2-Hilbert Spaces and Bimodules from Spans

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Abstract

If you believe that abstract nonsense is the language in which the foundations of the world are written, you want to know if the categorification of quantum mechanics is related to physics (e.g. to field theory and/or to string physics). This question was examined by J. Baez in a series of lectures [1], which recently culminated in a comprehensive exposition [2]. In this approach linear operators are categorified in terms of span categories. It would be nice if this were related to an approach which puts bimodule categories at the center of attention [4]. Here I try to work out that both points of view are closely related.

My main point is that if C has products and a terminal object "$\bullet$", then $\mathbf{Span}(C)$ decomposes in 2-Hilbert spaces $\mathrm{Hom}_{\mathbf{Span}(C)}(\bullet, -)$ which are left C-module categories, dual 2-Hilbert spaces $\mathrm{Hom}_{\mathbf{Span}(C)}(-, \bullet)$ which are right C-module categories, as well as C-linear operators $\mathrm{Hom}_{\mathbf{Span}(C)}(X, Y)$ between these. This is related to bimodules by Ostrik's theorem [5].

For more technical background see [3] and references given there.

1 Spans

Definition 1 *Let C be some category with pullbacks. Then $\mathbf{Span}(C)$ is the weak 2-category whose*

1. *objects are objects in C*

2. *morphisms $X \xrightarrow{(S,p_1,p_2)} Y$ are diagrams*

$$\begin{tikzcd} & S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

in C

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3. *2-morphisms*

$$\begin{tikzcd}
X \arrow[r, bend left, "{(S,p_1,p_2)}"] \arrow[r, bend right, "{(S',p_1',p_2')}"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y
\end{tikzcd}$$

are commuting diagrams

$$\begin{tikzcd}
& S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\alpha"] & \\
X & & Y \\
& S' \arrow[ul, "p_1'"] \arrow[ur, "p_2'"'] &
\end{tikzcd}$$

in C.

Horizontal composition of morphisms is defined by pullbacks

$$X \xrightarrow{S} Y \xrightarrow{S'} Z \;=\; X \xrightarrow{S \cdot S'} Z$$

with $S \cdot S'$ being the pullback

$$\begin{tikzcd}
& & S \cdot S' \arrow[dl] \arrow[dr] & & \\
& S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & S' \arrow[dl, "p_1'"'] \arrow[dr, "p_2'"] & \\
X & & Y & & Z
\end{tikzcd} .$$

In order to see what **Span**(C) is like, it is illustrative to consider **monads in Span**(C). Recall the definition of a monad:

Definition 2 *Let D be some (weak) 2-category. A* **monad** *in D is*

1. *an object X of D*
2. *an endomorphism*

$$X \xrightarrow{A} X$$

3. *2-morphisms*

$$\begin{tikzcd}
X \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "A"'] \arrow[r, phantom, "\Downarrow i"] & X
\end{tikzcd}$$

and

$$\begin{tikzcd} & X \arrow[dr, bend left, "A"] \arrow[d, phantom, "\Downarrow m"] & \\ X \arrow[ur, bend left, "A"] \arrow[rr, bend right, "A"'] & {} & X \end{tikzcd}$$

such that m is associative and i is a unit with respect to m.

Remark. Let $\boxed{1}$ be the 2-category with a single object, a single 1-morphism (the identity 1-morphism) and a single 2-morphism (the unique identity 2-morphism). Then a monad in D is the same as a lax 2-functor

$$\boxed{1} \longrightarrow D\,.$$

Before considering the first example, recall that a monoidal category can be regarded as a 2-category with a single object.

Definition 3 *Let C be any monoidal category. Denote by $\Sigma(C)$ the 2-category which is the* **suspension** *of C. This is the 2-category whose*

1. *set of objects contains only a single element* $\mathrm{Obj}(\Sigma) = \{\bullet\}$
2. *morphisms* $\bullet \xrightarrow{c} \bullet$ *are objects* $c \in \mathrm{Obj}(C)$
3. *2-morphisms* $\begin{tikzcd} \bullet \arrow[r, bend left=50, "c"] \arrow[r, bend right=50, "c'"'] \arrow[r, phantom, "\Downarrow\phi"] & \bullet \end{tikzcd}$ *are morphisms* $c \xrightarrow{\phi} c'$ *in* C

such that horizontal composition in $\Sigma(C)$ is the tensor product in C.

Example 1 *An algebra object in a monoidal category C is the same as a monad in $\Sigma(C)$.*

The next example is more interesting. It shows that $\mathbf{Span}(C)$ can be regarded as a way of generalizing the notion of a category internal to C.

Example 2 *A category internal to some category C is the same as a monad in* $\mathbf{Span}(C)$.

Proof. For illustration purposes, I'll spell this out in detail. By definition (def. 1 and def. 2), a monad in $\mathbf{Span}(C)$ is

1. an object $X \in \mathrm{Obj}(C)$ which we call $X \equiv \mathrm{Obj}(\mathcal{C})$,

2. a morphism $\mathrm{Obj}(\mathcal{C}) \xrightarrow{A} \mathrm{Obj}(\mathcal{C})$ in $\mathbf{Span}(C)$, i.e. an object $S \in \mathrm{Obj}(C)$ which we shall call $S \equiv \mathrm{Mor}(\mathcal{C})$, together with morphisms

$$\begin{tikzcd} & S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd} \equiv \begin{tikzcd} & \mathrm{Mor}(\mathcal{C}) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathrm{Obj}(\mathcal{C}) & & \mathrm{Obj}(\mathcal{C}) \end{tikzcd}$$

3. a 2-morphism

$$\begin{tikzcd} X \arrow[rr, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "A"'] \arrow[rr, phantom, "\Downarrow i"] & & X \end{tikzcd}$$

in $\mathbf{Span}(C)$, i.e. a commuting diagram

$$\begin{tikzcd} & \mathrm{Obj}(\mathcal{C}) \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] \arrow[dd, "i"] & \\ \mathrm{Obj}(\mathcal{C}) & & \mathrm{Obj}(\mathcal{C}) \\ & \mathrm{Mor}(\mathcal{C}) \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

in C

4. a 2-morphism

$$\begin{tikzcd} & X \arrow[dr, bend left, "A"] & \\ X \arrow[ur, bend left, "A"] \arrow[rr, bend right, "A"'] \arrow[rr, phantom, "\Downarrow m"] & & X \end{tikzcd}$$

in $\mathbf{Span}(C)$, i.e. a commuting diagram

$$\begin{tikzcd} & \mathrm{Mor}(\mathcal{C}) \, {}_t\!\times_s \mathrm{Mor}(\mathcal{C}) \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "m\equiv\circ"] & \\ \mathrm{Obj}(\mathcal{C}) & & \mathrm{Obj}(\mathcal{C}) \\ & \mathrm{Mor}(\mathcal{C}) \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

in C

such that i and $m = \circ$ satisfy associativity and unit laws. This is nothing but the definition of a category $\mathcal{C}$ internal to C. $\square$

2 Half-Spans

Now restrict attention to the special case of spans in a category C that has a *terminal object*. Let us denote this terminal object by "$\bullet$". Clearly, the category of endomorphisms of $\bullet$ in $\mathbf{Span}(C)$ is nothing but C itself:

$$\mathrm{Hom}_{\mathbf{Span}(C)}(\bullet, \bullet) \simeq C\,.$$

Hence, as is very well known, C must in fact be a monoidal category, with the tensor product being the horizontal composition on $\mathrm{Hom}_{\mathbf{Span}(C)}(\bullet, \bullet)$ induced by that in $\mathbf{Span}(C)$.

Definition 4 *Given a category C with* terminal object $\bullet$, *call*

$$\mathbf{LSpan}(C) \equiv \mathrm{Hom}_{\mathbf{Span}(C)}(--, \bullet)$$

the **category of left half spans** *and*

$$\mathbf{RSpan}(C) \equiv \mathrm{Hom}_{\mathbf{Span}(C)}(--, \bullet)$$

the **category of right half spans** *in C.*

In more detail, this definition implies the following.

1. An object of $\mathbf{RSpan}(C)$ is a morphism

$$\begin{tikzcd} S \arrow[dr, "p"] & \\ & X \end{tikzcd}$$

2. A morphism $(S, p, X) \xrightarrow{\alpha} (S', p', X)$ in $\mathbf{LSpan}(C)$ is a commuting diagram

$$\begin{tikzcd} S \arrow[dd, "\alpha"'] \arrow[dr, "p"] & \\ & X \\ S' \arrow[ur, "p'"'] & \end{tikzcd}$$

in C.

Similarly for $\mathbf{LSpan}(C)$. In fact, it is obvious that

$$\mathbf{LSpan}(C) \simeq \mathbf{RSpan}(C)\ .$$

There is an obvious notion of a (left or right) **module category** over any monoidal category C (see for instance [5] for some definitions and further references) and this is what naturally appear here:

Proposition 1 *Using horizontal composition in* $\mathbf{Span}(C)$, $\mathbf{RSpan}(C)$ *becomes a* **left module category** *over* C *and* $\mathbf{LSpan}(C)$ *becomes a* **right module category** *over* C.

Proof. Follows trivially from the fact that $C \simeq \mathrm{Hom}_{\mathbf{Span}(C)}(\bullet, \bullet)$, $\mathbf{RSpan}(C) = \mathrm{Hom}_{\mathbf{Span}(C)}(\bullet, -)$ $\mathbf{LSpan}(C) = \mathrm{Hom}_{\mathbf{Span}(C)}(-, \bullet)$ $\square$

Proposition 2 *Let* ${}_C\mathbf{Mod}$ *be the 2-category of left* C*-modules. There is 2-functor*

$$E : \mathbf{Span}(C) \to {}_C\mathbf{Mod}$$

which is injective on objects, morphisms and 2-morphisms.

Proof. This is completely analogous to definition 3 and propostion 1 in [5]. $\square$

Furthermore, composition in $\mathbf{Span}(C)$ induces the following pairing.

Definition 5 *Let*

$$\mathbf{RSpan}(C)\ {}_t\times_s\ \mathbf{LSpan}(C) \xrightarrow{\langle\cdot,\cdot\rangle} C$$

be given by composition in $\mathbf{Span}(C)$, *such that*

$$\left(\begin{array}{ccc} S & & \\ & \searrow^{p} & \\ & & X \end{array} \ ,\ \begin{array}{ccc} & & S' \\ & \swarrow^{p'} & \\ X & & \end{array} \right) \ \mapsto\ S \cdot S'\,,$$

where $S \cdot S'$ *is the pullback, as before.*

Phew, I am running out of time. Unless I made a mistake above, the upshot should be that $\mathbf{RSpan}(C)$ is a collection of 2-Hilbert spaces over C, $\mathbf{LSpan}(C)$ plays the role of a collection of dual spaces and $\mathrm{Hom}_{\mathbf{Span}(C)}(X, Y)$ that of C-linear operators from the 2-Hilbert space indexed by X to that indexed by Y.

Moreover, Ostrik's theorem relates all this to an equivalent formulation in terms of bimodules [5].

References

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