Notes on associated vector 2-bundles

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June 12, 2006

Abstract

Rough notes on some aspects of associated vector 2-bundles.

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1 Introduction

This are some notes on the general issue of 2-vector 2-bundles associated to principal 2-bundles.

Conventions.

- Torsors. In the following the word *torsor* always refers to *torsor over a point.* Our main motivation comes from parallel transport in 2-bundles. The 2-transport 2-functor will associate torsors to every object. The full 2-bundle regarded as a 2-torsor over a base space is then then the full image under the 2-transport of the collection of all points in base space.
- Connections. In the present context, a (2-)connection on a (2-)bundle shall always mean a transport (2-)functor which takes points in the base to the fibers above them and (2-)paths in the base to (2-)morphismsm of (2-)torsors. We shall loosely refer to this as a (2-)connection, though that term might maybe better be reserved for some infinitesimal notion of (2-)transport.

For the most part we do not care here about the specific ambient topos (sets, or topological spaces, or smooth spaces, etc.).

In outline, the complex of questions we shall be concerned with is the following.

1.1 Basic concepts in associated 2-bundles

Let G_2 be a monoidal category, usually a 2-group.

Let

$$\begin{tikzcd} P \arrow[d] \\ X \end{tikzcd}$$

be a principal G_2-2-bundle over a discrete category X. Some authors call this a G_2-torsor. Here I shall reserve the term **(right) G_2-torsor** for any category T equipped with a right G_2-action

$$\begin{tikzcd} T \times G_2 \arrow[r, "r"] & T \end{tikzcd}$$

up to coherent isomorphism

$$\begin{tikzcd} T \times G_2 \times G_2 \arrow[r, "T\times m"] \arrow[d, "r\times G_2"'] \arrow[dr, phantom, "\Swarrow"] & T \times G_2 \arrow[d, "r"] \\ T \times G_2 \arrow[r, "r"'] & T \end{tikzcd} ,$$

such that it is equivalent to G_2 as a right G_2 space, or, alternatively, such that

$$\begin{tikzcd} T \times G_2 \arrow[r, "(\mathrm{Id} \times r)"] & T \times T \end{tikzcd}$$

is an equivalence.

Then each fiber P_x, $x \in X$ of P is a G_2-torsor.

A linear representation of G_2 is a 2-functor

$$\rho_2 \quad : \qquad \Sigma(G_2) \qquad \to \qquad \mathrm{End}(V_2) \subset \mathrm{Mod}_C$$

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g", ""{name=U, below}] \arrow[r, bend right=50, "g'"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & \bullet & \mapsto & V_2 \arrow[r, bend left=50, "\rho_2(g)", ""{name=U2, below}] \arrow[r, bend right=50, "\rho_2(g')"', ""{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\rho_2(h)"] & V_2 \end{tikzcd} ,$$

for C some (usually braided) monoidal category C.

Given such a representation, we obtain a left action of G_2 on V_2 by setting

$$\begin{tikzcd} G_2 \times V_2 \arrow[r, "\rho_2 \times \mathrm{Id}"] \arrow[dr, "l"'] & \mathrm{Hom}(V_2) \times V_2 \arrow[d, "\mathrm{ev}"] \\ & V_2 \end{tikzcd} ,$$

where ev is the image of the identity under

$$\mathrm{Hom}(\mathrm{Hom}(V_2, V_2), \mathrm{Hom}(V_2, V_2)) \simeq \mathrm{Hom}(\mathrm{Hom}(V_2, V_2) \times V_2, V_2) \ .$$

The right C-module associated by ρ to T is the coequalizer

$$\begin{tikzcd} T \times G_2 \times V_2 \arrow[r, shift left, "r \times V_2"] \arrow[r, shift right, "T \times l"'] & T \times V_2 \arrow[r] & T \otimes_{G_2} V_2 \end{tikzcd} \ .$$

Coequalizers in Cat have been discussed for instance in [1]. Aspects of coequalizers in enriched categories are discussed in a later section.

1.2 2-reps induced from ordinary reps

1.2.1 Introduction.

In this subsection we present a method that induces from any ordinary finite-dimensional linear representation of an ordinary group a representation of its automorphism 2-group on 2-vector spaces. The induced representation is in terms of bimodules for the algebra generated by the representation of H.

We expect that a generalization of this method to infinite-dimensional representations will apply to the String(n)-2-group and will in fact reproduce, using the discussion from section 1.4 below, the construction of String-connections in terms of bimodules considered by Stolz and Teichner.

1.2.2 2-reps on bimodules over a representation algebra

Let H be any group and let

$$\begin{tikzcd}
\rho : \Sigma(H) & \rightarrow & \mathrm{Vect} \\
\bullet \arrow[d, "h"] & \mapsto & V \arrow[d, "\rho(h)"] \\
\bullet & & V
\end{tikzcd}$$

be any finite-dimensional representation on vector spaces. We want to construct from ρ a representation of the automorphism 2-group $\mathrm{Aut}_2(H)$ on 2-vector spaces

$$\tilde{\rho} : \mathrm{Aut}_{\mathrm{Cat}}(\Sigma(H)) \to \mathrm{Mod}_{\mathrm{Vect}} \,.$$

Recall that 2-morphisms in $\mathrm{Aut}_{\mathrm{Cat}}(\Sigma(H))$

$$\begin{tikzcd}
\Sigma(H) \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \Sigma(H)
\end{tikzcd}$$

are labeled by $g \in \mathrm{Aut}(H)$ and $h \in H$ with

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "g(f)"'] & \bullet \arrow[d, "g'(f)"] \\
\bullet \arrow[r, "h"'] & \bullet
\end{tikzcd}$$

for arbitrary $f \in H$. What we shall need below is the commutativity of the image of this diagram under ρ

$$\begin{tikzcd}
V \arrow[r, "\rho(h)"] \arrow[d, "\rho(g(f))"'] & V \arrow[d, "\rho\left(g'(f)\right)"] \\
V \arrow[r, "\rho(h)"'] & V
\end{tikzcd} \,. \tag{1}$$

In order to construct $\tilde{\rho}$ let now

$$\mathrm{End}(V) \supset A_\rho \equiv \langle \rho(h) \,|\, h \in H \rangle$$

be the subalgebra of the endomorphism algebra of V which is generated by the linear maps $\rho(h)$ for all $h \in H$. We obtain for each $g \in \mathrm{Aut}(H)$ an automorphism $\rho(g) \in \mathrm{Aut}(A_\rho)$ of this algebra by setting

$$\rho(g) : \rho(h) \mapsto \rho(g(h))$$

for all $h \in H$, and extended linearly to all of A_ρ.

Using this, for each $g \in \mathrm{Aut}(h)$ we define an A_ρ-bimodule

$$N_g \equiv \begin{tikzcd} A_\rho \arrow[r, dashed, "\rho(g)"] & A_\rho & A_\rho \arrow[l, dashed, "\mathrm{Id}"'] \end{tikzcd}$$

which, as an object in Vect, is A_ρ itself, with both the right and the left A_ρ action given by the product in A_ρ, but with the left action twisted by $\rho(g)$:

$$\begin{aligned} \rho(h) \cdot n &\equiv \rho(g(h)) \circ n \\ n \cdot \rho(h) &\equiv n \circ \rho(h)\ . \end{aligned} \tag{2}$$

for all $n \in N_g$.

For all bimodules of this form the tensor product over A_ρ corresponds to the composition of automorphisms

$$N_g \otimes_{A_\rho} N_{g'} = N_{g' \circ g}\ .$$

Let ${}_{A_\rho}\mathrm{Mod}$ be the category of left A_ρ-modules. Every A_ρ bimodule induces, by tensor multiplication on the left, an endofunctor

$$N_g \otimes_{A_\rho} ? \quad : \quad {}_{A_\rho}\mathrm{Mod} \longrightarrow {}_{A_\rho}\mathrm{Mod}\ .$$

By the above remark, we have hence obtained a representation of all identity 2-morphisms in $\mathrm{Aut}_2(H)$ on identity 2-morphisms in $\mathrm{Bim}(\mathrm{Vect}) \subset \mathrm{Mod}_{\mathrm{Vect}}$.

For each nontrivial 2-morphism

$$\begin{tikzcd} \Sigma(H) \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] & \Sigma(H) \arrow[Rightarrow, from=U, to=D, "h"] \end{tikzcd}$$

define an map of bimodules

$$\begin{aligned} \tilde\rho(h) : N_g &\rightarrow N_{g'} \\ n &\mapsto \rho(h) \circ n\,. \end{aligned}$$

This map trivially respects the right A_ρ-action. That it also respects the left

A_ρ action is a consequence of the commutativity of (1):

$$\begin{tikzcd}
(\rho(f), n) \arrow[r, mapsto, "\mathrm{Id}\times\tilde{\rho}(h)"] \arrow[dd, mapsto, "(2)"'] & (\rho(f), \rho(h)\circ n) \arrow[d, mapsto, "(2)"] & \\
 & \rho(g'(f))\circ\rho(h)\circ n \arrow[d, equal, "(1)"] & . \\
\rho(g(f))\circ n \arrow[r, mapsto, "\tilde{\rho}(h)"'] & \rho(h)\circ\rho(g(f))\circ n &
\end{tikzcd}$$

We obtain this way a representation 2-functor

$$\tilde{\rho} \quad : \quad \mathrm{Aut}_2(H) \quad \to \quad \mathrm{Mod}_{\mathrm{Vect}}$$

$$\begin{tikzcd}
\Sigma(H) \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & \Sigma(H) & \mapsto & {}_{A_\rho}\mathrm{Mod} \arrow[r, bend left=50, "N_g"{name=U2}] \arrow[r, bend right=50, "N_{g'}"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\tilde{\rho}(h)"] & {}_{A_\rho}\mathrm{Mod} & .
\end{tikzcd}$$

Example 1

Let $G_2 = (U(1) \to 1) = \Sigma(U(1))$. Let $\rho : \Sigma(U(1)) \to \mathrm{Vect}_{\mathbb{C}}$ be the defining 1-dimensional rep.

In this case we find $A_\rho = \mathbb{C}$, the complex numbers. The bimodule N_{Id} is just $\mathbb{C}$ itself, with the left and right $\mathbb{C}$-action given by multiplication of complex numbers. Endomorphisms of this bimodule are given by injecting $U(1)$ into $\mathbb{C}$ and multiplying in $\mathbb{C}$. The 2-vector space ${}_{\mathbb{C}}\mathrm{Mod} = \mathrm{Vect}_{\mathbb{C}}$ is 1-dimensional.

$$\tilde{\rho} \quad : \quad \Sigma(\Sigma(U(1))) \quad \to \quad \mathrm{Mod}_{\mathrm{Vect}}$$

$$\begin{tikzcd}
\Sigma(U(1)) \arrow[r, bend left=50, "\mathrm{Id}"{name=U}] \arrow[r, bend right=50, "\mathrm{Id}"'{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & \Sigma(U(1)) & \mapsto & \mathrm{Vect} \arrow[r, bend left=50, "\mathbb{C}"{name=U2}] \arrow[r, bend right=50, "\mathbb{C}"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\tilde{\rho}(h)"] & \mathrm{Vect} & .
\end{tikzcd}$$

1.3 Actions from representations

1.3.1 Introduction

In the ordinary (non-categorified) setup it is very obvious how to get a G-action on some vector space V given a representation of G on V. In fact, this is so very obvious that one hardly sees the difference.

But as soon as one categorifies, the difference becomes more pronounced. The right answer is still easy and elegant, but maybe deserves to be made explicit. It crucially depends on realizing elements of 2-vector spaces as maps into 2-vector spaces.

1.3.2 Actions from reps combined with Yoneda embedding

How do we get an action, given a representation?

As a motivation, reformulate the ordinary case like this:

Let G be any group and

$$\begin{tikzcd}
\rho \; : & \Sigma(G) & \to & \mathrm{Aut}(V) \subset \mathrm{Vect} \\
& \bullet \arrow[d, "g"] & \mapsto & V \arrow[d, "\rho(g)"] \\
& \bullet & & V
\end{tikzcd}$$

a linear representation. In order to get an action from this representation we use the identification

$$V \simeq \mathrm{Hom}(\mathbb{C}, V)$$

to set

$$\begin{tikzcd}
G \times V \arrow[r, "\simeq"] \arrow[drr, "l"'] & \Sigma(G) \times \mathrm{Hom}(\mathbb{C}, V) \arrow[r, "\rho\times\mathrm{Id}"] & \mathrm{Aut}(V) \times \mathrm{Hom}(\mathbb{C}, V) \arrow[d, "\circ"] \; . \\
& & \mathrm{Hom}(\mathbb{C}, V) \simeq V
\end{tikzcd}$$

On elements this looks like

$$(g,v) \mapsto \left(\begin{array}{c} \bullet \\ \Big\downarrow {\scriptstyle g} \\ \bullet \end{array} , \begin{array}{c} 1 \\ \Big\downarrow {\scriptstyle v} \\ V \end{array} \right) \mapsto \left(\begin{array}{c} V \\ \Big\downarrow {\scriptstyle \rho(g)} \\ V \end{array} , \begin{array}{c} 1 \\ \Big\downarrow {\scriptstyle v} \\ V \end{array} \right) \mapsto \begin{array}{c} 1 \\ \Big\downarrow {\scriptstyle v} \\ V \\ \Big\downarrow {\scriptstyle \rho(g)} \\ V \end{array} \; .$$

This trivial observation helps to understand how to proceed in the categorified case.

So let again C be a monoidal category, let V_2 be a C-module category, let G_2 be a 2-group and

$$\begin{tikzcd}
\rho \; : & \Sigma(G_2) & & \to & \mathrm{Aut}(V_2) \subset \mathrm{Mod}_C & \\
& \bullet \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & \bullet & \mapsto & V_2 \arrow[r, bend left=50, "\rho(g)"{name=U2}] \arrow[r, bend right=50, "\rho(g')"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\rho(h)"] & V_2
\end{tikzcd}$$

a C-linear representation. In order to obtain an action from this we use the identification

$$\begin{tikzcd}
V_2 & \simeq & \mathrm{Hom}_{\mathrm{Mod}_C}(C, V_2) & \\
v \arrow[d, "f"] & & & \\
v' & \mapsto & C \arrow[r, bend left=50, "\mathrm{Id}_v\otimes?"{name=U1}] \arrow[r, bend right=50, "\mathrm{Id}_w\otimes?"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "f\otimes?"] & V_2
\end{tikzcd} .$$

The action is then given by

$$\left(\begin{tikzcd} g \arrow[d, "h"] \\ g' \end{tikzcd}, \begin{tikzcd} v \arrow[d, "f"] \\ w \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} V_2 \arrow[r, bend left=50, "\rho(g)"{name=U2}] \arrow[r, bend right=50, "\rho(g')"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\rho(h)"] & V_2 \end{tikzcd}, \begin{tikzcd} C \arrow[r, bend left=50, "\mathrm{Id}_v\otimes?"{name=U3}] \arrow[r, bend right=50, "\mathrm{Id}_w\otimes?"'{name=D3}] \arrow[Rightarrow, from=U3, to=D3, "f"] & V_2 \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} C \arrow[r, bend left=50, "\mathrm{Id}_v\otimes?"{name=U4}] \arrow[r, bend right=50, "\mathrm{Id}_w\otimes?"'{name=D4}] \arrow[Rightarrow, from=U4, to=D4, "f"] & V_2 \arrow[r, bend left=50, "\rho(g)"{name=U5}] \arrow[r, bend right=50, "\rho(g')"'{name=D5}] \arrow[Rightarrow, from=U5, to=D5, "\rho(h)"] & V_2 \end{tikzcd}\right) .$$

More formally, the left action l is hence defined by

$$\begin{tikzcd}
G_2 \times V_2 \arrow[r, "\simeq"] \arrow[drr, "l"'] & \Sigma(G_2) \times \mathrm{Hom}(C, V_2) \arrow[r, "\rho\times\mathrm{Id}"] & \mathrm{Aut}(V_2) \times \mathrm{Hom}(C, V_2) \arrow[d, "\circ"] \\
& & \mathrm{Hom}(C, V_2) \simeq V_2
\end{tikzcd} .$$

Example 2

Continuing example 1, we can now derive which action the representation of $\Sigma(U(1))$ induces on Vect. We find that

$$\begin{tikzcd} C \arrow[r, bend left=50, "\mathrm{Id}_v\otimes?"{name=U6}] \arrow[r, bend right=50, "\mathrm{Id}_w\otimes?"'{name=D6}] \arrow[Rightarrow, from=U6, to=D6, "f"] & V_2 \end{tikzcd} \mapsto \begin{tikzcd} C \arrow[r, bend left=50, "\mathrm{Id}_v\otimes?"{name=U7}] \arrow[r, bend right=50, "\mathrm{Id}_w\otimes?"'{name=D7}] \arrow[Rightarrow, from=U7, to=D7, "f"] & V_2 \arrow[r, bend left=50, "\mathbb{C}"{name=U8}] \arrow[r, bend right=50, "\mathbb{C}"'{name=D8}] \arrow[Rightarrow, from=U8, to=D8, "\tilde{\rho}(h)"] & V_2 \end{tikzcd}$$

$\Sigma(U(1))$ acts trivially on objects (vector spaces) and acts by multiplication by a phase on morphisms (linear maps between vector spaces).

1.4 Associated connections

1.4.1 Introduction

Associating a vector bundle to a principal bundle involves a coequalizer construction. If the principal bundle carries a connection with parallel transport, this should induce a connection on the associated bundle. In order to realize this we need to be able to tensor not only torsors with vector spaces, but also morphisms of torsors with identity morphisms on vector spaces.

In the following it is spelled out what this should mean in general.

1.4.2 Ordinary associated parallel transport

Given a principal G-bundle

$$\begin{tikzcd} T \arrow[d] \\ X \end{tikzcd}$$

together with a representation

$$\begin{tikzcd} \rho \ : \ \Sigma(G) \arrow[r] & \mathrm{Aut}(V) \subset \mathrm{Vect} \\ \bullet \arrow[d, "g"] & V \arrow[d, "\rho(g)"] \\ \bullet & V \end{tikzcd}$$

where the middle row reads $\bullet \xrightarrow{g} \bullet \mapsto V \xrightarrow{\rho(g)} V$;

$$\begin{tikzcd} \bullet \arrow[d, "g"] & \mapsto & V \arrow[d, "\rho(g)"] \\ \bullet & & V \end{tikzcd}$$

we know how to obtain the associated vector bundle

$$\begin{tikzcd} T \otimes_G V \arrow[d] \\ X \end{tikzcd}$$

by applying the coequalizer

$$\begin{tikzcd} T_x \times G \times V \arrow[r, shift left, "r \times V"] \arrow[r, shift right, "T_x \times l"'] & T_x \times V \arrow[r, "p_x"] & T_x \otimes_G V \end{tikzcd}$$

fiberwise.

Now, suppose we are also given a connection with parallel transport

$$\mathrm{tra} \ : \ \mathcal{P}_1(X) \ \to \ \mathrm{Trans}(T) \subset \mathrm{Tor}(G)$$

$$\begin{tikzcd} x \arrow[d, "\gamma"] & \mapsto & T_x \arrow[d, "\mathrm{tra}(\gamma)"] \\ y & & T_y \end{tikzcd} .$$

We want to send the morphism on the right, living in $\mathrm{Trans}(T)$, to $\mathrm{Trans}(T \otimes_G V) \subset \mathrm{Vect}$.

It is clear how this works in terms of elements, but for categorification we need a diagrammatic construction. Hence consider the diagram

$$\begin{tikzcd}
T_x \times G \times V \arrow[r, shift left, "r\times V"] \arrow[r, shift right, "T_x\times l"'] \arrow[d, "\phi\times G\times V"] & T_x \times V \arrow[r, "p_x"] \arrow[d, "\phi\times V"] & T_x \otimes_G V \arrow[d, dashed, "\phi\otimes_G V"] \\
T_y \times G \times V \arrow[r, shift left, "r\times V"] \arrow[r, shift right, "T_y\times l"'] & T_y \times V \arrow[r, "p_y"'] & T_y \otimes_G V
\end{tikzcd}$$

p_x and p_y are the coequalizers of the horizontal rows. The square on the left commutes (this are really two squares, one involving the left action l of G on T, one involving the right action r on V) because ϕ is a morphism of torsors. Therefore

$$\begin{tikzcd}
T_x \times V \arrow[d, "\phi\times V"] & \\
T_y \times V \arrow[r, "p_y"'] & T_y \otimes_G V
\end{tikzcd}$$

coequalizes $\begin{tikzcd} T_x \times G \times V \arrow[r, shift left, "r\times V"] \arrow[r, shift right, "T_x\times l"'] & T_x \times V \end{tikzcd}$ and hence, by the universal property of p_x, the morphism

$$\begin{tikzcd}
T_x \otimes_G V \arrow[d, dashed, "\phi\otimes_G V"] \\
T_y \otimes_G V
\end{tikzcd}$$

exists uniquely. By uniqueness, the assignment

$$\phi \mapsto \phi \otimes_G V$$

is functorial:

$$\begin{tikzcd}
T_x \times G \times V \arrow[r, shift left, "r\times V"] \arrow[r, shift right, "T_x\times l"'] \arrow[d, "\phi\times G\times V"] & T_x \times V \arrow[r, "p_x"] \arrow[d, "\phi\times V"] & T_x \otimes_G V \arrow[d, dashed, "\phi\otimes_G V"] \arrow[dd, dashed, bend left=60, "(\lambda\circ\phi)\otimes_G V"] & . \\
T_y \times G \times V \arrow[r, shift left, "r\times V"] \arrow[r, shift right, "T_y\times l"'] \arrow[d, "\lambda\times G\times V"] & T_y \times V \arrow[r, "p_y"'] \arrow[d, "\lambda\times V"] & T_y \otimes_G V \arrow[d, dashed, "\lambda\otimes_G V"] & \\
T_z \times G \times V \arrow[r, shift left, "r\times V"] \arrow[r, shift right, "T_z\times l"'] & T_z \times V \arrow[r, "p_z"'] & T_z \otimes_G V &
\end{tikzcd}$$

Thus we have a functor

$$\begin{tikzcd}
?\otimes_G V \arrow[r, phantom, ":"] & \mathrm{Tor}(G) \arrow[r] & \mathrm{Vect} \\
 & T \arrow[d, "\phi"] \arrow[r, phantom, "\mapsto"] & T\otimes_G V \arrow[d, "\phi\otimes_G \mathrm{Id}_V"] & . \\
 & T' & T'\otimes_G V
\end{tikzcd}$$

Hence, given a connection $\mathrm{tra}:\mathcal{P}_1(X)\to\mathrm{Tor}(G)$ on a principal bundle, we obtain from the representation $\rho:\Sigma(G)\to\mathrm{Vect}$ a connection

$$\mathrm{tra}_\rho : \mathcal{P}_1(X) \xrightarrow{\mathrm{tra}} \mathrm{Tor}(G) \xrightarrow{?\otimes_G V} \mathrm{Vect}$$

on the associated vector bundle.

Example 3

For ordinary tensor products this is fancy machinery for something very trivial.

Pick once and for all elements $t_x\in T_x$, $t_y\in T_y$. Every element in $T_x\otimes_G V$ is then uniquely represented by some $v\in V$ as the class of $(t_x,v)\in T_x\times V$.

There is a unique $g\in G$ such that $T_x \xrightarrow{f} T_y$ is given by

$$f(t_x) = t_y\cdot g\,.$$

Thus $f\otimes_G \mathrm{Id}_V$ is given by

$$[(t_x,v)] \xrightarrow{f\otimes_G\mathrm{Id}_V} [(f(t_x),v)] = [(t_y\cdot g,v)] = [(t_y,\rho(g)(v))]\ .$$

Of course we knew this before. But now we can use the above equalizer diagrams to obtain from a representation of a 2-group G_2 on a 2-vector space V_2 a 2-functor

$$?\otimes_G V \quad : \quad \mathrm{Tor}(G_2) \quad \to \quad \mathrm{Mod}_C \quad .$$

This is the content of the next subsection.

1.4.3 Associated 2-transport

In light of the discussion in section 1.4.2 it is clear what we need in order to have an associated 2-transport. Namely, given a principal 2-transport

$$\mathrm{tra} \quad : \quad \mathcal{P}_2(X) \quad \to \quad \mathrm{Tor}_{G_2}$$

$$\begin{tikzcd}
x \arrow[r, bend left=50, "\gamma", ""{name=U, below}] \arrow[r, bend right=50, "\gamma'"', ""{name=D}] & y \arrow[r, phantom, "\mapsto"] & T_x \arrow[r, bend left=50, "\phi", ""{name=U2, below}] \arrow[r, bend right=50, "\phi'"', ""{name=D2}] & T_y
\arrow[Rightarrow, from=U, to=D, "\Sigma"]
\arrow[Rightarrow, from=U2, to=D2, "\lambda"]
\end{tikzcd}$$

we are looking for the associated 2-transport

$$\mathrm{tra}_\rho \quad : \quad \mathcal{P}_2(X) \quad \to \quad \mathrm{Tor}_{G_2} \quad \to \quad \mathrm{Mod}_C$$

$$\begin{tikzcd}
x \arrow[r, bend left=50, "\gamma", ""{name=U, below}] \arrow[r, bend right=50, "\gamma'"', ""{name=D}] & y \arrow[r, phantom, "\mapsto"] & T_x \arrow[r, bend left=50, "\phi", ""{name=U2, below}] \arrow[r, bend right=50, "\phi'"', ""{name=D2}] & T_y \arrow[r, phantom, "\mapsto"] & T_x \otimes_{G_2} V_2 \arrow[r, bend left=30, "\phi\otimes_{G_2} V_2", ""{name=U3, below}] \arrow[r, bend right=30, "\phi'\otimes_{G_2} V_2"', ""{name=D3}] & T_y \otimes_{G_2} V_2
\arrow[Rightarrow, from=U, to=D, "\Sigma"]
\arrow[Rightarrow, from=U2, to=D2, "\lambda"]
\arrow[Rightarrow, from=U3, to=D3, "\lambda\otimes_{G_2} V_2"]
\end{tikzcd}$$

which is uniquely determined by the right face of a tin can diagram of the following form

$$\begin{tikzcd}
T_x \times G \times V \arrow[r, shift left, "r\times V"] \arrow[r, shift right, "T_x\times l"'] \arrow[d, bend right=40, "\phi'\times G\times V"', ""{name=L1}] \arrow[d, bend left=40, "\phi\times G\times V", ""{name=R1, left}] \arrow[dr, phantom, "\swarrow"] & T_x \times V \arrow[r, "p_x"] \arrow[d, dotted, bend right=40, "\phi'\times V"', ""{name=L2}] \arrow[d, bend left=40, "\phi\times V", ""{name=R2, left}] \arrow[dr, phantom, "\swarrow"] & T_x \otimes_G V \arrow[d, dotted, bend right=40, "\phi'\otimes_G V"', ""{name=L3}] \arrow[d, dashed, bend left=40, "\phi\otimes_G V", ""{name=R3, left}] \\
T_y \times G \times V \arrow[r, shift left, "r\times V"] \arrow[r, shift right, "T_y\times l"'] & T_y \times V \arrow[r, "p_y"'] & T_y \otimes_G V
\arrow[Rightarrow, from=R1, to=L1, "\lambda\times G\times V"']
\arrow[Rightarrow, dotted, from=R2, to=L2, "\lambda\times V"']
\arrow[Rightarrow, dotted, from=R3, to=L3, "\lambda\otimes_G V"']
\end{tikzcd}.$$

So we'd better use a version of coequalizers in Cat which makes this true. . . .
This is discussed in section 1.5.4. The above tin can is in instance of (4), given there.

1.5 Colimits in Cat

1.5.1 Introduction.

Associating 2-vector bundles crucially depends on a notion of coequalizer in Cat. Strict coequalizers in Cat are explicitly constructed in [1]. It is not a priori clear, though, that strict coequalizers are sufficient for our needs.

We want to internalize the notion of limit and colimit in Cat. Certainly some australian category theorists know all about this. But I don't. So here I give some notes on how I would try to approach this.

1.5.2 Internal limits and colimits.

Let C be any category. A **diagram** in C is a functor

$$D : S \to C$$

from any small category S (a "**shape**") to C. Let

$$D_t : 1 \to C$$

be the diagram consisting of a single object $t \in \mathrm{Obj}(C)$. A **cone** c over a diagram D with **tip** t is a natural transformation

$$\begin{tikzcd}
 & 1 \arrow[dr, "D_c"] \arrow[d, phantom, "\Downarrow c"] & \\
S \arrow[ur] \arrow[rr, "D"'] & {} & C
\end{tikzcd} .$$

Given two cones

$$\begin{tikzcd}
 & 1 \arrow[dr, "D_t"] \arrow[d, phantom, "\Downarrow c"] & \\
S \arrow[ur] \arrow[rr, "D"'] & {} & C
\end{tikzcd} , \quad
\begin{tikzcd}
 & 1 \arrow[dr, "D_{t'}"] \arrow[d, phantom, "\Downarrow c'"] & \\
S \arrow[ur] \arrow[rr, "D"'] & {} & C
\end{tikzcd}$$

over the same diagram, a **morphism of cones**

$$c \xrightarrow{f} c'$$

is a natural transformation

$$\begin{tikzcd}
1 \arrow[r, bend left=50, "D_t"] \arrow[r, bend right=50, "D_{t'}"'] \arrow[r, phantom, "\Downarrow f"] & C
\end{tikzcd}$$

(hence nothing but a morphism $t \longrightarrow t'$ between the tips in C) such that

$$\begin{tikzcd}
 & 1 \arrow[dr, "D_t"] \arrow[d, phantom, "c \Downarrow"] & \\
S \arrow[ur] \arrow[rr, "D"'] & {} & C
\end{tikzcd} \quad = \quad
\begin{tikzcd}
 & 1 \arrow[dr, "D_{t'}" description] \arrow[dr, bend left=70, "D_t"] \arrow[dr, phantom, bend left=35, "\Swarrow f"] \arrow[d, phantom, "c' \Downarrow"] & \\
S \arrow[ur] \arrow[rr, "D"'] & {} & C
\end{tikzcd} .$$

We get a **category of cones** over the diagram D this way. The **limit** over D is (if it exists) the terminal object in that category.

Similarly for cocones and colimits.

So a cocone over D with tip t is a natural transformation

$$\begin{tikzcd}
S \arrow[rr, "D"] \arrow[dr] & \arrow[d, phantom, "c\Downarrow"] & C \\
& 1 \arrow[ur, "D_t"'] &
\end{tikzcd},$$

and a morphism of cocones

$$c \xrightarrow{f} c'$$

is a natural transformation

$$\begin{tikzcd}
1 \arrow[r, bend left=50, "D_t"{name=U}] \arrow[r, bend right=50, "D_{t'}"'{name=L}] & C
\arrow[from=U, to=L, Rightarrow, "f"]
\end{tikzcd}$$

such that

$$\begin{tikzcd}
S \arrow[rr, "D"] \arrow[dr] & \arrow[d, phantom, "c\Downarrow"] & C \\
& 1 \arrow[ur, "D_t"] \arrow[ur, bend right=60, "D_{t'}"'] \arrow[ur, phantom, bend right=30, "f"] &
\end{tikzcd}
=
\begin{tikzcd}
S \arrow[rr, "D"] \arrow[dr] & \arrow[d, phantom, "c'\Downarrow"] & C \\
& 1 \arrow[ur, "D_{t'}"'] &
\end{tikzcd}.$$

We get a **category of cocones** this way. The **colimit** over D (if it exists) is the initial object in this category.

1.5.3 Limits and colimits in 2Cat

Where we had natural transformation before we now have pseudonatural transformations. These have modifications going between them.

So a cocone is now a pseudonatural transformation

$$\begin{tikzcd}
S \arrow[rr, "D"] \arrow[dr] & \arrow[d, phantom, "c\Downarrow"] & C \\
& 1 \arrow[ur, "D_t"'] &
\end{tikzcd}.$$

Two of these may be related by a modification

$$\begin{tikzcd}
S \arrow[rr, "D"] \arrow[dr] & \arrow[d, phantom, "c\Downarrow"] & C \\
& 1 \arrow[ur, "D_t"'] &
\end{tikzcd}
\overset{\mu}{\to}
\begin{tikzcd}
S \arrow[rr, "D"] \arrow[dr] & \arrow[d, phantom, "c'\Downarrow"] & C \\
& 1 \arrow[ur, "D_t"'] &
\end{tikzcd}.$$

A morphisms of cocones is now a pseudonatural transformation

$$\begin{tikzcd} 1 \arrow[rr, bend left=50, "D_t"{name=U}] \arrow[rr, bend right=50, "D_{t'}"'{name=L}] & & C \arrow[Rightarrow, from=U, to=L, "f"] \end{tikzcd}$$

together with a specified isomodification

$$\begin{tikzcd} S \arrow[rr, "D"] \arrow[dr] & c\Downarrow & C & \overset{\mu_f}{\to} & S \arrow[rr, "D"] \arrow[dr] & c'\Downarrow & C \\ & 1 \arrow[ur, "D_t"{name=A}] \arrow[ur, bend right=60, "D_{t'}"'{name=B}] \arrow[Rightarrow, from=A, to=B, "f"] & & & & 1 \arrow[ur, "D_{t'}"'] & \end{tikzcd} .$$

We require this assigment to be natural with respect to modifications of c in the sense that

$$\begin{tikzcd} S \arrow[rr, "D"] \arrow[dr] & c\Downarrow & C & \overset{\mu_f}{\longrightarrow} & S \arrow[rr, "D"] \arrow[dr] & c'\Downarrow & C \\ & 1 \arrow[ur, "D_t"{name=A}] \arrow[ur, bend right=60, "D_{t'}"'{name=B}] \arrow[Rightarrow, from=A, to=B, "f"] & & & & 1 \arrow[ur, "D_{t'}"'] & \\ & \nu_f \downarrow & & & & \downarrow \nu & \\ S \arrow[rr, "D"] \arrow[dr] & c\Downarrow & C & \overset{\mu_{\tilde{f}}}{\longrightarrow} & S \arrow[rr, "D"] \arrow[dr] & \tilde{c}'\Downarrow & C \\ & 1 \arrow[ur, "D_t"{name=C}] \arrow[ur, bend right=60, "D_{t'}"'{name=E}] \arrow[Rightarrow, from=C, to=E, "\tilde{f}"] & & & & 1 \arrow[ur, "D_{t'}"'] & \end{tikzcd} \quad . \tag{3}$$

Cocones and morphisms between them form a category and the colimit (if it exists) is the initial object of that category.

1.5.4 Coequalizers in 2Cat

We now apply the above to coequalizers.

In this case, the shape in question is the small category

$$S = \left\{ \begin{tikzcd} \mathrm{A} \arrow[rr, equal] \arrow[dr, "l"'] & & \mathrm{A} \arrow[dl, "r"] \\ & \mathrm{B} & \end{tikzcd} \right\},$$

regarded as a 2-category with only identity 2-morphisms. A strict coequalizer would be a 2-functor on this shape which is just an ordinary functor, regarded as a 2-functor

$$\begin{tikzcd}
D(A) \arrow[rr, equal] \arrow[dr, "D(l)"'] & & D(A) \arrow[dl, "D(r)"] \\
& D(B) &
\end{tikzcd}$$

A cocone over this looks like

$$\begin{tikzcd}
D(A) \arrow[rr, equal] \arrow[dr, "D(l)"] \arrow[ddr, bend right=40, "c(A)"'] \arrow[ddr, phantom, bend right=15, "\overset{c(l)}{\Leftarrow}"] & & D(A) \arrow[dl, "D(r)"'] \arrow[ddl, bend left=40, "c(A)"] \arrow[ddl, phantom, bend left=15, "\overset{c(r)}{\Leftarrow}"] \\
& D(B) \arrow[d, "c(B)"] & \\
& t &
\end{tikzcd}$$

Strict coequalizers in 2Cat For the moment, restrict attention to the case where the 2-morphisms $c(l)$ and $c(r)$ above are taken to be identity 2-cells.

Then two cocones with the same tip look simply like

$$\begin{tikzcd}
D(A) \arrow[rr, equal] \arrow[dr, "D(l)"] & & D(A) \arrow[dl, "D(r)"'] & & D(A) \arrow[rr, equal] \arrow[dr, "D(l)"] & & D(A) \arrow[dl, "D(r)"'] \\
& D(B) \arrow[d, "c'(B)"] & & , & & D(B) \arrow[d, "\tilde{c}'(B)"] & \\
& t' & & & & t' &
\end{tikzcd}$$

and a modification of cocones $c' \xrightarrow{\nu} \tilde{c}'$ is nothing but a 2-morphism

$$\begin{tikzcd}
D(B) \arrow[d, bend right=50, "\tilde{c}'(B)"'] \arrow[d, bend left=50, "c'(B)"] \arrow[d, phantom, "\overset{\nu}{\Leftarrow}"] \\
t'
\end{tikzcd} .$$

By assumption (3), this comes with modifications μ_f and $\mu_{\tilde f}$ such that

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D(B) \arrow[r, "c(B)"] \arrow[d, bend right=60, "\tilde{c}'(B)"'] \arrow[d, bend left=40, "c'(B)"] \arrow[d, phantom, shift right=4ex, "\overset{\nu}{\Longleftarrow}"] \arrow[rd, phantom, "\overset{\mu_f}{\Swarrow}"] & t \arrow[d, dotted, bend right=40, "\tilde{f}(B)"'] \arrow[d, bend left=60, "f(B)"] \arrow[d, phantom, shift left=4ex, "\overset{\nu_f}{\Leftarrow}"] \\
t' \arrow[r, "\mathrm{Id}"'] & t'
\end{tikzcd} \tag{4}$$

2-commutes.

1.5.5 Enriched categories in terms of lax functors

Here I say a couple of words about how to think of enriched categories in terms of lax functors. Not directly relevant for the rest of these notes.

Let $\mathcal{V}$ be any bicategory (usually the suspension of a monoidal category). Let C be any $\mathcal{V}$-enriched category. This is the same as a lax functor

$$F_C : PC \to \mathcal{V} \,,$$

where PC is the pair groupoid of the set of objects of C.

More generally, let S be any category, then we can say that enriching S over $\mathcal{V}$ is specifying a lax functor

$$C : S \to \mathcal{V} \,.$$

These lax functors naturally live in a slice-2-catgeory, where the 1-morphisms

$$C \xrightarrow{f} C'$$

are given by 2-cells

$$\begin{tikzcd}
S \arrow[rr, "f_1"] \arrow[dr, "C"'] & \arrow[d, phantom, "\Swarrow f_2"] & S' \arrow[dl, "C'"] \\
& \mathcal{V} &
\end{tikzcd} ,$$

with horizontal composition

$$C \xrightarrow{f} C' \xrightarrow{g} C''$$

given by

$$\begin{tikzcd}
S \arrow[rr, "f_1"] \arrow[drr, "C"'] & \arrow[d, phantom, "\Swarrow f_2"] & S' \arrow[rr, "g_1"] \arrow[d, "C'"] & \arrow[d, phantom, "\Swarrow g_2"] & S'' \arrow[dll, "C''"] \\
& & \mathcal{V} & &
\end{tikzcd} ,$$

We find that f_2 is given by 2-cells

$$\begin{tikzcd}
C(s) \arrow[r, "C(s,s')"] \arrow[d, "f_2(s)"'] & C(s') \arrow[d, "f_2(s')"] \arrow[dl, phantom, "\overset{f_2(s,s')}{\swArrow}"] \\
C'(f_1(s)) \arrow[r, "C'(f_1(s),f_1(s'))"'] & C'(f_1(s'))
\end{tikzcd}$$

in $\mathcal{V}$ satisfying a couple of tin can equations expressing the compatibility with composition and units.

For the special case that the vertical 1-morphisms in the above are taken to be identities, this reproduces the standard definition of functors of enriched categories.

[Next I need to define 2-morphisms in the slice category and show that they reproduce natural transformations in the enriched setup.]

References

[1] M. Bednarczyk, A. Borzyszkowski, W. Pawlowski, *Generalized Congruences – Epimorphisms in* Cat, TAC, **5** 11 (1999) 266-280