Frobenius Algebra & 2d TFT

we would like to understand the following

Fact (roughly):

$$\boxed{2d\,TFT \simeq s\,Frob}$$

"2-dimensional topological quantum field theories are equivalent to 'special' Frobenius algebras."

the reason turns out to be:

Frobenius algebras know all about the triangulations of surfaces

First: <u>what is a QFT ??</u>

<u>recall:</u>

quantum mechanics is given by a

(Hilbert-) <u>space of states</u> H

a linear operator $\boxed{H \xrightarrow{\exp(it\Delta)} H}$

for every <u>time interval</u> $\boxed{\circ \xrightarrow{t} \circ}$

such that $\boxed{H \xrightarrow{\exp(it_1\Delta)} H \xrightarrow{\exp(it_2\Delta)} H = H \xrightarrow{\exp(i(t_1+t_2)\Delta)} H}$

this operator describes <u>time evolution</u> of states

<u>sophisticated reformulation:</u>

QM is a <u>functor</u>

$$U : 1dCob_{Riem} \longrightarrow Hilb$$

$$(\circ \xrightarrow{t} \circ) \longmapsto (H \xrightarrow{\exp(it\Delta)} H)$$

from 1-dimensional Riemannian cobordisms to Hilbert spaces an <u>n-dimensional cobordism</u> is simply a manifold (oriented) with boundary

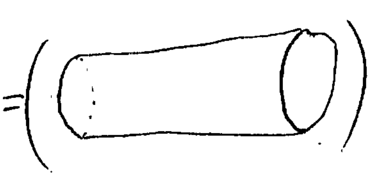
$$M \; : \; \partial M = B^+ \cup B^-$$

where B^+ is considered "ingoing"
and B^- is considered "outgoing"

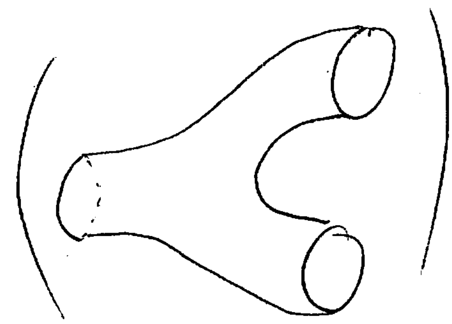
example:
- the interval <u>co</u>bounds two points

$$M = (\bullet\!\!-\!\!-\!\!-\!\!\bullet) \; ; \; \partial M = (\bullet^+ \cup \bullet^-)$$

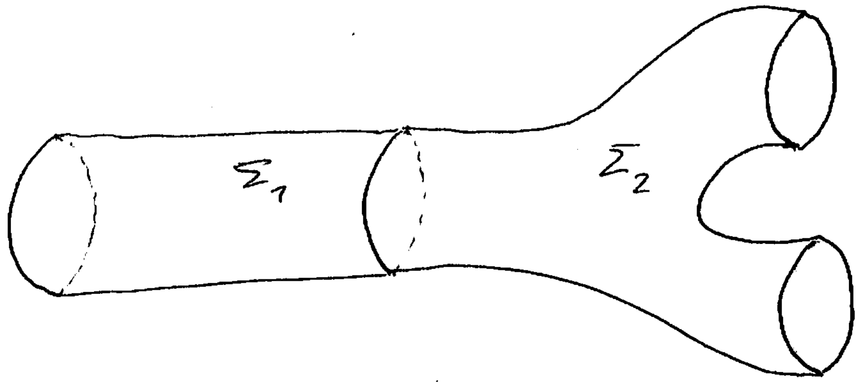
- the cylinder cobounds two circles

$M =$  $\partial M = (\bigcirc^+ \cup \bigcirc^-)$

- the "trinion" ("pair of pants") cobounds three circles

$M =$  $\partial M = (\bigcirc^+ \cup \bigcirc^- \cup \bigcirc^-)$

n-cobordisms form a category in the obvious way



$\in \mathrm{Mor}(2\mathrm{Cob})$

$$S_1 \xrightarrow{\Sigma_1} S_1 \xrightarrow{\Sigma_2} (S_1 \cup S_1)$$

we think of a

- 1-cobordism as a <u>worldline of a particle</u>
- 2-cobordism as a <u>worldsheet of a string</u>
- 3-cobordism as a <u>worldvolume of a membrane</u>

etc...

<u>so</u> we want to associate a <u>Hilbert space of strings</u> to the circle, and an operator of <u>string propagation</u> for a 2-dim cobordism

<u>so</u> we say:

<u>Def.</u>: An n-dimensional QFT is a functor

$$\boxed{U : n\mathrm{Cob}_S \longrightarrow \mathrm{Hilb}}$$

possibly extra structure

different flavors of QFT result
from different structure on the cobordisms:

a) $2\,\mathrm{Cob}_{\mathrm{top}}$: 2d-cobordisms up to diffeomorphism

$\rightsquigarrow$ "topological string theory"

2) $2\,\mathrm{Cob}_{\mathrm{conf}}$: 2d-cobordisms with conformal structure

$\rightsquigarrow$ 2d conformal field theory

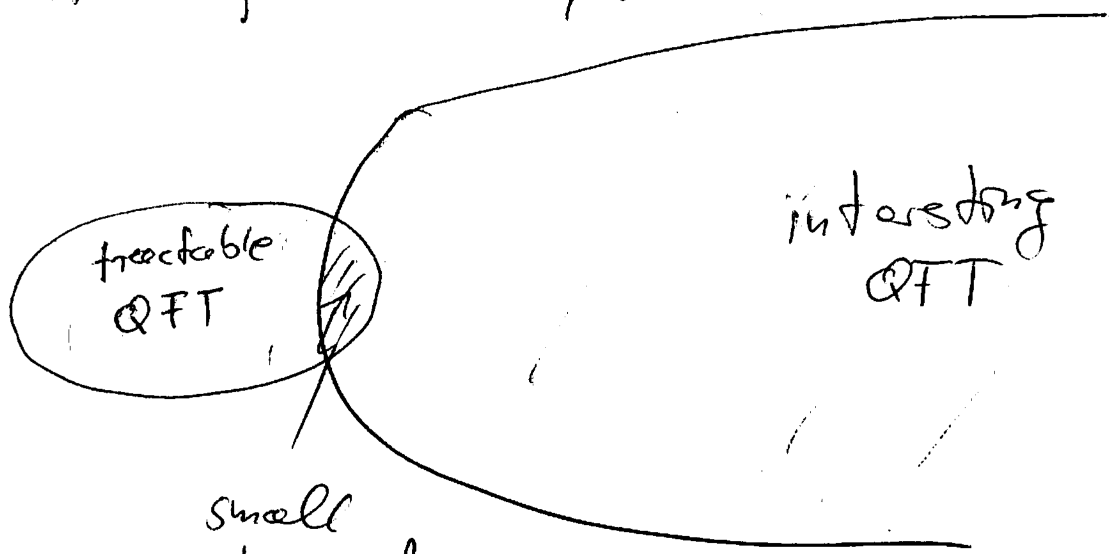
"physical strings"

3) $2\,\mathrm{Cob}_{\mathrm{Riem}}$: 2-d cobordisms with Riemannian structure

$\rightsquigarrow$ 2-dimensional QFT on curved background

etc.

main problem with the study of quantum field theory:



topological
↓
2d TFT is rather easy to understand (by the end of this discussion)

conformal
↓
2d CFT is already considerably richer and already much harder to get under control

but: Fuchs-Runkel-Schweigert say:
Fjelstad-

FFRS theorem:

$$2d\,R\,CFT = \text{complex analytic part} + \text{top. part}$$

$$= \underbrace{\text{vertex op. algebra}}_{\text{pretty much understood}} + \underbrace{\text{sewing constraints}}_{\text{purely topol. problem}}$$

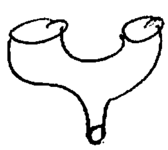
"rational" "special subclass of 2d CFT"

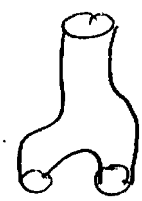
So: to understand 2d CFT
first understand 2d TFT:

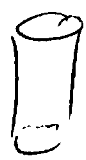
<u>global description of 2d TFT:</u>

$2\mathrm{Cob}_{top}$ is easy to understand, because it can be presented in terms of <u>generators and relations</u>

<u>generators</u>:


trinion

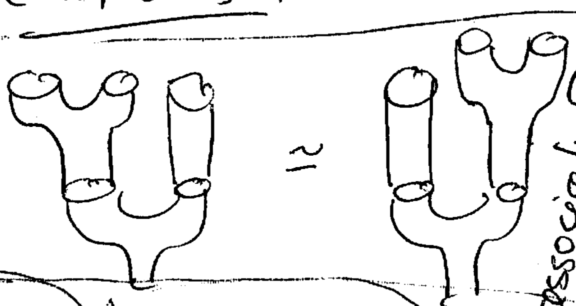

cotrinion

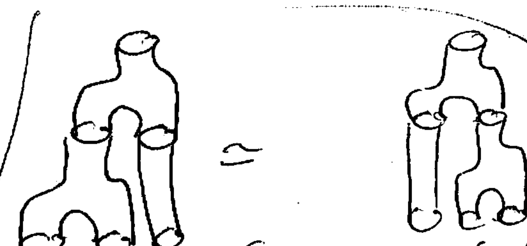

cylinder


cap

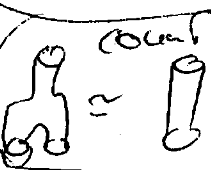

co cap

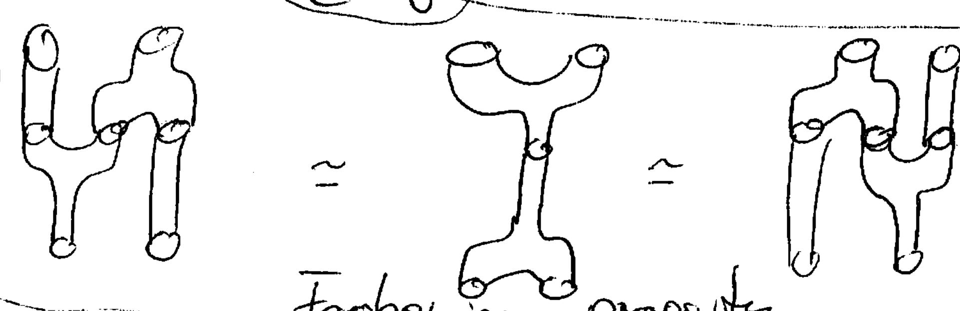
<u>relations</u>:

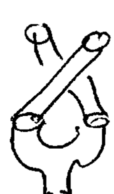

associativity


co associativity


unit


counit


Frobenius property


commutativity

≃ cop

this description is useful, because it allows us to also express the representations of 2Cob — the 2d TFTs — in terms of generators and relations:

the functor

$$U : 2Cob \longrightarrow Vect$$

sends the single object

$$\bigcirc = S^1$$

to some vector space

$$U(S^1) := A$$

it sends the trinion to some multiplication

$U($  $) = A \otimes A \xrightarrow{P} A$

and the cotrinion to a coproduct

$U($  $) = A \xrightarrow{\Delta} A \otimes A$

It sends the cap to a unit

$$\mathcal{U}(\bigcirc) = \mathbb{C} \xrightarrow{i} A$$

and the cocap to a counit

$$\mathcal{U}(\bigcirc) = A \xrightarrow{e} \mathbb{C}.$$

Moreover, to really be a functor, $\mathcal{U}$ must respect all the relations.

These say that the above operations make A

a commutative associative algebra with unit

and at the same time

a co-commutative coalgebra with counit

hence a bialgebra

such that product and coproduct satisfy the Frobenius property

$\Longrightarrow$ <u>A must be a Frobenius algebra</u>

this way one proves

monoidal functors $2\mathrm{Cob} \to \mathrm{Vect}$

$\simeq$

commutative Frobenius algebras A

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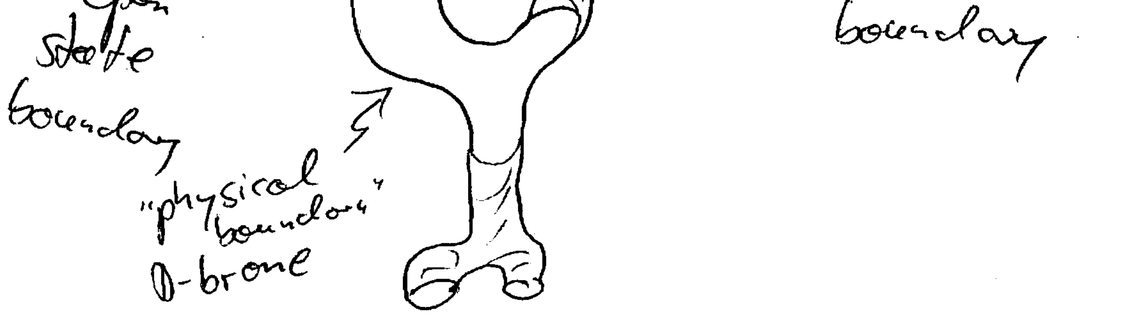
we can do something similar for "open-closed strings":

the category

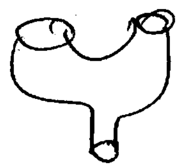
$2\mathrm{Cob}^{op}$

also contains cobordisms like

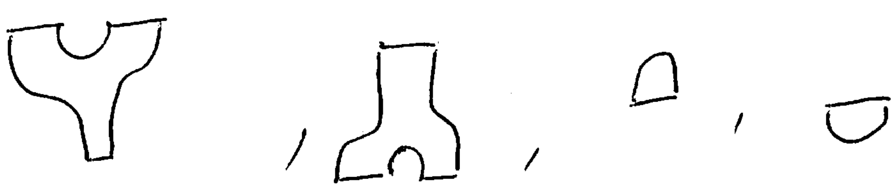


Moore and Segal said
and
Lauda and Pfeiffer fully proved
that <u>$2Cob^{op}_B$ generated</u> "<u>in the obvious way</u>"

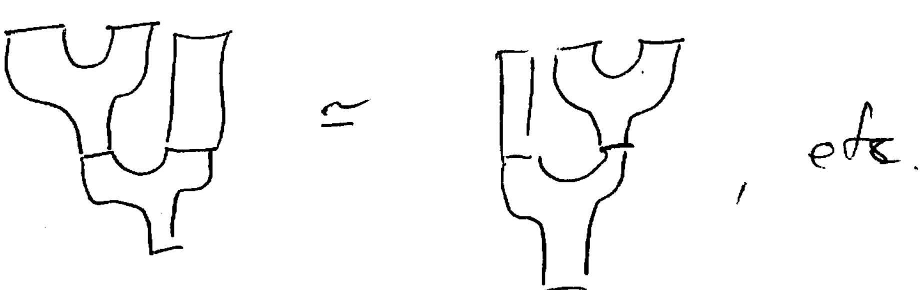
apart from the closed string worldsheet generators

 etc.

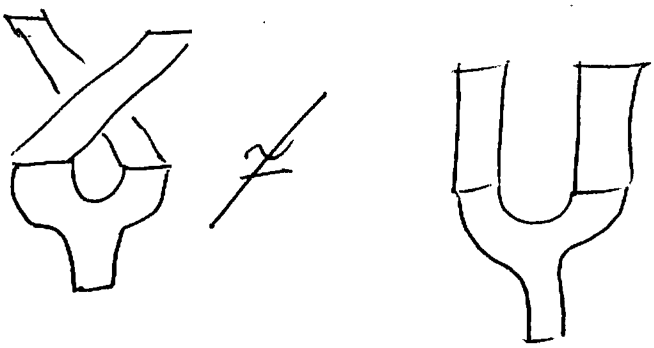
It also has all the corresponding open string worldsheet generators



with the obvious relations on them.



one relation that does <u>not</u> hold for the open string is commutativity



This is quickly seen by attempting to identify the boundaries and necessarily failing

It is clear then that a representation

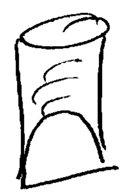
$$\mathcal{U} : 2\mathrm{Cob}^{op} \longrightarrow \mathrm{Vect}$$

has to associate

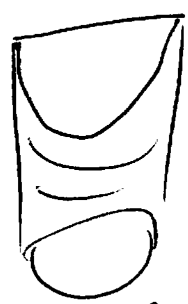
- a commutative Frobenius algebra to the circle $\bigcirc$

- a not-necessarily comm Frob. algebra to the interval but there are also two generators that relate the open and the closed string state:

the zipper

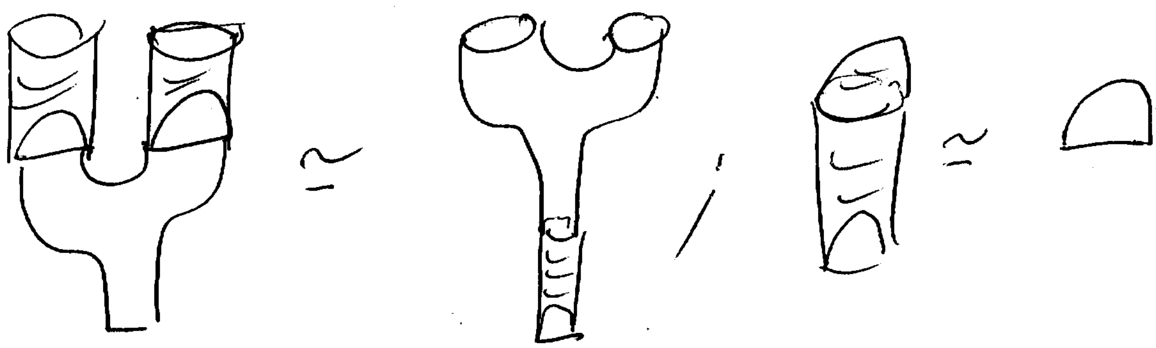


and the cozipper



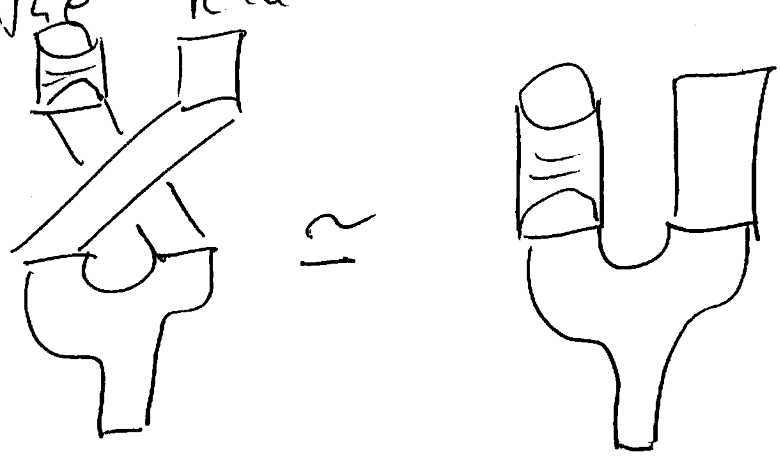
describing how an open string closes its ends to become a closed string, or a closed string splitting into an open string

one finds that the zipper gives
an algebra homomorphism from
the closed to the open algebra



co-similarly for the co zipper

in addition to all this, one finds
the relation



this says that the image of the zipper is
in the <u>center</u> of the open string algebra!

This is essentially due to

Moore & Segal

but Lauda & Pfeiffer made it

precise

They call the structure consisting of the closed and the open string Frobenius algebra together with the zipper morphism a "knowledgeable Frobenius algebra"

hence:

monoidal Functors $2\mathrm{Cob}^{op} \longrightarrow \mathrm{Vect}$

$\simeq$ knowledgeable Frobenius algebras next: "local QFT"

↓

state sum models

and

how all this can be understood from special ambidextrous adjunctions

main fact in the background:

a local 2d QFT is something describeable by a propagation 2-functor which is "locally trivializable"

to be explained later...