

# 2-Hilbert Spaces and Bimodules from Spans

Urs Schreiber\*

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## Abstract

If you believe that abstract nonsense is the language in which the foundations of the world are written, you want to know if the categorification of quantum mechanics is related to physics (e.g. to field theory and/or to string physics). This question was examined by J. Baez in a series of lectures [1], which recently culminated in a comprehensive exposition [2]. In this approach linear operators are categorified in terms of span categories. It would be nice if this were related to an approach which puts bimodule categories at the center of attention [4]. Here I try to work out that both points of view are closely related.

My main point is that if  $C$  has products and a terminal object “ $\bullet$ ”, then  $\mathbf{Span}(C)$  decomposes in 2-Hilbert spaces  $\mathbf{Hom}_{\mathbf{Span}(C)}(\bullet, -)$  which are left  $C$ -module categories, dual 2-Hilbert spaces  $\mathbf{Hom}_{\mathbf{Span}(C)}(-, \bullet)$  which are right  $C$ -module categories, as well as  $C$ -linear operators  $\mathbf{Hom}_{\mathbf{Span}(C)}(X, Y)$  between these. This is related to bimodules by Ostrik’s theorem [5].

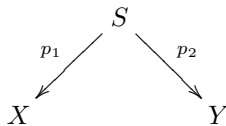
For more technical background see [3] and references given there.

## 1 Spans

**Definition 1** *Let  $C$  be some category with pullbacks. Then  $\mathbf{Span}(C)$  is the weak 2-category whose*

1. *objects are objects in  $C$*

2. *morphisms  $X \xrightarrow{(S, p_1, p_2)} Y$  are diagrams*

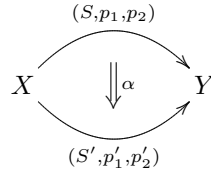


*in  $C$*

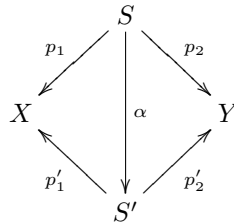
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\*E-mail: urs.schreiber at math.uni-hamburg.de

3. 2-morphisms



are commuting diagrams

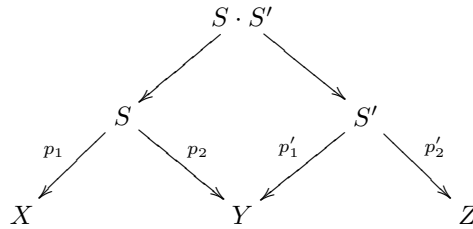


in  $C$ .

Horizontal composition of morphisms is defined by pullbacks

$$X \xrightarrow{S} Y \xrightarrow{S'} Z = X \xrightarrow{S \cdot S'} Z$$

with  $S \cdot S'$  being the pullback



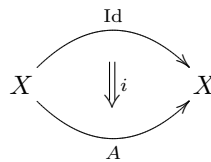
In order to see what  $\mathbf{Span}(C)$  is like, it is illustrative to consider **monads in  $\mathbf{Span}(C)$** . Recall the definition of a monad:

**Definition 2** Let  $D$  be some (weak) 2-category. A **monad** in  $D$  is

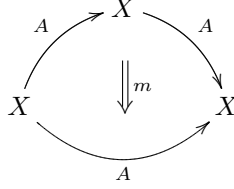
1. an object  $X$  of  $D$
2. an endomorphism

$$X \xrightarrow{A} X$$

3. 2-morphisms



and



such that  $m$  is associative and  $i$  is a unit with respect to  $m$ .

**Remark.** Let  $\boxed{1}$  be the 2-category with a single object, a single 1-morphism (the identity 1-morphism) and a single 2-morphism (the unique identity 2-morphism). Then a monad in  $D$  is the same as a lax 2-functor

$$\boxed{1} \longrightarrow D.$$

Before considering the first example, recall that a monoidal category can be regarded as a 2-category with a single object.

**Definition 3** Let  $C$  be any monoidal category. Denote by  $\Sigma(C)$  the 2-category which is the **suspension** of  $C$ . This is the 2-category whose

1. set of objects contains only a single element  $\text{Obj}(\Sigma) = \{\bullet\}$
2. morphisms  $\bullet \xrightarrow{c} \bullet$  are objects  $c \in \text{Obj}(C)$

3. 2-morphisms  $\bullet \begin{array}{c} \xrightarrow{c} \\ \Downarrow \phi \\ \xrightarrow{c'} \end{array} \bullet$  are morphisms  $c \xrightarrow{\phi} c'$  in  $C$

such that horizontal composition in  $\Sigma(C)$  is the tensor product in  $C$ .

**Example 1** An algebra object in a monoidal category  $C$  is the same as a monad in  $\Sigma(C)$ .

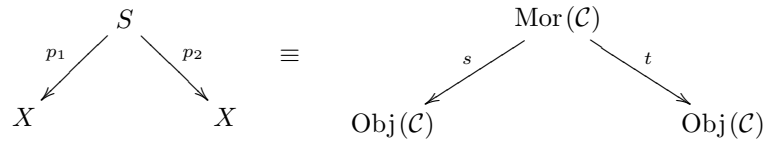
The next example is more interesting. It shows that  $\mathbf{Span}(C)$  can be regarded as a way of generalizing the notion of a category internal to  $C$ .

**Example 2** A category internal to some category  $C$  is the same as a monad in  $\mathbf{Span}(C)$ .

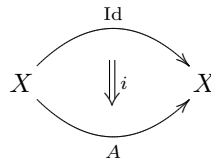
Proof. For illustration purposes, I'll spell this out in detail. By definition (def. 1 and def. 2), a monad in  $\mathbf{Span}(C)$  is

1. an object  $X \in \text{Obj}(C)$  which we call  $X \equiv \text{Obj}(C)$ ,

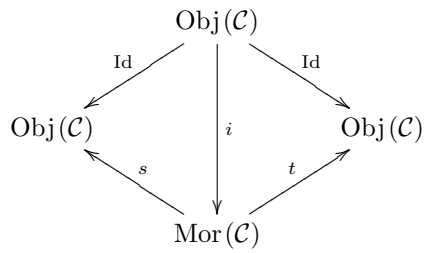
2. a morphism  $\text{Obj}(\mathcal{C}) \xrightarrow{A} \text{Obj}(\mathcal{C})$  in  $\mathbf{Span}(\mathcal{C})$ , i.e. an object  $S \in \text{Obj}(\mathcal{C})$  which we shall call  $S \equiv \text{Mor}(\mathcal{C})$ , together with morphisms



3. a 2-morphism

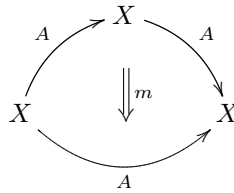


in  $\mathbf{Span}(\mathcal{C})$ , i.e. a commuting diagram

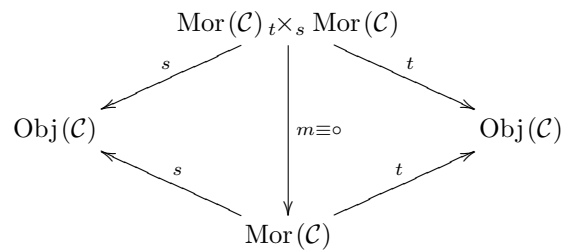


in  $\mathcal{C}$

4. a 2-morphism



in  $\mathbf{Span}(\mathcal{C})$ , i.e. a commuting diagram



in  $\mathcal{C}$

such that  $i$  and  $m = \circ$  satisfy associativity and unit laws. This is nothing but the definition of a category  $\mathcal{C}$  internal to  $C$ .  $\square$

## 2 Half-Spans

Now restrict attention to the special case of spans in a category  $C$  that has a *terminal object*. Let us denote this terminal object by “ $\bullet$ ”. Clearly, the category of endomorphisms of  $\bullet$  in  $\mathbf{Span}(C)$  is nothing but  $C$  itself:

$$\mathrm{Hom}_{\mathbf{Span}(C)}(\bullet, \bullet) \simeq C.$$

Hence, as is very well known,  $C$  must in fact be a monoidal category, with the tensor product being the horizontal composition on  $\mathrm{Hom}_{\mathbf{Span}(C)}(\bullet, \bullet)$  induced by that in  $\mathbf{Span}(C)$ .

**Definition 4** *Given a category  $C$  with terminal object  $\bullet$ , call*

$$\mathbf{LSpan}(C) \equiv \mathrm{Hom}_{\mathbf{Span}(C)}(--, \bullet)$$

*the category of left half spans and*

$$\mathbf{RSpan}(C) \equiv \mathrm{Hom}_{\mathbf{Span}(C)}(--, \bullet)$$

*the category of right half spans in  $C$ .*

In more detail, this definition implies the following.

1. An object of  $\mathbf{RSpan}(C)$  is a morphism

$$\begin{array}{ccc} S & & \\ & \searrow p & \\ & & X \end{array}$$

2. A morphism  $(S, p, X) \xrightarrow{\alpha} (S', p', X)$  in  $\mathbf{LSpan}(C)$  is a commuting diagram

$$\begin{array}{ccc} S & & \\ \alpha \downarrow & \searrow p & \\ & & X \\ & \nearrow p' & \\ S' & & \end{array}$$

in  $C$ .

Similarly for  $\mathbf{LSpan}(C)$ . In fact, it is obvious that

$$\mathbf{LSpan}(C) \simeq \mathbf{RSpan}(C) .$$

There is an obvious notion of a (left or right) **module category** over any monoidal category  $C$  (see for instance [5] for some definitions and further references) and this is what naturally appear here:

**Proposition 1** *Using horizontal composition in  $\mathbf{Span}(C)$ ,  $\mathbf{RSpan}(C)$  becomes a left module category over  $C$  and  $\mathbf{LSpan}(C)$  becomes a right module category over  $C$ .*

Proof. Follows trivially from the fact that  $C \simeq \text{Hom}_{\mathbf{Span}(C)}(\bullet, \bullet)$ ,  $\mathbf{RSpan}(C) = \text{Hom}_{\mathbf{Span}(C)}(\bullet, -)$   $\mathbf{LSpan}(C) = \text{Hom}_{\mathbf{Span}(C)}(-, \bullet)$   $\square$

**Proposition 2** *Let  ${}_C\mathbf{Mod}$  be the 2-category of left  $C$ -modules. There is 2-functor*

$$E : \mathbf{Span}(C) \rightarrow {}_C\mathbf{Mod}$$

*which is injective on objects, morphisms and 2-morphisms.*

Proof. This is completely analogous to definition 3 and proposition 1 in [5].  $\square$

Furthermore, composition in  $\mathbf{Span}(C)$  induces the following pairing.

**Definition 5** *Let*

$$\mathbf{RSpan}(C) \times_s \mathbf{LSpan}(C) \xrightarrow{\langle \cdot, \cdot \rangle} C$$

*be given by composition in  $\mathbf{Span}(C)$ , such that*

$$\left( \begin{array}{ccc} S & & S' \\ & \searrow p & \nearrow p' \\ & X & X \end{array} \right) \mapsto S \cdot S' ,$$

*where  $S \cdot S'$  is the pullback, as before.*

Phew, I am running out of time. Unless I made a mistake above, the upshot should be that  $\mathbf{RSpan}(C)$  is a collection of 2-Hilbert spaces over  $C$ ,  $\mathbf{LSpan}(C)$  plays the role of a collection of dual spaces and  $\text{Hom}_{\mathbf{Span}(C)}(X, Y)$  that of  $C$ -linear operators from the 2-Hilbert space indexed by  $X$  to that indexed by  $Y$ .

Moreover, Ostrik's theorem relates all this to an equivalent formulation in terms of bimodules [5].

## References

- [1] J. Baez, Categorifying Quantum Mechanics, lecture notes (2003), available at <http://math.ucr.edu/home/baez/categorifying.html>
- [2] J. Morton, Categorified Algebra and Quantum Mechanics, available as [math.QA/0601458](https://arxiv.org/abs/math/0601458)
- [3] J. Baez, Universal Algebra and Diagrammatic Reasoning, lecture notes (2006), available at <http://math.ucr.edu/home/baez/universal/>
- [4] U. Schreiber, FRS Formalism from 2-Transport, private notes, available at <http://www.math.uni-hamburg.de/home/schreiber/FRSfrom2Transport.pdf>
- [5] U. Schreiber, Module Categories and internal Bimodules, private notes, available at <http://www.math.uni-hamburg.de/home/schreiber/ModCat.pdf>