Derived categories in physics

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Homological mirror symmetry workshop
Erwin Schrodinger Institute
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Mathematics		Physics
Geometry: Gromov-Witten, Donaldson-Thomas, quantum cohomology, etc	⇔	Supersymmetric field theories
Homotopy, categories: derived categories, stacks, etc.	⇔	Renormalization group

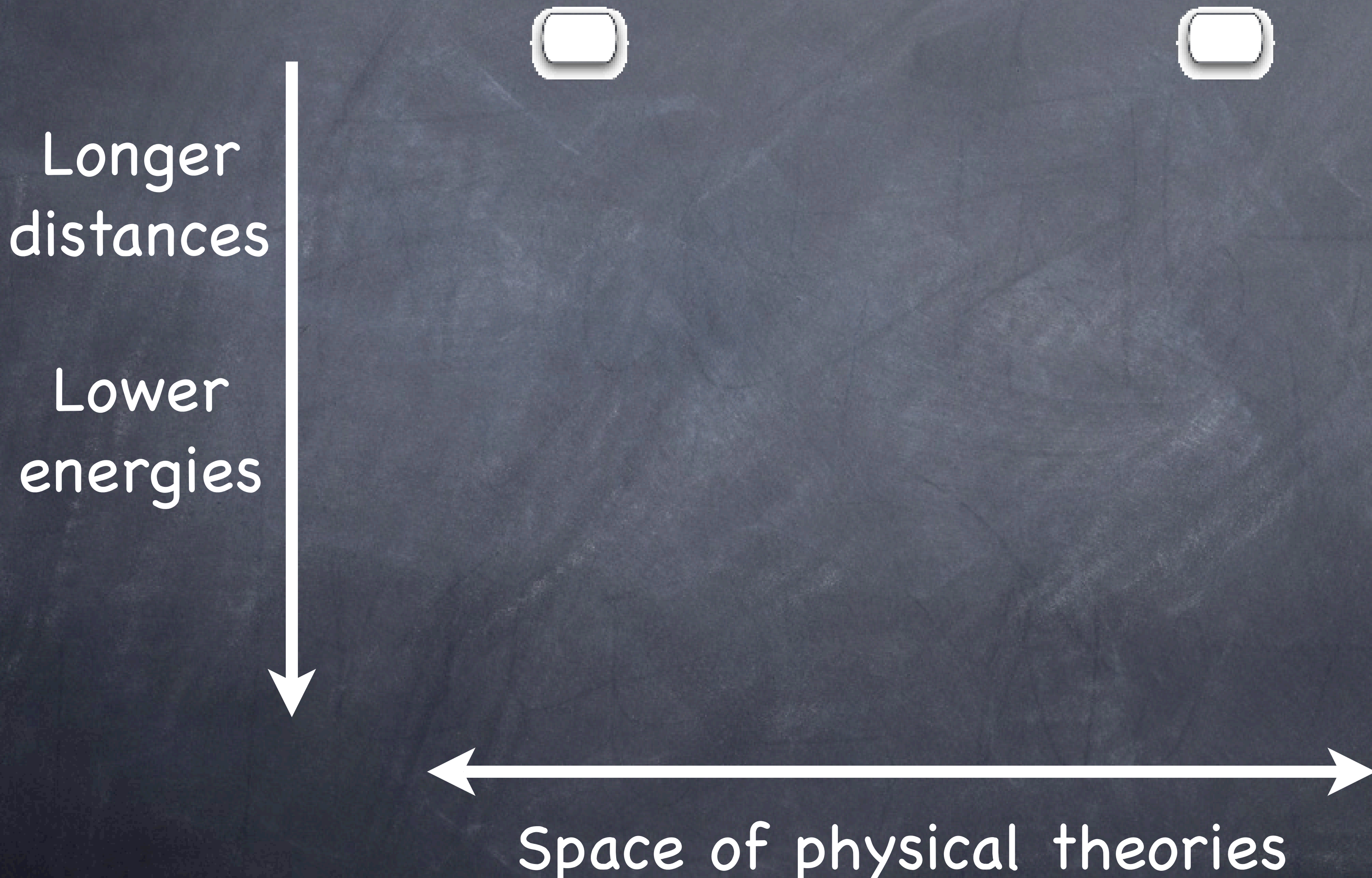
Renormalization group

What is it?

-- a **semi**group operation on an abstract space of physical theories.

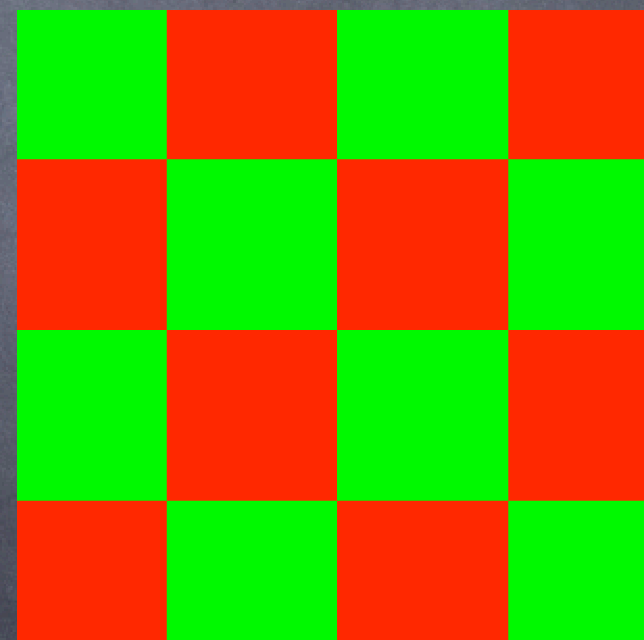
Given one QFT, new QFT's are constructed which are descriptions valid at longer and longer distances.

Renormalization group



Renormalization group

These pictures start out very different...
but as you stand farther & farther back...



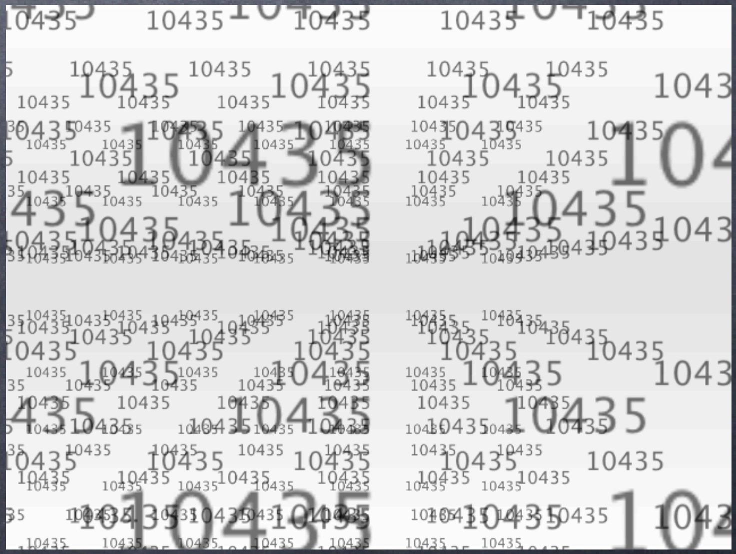
eventually they become the same.

Renormalization group

-- is a powerful tool, but unfortunately we really can't follow it completely explicitly in general.

-- can't really prove in any sense that two theories will flow under renormalization group to same point.

Instead, we do lots of calculations, perform lots of consistency tests, and if all works out, then we believe it.



Renormalization group

Applications I'll discuss today:

-- derived categories and ``D-branes''
(homological mirror symmetry)

-- stacks

Brief History of d.c.'s in physics

1994 -- Kontsevich introduces HMS; what is physics?

1995 -- Polchinski introduces D-branes

1996 -- Harvey, Moore suggest D-branes <-> sheaves

1998 -- Sen, Witten introduce antibranes, K theory

1999 -- E.S. proposes brane/antibrane systems <-> derived categories

2001 -- Douglas introduces pi-stability

2001 -- Aspinwall, Lawrence, Lazaroiu, Diaconescu....

Derived categories

Nowadays we believe that derived categories arise physically as a description of certain extended objects called ``D-branes.''

(ES '99, M Douglas '01, ...)

D-branes are, to lowest order, pairs:

-- submanifold of spacetime

-- vector bundle on submanifold

Model with sheaves $i_* \mathcal{E}$ $(\neq$  $)$

We think of the submfld as that swept out by the extended object as it propagates in spacetime.

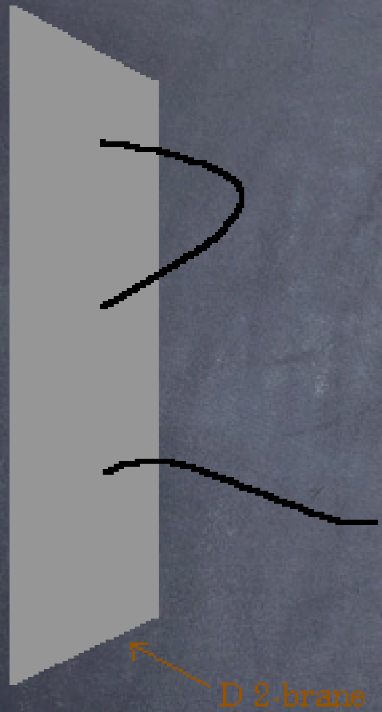
D-branes

Why are sheaves a good model for D-branes?

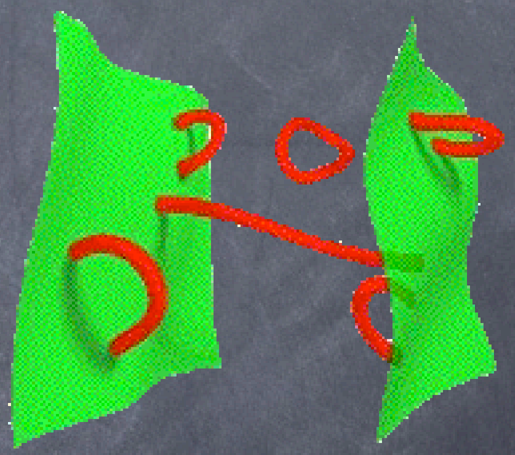
-- mathematical deformations of a sheaf
match
physical deformations of corresponding D-brane

More generally, can compute physical quantities (eg, massless spectra) using mathematical operations on sheaves (eg, Ext groups)

Computations in the conformal theory



D-branes are described by open strings, whose boundaries lie on the D-brane.



Massless states, which should correspond to Ext groups, will arise from zero-length open strings.

Computations in the conformal theory

Massless states are BRST-closed combinations of

$\phi^i, \phi^{\bar{\imath}}, \eta^{\bar{\imath}}, \theta_i$ modulo BRST-exact.

$$\begin{aligned} Q_{BRST} \cdot \phi^i &= 0, \quad Q_{BRST} \cdot \phi^{\bar{\imath}} \neq 0 \\ Q_{BRST} \cdot \eta^{\bar{\imath}} &= 0, \quad Q_{BRST} \cdot \theta_i = 0 \end{aligned}$$

States: $b(\phi)^{\alpha\beta}_{\bar{\imath}_1 \cdots \bar{\imath}_n}{}^{j_1 \cdots j_m} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$

$$Q_{BRST} \sim \overline{\partial} \qquad \eta^{\bar{\imath}} \sim d\overline{z}^{\bar{\imath}} \sim TS \qquad \theta_i \sim \mathcal{N}_{S/X}$$

States: $H^n(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^m \mathcal{N}_{S/X})$

Where are Ext's?

Computations in the conformal theory

A few complications:

The Freed-Witten anomaly:

To the sheaf $i_*\mathcal{E}$ one associates a D-brane on S with `bundle' $\mathcal{E} \otimes \sqrt{K_S}$ (instead of $\mathcal{E}$)

Open string analogue of Calabi-Yau condition:

$$\Lambda^{top}\mathcal{N}_{S\cap T/S} \otimes \Lambda^{top}\mathcal{N}_{S\cap T/T} \cong \mathcal{O}$$

(S Katz, ES, '02)

Computations in the conformal theory

More complications:

Boundary conditions on worldsheet fields:

$$\theta_i \;=\; F_{i\bar{\jmath}}\eta^{\bar{\jmath}}$$

(Abouelsaood et al '87)

Computations in the conformal theory

Taking into account such complications realizes a spectral sequence

$$H^n(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^m \mathcal{N}_{S/X}) \Longrightarrow \mathrm{Ext}_X^{n+m}(i_* \mathcal{E}, i_* \mathcal{F})$$

(Katz, ES '02)

Other sheaves?

Not all sheaves are of the form $i_* \mathcal{E}$ for some vector bundle $\mathcal{E}$. How to handle more general cases?

Partial answer: Higgs fields.
For each direction perpendicular to the D-brane worldvolume, there is a Higgs field.

Math'ly, given a D-brane w/ bundle $\mathcal{E}$
interpret $\Gamma(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{N}_{S/X})$
as defining a def' of the ring action on the module for $\mathcal{E}$
which gives more gen'l sheaves.

(Donagi, Ein Lazarsfeld '95;
T Gomez, ES '00;
Donagi, Katz, ES '03)

Other sheaves

Example: Describe skyscraper sheaf at $a \in \mathbf{C}$ starting with $C[x]/(x)$

Deform using Higgs field with vev a.

New module: $x \cdot \alpha = a\alpha$ where α is generator.

$$\Rightarrow (x-a)\cdot\alpha = 0 \Rightarrow \mathbf{C}[x]/(x-a)$$

Other sheaves

Example: Describe $\mathbf{C}[x]/(x^2)$
by single Higgs field acting on rank 2 bdle over pt
with vev

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Question: is this a good model of physics?

Other sheaves

On the worldsheet, giving a vev to a Higgs field adds a term to the boundary which deforms the BRST operator.

New BRST operator: (related to Kapustin's Wilson ops)

$$Q_{BRST} = \overline{\partial} + \Phi_1^i \theta_i - \Phi_2^i \theta_i$$

where the Φ^i are Higgs fields on either side of the open string.

Can show that cohomology of Q_{BRST} above
= Ext groups between corresponding sheaves

(Donagi, Katz, ES '03)

Derived categories

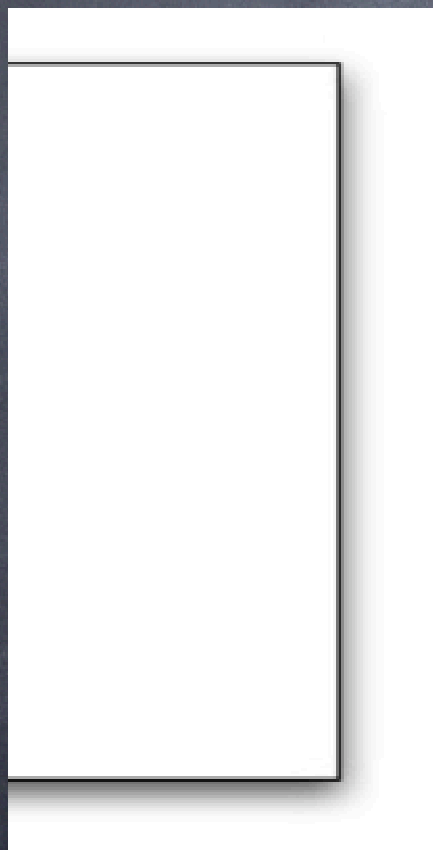
There's a lot more to derived categories than just, sheaves.

Where does the structure of complexes come from?

For that matter, where does the renormalization group enter?

First, in addition to D-branes, also have anti-D-branes....

Brane / antibrane annihilation



Derived categories

In add'n to antibranes, have ``tachyons'' which are represented by maps between the sheaves representing the branes & antibranes.

So the idea is going to be that given a complex

$$\cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \cdots$$

we relate it to a brane/antibrane system in which branes are $\mathcal{E}_i$ for i odd, say, other sheaves are antibranes, and the maps are tachyons.

(ES '99; Douglas '01)

Derived categories

Problem: We don't know how to associate branes to every possible sheaf.

Sol'n: So long as we're on a smooth cpx mfld, every (equiv' class of) object has a representative in terms of a complex of locally-free sheaves,

and we **do** know how to associate branes to those.

Derived categories

So, for any given (equiv' class of) object, pick a physically-realizable representative complex (at least one exists), and map it to branes/antibranes.

Problem: Such representatives are not unique, and different rep's lead to different physics.

Ex: $0 \longrightarrow \mathcal{E} \xrightarrow{=} \mathcal{E} \longrightarrow 0$ vs. 0

Derived categories

Sol'n: renormalization group flow

The proposal is that any two brane/antibrane systems representing quasi-isomorphic complexes flow to the same physical theory under the renormalization group.

Can't be shown explicitly, so must do lots of indirect checks.

Computations in the nonconformal theory

On the worldsheet, to describe tachyons, we add a term to the boundary, which has the effect of modifying the BRST operator, which becomes

$$Q_{BRST} = \overline{\partial} + \sum_i \phi_i^{\alpha\beta} \qquad \text{(schematically)}$$

Necessary for supersymmetry: $Q_{BRST}^2 = 0$

$$\Rightarrow \begin{cases} \overline{\partial}\phi^{\alpha\beta} = 0 & \text{-- maps are holomorphic} \\ \phi_i^{\alpha\beta}\phi_{i+1}^{\beta\gamma} = 0 & \text{-- condition for complex} \end{cases}$$

(Aspinwall, Lawrence '01)

Computations in the nonconformal theory

S'pose $f. = C. \longrightarrow D.$ is a chain homotopy

i.e. $f = \phi_D s - s\phi_C$ for $s_n : C_n \longrightarrow D_{n-1}$

Then $f = Qs$

-- BRST exact

Computations in the nonconformal theory

Ex: Compute $\mathrm{Ext}^n_{\mathbf{C}}\left(\mathcal{O}_D, \mathcal{O}\right)$

$$0 \longrightarrow \mathcal{O}(-D) \stackrel{\phi}{\longrightarrow} \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Boundary (R-sector) states are of the form

$$b^{\alpha\beta}_{0\bar{\imath}_1\cdots\bar{\imath}_n} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \sim H^n(\mathcal{O}(-D)^\vee \otimes \mathcal{O})$$

$$b^{\alpha\beta}_{1\bar{\imath}_1\cdots\bar{\imath}_n} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \sim H^n(\mathcal{O}^\vee \otimes \mathcal{O})$$

Computations in the nonconformal theory

Ex, cont'd

Degree 1 states: $b_0 + b_{1\bar{\imath}}\eta^{\bar{\imath}}$

BRST closed:	BRST exact:
$\overline{\partial} b_0 = -\phi(b_{1\bar{\imath}} d\overline{z}^{\bar{\imath}})$	$b_0 = \phi a$
$\overline{\partial}(b_{1\bar{\imath}} d\overline{z}^{\bar{\imath}}) = 0$	$b_{1\bar{\imath}} d\overline{z}^{\bar{\imath}} = \overline{\partial} a$

$$\Rightarrow b_0 \bmod \mathrm{Im}\,\phi \in H^0\left(D, \mathcal{O}(-D)^\vee|_D \otimes \mathcal{O}|_D\right) = \mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O})$$

Computations in the nonconformal theory

Ex, cont'd

Conversely, given an element of

$$\mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O}) = H^0\left(D, \mathcal{O}(-D)^\vee|_D \otimes \mathcal{O}_D\right)$$

we can define b_0 and b_1 using the long exact seq'

$$\cdots \longrightarrow H^0(\mathcal{O}) \longrightarrow H^0(\mathcal{O}(D)) \longrightarrow H^0(D, \mathcal{O}(D)|_D) \xrightarrow{\delta} H^1(\mathcal{O}) \longrightarrow \cdots$$

b_1 is the image under δ

b_0 is the lift to an element of $C^\infty(\mathcal{O}(D))$

Computations in the nonconformal theory

More generally, it can be shown that Ext groups can be obtained in this fashion.

Thus, massless spectra can be counted in the nonconformal theory, and they match massless spectra of corresponding conformal theory: both counted by Ext's

-- a nice test of presentation-independence of RG

Computations in the nonconformal theory

Grading:

The tachyon T is a degree zero operator.
We add [G,T] to the boundary.
G has $U(1)_R$ charge -1,
so [G,T] has charge -1.

Nec' condition for susy: boundary ops must be neutral under $U(1)_R$

Thus, the Noether charge has b.c. s.t. grading shifts by one.

Bondal-Kapranov

Why should maps between branes & antibranes
unravel into a **complex**,
as opposed to something with more gen'l maps?

$$
\begin{tikzcd}
\mathcal{E}_0 \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E}_2 \arrow[l, bend right] \arrow[r] & \mathcal{E}_3 \arrow[r] & \mathcal{E}_4 \arrow[r] & \mathcal{E}_5 \arrow[ll, bend right]
\end{tikzcd}
$$

In fact, you can....

Bondal-Kapranov

If we add a boundary operator O of deg n,
then [G, O] has charge n-1,
so the boundaries it lies between must have relative
$U(1)_R$ charge 1-n,
and so gives rise to the `unusual' maps displayed.

Bondal-Kapranov

The BRST operator is deformed:

$$Q_{BRST} = \overline{\partial} + \sum_i \phi_i^{\alpha\beta}$$

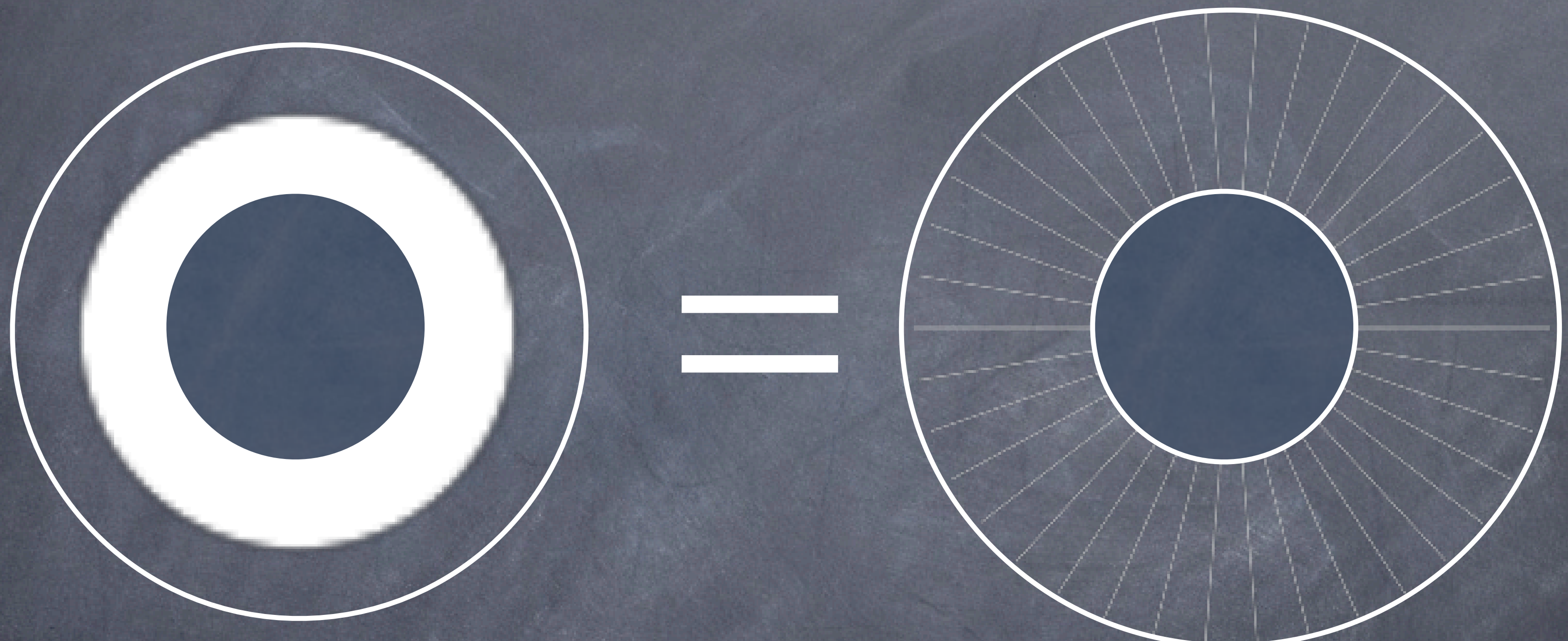
and demanding $Q^2_{BRST} = 0$ implies

$$\sum_i \overline{\partial}\phi_i + \sum_{i,j} \phi_i \cdot \phi_j = 0$$

which is the condition of [BK] for a "generalized complex"

(Bondal, Kapranov '91; Lazaroiu '01)

Cardy condition



$$\int_M \mathrm{ch}(\mathcal{E})^* \wedge \mathrm{ch}(\mathcal{F}) \wedge td(TM) = \sum_i (-)^i \dim \mathrm{Ext}^i_M(\mathcal{E}, \mathcal{F})$$

(A. Caldararu)

Open problems

* Bundles of rank > 1

* Open string anomaly cancellation implies that can only have B model open strings between **some** D-branes, & not others

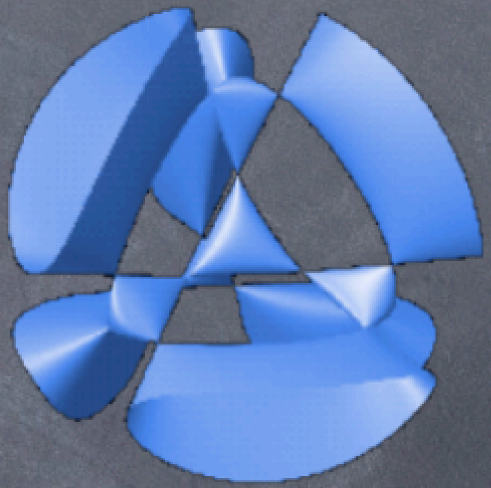
* Physics for more general sheaves?

Although we now have most of the puzzle pieces, a complete comprehensive physical understanding still does not quite exist.

On to the second application of the renormalization group....

T Pantev and I have been studying what it means to compactify a string on a stack.

After all, stacks are a mild generalization of spaces....



Why bother?

-- to understand the most general possible string compactifications

-- in certain formal constructions, they sometimes appear as mirrors to spaces

Stacks

How to make sense of strings on stacks concretely?

Well, stacks can be thought of as local orbifolds, which in patches look like quotients by finite not-necessarily-effective groups.

Problem: in physics, only global quotients are known to define CFT's, and only effectively-acting quotients are well-understood.

Stacks

How to make sense of strings on stacks concretely?

Most (smooth, Deligne-Mumford) stacks can be presented as a global quotient

$$[X/G]$$

for X a space and G a group.

To such a presentation, associate a ``G-gauged sigma model on X."

Problem: such presentations not unique

Stacks

If to [X/G] we associate ``G-gauged sigma model,"
then:

$[\mathbf{C}^2/\mathbf{Z}_2]$ defines a 2d theory with a symmetry called conformal invariance

$[X/\mathbf{C}^\times]$ defines a 2d theory w/o conformal invariance

Same stack, different physics!

Potential presentation-dependence problem:
fix with renormalization group flow
(Can't be checked explicitly, though.)

Stacks

One would like to at least check that massless spectra are presentation-independent.

Problem: massless spectra only computable for global quotients by finite groups,
and only well-understood for
global quotients by finite effectively-acting gps

So: no way to tell if massless spectra are the same across presentations

The problems here are analogous to the derived-categories-in-physics program.

There, to a given object in a derived category, one picks a representative with a physical description (as branes/antibranes/tachyons).

Alas, such representatives are not unique.

It is conjectured that different representatives give rise to the same low-energy physics, via boundary renormalization group flow.

Only indirect tests possible, though.

Stacks

First indirect test: do deformations of stacks match deformations of corresponding CFT's?

In every other known example of geometry applied to physics, math deformations match physics def's.

Stacks **fail** this test, even in basic cases:

* $[\mathbf{C}^2/\mathbf{Z}_2]$ is rigid
* corresponding physical theory has def's

Could this signal presentation-dependence?

Stacks

To justify that stacks are relevant physically, as opposed to some other mathematics, one has to understand this deformation theory issue, as well as conduct tests for presentation-dependence.

This was the subject of several papers.

For the rest of today's talk,
I want to focus on special kinds of stacks, namely, gerbes.

(= quotient by noneffectively-acting group)

Gerbes

Gerbes have add'l problems when viewed from this physical perspective.

Example: The naive massless spectrum calculation contains multiple dimension zero operators, which manifestly violates cluster decomposition, one of the foundational axioms of quantum field theory.

There is a single known loophole: if the target space is disconnected. We **think** that's what's going on....

Decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

Claim

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K\right]\right)$$

(together with some B field), where

$\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G - \mathrm{gerbe\ on}\ X) \, = \, \mathrm{CFT}\left(\coprod_{\hat{G}} (X, B) \right)$$

where the B field is determined by the image of

$$H^2(X, Z(G)) \stackrel{Z(G) \to U(1)}{\longrightarrow} H^2(X, U(1))$$

Banded Example:

Consider $[X/D_4]$ where the center acts trivially.

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow D_4 \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$$

One of the effective orbifolds has vanishing discrete torsion, the other has nonvanishing discrete torsion.

(Using the relationship between discrete torsion and B fields first worked out by ES, '00.)

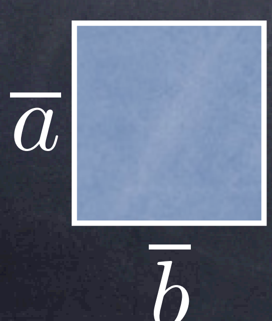
Check genus one partition functions:

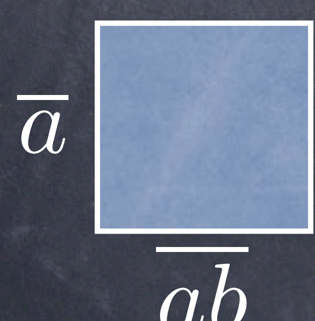
$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

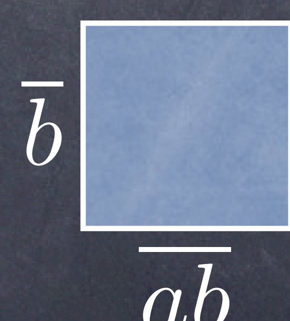
$$\mathbf{Z}_2 \times \mathbf{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$

$$Z(D_4) = \frac{1}{|D_4|} \sum_{g,h \in D_4, gh=hg} Z_{g,h}$$

Each of the $Z_{g,h}$ twisted sectors that appears, is the same as a $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector, appearing with multiplicity $|\mathbf{Z}_2|^2 = 4$ except for the







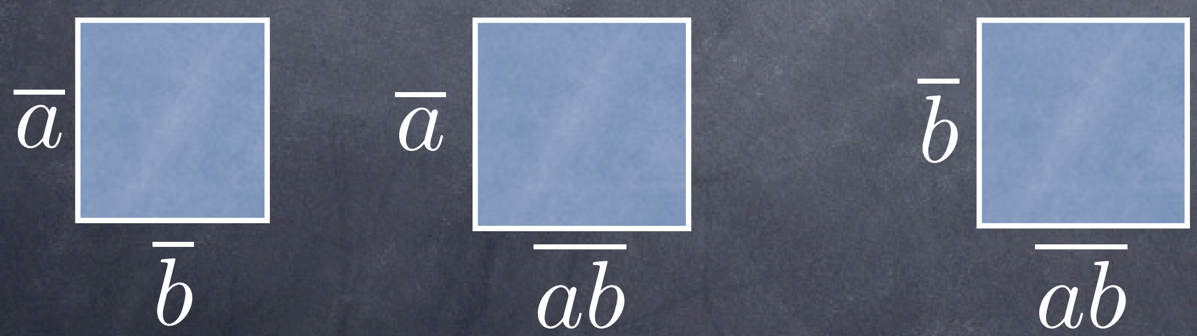
sectors.

Partition functions, cont'd

$$
\begin{aligned}
Z(D_4) &= \frac{|\mathbf{Z}_2 \times \mathbf{Z}_2|}{|D_4|} |\mathbf{Z}_2|^2 \left(Z(\mathbf{Z}_2 \times \mathbf{Z}_2) \; - \; (\text{some twisted sectors}) \right) \\
&= 2 \left(Z(\mathbf{Z}_2 \times \mathbf{Z}_2) \; - \; (\text{some twisted sectors}) \right)
\end{aligned}
$$

(In ordinary QFT, ignore multiplicative factors, but string theory is a 2d QFT coupled to gravity, and so numerical factors are important.)

Discrete torsion acts as a sign on the



twisted sectors

so we see that $Z([X/D_4]) = Z\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$

with discrete torsion in one component.

A quick check of this example comes from comparing massless spectra:

Spectrum for $[T^6/D_4]$:

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 0 & & 0 & & \\ & 0 & & 54 & & 0 & \\ 2 & & 54 & & 54 & & 2 \\ & 0 & & 54 & & 0 & \\ & & 0 & & 0 & & \\ & & & 2 & & & \end{array}$$

and for each $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 51 & & 51 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 51 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 51 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Sum matches. ✓

Nonbanded example:

Consider $[X/\mathbf{H}]$ where $\mathbf{H}$ is the eight-element group of quaternions, and a $\mathbf{Z}_4$ acts trivially.

$$1 \longrightarrow < i > (\cong \mathbf{Z}_4) \longrightarrow \mathbf{H} \longrightarrow \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}([X/\mathbf{H}]) = \mathrm{CFT}\left([X/\mathbf{Z}_2] \coprod [X/\mathbf{Z}_2] \coprod X \right)$$

Straightforward to show that this is true at the level of partition functions, as before.

Another class of examples:
global quotients by nonfinite groups

The banded $\mathbf{Z}_k$ gerbe over $\mathbf{P}^N$
with characteristic class $-1 \mod k$
can be described mathematically as the quotient

$$\left[\frac{ \mathbf{C}^{N+1} - \{0\} }{ \mathbf{C}^\times } \right]$$

where the $\mathbf{C}^\times$ acts as rotations by k times

which physically can be described by a U(1) susy gauge theory with N+1 chiral fields, of charge k

How can this be different from ordinary $\mathbf{P}^N$ model?

The difference lies in nonperturbative effects. (Perturbatively, having nonminimal charges makes no difference.)

Example: Anomalous global U(1)'s

$$\mathbf{P}^{N-1}: \quad U(1)_A \mapsto \mathbf{Z}_{2N}$$
$$\mathrm{Here}: \quad U(1)_A \mapsto \mathbf{Z}_{2kN}$$

Example: A model correlation functions

$$\mathbf{P}^{N-1}: \quad < X^{N(d+1)-1} > = q^d$$
$$\mathrm{Here}: \quad < X^{N(kd+1)-1} > = q^d$$

Example: quantum cohomology

$$\mathbf{P}^{N-1}: \quad \mathbf{C}[x]/(x^N - q)$$
$$\mathrm{Here}: \quad \mathbf{C}[x]/(x^{kN} - q)$$

Different physics

General argument:

Compact worldsheet:

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then Φ charge Q implies
$$\Phi \in \Gamma(L^{\otimes Q})$$

Different bundles => different zero modes
=> different anomalies => different physics

Noncompact worldsheet:

If electrons have charge k, then instantons have charge 1/k, and result is identical to ordinary case.

S'pose add massive fields of charge ± 1

Can determine instanton num's by periodicity of theta angle, which acts like electric field in 2d.

If everything has charge k, then theta angle has periodicty $2\pi k$ and we're back to ordinary case.

But, existence of massive fields of unit charge means theta angle has periodicity 2π, which is the new case.

(J Distler, R Plesser)

4d analogues

* $SU(n)$ vs $SU(n)/\mathbf{Z}_n$ gauge theories
(crucial for Kapustin-Witten's geom' Langlands pic)
* Spin(n) vs SO(n) gauge theories

M. Strassler has studied Seiberg duality in this context, has exs of Spin(n) gauge theories with

$\mathbf{Z}_2$ monopoles
(distinguishing Spin(n) from SO(n) nonpert'ly)
Seiberg dual to
Spin(n) gauge theory w/ massive spinors
(distinguishing Spin(n) from SO(n) pert'ly)

Back to 2d.....

K theory implications

This equivalence of CFT's implies a statement about K theory (thanks to D-branes).

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

If G acts trivially on X
then the ordinary H-equivariant K theory of X

is the same as

twisted K-equivariant K theory of $X \times \hat{G}$

* Can be derived just within K theory
* Provides a check of the decomposition conjecture

D-branes and sheaves

D-branes in the topological B model can be described with sheaves and, more gen'ly, derived categories.

This also is consistent with the decomp' conjecture:

Math fact:

A sheaf on a banded G-gerbe

is the same thing as

a twisted sheaf on the underlying space,
twisted by image of an element of $H^2(X,Z(G))$

which is consistent with the way D-branes should behave according to the conjecture.

D-branes and sheaves

Similarly, massless states between D-branes should be counted by Ext groups between the corresponding sheaves.

Math fact:

Sheaves on a banded G-gerbe decompose according to irrep' of G,
and sheaves associated to distinct irreps have vanishing Ext groups between them.

Consistent w/ idea that sheaves associated to distinct reps should describe D-branes on different components of a disconnected space.

Gromov-Witten prediction

Notice that there is a prediction here for Gromov-Witten theory of gerbes:

GW of $[X/H]$

should match

GW of $\left[(X \times \hat{G})/K \right]$

Works in basic cases:
BG (T Graber), other exs (J Bryan)

Quantum cohomology

One of the results of our analysis of stacks is a generalization of Batyrev's conjecture for quantum cohomology to toric stacks (Borisov, Chen, Smith, '04)

In physics, Batyrev's conjecture has a precise meaning -- it's the quantum cohomology ring in the UV (GLSM) theory, and it can be extracted from the 2d effective action of the gauge theory, w/o any explicit mention of rat'l curves.

Quantum cohomology

Specifically, we found that old results of Morrison-Plesser generalize from toric varieties to toric stacks.

Let the toric stack be described in the form

$$\left[\frac{\mathbf{C}^N - E}{(\mathbf{C}^\times)^n} \right]$$

E some exceptional set

Q_i^a the weight of the ith vector under ath $\mathbf{C}^\times$

then the quantum cohomology ring is of the form

$\mathbf{C}[\sigma_1, \cdots, \sigma_n]$ modulo the relations

$$\prod_{i=1}^{N} \left(\sum_{b=1}^{n} Q_i^b \sigma_b \right)^{Q_i^a} = q_a$$

(ES, T Pantev, '05)

Quantum cohomology

Ex: Quantum cohomology ring of $\mathbf{P}^N$ is

$$\mathbf{C}[x]/(x^{N+1} - q)$$

Quantum cohomology ring of $\mathbf{Z}_k$ gerbe over $\mathbf{P}^N$ with characteristic class $-n \bmod k$ is

$$\mathbf{C}[x,y]/(y^k - q_2,\ x^{N+1} - y^n q_1)$$

Aside: these calculations give us a check of the massless spectrum -- in physics, can derive q.c. ring w/o knowing massless spectrum.

Quantum cohomology

We can see the decomposition conjecture in the quantum cohomology rings of toric stacks.

Ex: Q.c. ring of a $\mathbf{Z}_k$ gerbe on $\mathbf{P}^N$ is given by

$$\mathbf{C}[x,y]/(y^k - q_2,\ x^{N+1} - y^n q_1)$$

In this ring, the y's index copies of the quantum cohomology ring of $\mathbf{P}^N$ with variable q's.

The gerbe is banded, so this is exactly what we expect -- copies of $\mathbf{P}^N$, variable B field.

Quantum cohomology

More generally, a gerbe structure is indicated from this quotient description whenever $\mathbf{C}^{\times}$ charges are nonminimal.

In such a case, from our generalization of Batyrev's conjecture, at least one rel'n will have the form

$$p^k = q$$

where p is a rel'n in q.c. of toric variety, and k is the nonminimal part.

Can rewrite this in same form as for gerbe on $\mathbf{P}^N$, and in this fashion can see our decomp' conj' in our gen'l of Batyrev's q.c.

Mirrors to stacks

There exist mirror constructions for any model realizable as a 2d abelian gauge theory.

For toric stacks (BCS '04), there is such a description.

Standard mirror constructions now produce character-valued fields, a new effect, which ties into the stacky fan description of (BCS '04).

(ES, T Pantev, '05)

Toda duals

Ex: The ``Toda dual'' of $\mathbf{P}^N$ is described by the holomorphic function

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \exp(Y_1 + \cdots + Y_N)$$

The analogous duals to $\mathbf{Z}_k$ gerbes over $\mathbf{P}^N$ are described by

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \Upsilon^n \exp(Y_1 + \cdots + Y_N)$$

where Υ is a character-valued field

(ES, T Pantev, '05)

Gerby quintic

In the same language, the LG-point mirror to the quintic hypersurface in a $\mathbf{Z}_k$ gerbe over $\mathbf{P}^4$ is described by (an orbifold of) the superpotential

$$W = x_0^5 + \cdots + x_4^5 + \psi \Upsilon x_0 x_1 x_2 x_3 x_4$$

where ψ is the ordinary cpx structure parameter

Υ is a discrete (character)-valued field

How to interpret this?

Gerby quintic

In terms of the path integral measure,

$$\int [\mathcal{D}x_i, \Upsilon] = \int [\mathcal{D}x_i] \sum_{\Upsilon} = \sum_{\Upsilon} \int [\mathcal{D}x_i]$$

so having a discrete-valued field is equivalent to summing over contributions from different theories,

or, equiv'ly,

summing over different components of the target space.

Gerby quintic

Mirror map: $B + iJ = -\frac{5}{2\pi i}\log(5\psi) + \cdots$

So shifting ψ by phases has precisely the effect of shifting the B field, exactly as the decomposition conjecture predicts for this case.

So far, we've argued that

CFT(string on gerbe) = CFT(string on spaces)

and outlined several families of tests.

Physically, we're interpreting this as **T-duality**.

* CFT's same on both sides

* Sometimes can be understood as a Fourier-Mukai transform.

Summary

* renormalization (semi)group

* derived categories & D-branes

* stacks