On the String 2-Group

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Abstract

We try to convey the main idea for

1. - what the String-group String_G is
   - and how it is the nerve of a 2-group Str_G;
2. as well as
   - what a Str_G-2-bundle is
   - and how it is "the same" as a String_G-bundle.

The first point is due to [10, 9], which will be reviewed in section 2. The second point has been addressed in [8] using the language of bundle gerbes. In section 3 we review this, using a 2-functorial language which is natural with respect to the 2-group nature of Str_G.

A commented bibliography is given in section 1.

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1 Motivation

The main motivation for the following discussion has its origin in theoretical physics.

Elementary particles with spin are described by sections of spin bundles. From the physical point of view, the necessity of a spin structure on spacetime may be deduced from a certain global anomaly for the path integral of a single, point-like, fermion. The path integral (albeit a somewhat heuristic device) can be regarded as a single valued function on the space of configurations of the particle, only if the (first and) second Stiefel-Whitney class of spacetime vanishes. In other words, if spacetime admits a spin struture.

It is possible to generalize this argument to the case where the fermion is line-like (better maybe: loop-like). (In theoretical physics such a hypothetical object is called a superstring.) It was found that in this case there is another obstruction, which this time is measured by the first Pontryagin class of spacetime [1]. This is interpreted as saying that the *loop space* over spacetime admits a spin structure (see also [2]). In fact, this condition is intimately related to the famous Green-Schwarz anomaly cancellation that has been one of the main reasons why physicists considered superstrings a promising idea to pursue.

As for the pointlike fermion, this situation may be reformulated in terms of lifts of bundles. In fact, two different lifting problems are known to be related to this, one on base space, the other on loop space.

Assuming we started with a $\mathrm{Spin}\,(n)$-bundle $E \to X$ over base space X, we may take loops everywhere and obtain a bundle on the loop space of X with structure group the loop group of $\mathrm{Spin}\,(n)$. The first Pontryagin class of X obstructs the lift of this $\Omega\mathrm{Spin}\,(n)$-bundle along the exact sequence

$$1 \to U\,(1) \to \hat{\Omega}_k \mathrm{Spin}\,(n) \to \mathrm{Spin}\,(n) \to 1\,,$$

where $\hat{\Omega}_k\mathrm{Spin}\,(n)$ is the canonical central extension of $\mathrm{Spin}\,(n)$ (known as the Kac-Moody central extension).

An equivalent lift exists down on X itself. There is a topological group called $\mathrm{String}(n)$, which is a 3-connected cover of $\mathrm{Spin}(n)$. The Pontryagin class is the obstruction controlling the lift of $\mathrm{Spin}(n)$-bundles to $\mathrm{String}(n)$-bundles [5, 2].

However, both of these geometric interpretations are comparatively unwieldy. The first due to the fact that it deals with loop spaces, the second because it deals with the topological group $\mathrm{String}\,(n)$.

We would like to see if there is maybe a third way to describe this situation, a way possibly more natural and possibly such that it unifies the above two perspectives.

For several reasons one may suspect that this third way is naturally formulated in terms of categorical algebra. Indeed, it can be shown that $\mathrm{String}(n)$ is nothing but the geometric realization of the nerve of a certain category with group structure, called $\mathrm{Str}\,(n)$ – a (Fréchet Lie) 2-group built from precisely the Kac-Moody central extension of $\mathrm{Spin}\,(n)$. This is the content of section 2 [10, 9].

Moreover, the obstruction to lifting a Spin(n)-bundle to a String(n)-bundle is the same as that for lifting it to a 2-bundle (gerbe) with structure 2-group Str(n). This is the content of section 3. Following [8] we there argue that generally 2-bundles have the same classification as bundles whose structure group is the nerve of the original structure 2-group. For the special case of String(n) this result is also discussed, from various points of view, in [15, 16].

Literature. As we mentioned above, theoretical physicists have known for a long time (using a method called "global anomaly cancellation in path integrals") that the presence of spinors on loop spaces over X is obstructed by the first Pontryagin class of vector bundles on X. This was first noticed in [1].

The first record of the geometric interpretation of this class as the obstruction to lifting Spin(n) bundles to String(n) bundles that I am aware of is [5]. This is part of a project aimed at giving a rigorous formulation of some concepts used in string physics, motivated by the desire to find a geometric realization of elliptic cohomology. Therein the authors construct a realization of String(n) based on von Neumann algebra factors.

Independently of that, Lie 2-groups and Lie 2-algebras were studied in [11, 12]. There it was found that the simplest class of nontrivial semistrict Lie 2-algebras (usually called $\mathfrak{g}_k$ for $\mathfrak{g}$ any ordinary Lie algebra and k any integer) had interesting properties, but could not directly be integrated to Lie 2-groups.

When it turned out [10] that $\mathfrak{g}_k$ was equivalent to a Fréchet Lie 2-algebra which did appear as the 2-algebra of a strict Fréchet Lie 2-group, the announcement of a talk by by André Henriques on a relation between $\mathfrak{g}_k$ and String_G suggested that String_G was nothing but the nerve of this 2-group. This is one of the two main results discussed here. Meanwhile André Henriques has developed a systematic theory for integration of semistrict Lie n-algebras [9] which contains the relation of $\mathfrak{spin}(n)_1$ to String(n) as a special case.

2-groups naturally appear as structure groups for higher order bundles [13, 17] and have, more or less implicitly in their incarnation as crossed modules, been realized as structure 2-groups for nonabelian gerbes [7]. It was, therefore, conjectured already in [10] that Str_G-2-bundles are closely related to String_G-bundles.

Informal notes by Danny Stevenson on this claim existed early on [15]. A paper which claims this result, and in fact its generalization to arbitrary 2-groups, was announced by Branislav Jurčo in Oberwolfach and has meanwhile bee made available in preprint form [8]. A full detailed proof, has, however, not appeared in print yet. A discussion of the same result in the context of the theory of group stacks is given in [16].

2 The String 2-Group

The topological group String_G, to be defined below in def. 4, appears to be rather unwieldy. But that turns out to be a matter of perspective. When regarded from the proper point of view, String_G looks essentially like a familiar (albeit infinite dimensional) Lie group. That point of view is the point of view of 2-groups.

2.1 2-Groups

Recalling that a group is a groupoid with a single object, we say that

Definition 1 (see [11]) *A* **2-group** *is a 2-groupoid (bigroupoid) with a single object.*

We call a 2-groupoid **strict** if all its 1-morphisms are isomorphisms (instead of just equivalences) which, furthermore, compose strictly associatively (meaning that the associator is the identity). Consequently, a **strict 2-group** is defined to be a strict 2-groupoid with a single object.

Strict 2-groups turn out to have a useful description in terms of crossed modules.

Definition 2 *A* **crossed module** *of groups is a pair* (G_0, G_1) *of groups, together with homomorphisms*

$$G_1 \xrightarrow{t} G_0 \xrightarrow{\alpha} \mathrm{Aut}(G_1)$$

such that t is equivariant with respect to the action induced by α, i.e. such that

$$\begin{tikzcd} G_1 \arrow[rr, "\mathrm{Ad}"] \arrow[dr, "t"'] & & \mathrm{Aut}(G_1) \\ & G_0 \arrow[ur, "\alpha"'] & \end{tikzcd}$$

$$\Leftrightarrow \alpha(t(h))(h') = h\, h'\, h^{-1}$$

and such that

$$t(\alpha(g)(h)) = g\, t(h)\ g^{-1}\,.$$

Namely we have

Theorem 1 (classic, probably first recorded in [4]) *The 2-category of 2-groups is equivalent to the 2-category of crossed modules.*

This equivalence is induced by identifying G_0 with the set of morphisms

$$\mathrm{Mor}_1 = \left\{ \bullet \xrightarrow{g} \bullet \,|g \in G_0 \right\}$$

of the 2-groupoid; G_1 with the kernel of the source map, i.e. with those 2-morphisms starting at the identity

$$\left\{ \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"{name=U}] \arrow[r, bend right=50, "t(h)"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd} \;\middle|\; h \in G_1 \right\}\,;$$

and the set of all morphisms with the semidirect product $G_1 \ltimes G_0$ as

$$\mathrm{Mor}_2 = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd} \;\middle|\; h \in G_1, g \in G_0 \right\} .$$

The main fact to keep in mind, especially for the discussion in section 3, is the following rule for horizontal and vertical composition of 2-group elements (their precise form depends on some conventions that we chose to fix):

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_1'"'] \arrow[r, phantom, "\Downarrow h_1"] & \bullet \arrow[r, bend left=50, "g_2"] \arrow[r, bend right=50, "g_2'"'] \arrow[r, phantom, "\Downarrow h_2"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1 \cdot g_2"] \arrow[r, bend right=50, "g_1' \cdot g_2'"'] \arrow[r, phantom, "\Downarrow h_1 \cdot \alpha(g_1)(h_2)"] & \bullet \end{tikzcd}$$

and

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=50, "g_3"'] \arrow[r, phantom, shift left=2ex, "\Downarrow h_1"] \arrow[r, phantom, shift right=2ex, "\Downarrow h_2"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_3"'] \arrow[r, phantom, "\Downarrow h_2 \cdot h_1"] & \bullet \end{tikzcd} ,$$

where the dot on the right hand side indicates the ordinary product in the respective group.

For us the identification of crossed modules with strict 2-groups serves two purposes.

1. Thinking of (topological) crossed modules as 2-groups suggests that there is naturally a topological group associated to them, namely the realization of the nerve of the corresponding 2-group. This is crucial for theorem 2 (p. 9).

2. The 2-group notation provides a useful graphical calculus that easily deals with the otherwise pretty obscure higher order nonabelian Čech cocycles that represent higher order bundles (gerbes). This is useful in particular for observation 6 (p. 12) and observation 7 (p. 15). It is even more useful in the proof of theorem 3 (p. 19), which will be given elsewhere.

Example 1

The two standard classes of examples for strict 2-groups and crossed modules are the following:

- Let G be any group, regarded as a groupoid with a single object. Then the automorphism functor 2-category $\mathrm{Aut}_{Cat}(G)$ is a 2-group. It corresponds to the crossed module

$$G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G) \xrightarrow{\mathrm{Id}} \mathrm{Aut}(G)\ .$$

- Every central extension

$$1 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow 1$$

  with the usual action of G on H defines a crossed module.

The crucial example for our present purpose is a slight modification of this, which we now turn to.

2.2 String_G

Example 2

Let G be any simply-connected compact simple Lie group. Let PG be the group of picewise smooth based paths in G; and let ΩG be the group of piecewise smooth based loops in G. ΩG has a canonical central extension, the Kac-Moody central extension $\hat{\Omega} G$ (here considered only at level 1). There is an obvious homomorphism $\hat{\Omega} G \to PG$ and an action of PG on ΩG by pointwise conjugation. It is a less trivial result ([10], based on [2, 3]) that this action lifts to an action α on $\hat{\Omega} G$ and that we get a crossed module

$$\hat{\Omega} G \xrightarrow{t} PG \xrightarrow{\alpha} \mathrm{Aut}(\hat{\Omega} G)\,.$$

We shall address here the corresponding 2-group as the string 2-group.

Definition 3 *The 2-group corresponding to the crossed module in example 2 is the* **string 2-group**

$$\mathrm{Str}_g \equiv (\hat{\Omega} G \to PG)\,.$$

This terminology is motivated from the fact (theorem 2 below) that the geometric realization of the nerve of Str_G is a topological group known as the string group String_G.

Definition 4 *The* **string group** String_G *of a simple, simply connected, compact topological group G is (a model for) the 3-connected topological group with the same homotopy groups as G, except*

$$\pi_3(\mathrm{String}_G) = 0\,,$$

which, furthermore, fits into the exact sequence

$$1 \longrightarrow (BU(1) \simeq K(\mathbb{Z},2)) \longrightarrow \mathrm{String}_G \longrightarrow G \longrightarrow 1$$

of topological groups.

The string group proper is obtained by setting $G = \mathrm{Spin}(n)$.

$$\mathrm{String}(n) \equiv \mathrm{String}_{\mathrm{Spin}(n)}\,.$$

Remark - Homotopy groups and the boundary map. The way to see that such a group is a plausible candidate for something generalizing the Spin-group, which, recall, fits into the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 1 \,,$$

is to note that the first few homotopy groups π_k of $O(n)$ are

$k =$	0	1	2	3	4	5	6	7
$\pi_k(O(n)) =$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$

Starting with $O(n)$, we can successively "kill" the lowest nonvanishing homotopy groups, thus obtaining first $SO(n)$ (the connected component), then $\mathrm{Spin}(n)$ (the universal cover) and finally $\mathrm{String}(n)$ (the 3-connected cover). Notice that with π_3 vanishing, $\mathrm{String}(n)$ cannot be a Lie group – but it can be a Lie 2-group.

Usually (see [5]), the definition of String_G includes also a condition on the boundary map $\pi_3(G) \xrightarrow{\partial} \pi_2(K(\mathbb{Z},2))$. Our definition above is really geared towards the application where $G = \mathrm{Spin}(n)$, for which we find it more natural.

Namely, recall that every short exact sequence of topological groups

$$0 \to A \to B \to C \to 0 \,,$$

which happens to be a fibration, gives rise to a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(B) \longrightarrow \pi_n(C) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots \,.$$

In our case this becomes

$$\cdots \longrightarrow \pi_n(K(\mathbb{Z},2)) \longrightarrow \pi_n(\mathrm{String}_G) \longrightarrow \pi_n(G) \xrightarrow{\partial} \pi_{n-1}(K(\mathbb{Z},2)) \longrightarrow \cdots \,.$$

Demanding that $\pi_3(\mathrm{String}_G) = 0$ and assuming that also $\pi_2(\mathrm{String}_G) = 0$ (which we noticed above is the case for $G = \mathrm{Spin}(n)$) implies that we find inside this long exact sequence the short exact sequence

$$0 \longrightarrow (\pi_3(G) \simeq \mathbb{Z}) \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0 \,.$$

But this implies that the boundary map ∂ here is an isomorphism, hence that it acts on $\mathbb{Z}$ either by multiplication with $k = 1$ or $k = -1$. (This number is really the "level" governing this construction. If I find the time I will explain this later.)

In [5] this logic is applied the other way around. Instead of demanding that $\pi_3(\mathrm{String}_G) = 0$ it is demanded that the boundary map

$$\pi_3(G) \xrightarrow{\partial} \mathbb{Z}$$

is given by multiplication with the level, namely a specified element in $H^4(BG)$.

In any case, it turns out that vaguely similar sequences as in def. 4 appear naturally in the study of 2-groups:

Observation 1 *For every strict 2-group* $(G_1 \xrightarrow{t} G_0)$ *we have*

$$\begin{tikzcd} 1 \arrow[r] & (G_1 \to \mathrm{im}(t)) \arrow[r] & (G_1 \to G_0) \arrow[r] \arrow[d, equal] & (1 \to G_0/\mathrm{im}(t)) \arrow[r] & 1 \\ 1 \arrow[r] & (\ker(t) \to 1) \arrow[r] & (G_1 \to G_0) \arrow[r] & (\mathrm{im}(t) \to G_0) \arrow[r] & 1 \end{tikzcd} ,$$

where the horizontal sequences are exact.

This simple observation on the nature of strict 2-groups is the basis of the theorem in [10]. It was explicitly stated in this general form in section 4.5 of [8]. We here present it in a way that prefers categories over simplicial sets.

Recall that all morphisms here are morphisms of 2-groups, i.e. functors that respect the group structure (possibly weakly). Exactness of the sequences means that these functors induce exact sequences on objects and on morphisms.

When dealing with such sequences, it is helpful to be aware of

Observation 2 *For every crossed module* $G_1 \xrightarrow{t} G_0 \xrightarrow{\alpha} \mathrm{Aut}(G_1)$ *we have*

- $\mathrm{im}(t) \subset G$ *is a normal subgroup,*
- $\ker(t) \subset Z(G_1)$.

The 2-group sequences in obs. 1 can be reformulated in terms of topological groups.

Nerves. Categories may, equivalently, be regarded as simplical objects, whose n-simplices are given by collections of n composable morphisms of the category. The simplicial object associated to a category this way is called its **nerve**. Functors between categories then correspond to simplicial maps between their nerves, and natural transformations to homotopies of maps.

All the categories we encounter here are actually **topological categories**. Their sets of objects and morphisms form topological spaces, and all maps between these are continuous. The nerve of a topological category may naturally be turned into a topological space simply by filling all its abstract n-simplices with the standard n-simplex in $\mathbb{R}^n$. The resulting topological space is known as the **geometric realization of the nerve**.

Definition 5 *For* C *a category, we call* $|C|$ *the topological space which is the geometric realization of the nerve of* C.

In fact $C \mapsto |C|$ is a functor from the category of topological categories to that of topological spaces. When applied to our strictly exact sequences of topological group-like categories as above , it returns exact sequences of topological groups.

It is hence reasonable to expect that **the nerve functor translates between the string 2-group and the string group**. Indeed, consider

Example 3

For the string 2-group $\mathrm{Str}_G = (\hat{\Omega} G \to PG)$ (def. 4) obs. 1 yields

$$
\begin{tikzcd}
1 \arrow[r] & (\hat{\Omega} G \to \Omega G) \arrow[r] & \mathrm{Str}_G \arrow[r] \arrow[d, "="] & (1 \to G) \arrow[r] & 1 \\
1 \arrow[r] & (U(1) \to 1) \arrow[r] & \mathrm{Str}_G \arrow[r] & (\Omega G \to PG) \arrow[r] & 1
\end{tikzcd} .
$$

Applying the nerve functor $|\cdot|$ to this gives an exact sequence of topological groups

$$
\begin{tikzcd}
1 \arrow[r] & |(\hat{\Omega} G \to \Omega G)| \arrow[r] \arrow[d, "\sim"] & |\mathrm{Str}_G| \arrow[r] \arrow[d, "="] & G \arrow[r] & 1 \\
1 \arrow[r] & BU(1) \arrow[r] & |\mathrm{Str}_G| \arrow[r] & |(\Omega G \to PG)| \arrow[r] & 1
\end{tikzcd}
$$

with an additional vertical equivalence $|(\hat{\Omega} G \to \Omega G)| \simeq BU(1) \simeq K(\mathbb{Z}, 2)$ [10]. Therefore

$$
0 \longrightarrow K(\mathbb{Z}, 2) \longrightarrow |\mathrm{Str}_G| \longrightarrow G \longrightarrow 0
$$

is an exact sequence of topological groups. This is one of the two defining properties of the string group, def. 4. That $|\mathrm{Str}_G|$ also satisfies the other defining property is

Theorem 2 ([10])

$$
\mathrm{String}_G \simeq |\mathrm{Str}_G| .
$$

3 String 2-Bundles

Next we want to understand the gauge theory of String_G.

3.1 2-Bundles

In all of the following, X denotes some topological space and $\mathcal{U} \xrightarrow{p} X$ a choice of good covering (i.e. a covering by open contractible sets such that all their finite intersections are contractible). We write $\mathcal{U} = \bigsqcup_{i \in I} U_i$.

Definition 6 *The* **Čech groupoid** $\check{C}_1(\mathcal{U})$ *of the good covering* $\mathcal{U} \to X$ *is the groupoid*

- *whose objects are points* $(x, i) \in \mathcal{U}$*,*
- *which has a unique morphism* $(x, i) \longrightarrow (x, j)$ *for every pair of objects in the same* p*-fiber.*

As a guide for how to proceed, we record the following easy

Observation 3

- *In* $\check{C}_1(\mathcal{U})$ *we have the "triangle identities"*

$$\begin{tikzcd} & (x,j) \arrow[dr] \arrow[d, phantom, "\|"] & \\ (x,i) \arrow[ur] \arrow[rr] & & (x,k) \end{tikzcd} \quad ,$$

- *a local trivialization of a* **principal** G**-bundle** $P \to X$ *with respect to* $\mathcal{U}$*, hence a choice of transition functions, is a functor*

$$\check{C}_1(\mathcal{U}) \xrightarrow{E} G \ ,$$

- *a gauge transformation of such a trivialization is a natural transformation*

$$\begin{tikzcd} \check{C}_1(\mathcal{U}) \arrow[r, bend left, "E"] \arrow[r, bend right, "E'"'] \arrow[r, phantom, "\Downarrow"] & G \end{tikzcd} \ ,$$

- *hence isomorphism classes of* G*-bundles on* X *are isomorphism classes in the functor category*
$$[\check{C}_1(\mathcal{U}), G]\,.$$

This way of looking at bundles makes it particularly easy to consider certain generalizations. Let $\mathcal{K} = (G_1 \to G_0)$ be a strict 2-group, but now regarded as a monoidal 1-category.

Observation 4

- *Functors* $\check{C}_1(\mathcal{U}) \xrightarrow{E} \mathcal{K}$ *define local trivializations of certain* G_1*-bibundles. Following [8] we call these bibundles* **crossed module (1-)bundles**.

- *Being bibundles, these bundles have a tensor product which in terms of their local trivializations is given by the product functor* $E \cdot E'$*, defined by*

$$\begin{tikzcd} \check{C}_1(\mathcal{U}) \arrow[rr, "E\cdot E'"] \arrow[dr, "E\times E'"'] & & \mathcal{K} \\ & \mathcal{K}\times\mathcal{K} \arrow[ur, "m"'] & \end{tikzcd}$$

Observation 5 ([8]) *The classifying space for crossed module* $\mathcal{K}$*-1-bundles is* $|\mathcal{K}|$*.*

Proof. A proof is indicated in section 4.3 of [8]. The result is due to Larry Breen.

Here is a simple plausibility argument for seeing why this is true.

By construction, crossed module bundles correspond to equivalence classes in the 1-functor category

$$[\check{\mathrm{C}}_1(\mathcal{U}), \mathcal{K}] .$$

Applying the nerve functor to this, we find that equivalence classes of maps

$$[|\check{\mathrm{C}}_1(\mathcal{U})|, |\mathcal{K}|] .$$

But for every Čech groupoid $\check{\mathrm{C}}_1(\mathcal{U})$ of a good covering $\mathcal{U} \to X$, we have

$$|\check{\mathrm{C}}(\mathcal{U})| \simeq X .$$

Hence we are left with classes in

$$[X, |\mathcal{K}|] .$$

□

Bibundles of this kind arise as morphisms between higher order structures, which we now turn to.

We want to generalize the above functorial description from bundles to some structure which can accomodate a 2-group in place of the ordinary structure group. Without going into any details on how to motivate this, consider the following

Definition 7 *The* **Čech 2-groupoid** $\check{C}_2(\mathcal{U})$ *of the good covering* $\mathcal{U} \to X$ *is the 2-groupoid*

- *whose objects are points* $(x, i) \in \mathcal{U}$*,*
- *whose 1-morphisms are generated from unique morphisms* $(x,i) \longrightarrow (x,j)$ *for every pair of objects in the same p-fiber,*
- *which has a unique 2-morphism between any pair of parallel 1-morphisms, hence in particular a unique 2-morphism*

$$\begin{tikzcd} & (x,j) \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ (x,i) \arrow[ur] \arrow[rr] & {} & (x,k) \end{tikzcd} ,$$

for every triple of objects in the same p-fiber.

For our present purposes we can simplify life by imposing the additonal condition that all 1-morphisms be strictly invertible, i.e. that

$$\begin{tikzcd} & (x,j) \arrow[dr] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \\ (x,i) \arrow[ur] \arrow[rr] & {} & (x,i) \end{tikzcd} ,$$

Notice that, in analogy to the triangle equations for ordinary Čech groupoids, in the Čech 2-groupoid we now have **tetrahedron equations** of the form

$$
\begin{tikzcd}
(x,j) \arrow[r] \arrow[dr, phantom, near start, "\Searrow"] & (x,k) \arrow[d] \\
(x,i) \arrow[u] \arrow[ur] \arrow[r] \arrow[ur, phantom, shift right=4ex, "\Downarrow"] & (x,l)
\end{tikzcd}
=
\begin{tikzcd}
(x,j) \arrow[r] \arrow[dr] \arrow[dr, phantom, shift left=4ex, "\Swarrow"] & (x,k) \arrow[d] \\
(x,i) \arrow[u] \arrow[r] \arrow[ur, phantom, "\Downarrow"] & (x,l)
\end{tikzcd}
.
$$

Of course we are interested in 2-functors from $\check{C}_2(\mathcal{U})$ to some 2-group.[1] Matching the terminology on the group side suggests to address these 2-functors as (local trivializations of) 2-bundles. As shown below, the reader more comfortable with gerbes can equally well think of them as locally trivialized gerbes.

Observation 6 *Let $\mathcal{K} = (G_1 \to G_0)$ be a (strict) 2-group.*

- *2-functors* $\check{C}_2(\mathcal{U}) \xrightarrow{E} \mathcal{K}$ *are in bijection with pairs of functions*

$$g : \mathcal{U}^{[2]} \to G_0$$

and

$$f : \mathcal{U}^{[3]} \to G_1$$

satifying the cocycle equations

$$g_{ij}(x)\, g_{jk}(x) = t(f_{ijk}(x))\, g_{ik}(x)$$

and

$$f_{ijk}(x)\, f_{ikl}(x) = \alpha(g_{ij}(x))(f_{jkl}(x))\, f_{ijl}(x)$$

for all x and all i, j, k, l.

- *morphisms of 2-functors*

$$
\begin{tikzcd}
\check{C}_2(\mathcal{U}) \arrow[r, bend left=50, "E"{name=U}] \arrow[r, bend right=50, "E'"'{name=D}] & \mathcal{K}
\arrow[Rightarrow, from=U, to=D, "\lambda"]
\end{tikzcd}
$$

are in bijection with maps

$$h : \mathcal{U} \to G_0$$

and

$$j : \mathcal{U}^{[2]} \to G_1$$

[1] The reader may alternatively think of pseudofunctors from $\check{C}_1(\mathcal{U})$ to some 2-group. We prefer here to work, equivalently, with (strict) 2-functors on $\check{C}_2(\mathcal{U})$.

satisfying the coboundary equations

$$h_i(x)\, g'_{ij}(x) = t(j_{ij})\, g_{ij}(x)\, h_j(x)$$

and

$$f'_{ijk}(x)\, j_{ij}(x)\, \alpha(g_{ij}(x))(j_{jk}(x)) = j_{ik}(x)\, f_{ijk}(x)\ ,$$

- *2-morphisms of 2-functors*

$$\begin{tikzcd}[column sep=large] & \check{C}_2(\mathcal{U}) \arrow[dd, bend right=70, "E"'] \arrow[dd, bend left=70, "E'"] \arrow[dd, phantom, "\overset{\lambda}{\Longrightarrow}\ \rho\Downarrow\ \underset{\lambda'}{\Longrightarrow}"] & \\ & & \\ & \mathcal{K} & \end{tikzcd}$$

are in bijection with maps

$$k : \mathcal{U} \to G_1$$

satisfying the second order coboundary equation

$$k_j(x)\, j'_{ij}(x) = j_{ij}(x)\, \alpha(g_{ij}(x))(k_j(x))\ .$$

Proof. These formidable equations are nothing but the tin can equations for pseudonatural transformations and modifications of these, translated from the language of 2-groups to that of crossed modules.

- The 2-functor is determined by its image on elementary Čech triangles and tetrahedra. The image of a triangle is

$$\begin{tikzcd} & \bullet \arrow[dr, "g_{jk}(x)"] & \\ \bullet \arrow[ur, "g_{ij}(x)"] \arrow[rr, "g_{ik}(x)"'] \arrow[rr, phantom, shift left=4, "\Downarrow f_{ijk}(x)"] & & \bullet \end{tikzcd} .$$

By (2-)functoriality it follows that the image of a tetrahedron satisfies

$$\begin{tikzcd} \bullet \arrow[r, "g_{jk}(x)"] & \bullet \arrow[d, "g_{kl}(x)"] \\ \bullet \arrow[u, "g_{ij}(x)"] \arrow[ur, "g_{ik}(x)"] \arrow[r, "g_{il}(x)"'] & \bullet \end{tikzcd} \begin{array}{c} \Downarrow f_{ijk}(x) \\ f_{jkl}(x)\Downarrow \end{array} \quad = \quad \begin{tikzcd} \bullet \arrow[r, "g_{jk}(x)"] \arrow[dr, "g_{jl}(x)"] & \bullet \arrow[d, "g_{kl}(x)"] \\ \bullet \arrow[u, "g_{ij}(x)"] \arrow[r, "g_{il}(x)"'] & \bullet \end{tikzcd} \begin{array}{c} f_{jkl}(x)\Downarrow \\ \Downarrow f_{ijl}(x) \end{array} .$$

- The tin can equations for the morphisms between our 2-functors look like

$$
\begin{tikzcd}
& \bullet \arrow[dr, "g_{jk}"] \arrow[d, phantom, "\Downarrow f_{ijk}"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"] \arrow[dd, "h_i"'] \arrow[ddrr, phantom, "\Swarrow j_{ik}"] & & \bullet \arrow[dd, "h_k"] \\
& & \\
\bullet \arrow[rr, "g'_{ik}"'] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
& \bullet \arrow[dr, "g_{jk}"] \arrow[dd, "h_j"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[ddd, "h_i"'] \arrow[dr, phantom, "\Swarrow j_{ij}"] & & \bullet \arrow[ddd, "h_k"] \arrow[dl, phantom, "\Swarrow j_{jk}"'] \\
& & \\
& \bullet \arrow[dr, "g'_{jk}"] \arrow[d, phantom, "\Downarrow f'_{ijk}"] & \\
\bullet \arrow[ur, "g'_{ij}"] \arrow[rr, "g'_{ik}"'] & & \bullet
\end{tikzcd}
$$

(where we suppress the dependence of everything on x for sake of readability).

- The tin can equations for a 2-morphism of 1-morphisms of our 2-functors look like

$$
\begin{tikzcd}
\bullet \arrow[r, "g_{ij}"] \arrow[d, "h_i"'] \arrow[dr, phantom, "\Swarrow j_{ij}"] & \bullet \arrow[d, "h_j"'] \arrow[d, bend left=60, "h'_j"] \arrow[d, phantom, shift left=3, "\overset{k_j}{\Leftarrow}"] \\
\bullet \arrow[r, "g'_{ij}"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "g_{ij}"] \arrow[d, "h'_i"] \arrow[d, bend right=60, "h'_i"'] \arrow[d, phantom, shift right=3, "\overset{k_i}{\Leftarrow}"] \arrow[dr, phantom, "\Swarrow j'_{ij}"] & \bullet \arrow[d, "h'_j"] \\
\bullet \arrow[r, "g'_{ij}"'] & \bullet
\end{tikzcd}
$$

Applying the dictionary between 2-groups and crossed modules, one finds that these diagrams are equivalent to the advertized formulas. □

The above formulas, sometimes called "nonabelian Čech cocycle conditions" are known to describe local trivializations of gerbes. We here find it convenient to address them as local trivialization and transition data for 2-bundles with structure 2-group $\mathcal{K}$. Since in the present context we will not use any global notion of 2-bundles, let us make the following

Definition 8 *Given a topological space X, fix once and for all a good covering $\mathcal{U} \to X$. Let $\mathcal{K}$ be a strict 2-group. In this text, we say that*

- *a (strict) 2-functor $\check{C}_2(\mathcal{U}) \to \mathcal{K}$ is a* **principal $\mathcal{K}$ 2-bundle** *on X;*

- *the 2-functor 2-category category $[\check{C}_2(\mathcal{U}), \mathcal{K}]$ is the* **2-category of 2-bundles**.

The 2-category of 2-functors does depend on our choice of covering, but its equivalence classes of objects do not.

3.2 Transition Bundles

Our goal is to relate Str_G-2-bundles to String_G-(1-)bundles. We are led by the following

Observation 7 *The category of automorphims of trivial $\mathcal{K}$-2-bundles on X is isomorphic to the category of crossed module 1-bundles on X. In formulas, for E_0 trivial we have*

$$\mathrm{Aut}_{[\check{C}_2(\mathcal{U}),\mathcal{K}]}(E_0) \simeq [\check{C}_1(\mathcal{U}),\mathcal{K}]\,.$$

(Note that on the left hand side $\mathcal{K}$ is regarded as a 2-groupoid with a single object, while on the right hand side it is regarded as a monoidal 1-category).

Here we use the obvious

Definition 9 *Let $1 \xrightarrow{i} \mathcal{K}$ be the unique injection of the trivial 2-group into $\mathcal{K}$. A 2-bundle $E : \check{C}_2(\mathcal{U}) \to \mathcal{K}$*

- *is i-**trivial** if*

$$\begin{tikzcd} \check{C}_2(\mathcal{U}) \arrow[r, "="] \arrow[d] \arrow[dr, phantom, "\swarrow_{\mathrm{Id}}"] & \check{C}_2(\mathcal{U}) \arrow[d, "E"] \\ 1 \arrow[r] & \mathcal{K} \end{tikzcd}$$

- *is i-**trivializable** if*

$$\exists \quad \begin{tikzcd} \check{C}_2(\mathcal{U}) \arrow[r, "="] \arrow[d] \arrow[dr, phantom, "\swarrow_{\sim}"] & \check{C}_2(\mathcal{U}) \arrow[d, "E"] \\ 1 \arrow[r] & \mathcal{K} \end{tikzcd}$$

- *is p-**local** i-**trivializable** if*

$$\exists \quad \begin{tikzcd} \mathrm{dom}(p) \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\swarrow_{\sim}"] & \check{C}_2(\mathcal{U}) \arrow[d, "E"] \\ 1 \arrow[r] & \mathcal{K} \end{tikzcd} \quad .$$

(In general we will demand that p is surjective in some suitable sense.)

Now consider the
Proof.
of obs. 7. The image under trivial E_0 of a $\check{\mathrm{C}}_2(\mathcal{U})$ triangle is

$$\begin{tikzcd} & \bullet \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \\ \bullet \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}"'] & {} & \bullet \end{tikzcd} \quad .$$

An automorphism of E is an assigment

$$
(x,i) \longrightarrow (x,j) \quad \mapsto \quad
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "h_i"'] \arrow[dr, phantom, "\swarrow j_{ij}"] & \bullet \arrow[d, "h_j"] \\
\bullet \arrow[r, "\mathrm{Id}"'] & \bullet
\end{tikzcd}
$$

which is functorial in the sense that

$$
(x,i) \longrightarrow (x,j) \longrightarrow (x,k) \quad \mapsto \quad
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "h_i"'] \arrow[dr, phantom, "\swarrow j_{ij}"] & \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "h_j"] \arrow[dr, phantom, "\swarrow j_{jk}"] & \bullet \arrow[d, "h_k"] \\
\bullet \arrow[r, "\mathrm{Id}"'] & \bullet \arrow[r, "\mathrm{Id}"'] & \bullet
\end{tikzcd}
$$

and which satisfies the tin can equation

$$
\begin{tikzcd}
 & \bullet \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \\
\bullet \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}"] \arrow[dd, "h_i"'] & \arrow[dd, phantom, "\swarrow j_{ik}"] & \bullet \arrow[dd, "h_k"] \\
 & & \\
\bullet \arrow[rr, "\mathrm{Id}"'] & & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \bullet \arrow[dr, "\mathrm{Id}"] \arrow[dd, "h_j"] & \\
\bullet \arrow[ur, "\mathrm{Id}"] \arrow[dd, "h_i"'] \arrow[r, phantom, "\swarrow j_{ij}"] & \arrow[r, phantom, "\swarrow j_{jk}"] & \bullet \arrow[dd, "h_k"] \\
 & \bullet \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \\
\bullet \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}"'] & & \bullet
\end{tikzcd}
\, .
$$

Clearly, this is a 1-functor $\check{\mathrm{C}}_1(\mathcal{U}) \to \mathcal{K}$ and hence a crossed module 1-bundle, according to obs. 4.

Moreover, a 2-morphism of an automorphism of trivial E comes with tin can equations of the form

$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "h_i"'] \arrow[dr, phantom, "\swarrow j_{ij}"] & \bullet \arrow[d, "h_j"'] \arrow[d, bend left=70, "h'_j"] \arrow[d, phantom, shift left=3, "\overset{k_j}{\Leftarrow}"] \\
\bullet \arrow[r, "\mathrm{Id}"'] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "h'_i"] \arrow[d, bend right=70, "h'_i"'] \arrow[d, phantom, shift right=3, "\overset{k_i}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow j'_{ij}"] & \bullet \arrow[d, "h'_j"] \\
\bullet \arrow[r, "\mathrm{Id}"'] & \bullet
\end{tikzcd}
\, .
$$

These correspond to natural transformations between the above 1-functors. □

In the case of abelian gerbes, the bundles we obtain this way are familiar as **transition bundles**. Like a (1-)bundle can be described in terms its transition functions (0-bundles), a gerbe (2-bundle) can be described in terms of its transition 1-bundles. This gives rise to the notion of **bundle gerbe**. For the abelian case these bundle gerbes were introduced in [6]. The nonabelian bundle gerbe, which we will rederive now from our functorial perspective, was studied in [7].

A general 2-bundle is neither trivial, nor trivializable. But every 2-bundle (in the sense defined above) is locally trivializable, in the sense of def. 9.

Namely, we can always find a surjection $Y \to X$, such that our 2-bundle pulled back to Y becomes equivalent to the trivial 2-bundle on Y. Here the precise details of the equivalence are important. These specifiy precisely the transition 1-bundle on $Y^{[2]}$.

Since we have worked with good coverings all along, let us not bother with the general case here but assume for simplicity that $Y \to X$ is itself a good covering of X by open sets $\{Y_r\}_r$. We can pull back our 2-bundle to the Čech of $\mathcal{U}$ *relative* to Y:

Definition 10 *Let $\check{C}_2(\mathcal{U}, Y)$ be the 2-groupoid*

- *whose objects are points $((x,i),r) \in U_i \cap Y_r$;*
- *whose 1-morphisms are generated from unique morphisms* $((x,i),r) \longrightarrow ((x,j),r)$ *;*
- *which has a unique 2-morphism between any pair of parallel 1-morphisms, hence in particular a unique 2-morphism*

$$\begin{tikzcd} & ((x,j),r) \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ ((x,i),r) \arrow[ur] \arrow[rr] & & ((x,k),r) \end{tikzcd} ,$$

for every triple of objects in the same fiber.

Restricted to each Y_i (since it is contractible) our 2-bundle becomes trivializable, hence we can a find a p-local i-trivialization

$$\begin{tikzcd} \check{C}_2(\mathcal{U}, Y) \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\overset{t}{\swarrow}\sim"] & \check{C}_2(\mathcal{U}) \arrow[d, "E"] \\ 1 \arrow[r] & \mathcal{K} \end{tikzcd}$$

with p the obvious morphism from $\check{C}_2(\mathcal{U}, Y)$ to $\check{C}(\mathcal{U})$.

Given such a p-local i-trivialization, we can apply a general construction [18] to obtain local transition data: the composite 2-morphism

$$\begin{tikzcd}
 & & & 1 \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\bar{t}"] & \\
\check{\mathrm{C}}_2(\mathcal{U},Y)^{[2]} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \check{\mathrm{C}}_2(\mathcal{U},Y) \arrow[r, "p"] \arrow[urr] \arrow[drr] & \check{\mathrm{C}}_2(\mathcal{U}) \arrow[r, "E"] & \arrow[d, phantom, "\Downarrow t"] & \mathcal{K} \\
 & & & 1 \arrow[ur, "i"'] &
\end{tikzcd}$$

gives the transition from the trivialization over Y_r to that over Y_s. We call this 2-morphism γ

$$\begin{tikzcd}
 & & 1 \arrow[dr, "i"] & \\
\check{\mathrm{C}}_2(\mathcal{U},Y)^{[2]} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \check{\mathrm{C}}_2(\mathcal{U},Y) \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow\gamma"] & & \mathcal{K} \\
 & & 1 \arrow[ur, "i"'] &
\end{tikzcd}$$

and find [18] on triple Y-intersections morphisms

$$\begin{tikzcd}
 & E_0 \arrow[dr, "p_{23}^*\gamma"] \arrow[d, phantom, "\Downarrow\phi"] & \\
E_0 \arrow[ur, "p_{12}^*\gamma"] \arrow[rr, "p_{13}^*\gamma"'] & & E_0
\end{tikzcd}$$

which satisfy a tetrahedron equation on quadruple overlaps.

What we obtain this way is a Čech-simplex structure very similar to the one we started with. The difference is, where before we had a 2-functor in

$$[\check{\mathrm{C}}_2(\mathcal{U}), \mathcal{K}]$$

from the Čech 2-groupoid to a 2-group, we now have a 2-functor taking values in

$$\mathrm{Aut}_{[\check{\mathrm{C}}_2(\mathcal{U}),\mathcal{K}]}(E_0)\,,$$

automorphisms of the trivial 2-bundle. According to obs. 7 these automorphisms are nothing but crossed module bundles. Hence γ, living on $Y^{[2]}$, is the transition bundle (the "bundle gerbe") of our 2-bundle.

Observation 8 ([8])

- *Given a principal $\mathcal{K}$-2-bundle on X, its crossed module $\mathcal{K}$-transition bundle γ on $Y^{[2]}$ descends to a $|\mathcal{K}|$-bundle on X.*

- *Conversely, every $|\mathcal{K}|$-bundle on X gives rise to a transition bundle on $Y^{[2]}$, which defines a principal $\mathcal{K}$-2-bundle on X.*

A detailed
Proof.
of this exists, but has not been published yet. The idea is to pass from cocycle representatives of $\mathcal{K}$ crossed module transition bundles on double overlaps to the corresponding classifying functions with values in $|\mathcal{K}|$ by taking nerves. The crucial point is that one can show that there exists a choice of gauge such that these classifying functions satisfy the Čech 1-cocycle condition, hence that they represent a $|\mathcal{K}|$-bundle. □

Furthermore, one can check that this construction respects morphisms of 2-bundles and 1-bundles. Under suitable conditions we hence have

Theorem 3 *Equivalence classes of $\mathcal{K}$-2-bundles on X are in bijection with equivalence classes of $|\mathcal{K}|$-bundles on X.*

Our desired insight into the relation between Str_G-2-bundles and String_G-1-bundles is now nothing but a special case of this general result.

Corollary 1 *Equivalence classes of Str_G-2-bundles on X are in bijection with equivalence classes of String_G-bundles on X.*

In particular, since it is known that the existence of String(n)-bundles is obstructed by the first Pontryagin class, this implies that the same obstruction governs Str_G-2-bundles. For the case of Str_G independent arguments for this special result have been noticed in [15, 16].

Two conclude, we may close the circle begun in section 1 and state:

- A point-like fermion may propagate on X whenever the frame bundle of X lifts to a Spin-bundle.
- A string-like fermion may propagate on X whenever the Spin-bundle lifts to a Str-2-bundle.

4 Outlook: String Connections

Given the origin of String_G in the study of propagation of strings, it should not come as a surprise that for certain applications the most interesting thing one would want to do with a Str_G 2-bundle is to equip it with some notion of connection [5].

The 2-functorial way of talking about 2-bundles, which we have used, is designed in such a way as to have a natural extension to a concept that captures a notion of parallel transport in 2-bundles [18].

To see this, first consider the following

Definition 11 *Denote by* $\mathcal{P}_1\check{C}_1(\mathcal{U})$ *the groupoid of thin* **paths in the Čech groupoid**. *This is the groupoid freely generated by the morphisms in* $\check{C}_1(\mathcal{U})$ *together with all thin homotopy classes of paths in* $\mathcal{U}$*, divided out by some natural relations.*

This gives rise to a straightforward enhancement of observation 3.

Observation 9

- *Functors* $\mathrm{tra} : \mathcal{P}_1\check{C}_1(\mathcal{U}) \to G$ *correspond to local trivializations (with respect to* $\mathcal{U}$*) of principal* G**-bundles with connection** *on* X*.*

- *Equivalence classes in the functor category*

$$[\mathcal{P}_1\check{C}_1(\mathcal{U}), G]$$

  are in bijection with G*-bundles with connection on* X*.*

More precisely, this bijection holds when everything in sight is smooth. The definition of a bundle with connection as an equivalence class in the functor category $[\mathcal{P}_1\check{\mathrm{C}}_1(\mathcal{U}), G]$ makes sense much more generally. It defines a notion of **bundle with parallel transport**. (And one can give global definitions along the same lines which do not make reference to a choice of covering.)

There is now nothing more straightforward than adapting the same idea to 2-bundles.

Definition 12 *Denote by* $\mathcal{P}_2\check{C}_2(\mathcal{U})$ *the 2-groupoid of thin* **2-paths in the Čech 2-groupoid**. *This is the 2-groupoid freely generated by the morphisms in* $\check{C}_2(\mathcal{U})$ *together with all thin homotopy classes of 2-paths in* $\mathcal{U}$*, divided out by some natural relations.*

This allows us to *define* parallel transport in 2-bundles:

Definition 13

- *A (locally trivialized with respect to* $\mathcal{U}$*)* $\mathcal{K}$**-2-bundle with parallel transport** *is a 2-functor*

$$\mathrm{tra} : \mathcal{P}_2\check{C}_2(\mathcal{U}) \to \mathcal{K}\,;$$

It can been shown that, in the smooth case and for $\mathcal{K} = \mathrm{Aut}(G)$, using the locally trivialized notion of 2-bundle as above[2]

Theorem 4 $\mathrm{Aut}(G)$ *2-bundles with parallel transport are the same as fake flat* G*-gerbes with connection.*

[2] What has explicitly been shown in [17] is only that a 2-functor $[\mathcal{P}_2\check{\mathrm{C}}_2(\mathcal{U}), \mathcal{K}]$ corresponds to a local trivialization of a fake flat G-gerbe with connection. The full statement requires that also the 1-morphisms and 2-morphisms match. But this follows from simply writing down the relevant tin can diagrams as in obs. 6.

Fake flatness is a certain condition on the 2-curvature that is necessary in order for 2-paths to lift from X, i.e. which is implied by tra being a 2-functor and hence a morphism of 2-groupoids (compare [14]).

We may apply the above to the case $\mathcal{K} = \mathrm{Str}_G$ and thus obtain a 2-bundle with "**String connection**" in a rather explicit way. Such a conception of String connection, not least due to its 2-functorial nature, superficially looks like it should be closely related to the notion of String connection promoted in [5]. It is not clear yet (to me) how both concepts are related in detail.

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