain of an n-transport
$\mathrm{tra} : \mathcal{P} \to T$	an n-transport n-functor
$T' \xrightarrow{i} T$	an injection of n-transport codomains
$\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P}$	a surjection of n-transport domains

Whenever it matters, we here take $n = 2$.

1 Trivialization

Given any n-transport functor, it is often desireable to study its global and its local properties seperately. If the functor is locally trivializable in some suitable sense, we may express its global behaviour by gluing of local data.

Definition 1 *Given a transport* $\mathrm{tra} : \mathcal{P} \to T$ *as well as a morphism*

$$T' \xrightarrow{i} T$$

of codomains, we say that tra *is* **trivial** *with respect to* i*, or* i**-trivial** *iff there exists* $\mathrm{tra}^i : \mathcal{P} \to T'$ *such that*

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{tra}^i"'] \arrow[dr, phantom, "//"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

We say that tra *is* i**-trivializable** *iff there is a* **trivialization** t

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{tra}^i"'] \arrow[dr, phantom, "\overset{t}{\Swarrow}\sim"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

Finally, given a morphism

$$\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P} ,$$

we say that tra *is* p**-locally** i**-trivializable** *iff there is* t *such that*

$$\begin{tikzcd} \mathcal{P}_{\mathcal{U}} \arrow[r, "p"] \arrow[d, "\mathrm{tra}^i"'] \arrow[dr, phantom, "t \swArrow \sim"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

Here $\stackrel{\sim}{\Longrightarrow}$ denotes an adjoint equivalence.

For many applications one is interested in a weaker notion of trivialization, where $\stackrel{\sim}{\Longrightarrow}$ is just a special ambidextrous adjunction. (The relevant definitions are assembled in section **??**.) We shall speak of **generalized trivializations** if t is just required to be a special ambidextrous adjunction.

Proper Local Trivialization. An i-trvialization is a pullback cone of

$$\begin{tikzcd} & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

It need not, however, in general be the pullback itself (the universal pullback cone), which might not even exist. Rather, we are interested in those p-local trivializations, where we can weakly invert p, in the sense that they admit a **section**

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "s"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\swArrow \sim"] & \\ \mathcal{P}_{\mathcal{U}} \arrow[rr, "p"'] & & \mathcal{P} \end{tikzcd}$$

of $\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P}$.

Definition 2 *We call the transport* $\mathrm{tra} : \mathcal{P} \to T$ **properly** p*-locally* i*-trivializable* *if a 2-morphism*

$$\begin{tikzcd}
 & \mathcal{P} \arrow[dl, "s"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\swarrow\!\!\swarrow_{\sim}"] & \\
\mathcal{P}_{\mathcal{U}} \arrow[rr, "p"] \arrow[d, "\mathrm{tra}^i"'] & & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\
T' \arrow[rr, "i"'] \arrow[urr, phantom, "\swarrow\!\!\swarrow_{\sim}"] & & T
\end{tikzcd}$$

exists.

Hence a properly p-locally i-trivializable n-transport factors (weakly) through an i-trivial transport. A major aspect of the study of n-transport is the determination of proper local trivializations. Proper local trivializations provide what is often called the **local data** of parallel transport.

Pullback. We have seen that local trivialization of transport is related to a pullback cone. On the other hand, what one would want to call *pullback of transport*

$$\begin{tikzcd} \mathcal{P}' \arrow[r, "p"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ & T \end{tikzcd}$$

is just composition of morphisms. There is not any sensible universal property that would complete this diagram to a square.

Note that this composition by itself already induces ordinary pullback of the bundles induced by the transport functor, since $E'_x = (\mathrm{tra} \circ p)(x) = \mathrm{tra}(p(x)) = E_{p(x)}$.

In certain situations, however, we may want to demand that pulled back transport factors as

$$\begin{tikzcd} \mathcal{P}' \arrow[r, "p"] \arrow[d, "\mathrm{tra}'"'] \arrow[dr, phantom, "\stackrel{\sim}{\swarrow}"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} ,$$

for specified T'. For instance if $T = \mathrm{Trans}(E)$ is the transport n-groupoid of an n-bundle $E \to M$ and $\mathrm{tra} : \mathcal{P}_n(M) \to \mathrm{Trans}(E)$ is a smooth transport on smooth n-paths in M, and if $M' \xrightarrow{f} M$ is a smooth map, then we may want to factor

$$\begin{tikzcd} \mathcal{P}_n(M') \arrow[r, "f"] \arrow[d, "f^*\mathrm{tra}"'] \arrow[dr, phantom, "\stackrel{\sim}{\swarrow}"] & \mathcal{P}_n(M) \arrow[d, "\mathrm{tra}"] \\ \mathrm{Trans}(f^*E) \arrow[r] & \mathrm{Trans}(E) \end{tikzcd} .$$

2 Transition

Trivialization allows to relate transport with codomain T to transport with some codomain T'. Under suitable conditions we may forget about T altogether and perform transitions entirely within T'.

Definition 3 *Given a p-local i-trivialization*

$$
\begin{tikzcd}
\mathcal{P}_{\mathcal{U}} \arrow[r, "p"] \arrow[d, "\mathrm{tra}_{\mathcal{U}}"'] \arrow[dr, phantom, "\overset{\sim}{\swarrow} t"] & \mathcal{P} \arrow[d, "\mathrm{tra}"] \\
T' \arrow[r, "i"'] & T
\end{tikzcd}
$$

we call

$$
\begin{tikzcd}
 & & T' \arrow[dr, "i"] & \\
\mathcal{P}_{\mathcal{U}}^{[2]} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_{\mathcal{U}} \arrow[ur, "\mathrm{tra}_{\mathcal{U}}"] \arrow[dr, "\mathrm{tra}_{\mathcal{U}}"'] \arrow[rr, phantom, "\Downarrow g"] & & T \\
 & & T' \arrow[ur, "i"'] &
\end{tikzcd}
\quad \stackrel{\phi}{\equiv} \quad
\begin{tikzcd}
 & & T' \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow \bar{t}"] & \\
\mathcal{P}_{\mathcal{U}}^{[2]} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_{\mathcal{U}} \arrow[ur, "\mathrm{tra}_{\mathcal{U}}"] \arrow[dr, "\mathrm{tra}_{\mathcal{U}}"'] \arrow[r, "p"] & \mathcal{P} \arrow[r, "\mathrm{tra}"] \arrow[d, phantom, "\Downarrow t"] & T \\
 & & T' \arrow[ur, "i"'] &
\end{tikzcd}
$$

the induced p-**local** *i*-**transition**.

Notice that this isomorphism is a 3-morphism in the 3-category $2\mathbf{Cat}$ of 2-categories, hence a 2-morphisms in the Hom-2-category $\mathrm{Hom}_{2\mathbf{Cat}}\left(\mathcal{P}_{\mathcal{U}}^{[n]}, T\right)$. For convenience, all diagrams in the following live in this Hom-2-category.

Let us write

$$
A \xrightarrow{p^*\mathrm{tra}} C \quad \equiv \quad A \xrightarrow{p} B \xrightarrow{\mathrm{tra}} C \ .
$$

Then the above equation becomes an identity 2-morphism in $\mathrm{Hom}_{2\mathbf{Cat}}\left(\mathcal{P}_{\mathcal{U}}^{[2]}, T\right)$

$$
\begin{tikzcd}
 & \begin{matrix} p_1^* p^* \mathrm{tra} \\ = p_2^* p^* \mathrm{tra} \end{matrix} \arrow[dr, "p_2^* t"] \arrow[d, phantom, "\sim \Downarrow \phi"] & \\
p_1^* \mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_1^* \bar{t}"] \arrow[rr, "g"'] & & p_2^* \mathrm{tra}_{\mathcal{U}}
\end{tikzcd} \ .
$$

2.1 Transition Tetrahedra

The point of ($T' \xrightarrow{i} T$)-transition is that it allows us to forget about T and work entirely in terms of T'. In order to do so, we combine three ϕ-triangles to a single transition triangle.

Definition 4 *Given a transition, we can construct* **transition triangles**

$$
\begin{tikzcd}
 & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[dr, "p_{23}^*g"] \arrow[d, phantom, "f\Downarrow"] & \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_{12}^*g"] \arrow[rr, "p_{13}^*g"'] & {} & p_3^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
\equiv
\begin{tikzcd}
 & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[ddr, "p_{23}^*g"] \arrow[d, bend left, "p_2^*\bar{t}"] \arrow[d, phantom, "\Downarrow"] & \\
{} \arrow[r, phantom, "p_{12}^*\bar{\phi}\Rightarrow"] & p^*\mathrm{tra} \arrow[u, bend left, "p_2^*t"] \arrow[dr, "p_3^*t"] \arrow[d, phantom, "p_{13}^*\phi\Downarrow"] & {} \arrow[l, phantom, "\Leftarrow p_{23}^*\bar{\phi}"'] \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[uur, "p_{12}^*g"] \arrow[ur, "p_1^*\bar{t}"] \arrow[rr, "p_{13}^*g"'] & {} & p_3^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
.
$$

and

$$
\begin{tikzcd}
 & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[dr, "p_{23}^*g"] \arrow[d, phantom, "\bar{f}\Uparrow"] & \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_{12}^*g"] \arrow[rr, "p_{13}^*g"'] & {} & p_3^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
\equiv
\begin{tikzcd}
 & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[ddr, "p_{23}^*g"] \arrow[d, bend left, "p_2^*\bar{t}"] \arrow[d, phantom, "\Uparrow"] & \\
{} \arrow[r, phantom, "p_{12}^*\phi\Leftarrow"] & p^*\mathrm{tra} \arrow[u, bend left, "p_2^*t"] \arrow[dr, "p_3^*t"] \arrow[d, phantom, "p_{13}^*\bar{\phi}\Uparrow"] & {} \arrow[l, phantom, "\Rightarrow p_{23}^*\phi"'] \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[uur, "p_{12}^*g"] \arrow[ur, "p_1^*\bar{t}"] \arrow[rr, "p_{13}^*g"'] & {} & p_3^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
$$

On $\mathcal{P}_{\mathcal{U}}^{[4]}$ four of these triangles form a tetrahedron. This tetrahedron 2-commutes. Equivalently, cutting the tetrahedron along four sides yields an equation between the 2-morphisms on each of the two pieces.

Proposition 1 *The transition triangles satisfy the* **tetrahedron law**

$$
\begin{tikzcd}
p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_{23}^*g"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_{34}^*g"] \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[u, "p_{12}^*g"] \arrow[ur, "p_{13}^*g"] \arrow[r, "p_{14}^*g"'] & p_4^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
\quad (p_{123}^*f \Rightarrow,\ \Downarrow p_{134}^*f)
=
\begin{tikzcd}
p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_{23}^*g"] \arrow[dr, "p_{24}^*g"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_{34}^*g"] \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[u, "p_{12}^*g"] \arrow[r, "p_{14}^*g"'] & p_4^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
\quad (p_{234}^*f \Leftarrow,\ p_{124}^*f \Downarrow)
.
$$

Proof. See figure 1 on p. 17. $\square$

For the present purpose we assume t to be an adjoint equivalence. In general, it may be just a special ambidextrous adjunction. (See section **??** for the definitions.) In this case f and $\bar{f}$ need not be mutually inverse. They will however still both satisfy a tetrahedron law of their own, as well as a compatibility condition involving both of them (a Frobenius property).

2.2 2-Category of Transitions

Transport 2-functors live in a 2-category $\mathrm{Hom}(\mathcal{P}, T)$. When we replace a 2-transport by its local trivialization and transition, we likewise want these local data to live in a 2-category.

We define 2-categories of local trivialization data and local transition data, $\mathbf{Triv}_{p,i}(\mathcal{P}, T)$ and $\mathbf{Trans}_{p,i}(\mathcal{P}, T)$, and construct injective morphisms

$$\mathrm{Hom}(\mathcal{P}, T) \longrightarrow \mathbf{Triv}_{p,i}(\mathcal{P}, T) \longrightarrow \mathbf{Trans}_{p,i}(\mathcal{P}, T)\ .$$

The categories $\mathbf{Trans}_{p,i}(\mathcal{P}, T)$, will be shown, for special choices of p and i, to coincide with well-known 2-categories of higher order structures, like for instance that of bundle gerbes with connection and curving.

The injective morphisms above allow us to inject any diagram involving globally defined transport functors into their transition data 2-category. This is useful for instance for expressing equivariant structures on globally defined transport in terms of local data.

The reader familar with "walking structures" in category theory will notice that our construction of trivialization and transition data in section 2 can be encoded in a 2-functor from a "walking trivialization" into $\mathrm{Hom}(\mathcal{P}_{\mathcal{U}}, T)$. From that point of view, the 2-category $\mathbf{Triv}_{p,i}(\mathcal{P}, T)$ defined now is nothing but the 2-functor 2-category of such 2-functors on the walking trivialization, hence a canonical entity. We will however spell out the definition explicitly.

We start by describing the category of transition data, since its structure is simpler.

Definition 5 *The* **2-category**

$$\mathbf{Trans}_{p,i}$$

of ($\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P}$)-local ($T' \xrightarrow{i} T$)-transitions *is the 2-category defined as follows.*

1. *objects are triangles*
$$\begin{tikzcd} & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[dr, "p_{23}^*g"] & \\ p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_{12}^*g"] \arrow[rr, "p_{13}^*g"'] & \arrow[u, phantom, "\Downarrow f"] & p_1^*\mathrm{tra}_{\mathcal{U}} \end{tikzcd}$$
which satisfy the tetrahedron

law

$$
\begin{tikzcd}
p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_{23}^*g"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_{34}^*g"] \arrow[dl, phantom, "\Downarrow p_{134}^*f"', shift right=2ex] \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[u, "p_{12}^*g"] \arrow[ur, "p_{13}^*g" description] \arrow[r, "p_{14}^*g"'] \arrow[u, phantom, "\Searrow p_{123}^*f"', shift right=4ex] & p_4^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
=
\begin{tikzcd}
p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_{23}^*g"] \arrow[dr, "p_{24}^*g" description] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_{34}^*g"] \arrow[dl, phantom, "p_{234}^*f \Swarrow", shift left=2ex] \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[u, "p_{12}^*g"] \arrow[r, "p_{14}^*g"'] \arrow[ur, phantom, "p_{124}^*f \Downarrow"', shift right=2ex] & p_4^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
.
$$

2. 1-morphisms $f \xrightarrow{\epsilon} f'$ *are choices of 2-cells*

$$
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p_1^*h"'] \arrow[dr, phantom, "\Swarrow \epsilon_g"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_3^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "g'"'] & p_3^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

satisfying

$$
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, "p_{13}^*g"] \arrow[d, "p_1^*h"'] \arrow[drr, phantom, "\Swarrow p_{13}^*\epsilon_g"] & & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_3^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "p_{13}^*g'"] \arrow[dr, "p_{12}^*g'"'] & \arrow[d, phantom, "\Downarrow \bar{f}'"] & p_3^*\mathrm{tra}'_{\mathcal{U}} \\
& p_2^*\mathrm{tra}' \arrow[ur, "p_{23}^*g'"'] &
\end{tikzcd}
=
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, "p_{13}^*g"] \arrow[dr, "p_{12}^*g" description] \arrow[dd, "p_1^*h"'] & \arrow[d, phantom, "\Downarrow \bar{f}"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[dd, "p_3^*h"] \\
\arrow[r, phantom, "\Swarrow p_{12}^*\epsilon_g"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_{23}^*g" description] \arrow[dd, "p_2^*h" description] \arrow[r, phantom, "\Swarrow p_{23}^*\epsilon_g"] & \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[dr, "p_{12}^*g'" description] & & p_3^*\mathrm{tra}'_{\mathcal{U}} \\
& p_2^*\mathrm{tra}'_{\mathcal{U}} \arrow[ur, "p_{23}^*g'" description] &
\end{tikzcd}
.
$$

3. 2-morphisms

$$
\begin{tikzcd}
f \arrow[r, bend left=50, "\epsilon_1"{name=U}] \arrow[r, bend right=50, "\epsilon_2"'{name=D}] & f' \arrow[from=U, to=D, Rightarrow, "E"]
\end{tikzcd}
$$

are given by 2-cells

$$\begin{tikzcd} \mathrm{tra}_{\mathcal{U}} \arrow[rr, bend left, "h_1"] \arrow[rr, bend right, "h_2"'] \arrow[rr, phantom, "\Downarrow E_g"] & & \mathrm{tra}'_{\mathcal{U}} \end{tikzcd}$$

satisfying

$$\begin{tikzcd} p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p_1^*h_1"] \arrow[d, bend right=60, "p_1^*h_2"'] \arrow[d, phantom, shift right=4, "\Leftarrow p_1^*E_h"] \arrow[dr, phantom, "\swarrow \epsilon_{g_1}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h_1"] \\ p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "g'"'] & p_2^*\mathrm{tra}'_{\mathcal{U}} \end{tikzcd} = \begin{tikzcd} p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p_1^*h_2"] \arrow[dr, phantom, "\swarrow \epsilon_{g_2}"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h_2"] \arrow[d, bend left=60, "p_2^*h_1"] \arrow[d, phantom, shift left=4, "\Leftarrow p_2^*E_h"] \\ p^*\mathrm{tra}' \arrow[r, "g'"'] & \mathrm{tra}'_{\mathcal{U}} \end{tikzcd} .$$

The 2-category of trivialization data is similar, but takes into account the local triviallizations that the transitions are constructed from.

Definition 6 *The* **2-category**

$$\mathbf{Triv}_{p,i}$$

of $\left(\mathcal{P}_{\mathcal{U}} \xrightarrow{p} \mathcal{P} \right)$**-local** $\left(T' \xrightarrow{i} T \right)$**-trivializations** *is the 2-category defined as follows.*

1. *objects are local p-local i-trivializations* $\mathcal{G} = (\mathrm{tra}_{\mathcal{U}}, t, \phi)$ *of 2-transport 2-functors* $\mathcal{P} \longrightarrow T$

2. *a morphism* $\mathcal{G} \xrightarrow{\epsilon} \mathcal{G}'$ *is a morphism*

$$\mathrm{tra} \xrightarrow{f} \mathrm{tra}'$$

together with a map

$$\epsilon : \{t, \bar{t}, g\} \longrightarrow \mathrm{Mor}_2(\mathrm{Hom}_{2\mathbf{Cat}}(\mathcal{P}_{\mathcal{U}}, T))$$

given by

$$t \mapsto \begin{tikzcd} p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f"'] \arrow[dr, phantom, "\swarrow \epsilon_t"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h"] \\ p^*\mathrm{tra}' \arrow[r, "t'"'] & \mathrm{tra}'_{\mathcal{U}} \end{tikzcd} \qquad \bar{t} \mapsto \begin{tikzcd} \mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow \epsilon_{\bar{t}}"] & p^*\mathrm{tra} \arrow[d, "p^*f"] \\ \mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"'] & p^*\mathrm{tra}' \end{tikzcd}$$

$$
g \quad \mapsto \quad \begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p_1^*h"'] \arrow[dr, phantom, "\swArrow \epsilon_g"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "g'"'] & p_2^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

such that all relevant tin can equations hold, displayed in figure 2. Composition is by vertical composition of the above 2-morphisms.

3. *a 2-morphism between 1-morphisms between p-local i-trivializations*

$$
\begin{tikzcd}
\mathcal{G} \arrow[r, bend left, "\epsilon_1"] \arrow[r, bend right, "\epsilon_2"'] \arrow[r, phantom, "\Downarrow E"] & \mathcal{G}'
\end{tikzcd}
$$

is a "modification of the above pseudonatural transformations" in the sense that it is a map

$$
E : \ \{h, f\} \longrightarrow \mathrm{Mor}_2\left(\mathrm{Hom}_{\mathbf{2Cat}}\left(\mathcal{P}_{\mathcal{U}}, T\right)\right)
$$

given by

$$
h \quad \mapsto \quad \begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, bend left, "h_1"] \arrow[r, bend right, "h_2"'] \arrow[r, phantom, "\Downarrow E_h"] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

and

$$
f \quad \mapsto \quad \begin{tikzcd}
\mathrm{tra} \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow E_f"] & \mathrm{tra}'
\end{tikzcd}
$$

such that the modification tin can equations displayed in figure 3 hold. Horizontal and vertical composition of 2-morphisms is horizontal and vertical 2-morphisms of the above 2-morphisms in $\mathrm{Mor}_2\left(\mathrm{Hom}_{\mathbf{2Cat}}\left(\mathcal{P}_{\mathcal{U}}\right), T\right)$.

Proposition 2 *By combining the three tin can equations in figure 2, and using* *def. 4, one obtains in addition this tin can equation*

$$
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, "p_{13}^*g"] \arrow[d, "p_1^*h"'] \arrow[drr, phantom, "\Swarrow p_{13}^*\epsilon_g"] & & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_3^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "p_{13}^*g'"] \arrow[dr, "p_{12}^*g'"'] & \arrow[d, phantom, "\Downarrow \bar{f}'"] & p_3^*\mathrm{tra}'_{\mathcal{U}} \\
& p_2^*\mathrm{tra}' \arrow[ur, "p_{23}^*g'"'] &
\end{tikzcd}
=
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, "p_{13}^*g"] \arrow[dr, "p_{12}^*g"'] \arrow[dd, "p_1^*h"'] & \arrow[d, phantom, "\Downarrow \bar{f}"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[dd, "p_3^*h"] \\
\arrow[dr, phantom, "\Swarrow p_{12}^*\epsilon_g"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_{23}^*g"'] \arrow[dd, "p_2^*h"] \arrow[r, phantom, "\Swarrow p_{23}^*\epsilon_g"] & \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[dr, "p_{12}^*g'"'] & & p_3^*\mathrm{tra}'_{\mathcal{U}} \\
& p_2^*\mathrm{tra}'_{\mathcal{U}} \arrow[ur, "p_{23}^*g'"'] &
\end{tikzcd}
\quad . \qquad (6)
$$

Proof. Redraw f' on the left and side according to def. 4, then use the equations in figure 2 successively to pass the ϵ through to the other side of the transition triangle f'. $\square$

The main proposition involving these two definitions is

Proposition 3 *There are injections*

$$\mathrm{Hom}(\mathcal{P}, T) \longrightarrow \mathbf{Triv}_{p,i}(\mathcal{P}, T) \longrightarrow \mathbf{Trans}_{p,i}(\mathcal{P}, T) \ .$$

The second arrow is obtained simply by forgetting all trivialization data. The existence of the first arrow is now demonstrated by explicit construction.

Lemma 1 *Let* tra *and* tra$'$ *be transport 2-functors with p-local i-trivializations* $\mathcal{G}$ *and* $\mathcal{G}'$*, respectively. For every morphism*

$$\mathrm{tra} \xrightarrow{f} \mathrm{tra}'$$

there is (at least) one morphism

$$\mathcal{G} \xrightarrow{\epsilon(f)} \mathcal{G}'$$

in the 2-category of pre-trivializations.

Proof. We explicitly construct the morphism $\mathcal{G} \xrightarrow{\epsilon(f)} \mathcal{G}'$ in the obvious way by setting

$$
t \quad \mapsto \quad
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f"'] \arrow[dr, phantom, "\swArrow\epsilon_t"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h"] \\
p^*\mathrm{tra}' \arrow[r, "t'"'] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[dr, "\mathrm{Id}"] \arrow[ddd, "p^*f"'] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "\bar{t}"] \\
& p^*\mathrm{tra} \arrow[d, "p^*f"] \\
& p^*\mathrm{tra}' \arrow[d, "t'"] \\
p^*\mathrm{tra}' \arrow[r, "t'"'] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

and

$$
\bar{t} \quad \mapsto \quad
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h"'] \arrow[dr, phantom, "\swArrow\epsilon_{\bar{t}}"] & p^*\mathrm{tra} \arrow[d, "p^*f"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"'] & p^*\mathrm{tra}'
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "\bar{t}"'] & p^*\mathrm{tra} \arrow[ddd, "p^*f"] \\
p^*\mathrm{tra} \arrow[d, "p^*f"'] \arrow[r, phantom, "\swArrow\mathrm{Id}"] & {} \\
p^*\mathrm{tra}' \arrow[d, "t"'] \arrow[dr, "\mathrm{Id}"] & \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"'] & p^*\mathrm{tra}'
\end{tikzcd}
.
$$

Then we define

$$
g \quad \mapsto \quad
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p_1^*h"'] \arrow[dr, phantom, "\swArrow\epsilon_g"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "g'"'] & p_2^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

to be the unique solution of the tin can equation (1).

We then need to check that the remaining tin can equations (2) and (3) are satisfied. This turns out to be a consequence of the triangle identities and the speciality condition satisfied by the special ambidextrous adjunction between t

and $\bar{t}$. The zig-zag identity of the adjunction implies that

$$
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[rr, bend left, "\mathrm{Id}"] \arrow[d, "\bar{t}"'] \arrow[rr, phantom, bend left=20, "\Downarrow"] & p^*\mathrm{tra} \arrow[r, "t"] \arrow[ddd, "p^*f"] \arrow[dr, "\mathrm{Id}"] \arrow[r, phantom, shift left=4ex, ""] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "\bar{t}"] \arrow[dl, phantom, "\Swarrow"] \\
p^*\mathrm{tra} \arrow[d, "p^*f"'] \arrow[r, phantom, "\Swarrow \mathrm{Id}"] & & p^*\mathrm{tra} \arrow[d, "p^*f"] \arrow[l, phantom, "\Swarrow \mathrm{Id}"] \\
p^*\mathrm{tra}' \arrow[d, "t'"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, shift left=4ex, "\Swarrow"] & & p^*\mathrm{tra}' \arrow[d, "t'"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"] \arrow[rr, bend right, "\mathrm{Id}"'] \arrow[rr, phantom, bend right=20, "\Downarrow"] & p^*\mathrm{tra}' \arrow[r, "t'"] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\mathrm{Id}"] \arrow[d, "h"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\mathrm{Id}"'] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
.
$$

This is equivalent to (2). The speciality property implies that

$$
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[rr, bend left, "\mathrm{Id}"] \arrow[rr, phantom, bend left=20, "\Downarrow"] \arrow[dr, "\mathrm{Id}"] \arrow[ddd, "p^*f"'] & \mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "t"] \arrow[dl, phantom, "\Swarrow"] & p^*\mathrm{tra} \arrow[ddd, "p^*f"] \\
 & p^*\mathrm{tra} \arrow[d, "p^*f"] \arrow[dl, phantom, "\Swarrow \mathrm{Id}"] \arrow[r, phantom, "\Swarrow \mathrm{Id}"] & \\
 & p^*\mathrm{tra}' \arrow[d, "t'"] \arrow[dr, "\mathrm{Id}"] & \\
p^*\mathrm{tra}' \arrow[r, "t'"] \arrow[rr, bend right, "\mathrm{Id}"'] \arrow[rr, phantom, bend right=20, "\Downarrow"] & \mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"] \arrow[ur, phantom, "\Swarrow"] & p^*\mathrm{tra}'
\end{tikzcd}
=
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "\mathrm{Id}"] \arrow[d, "p^*f"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & p^*\mathrm{tra} \arrow[d, "p^*f"] \\
p^*\mathrm{tra}' \arrow[r, "\mathrm{Id}"'] & p^*\mathrm{tra}'
\end{tikzcd}
.
$$

This is equivalent to (3). $\square$

Corollary 1 *Let* tra *be a transport 2-functor with two p-local i-trivializations $\mathcal{G}$ and $\mathcal{G}'$. There is (at least) one morphism*

$$
\mathcal{G} \xrightarrow{\epsilon(\mathcal{G},\mathcal{G}')} \mathcal{G}' \ .
$$

Proof. Set $f = \mathrm{Id}$ in the above proposition. $\square$

We shall eventually show that this morphism is an equivalence.

Lemma 2 *The above map of 1-morphisms respects composition weakly.*

Proof. We construct an invertible 2-morphism between the composite image and the image of the composite as displayed in 4. The required associativity condition is easily checked. $\square$

Lemma 3 *Let* tra *and* tra$'$ *be transport 2-functors with p-local i-trivializations* $\mathcal{G}$ *and* $\mathcal{G}'$*, respectively. For every 2-morphisms of transport 2-functors*

$$\begin{tikzcd} \mathrm{tra} \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[rr, phantom, "\Downarrow\mathcal{A}"] & & \mathrm{tra}' \end{tikzcd}$$

there is (at least) one 2-morphism

$$\begin{tikzcd} \mathcal{G} \arrow[rr, bend left, "\epsilon(f_1)"] \arrow[rr, bend right, "\epsilon(f_2)"'] \arrow[rr, phantom, "\Downarrow E(\mathcal{A})"] & & \mathcal{G}' \end{tikzcd}$$

of local pre-trivializations.

Proof. We construct such a 2-morphism in an obvious way and check its properties. Set

$$h \quad \mapsto \quad \begin{tikzcd} \mathrm{tra}_{\mathcal{U}} \arrow[rr, bend left, "h_1"] \arrow[rr, bend right, "h_2"'] \arrow[rr, phantom, "\Downarrow E_h"] & & \mathrm{tra}'_{\mathcal{U}} \end{tikzcd} \quad \equiv \quad \begin{tikzcd} \mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] & p^*\mathrm{tra} \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[rr, phantom, "\Downarrow p^*\mathcal{A}"] & & p^*\mathrm{tra}' \arrow[r, "t'"] & \mathrm{tra}'_{\mathcal{U}} \end{tikzcd}$$

and

$$f \quad \mapsto \quad \begin{tikzcd} \mathrm{tra} \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[rr, phantom, "\Downarrow E_f"] & & \mathrm{tra}' \end{tikzcd} \quad \equiv \quad \begin{tikzcd} \mathrm{tra} \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[rr, phantom, "\Downarrow\mathcal{A}"] & & \mathrm{tra}' \end{tikzcd} \; .$$

This trivially satisfies the equation

$$
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[dr, "\mathrm{Id}"] \arrow[ddd, "p^*f_1"] \arrow[ddd, bend right=60, "p^*f_2"'] \arrow[ddd, phantom, shift right=8ex, "\Leftarrow p^*\mathcal{A}"] \arrow[dddr, phantom, "\Swarrow \mathrm{Id}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "\bar t"] \arrow[dl, phantom, shift right=1ex, "\Swarrow"] \\
& p^*\mathrm{tra} \arrow[d, "p^*f_1"] \\
& p^*\mathrm{tra}' \arrow[d, "t'"] \\
p^*\mathrm{tra}' \arrow[r, "t'"'] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[dr, "\mathrm{Id}"] \arrow[ddd, "p^*f_2"] \arrow[dddr, phantom, "\Swarrow \mathrm{Id}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "\bar t"] \arrow[dl, phantom, shift right=1ex, "\Swarrow"] \\
& p^*\mathrm{tra} \arrow[d, "p^*f_2"'] \arrow[d, bend left=60, "p^*f_1"] \arrow[d, phantom, shift left=2ex, "\Leftarrow p^*\mathcal{A}"] \\
& p^*\mathrm{tra}' \arrow[d, "t'"] \\
p^*\mathrm{tra}' \arrow[r, "t'"'] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

equivalent to (4) and the equation

$$
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar t"] \arrow[d, "\bar t"'] & p^*\mathrm{tra} \arrow[ddd, "p^*f_1"] \\
p^*\mathrm{tra} \arrow[d, "p^*f_1"] \arrow[d, bend right=60, "p^*f_2"'] \arrow[d, phantom, shift right=2ex, "\Leftarrow p^*\mathcal{A}"] \arrow[r, phantom, "\Swarrow \mathrm{Id}"] & \\
p^*\mathrm{tra}' \arrow[d, "t"'] \arrow[dr, "\mathrm{Id}"] \arrow[dr, phantom, shift right=3ex, "\Swarrow"] & \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar t'"'] & p^*\mathrm{tra}'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar t"] \arrow[d, "\bar t"'] & p^*\mathrm{tra} \arrow[ddd, "p^*f_2"'] \arrow[ddd, bend left=60, "p^*f_1"] \arrow[ddd, phantom, shift left=4ex, "\Leftarrow p^*\mathcal{A}"] \\
p^*\mathrm{tra} \arrow[d, "p^*f_2"] \arrow[r, phantom, "\Swarrow \mathrm{Id}"] & \\
p^*\mathrm{tra}' \arrow[d, "t"'] \arrow[dr, "\mathrm{Id}"] \arrow[dr, phantom, shift right=3ex, "\Swarrow"] & \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar t'"'] & p^*\mathrm{tra}'
\end{tikzcd}
$$

equivalent to (5). □

The assignment of 2-morphisms is easily seen to be 2-functorial. In summary, this establishes 2-functors as stated in prop. 3 on p. 12.

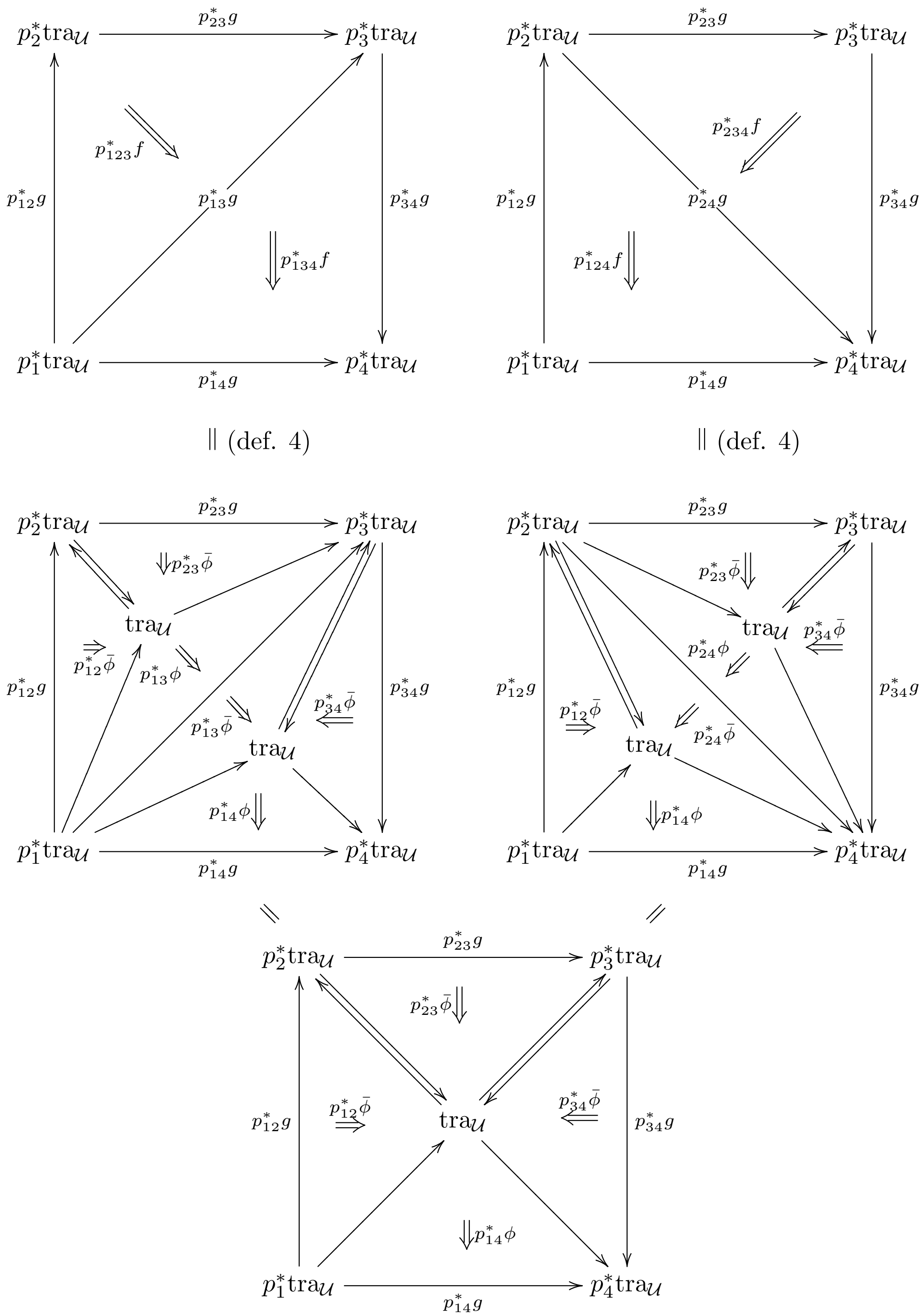

Figure 1: **Proof of the tetrahedron law** stated in prop. 1 on p. 7. Antiparallel arrows are shorthand for an equivalence.

(a) tin can based on the transition modification

$$
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, "g"] \arrow[d, "p_1^*h"'] \arrow[drr, phantom, "\Swarrow \epsilon_g"] & & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "g'"] \arrow[dr, "p_1^*\bar{t}'"'] & {} \arrow[d, phantom, "\Downarrow \phi'"] & p_2^*\mathrm{tra}'_{\mathcal{U}} \\
& p_1^*p^*\mathrm{tra} \arrow[ur, "p_2^*t'"'] &
\end{tikzcd}
=
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_1^*\bar{t}"] \arrow[d, "p_1^*h"'] \arrow[rr, bend left=50, "g"] \arrow[rr, phantom, bend left=25, "\Downarrow \phi"] \arrow[dr, phantom, "\Swarrow p_1^*\epsilon_{\bar{t}}"] & p_1^*p^*\mathrm{tra} \arrow[r, "p_2^*t"] \arrow[d, "p_1^*p^*f"] \arrow[dr, phantom, "\Swarrow p_2^*\epsilon_t"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "p_1^*\bar{t}'"] & p_1^*p^*\mathrm{tra}' \arrow[r, "p_2^*t'"] & p_2^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\qquad (1)
$$

(b) tin can based on the unit on $t \circ \bar{t}$

$$
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[rr, "\mathrm{Id}"] \arrow[d, "h"'] \arrow[drr, phantom, "\Swarrow \mathrm{Id}"] & & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "\mathrm{Id}"] \arrow[dr, "\bar{t}'"'] & {} \arrow[d, phantom, "\Downarrow"] & \mathrm{tra}'_{\mathcal{U}} \\
& p^*\mathrm{tra}' \arrow[ur, "t'"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h"'] \arrow[rr, bend left=50, "\mathrm{Id}"] \arrow[rr, phantom, bend left=25, "\Downarrow"] \arrow[dr, phantom, "\Swarrow \epsilon_{\bar{t}}"] & p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f"] \arrow[dr, phantom, "\Swarrow \epsilon_t"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"] & p^*\mathrm{tra}' \arrow[r, "t'"] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\qquad (2)
$$

(c) tin can based on the unit on $\bar{t} \circ t$

$$
\begin{tikzcd}
p^*\mathrm{tra} \arrow[rr, "\mathrm{Id}"] \arrow[d, "p^*f"'] \arrow[drr, phantom, "\Swarrow \mathrm{Id}"] & & p^*\mathrm{tra} \arrow[d, "p^*f"] \\
p^*\mathrm{tra}' \arrow[rr, "\mathrm{Id}"] \arrow[dr, "t'"'] & {} \arrow[d, phantom, "\Downarrow"] & p^*\mathrm{tra}' \\
& \mathrm{tra}'_{\mathcal{U}} \arrow[ur, "t'"'] &
\end{tikzcd}
=
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f"'] \arrow[rr, bend left=50, "\mathrm{Id}"] \arrow[rr, phantom, bend left=25, "\Downarrow"] \arrow[dr, phantom, "\Swarrow \epsilon_t"] & \mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h"] \arrow[dr, phantom, "\Swarrow \epsilon_{\bar{t}}"] & p^*\mathrm{tra} \arrow[d, "p^*f"] \\
p^*\mathrm{tra}' \arrow[r, "t'"] & \mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"] & p^*\mathrm{tra}'
\end{tikzcd}
\quad . \qquad (3)
$$

Figure 2: The **tin can equations satisfied by 1-morphisms in $\mathbf{Triv}_{p,i}$**, defined in def. 6 on p. 10.

$$
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f_1"] \arrow[d, bend right=60, "p^*f_2"'] \arrow[d, phantom, shift right=18, "\Leftarrow p^*E_f"] \arrow[dr, phantom, "\swarrow \epsilon_{t1}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h_1"] \\
p^*\mathrm{tra}' \arrow[r, "t'"'] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
=
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[d, "p^*f_2"] \arrow[dr, phantom, "\swarrow \epsilon_{t2}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "h_2"] \arrow[d, bend left=60, "h_1"] \arrow[d, phantom, shift left=18, "\Leftarrow E_h"] \\
p^*\mathrm{tra}' \arrow[r, "t"'] & \mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
\tag{4}
$$

$$
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h_1"] \arrow[d, bend right=60, "h_2"'] \arrow[d, phantom, shift right=18, "\Leftarrow E_h"] \arrow[dr, phantom, "\swarrow \epsilon_{\bar{t}1}"] & p^*\mathrm{tra} \arrow[d, "p^*f_1"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"'] & p^*\mathrm{tra}'
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "h_2"] \arrow[dr, phantom, "\swarrow \epsilon_{\bar{t}2}"] & p^*\mathrm{tra} \arrow[d, "p^*f_2"] \arrow[d, bend left=60, "p^*f_1"] \arrow[d, phantom, shift left=18, "\Leftarrow p^*E_f"] \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}"'] & p^*\mathrm{tra}'
\end{tikzcd}
\tag{5}
$$

Figure 3: The **tin can equations satisfied by 2-morphisms in $\mathbf{Triv}_{p,i}$**, defined in def. 6.

$$
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[dr, "\mathrm{Id}" description] \arrow[ddd, "p^*f_1"'] \arrow[ddr, phantom, "\Swarrow\mathrm{Id}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "\bar{t}"] \arrow[dl, phantom, "\Swarrow", near start] \\
 & p^*\mathrm{tra} \arrow[d, "p^*f_1"] \\
 & p^*\mathrm{tra}' \arrow[d, "t'"] \arrow[dd, bend left=60, "\mathrm{Id}"] \\
p^*\mathrm{tra}' \arrow[r, "t'" description] \arrow[dr, "\mathrm{Id}" description] \arrow[ddd, "p^*f_2"'] \arrow[ddr, phantom, "\Swarrow\mathrm{Id}"] & \mathrm{tra}'_{\mathcal{U}} \arrow[d, "\bar{t}'"] \arrow[r, phantom, "\Leftarrow", near start] \arrow[dl, phantom, "\Swarrow", near start] & {} \\
 & p^*\mathrm{tra}' \arrow[d, "p^*f_2"] \\
 & p^*\mathrm{tra}'' \arrow[d, "t''"] \\
p^*\mathrm{tra}'' \arrow[r, "t''"'] & \mathrm{tra}''_{\mathcal{U}}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
p^*\mathrm{tra} \arrow[r, "t"] \arrow[dr, "\mathrm{Id}" description] \arrow[ddd, "p^*(f_1\circ f_1)"'] \arrow[ddr, phantom, "\Swarrow\mathrm{Id}"] & \mathrm{tra}_{\mathcal{U}} \arrow[d, "\bar{t}"] \arrow[dl, phantom, "\Swarrow", near start] \\
 & p^*\mathrm{tra} \arrow[d, "p^*(f_2\circ f_1)"] \\
 & p^*\mathrm{tra}'' \arrow[d, "t''"] \\
p^*\mathrm{tra}'' \arrow[r, "t''"'] & \mathrm{tra}''_{\mathcal{U}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "\bar{t}"'] & p^*\mathrm{tra} \arrow[ddd, "p^*f_1"] \\
p^*\mathrm{tra} \arrow[d, "p^*f_1"'] \arrow[r, phantom, "\Swarrow\mathrm{Id}"] & {} \\
p^*\mathrm{tra}' \arrow[d, "t"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Swarrow", shift left=4ex] & \\
\mathrm{tra}'_{\mathcal{U}} \arrow[r, "\bar{t}'"'] \arrow[d, "\bar{t}'"'] & p^*\mathrm{tra}' \arrow[ddd, "p^*f_2"] \\
p^*\mathrm{tra}' \arrow[d, "p^*f_2"'] \arrow[r, phantom, "\Swarrow\mathrm{Id}"] & {} \\
p^*\mathrm{tra}'' \arrow[d, "t"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Swarrow", shift left=4ex] & \\
\mathrm{tra}''_{\mathcal{U}} \arrow[r, "\bar{t}'"'] & p^*\mathrm{tra}''
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathrm{tra}_{\mathcal{U}} \arrow[r, "\bar{t}"] \arrow[d, "\bar{t}"'] & p^*\mathrm{tra} \arrow[dddddd, "p^*(f_2\circ f_1)"] \\
 & p^*\mathrm{tra} \arrow[d, "p^*f_1"'] & \\
 & p^*\mathrm{tra}' \arrow[dl, "t'"'] \arrow[dd, "\mathrm{Id}" description] & \\
\mathrm{tra}'_{\mathcal{U}} \arrow[dr, "\bar{t}'"'] \arrow[r, phantom, "\Leftarrow"] & {} \arrow[r, phantom, "\Swarrow\mathrm{Id}"] & {} \\
 & p^*\mathrm{tra}' \arrow[d, "p^*f_2"'] & \\
 & p^*\mathrm{tra}'' \arrow[d, "t"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Swarrow", shift left=4ex] & \\
 & \mathrm{tra}''_{\mathcal{U}} \arrow[r, "\bar{t}'"'] & p^*\mathrm{tra}''
\end{tikzcd}
$$

Figure 4: **Proof that the injection** $\mathrm{Hom}(\mathcal{P}, T) \to \mathbf{Triv}_{p,i}(\mathcal{P}, T)$ **respects composition weakly**, as stated in prop. 2.