

# Notes on associated vector 2-bundles

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## **Abstract**

Rough notes on some aspects of associated vector 2-bundles.

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# 1 Introduction

This are some notes on the general issue of 2-vector 2-bundles associated to principal 2-bundles.

## Conventions.

- Torsors. In the following the word *torsor* always refers to *torsor over a point*. Our main motivation comes from parallel transport in 2-bundles. The 2-transport 2-functor will associate torsors to every object. The full 2-bundle regarded as a 2-torsor over a base space is then then the full image under the 2-transport of the collection of all points in base space.
- Connections. In the present context, a (2-)connection on a (2-)bundle shall always mean a transport (2-)functor which takes points in the base to the fibers above them and (2-)paths in the base to (2-)morphisms of (2-)torsors. We shall loosely refer to this as a (2-)connection, though that term might maybe better be reserved for some infinitesimal notion of (2-)transport.

For the most part we do not care here about the specific ambient topos (sets, or topological spaces, or smooth spaces, etc.).

In outline, the complex of questions we shall be concerned with is the following.

## 1.1 Basic concepts in associated 2-bundles

Let  $G_2$  be a monoidal category, usually a 2-group.

Let

$$\begin{array}{c} P \\ \downarrow \\ X \end{array}$$

be a principal  $G_2$ -2-bundle over a discrete category  $X$ . Some authors call this a  $G_2$ -torsor. Here I shall reserve the term **(right)  $G_2$ -torsor** for any category  $T$  equipped with a right  $G_2$ -action

$$T \times G_2 \xrightarrow{r} T$$

up to coherent isomorphism

$$\begin{array}{ccc} T \times G_2 \times G_2 & \xrightarrow{T \times m} & T \times G_2 \\ \downarrow r \times G_2 & \Downarrow & \downarrow r \\ T \times G_2 & \xrightarrow{r} & T \end{array} ,$$

such that it is equivalent to  $G_2$  as a right  $G_2$  space, or, alternatively, such that

$$T \times G_2 \xrightarrow{(\text{Id} \times r)} T \times T$$

is an equivalence.

Then each fiber  $P_x$ ,  $x \in X$  of  $P$  is a  $G_2$ -torsor.

A linear representation of  $G_2$  is a 2-functor

$$\rho_2 : \Sigma(G_2) \rightarrow \text{End}(V_2) \subset \text{Mod}_C$$

$$\begin{array}{ccc} \bullet & \begin{array}{c} \xrightarrow{g} \\ \Downarrow h \\ \xrightarrow{g'} \end{array} & \bullet \\ \bullet & \begin{array}{c} \xrightarrow{\rho_2(g)} \\ \Downarrow \rho_2(h) \\ \xrightarrow{\rho_2(g')} \end{array} & \bullet \end{array} ,$$

for  $C$  some (usually braided) monoidal category  $C$ .

Given such a representation, we obtain a left action of  $G_2$  on  $V_2$  by setting

$$\begin{array}{ccc} G_2 \times V_2 & \xrightarrow{\rho_2 \times \text{Id}} & \text{Hom}(V_2) \times V_2 \\ & \searrow l & \downarrow \text{ev} \\ & & V_2 \end{array}$$

where  $\text{ev}$  is the image of the identity under

$$\text{Hom}(\text{Hom}(V_2, V_2), \text{Hom}(V_2, V_2)) \simeq \text{Hom}(\text{Hom}(V_2, V_2) \times V_2, V_2) .$$

The right  $C$ -module associated by  $\rho$  to  $T$  is the coequalizer

$$T \times G_2 \times V_2 \begin{array}{c} \xrightarrow{r \times V_2} \\ \xrightarrow{T \times l} \end{array} T \times V_2 \longrightarrow T \otimes_{G_2} V_2 .$$

Coequalizers in  $\text{Cat}$  have been discussed for instance in [1]. Aspects of coequalizers in enriched categories are discussed in a later section.

## 1.2 2-reps induced from ordinary reps

### 1.2.1 Introduction.

In this subsection we present a method that induces from any ordinary finite-dimensional linear representation of an ordinary group a representation of its automorphism 2-group on 2-vector spaces. The induced representation is in terms of bimodules for the algebra generated by the representation of  $H$ .

We expect that a generalization of this method to infinite-dimensional representations will apply to the  $\text{String}(n)$ -2-group and will in fact reproduce, using the discussion from section 1.4 below, the construction of String-connections in terms of bimodules considered by Stolz and Teichner.

### 1.2.2 2-reps on bimodules over a representation algebra

Let  $H$  be any group and let

$$\begin{array}{ccc} \rho : \Sigma(H) & \rightarrow & \text{Vect} \\ \bullet & & V \\ \downarrow h & \mapsto & \downarrow \rho(h) \\ \bullet & & V \end{array}$$

be any finite-dimensional representation on vector spaces. We want to construct from  $\rho$  a representation of the automorphism 2-group  $\text{Aut}_2(H)$  on 2-vector spaces

$$\tilde{\rho} : \text{Aut}_{\text{Cat}}(\Sigma(H)) \rightarrow \text{Mod}_{\text{Vect}} .$$

Recall that 2-morphisms in  $\text{Aut}_{\text{Cat}}(\Sigma(H))$

$$\begin{array}{ccc} & g & \\ \Sigma(H) & \begin{array}{c} \curvearrowright \\ \Downarrow h \\ \curvearrowleft \end{array} & \Sigma(H) \\ & g' & \end{array}$$

are labeled by  $g \in \text{Aut}(H)$  and  $h \in H$  with

$$\begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ g(f) \downarrow & & \downarrow g'(f) \\ \bullet & \xrightarrow{h} & \bullet \end{array}$$

for arbitrary  $f \in H$ . What we shall need below is the commutativity of the image of this diagram under  $\rho$

$$\begin{array}{ccc} V & \xrightarrow{\rho(h)} & V \\ \rho(g(f)) \downarrow & & \downarrow \rho(g'(f)) \\ V & \xrightarrow{\rho(h)} & V \end{array} . \quad (1)$$

In order to construct  $\tilde{\rho}$  let now

$$\text{End}(V) \supset A_\rho \equiv \langle \rho(h) \mid h \in H \rangle$$

be the subalgebra of the endomorphism algebra of  $V$  which is generated by the linear maps  $\rho(h)$  for all  $h \in H$ . We obtain for each  $g \in \text{Aut}(H)$  an automorphism  $\rho(g) \in \text{Aut}(A_\rho)$  of this algebra by setting

$$\rho(g) : \rho(h) \mapsto \rho(g(h))$$

for all  $h \in H$ , and extended linearly to all of  $A_\rho$ .

Using this, for each  $g \in \text{Aut}(H)$  we define an  $A_\rho$ -bimodule

$$N_g \equiv A_\rho \overset{\rho(g)}{\rhd} A_\rho \overset{\text{Id}}{\lhd} A_\rho$$

which, as an object in  $\text{Vect}$ , is  $A_\rho$  itself, with both the right and the left  $A_\rho$  action given by the product in  $A_\rho$ , but with the left action twisted by  $\rho(g)$ :

$$\begin{aligned} \rho(h) \cdot n &\equiv \rho(g(h)) \circ n \\ n \cdot \rho(h) &\equiv n \circ \rho(h) . \end{aligned} \quad (2)$$

for all  $n \in N_g$ .

For all bimodules of this form the tensor product over  $A_\rho$  corresponds to the composition of automorphisms

$$N_g \otimes_{A_\rho} N_{g'} = N_{g' \circ g} .$$

Let  ${}_{A_\rho}\text{Mod}$  be the category of left  $A_\rho$ -modules. Every  $A_\rho$  bimodule induces, by tensor multiplication on the left, an endofunctor

$$N_g \otimes_{A_\rho} ? : {}_{A_\rho}\text{Mod} \longrightarrow {}_{A_\rho}\text{Mod} .$$

By the above remark, we have hence obtained a representation of all identity 2-morphisms in  $\text{Aut}_2(H)$  on identity 2-morphisms in  $\text{Bim}(\text{Vect}) \subset \text{Mod}_{\text{Vect}}$ .

For each nontrivial 2-morphism

$$\begin{array}{ccc} & g & \\ & \curvearrowright & \\ \Sigma(H) & & \Sigma(H) \\ & \Downarrow h & \\ & \curvearrowleft & \\ & g' & \end{array}$$

define an map of bimodules

$$\begin{aligned} \tilde{\rho}(h) : N_g &\rightarrow N_{g'} \\ n &\mapsto \rho(h) \circ n . \end{aligned}$$

This map trivially respects the right  $A_\rho$ -action. That it also respects the left

$A_\rho$  action is a consequence of the commutativity of (1):

$$\begin{array}{ccc}
 (\rho(f), n) & \xrightarrow{\text{Id} \times \tilde{\rho}(h)} & (\rho(f), \rho(h) \circ n) \\
 \downarrow (2) & & \downarrow (2) \\
 & & \rho(g'(f)) \circ \rho(h) \circ n \\
 & & \parallel (1) \\
 \rho(g(f)) \circ n & \xrightarrow{\tilde{\rho}(h)} & \rho(h) \circ \rho(g(f)) \circ n
 \end{array}$$

We obtain this way a representation 2-functor

$$\tilde{\rho} : \text{Aut}_2(H) \rightarrow \text{Mod}_{\text{Vect}}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \Sigma(H) & \begin{array}{c} \xrightarrow{g} \\ \Downarrow h \\ \xrightarrow{g'} \end{array} & \Sigma(H) \\
 & & 
 \end{array} & \mapsto & \begin{array}{ccc}
 {}_{A_\rho}\text{Mod} & \begin{array}{c} \xrightarrow{N_g} \\ \Downarrow \tilde{\rho}(h) \\ \xrightarrow{N_{g'}} \end{array} & {}_{A_\rho}\text{Mod} \\
 & & 
 \end{array}
 \end{array}$$

### Example 1

Let  $G_2 = (U(1) \rightarrow 1) = \Sigma(U(1))$ . Let  $\rho : \Sigma(U(1)) \rightarrow \text{Vect}_{\mathbb{C}}$  be the defining 1-dimensional rep.

In this case we find  $A_\rho = \mathbb{C}$ , the complex numbers. The bimodule  $N_{\text{Id}}$  is just  $\mathbb{C}$  itself, with the left and right  $\mathbb{C}$ -action given by multiplication of complex numbers. Endomorphisms of this bimodule are given by injecting  $U(1)$  into  $\mathbb{C}$  and multiplying in  $\mathbb{C}$ . The 2-vector space  ${}_{\mathbb{C}}\text{Mod} = \text{Vect}_{\mathbb{C}}$  is 1-dimensional.

$$\tilde{\rho} : \Sigma(\Sigma(U(1))) \rightarrow \text{Mod}_{\text{Vect}}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \Sigma(U(1)) & \begin{array}{c} \xrightarrow{\text{Id}} \\ \Downarrow h \\ \xrightarrow{\text{Id}} \end{array} & \Sigma(U(1)) \\
 & & 
 \end{array} & \mapsto & \begin{array}{ccc}
 \text{Vect} & \begin{array}{c} \xrightarrow{\mathbb{C}} \\ \Downarrow \tilde{\rho}(h) \\ \xrightarrow{\mathbb{C}} \end{array} & \text{Vect} \\
 & & 
 \end{array}
 \end{array}$$

## 1.3 Actions from representations

### 1.3.1 Introduction

In the ordinary (non-categorified) setup it is very obvious how to get a  $G$ -action on some vector space  $V$  given a representation of  $G$  on  $V$ . In fact, this is so very obvious that one hardly sees the difference.

But as soon as one categorifies, the difference becomes more pronounced. The right answer is still easy and elegant, but maybe deserves to be made explicit. It crucially depends on realizing elements of 2-vector spaces as maps into 2-vector spaces.

### 1.3.2 Actions from reps combined with Yoneda embedding

How do we get an action, given a representation?

As a motivation, reformulate the ordinary case like this:

Let  $G$  be any group and

$$\rho : \Sigma(G) \rightarrow \text{Aut}(V) \subset \text{Vect}$$

$$\begin{array}{ccc} \bullet & & V \\ \downarrow g & \mapsto & \downarrow \rho(g) \\ \bullet & & V \end{array}$$

a linear representation. In order to get an action from this representation we use the identification

$$V \simeq \text{Hom}(\mathbb{C}, V)$$

to set

$$\begin{array}{ccc} G \times V & \xrightarrow{\simeq} & \Sigma(G) \times \text{Hom}(\mathbb{C}, V) \xrightarrow{\rho \times \text{Id}} \text{Aut}(V) \times \text{Hom}(\mathbb{C}, V) . \\ & \searrow l & \downarrow \circ \\ & & \text{Hom}(\mathbb{C}, V) \simeq V \end{array}$$

On elements this looks like

$$(g, v) \mapsto \left( \begin{array}{c} \bullet \\ \downarrow g \\ \bullet \end{array}, \begin{array}{c} 1 \\ \downarrow v \\ V \end{array} \right) \mapsto \left( \begin{array}{c} V \\ \downarrow \rho(g) \\ V \end{array}, \begin{array}{c} 1 \\ \downarrow v \\ V \end{array} \right) \mapsto \begin{array}{c} 1 \\ \downarrow v \\ V \\ \downarrow \rho(g) \\ V \end{array} .$$

This trivial observation helps to understand how to proceed in the categorified case.

So let again  $C$  be a monoidal category, let  $V_2$  be a  $C$ -module category, let  $G_2$  be a 2-group and

$$\rho : \Sigma(G_2) \rightarrow \text{Aut}(V_2) \subset \text{Mod}_C$$

$$\begin{array}{ccc} \bullet & & V_2 \\ \begin{array}{c} \curvearrowright g \\ \Downarrow h \\ \curvearrowleft g' \end{array} & \mapsto & \begin{array}{c} \curvearrowright \rho(g) \\ \Downarrow \rho(h) \\ \curvearrowleft \rho(g') \end{array} \\ \bullet & & V_2 \end{array}$$

a  $C$ -linear representation. In order to obtain an action from this we use the identification

$$V_2 \simeq \text{Hom}_{\text{Mod}_C}(C, V_2)$$

$$\begin{array}{ccc} v & & \\ \downarrow f & \mapsto & \begin{array}{ccc} C & \begin{array}{c} \text{Id}_v \otimes ? \\ \Downarrow f \\ \text{Id}_w \otimes ? \end{array} & V_2 \\ v' & & \end{array} \end{array}$$

The action is then given by

$$\left( \begin{array}{c} g \\ \downarrow h \\ g' \end{array}, \begin{array}{c} v \\ \downarrow f \\ w \end{array} \right) \mapsto \left( \begin{array}{ccc} \begin{array}{ccc} V_2 & \begin{array}{c} \rho(g) \\ \Downarrow \rho(h) \\ \rho(g') \end{array} & V_2 \\ C & \begin{array}{c} \text{Id}_v \otimes ? \\ \Downarrow f \\ \text{Id}_w \otimes ? \end{array} & V_2 \end{array}, \begin{array}{ccc} C & \begin{array}{c} \text{Id}_v \otimes ? \\ \Downarrow f \\ \text{Id}_w \otimes ? \end{array} & V_2 \end{array} \right) \mapsto \left( \begin{array}{ccc} C & \begin{array}{c} \text{Id}_v \otimes ? \\ \Downarrow f \\ \text{Id}_w \otimes ? \end{array} & V_2 \\ V_2 & \begin{array}{c} \rho(g) \\ \Downarrow \rho(h) \\ \rho(g') \end{array} & V_2 \end{array} \right)$$

More formally, the left action  $l$  is hence defined by

$$\begin{array}{ccc} G_2 \times V_2 & \xrightarrow{\simeq} & \Sigma(G_2) \times \text{Hom}(C, V_2) \xrightarrow{\rho \times \text{Id}} \text{Aut}(V_2) \times \text{Hom}(C, V_2) \\ & \searrow l & \downarrow \circ \\ & & \text{Hom}(C, V_2) \simeq V_2 \end{array}$$

## Example 2

Continuing example 1, we can now derive which action the representation of  $\Sigma(U(1))$  induces on  $\text{Vect}$ . We find that

$$\begin{array}{ccc} \begin{array}{ccc} C & \begin{array}{c} \text{Id}_v \otimes ? \\ \Downarrow f \\ \text{Id}_w \otimes ? \end{array} & V_2 \end{array} & \mapsto & \begin{array}{ccc} C & \begin{array}{c} \text{Id}_v \otimes ? \\ \Downarrow f \\ \text{Id}_w \otimes ? \end{array} & V_2 \\ & & \begin{array}{ccc} C & \begin{array}{c} \text{Id}_v \otimes ? \\ \Downarrow \bar{\rho}(h) \\ \text{Id}_w \otimes ? \end{array} & V_2 \end{array} \end{array}$$

$\Sigma(U(1))$  acts trivially on objects (vector spaces) and acts by multiplication by a phase on morphisms (linear maps between vector spaces).

## 1.4 Associated connections

### 1.4.1 Introduction

Associating a vector bundle to a principal bundle involves a coequalizer construction. If the principal bundle carries a connection with parallel transport,



this should induce a connection on the associated bundle. In order to realize this we need to be able to tensor not only torsors with vector spaces, but also morphisms of torsors with identity morphisms on vector spaces.

In the following it is spelled out what this should mean in general.

### 1.4.2 Ordinary associated parallel transport

Given a principal  $G$ -bundle

$$\begin{array}{c} T \\ \downarrow \\ X \end{array}$$

together with a representation

$$\rho : \Sigma(G) \rightarrow \text{Aut}(V) \subset \text{Vect}$$

$$\begin{array}{ccc} \bullet & & V \\ \downarrow g & \mapsto & \downarrow \rho(g) \\ \bullet & & V \end{array}$$

we know how to obtain the associated vector bundle

$$\begin{array}{c} T \otimes_G V \\ \downarrow \\ X \end{array}$$

by applying the coequalizer

$$T_x \times G \times V \begin{array}{c} \xrightarrow{r \times V} \\ \xrightarrow{T_x \times l} \end{array} T_x \times V \xrightarrow{p_x} T_x \otimes_G V$$

fiberwise.

Now, suppose we are also given a connection with parallel transport

$$\text{tra} : \mathcal{P}_1(X) \rightarrow \text{Trans}(T) \subset \text{Tor}(G)$$

$$\begin{array}{ccc} x & & T_x \\ \downarrow \gamma & \mapsto & \downarrow \text{tra}(\gamma) \\ y & & T_y \end{array} .$$

We want to send the morphism on the right, living in  $\text{Trans}(T)$ , to  $\text{Trans}(T \otimes_G V) \subset \text{Vect}$ .

It is clear how this works in terms of elements, but for categorification we need a diagrammatic construction. Hence consider the diagram

$$\begin{array}{ccccc}
 T_x \times G \times V & \xrightarrow[r \times V]{T_x \times l} & T_x \times V & \xrightarrow{p_x} & T_x \otimes_G V \\
 \downarrow \phi \times G \times V & & \downarrow \phi \times V & & \downarrow \phi \otimes_G V \\
 T_y \times G \times V & \xrightarrow[r \times V]{T_y \times l} & T_y \times V & \xrightarrow{p_y} & T_y \otimes_G V
 \end{array}$$

$p_x$  and  $p_y$  are the coequalizers of the horizontal rows. The square on the left commutes (this are really two squares, one involving the left action  $l$  of  $G$  on  $T$ , one involving the right action  $r$  on  $V$ ) because  $\phi$  is a morphism of torsors. Therefore

$$\begin{array}{ccc}
 T_x \times V & & \\
 \downarrow \phi \times V & & \\
 T_y \times V & \xrightarrow{p_y} & T_y \otimes_G V
 \end{array}$$

coequalizes  $T_x \times G \times V \xrightarrow[r \times V]{T_x \times l} T_x \times V$  and hence, by the universal property of  $p_x$ , the morphism

$$\begin{array}{c}
 T_x \otimes_G V \\
 \vdots \\
 \phi \otimes_G V \\
 \vdots \\
 T_y \otimes_G V
 \end{array}$$

exists uniquely. By uniqueness, the assignment

$$\phi \mapsto \phi \otimes_G V$$

is functorial:

$$\begin{array}{ccccc}
T_x \times G \times V & \xrightarrow[r \times V]{T_x \times l} & T_x \times V & \xrightarrow{p_x} & T_x \otimes_G V \\
\downarrow \phi \times G \times V & & \downarrow \phi \times V & & \downarrow \phi \otimes_G V \\
T_y \times G \times V & \xrightarrow[r \times V]{T_y \times l} & T_y \times V & \xrightarrow{p_y} & T_y \otimes_G V \quad (\lambda \circ \phi) \otimes_G V \\
\downarrow \lambda \times G \times V & & \downarrow \lambda \times V & & \downarrow \lambda \otimes_G V \\
T_z \times G \times V & \xrightarrow[r \times V]{T_z \times l} & T_z \times V & \xrightarrow{p_z} & T_z \otimes_G V
\end{array}$$

Thus we have a functor

$$? \otimes_G V : \text{Tor}(G) \rightarrow \text{Vect}$$

$$\begin{array}{ccc}
T & & T \otimes_G V \\
\downarrow \phi & \mapsto & \downarrow \phi \otimes_G \text{Id}_V \\
T' & & T' \otimes_G V
\end{array}$$

Hence, given a connection  $\text{tra} : \mathcal{P}_1(X) \rightarrow \text{Tor}(G)$  on a principal bundle, we obtain from the representation  $\rho : \Sigma(G) \rightarrow \text{Vect}$  a connection

$$\text{tra}_\rho : \mathcal{P}_1(X) \xrightarrow{\text{tra}} \text{Tor}(G) \xrightarrow{? \otimes_G V} \text{Vect}$$

on the associated vector bundle.

### Example 3

For ordinary tensor products this is fancy machinery for something very trivial.

Pick once and for all elements  $t_x \in T_x$ ,  $t_y \in T_y$ . Every element in  $T_x \otimes_G V$  is then uniquely represented by some  $v \in V$  as the class of  $(t_x, v) \in T_x \times V$ .

There is a unique  $g \in G$  such that  $T_x \xrightarrow{f} T_y$  is given by

$$f(t_x) = t_y \cdot g.$$

Thus  $f \otimes_G \text{Id}_V$  is given by

$$[(t_x, v)] \xrightarrow{f \otimes_G \text{Id}_V} [(f(t_x), v)] = [(t_y \cdot g, v)] = [(t_y, \rho(g)(v))] .$$

Of course we knew this before. But now we can use the above equalizer diagrams to obtain from a representation of a 2-group  $G_2$  on a 2-vector space  $V_2$  a 2-functor

$$? \otimes_G V : \text{Tor}(G_2) \rightarrow \text{Mod}_G .$$

This is the content of the next subsection.

### 1.4.3 Associated 2-transport

In light of the discussion in section 1.4.2 it is clear what we need in order to have an associated 2-transport. Namely, given a principal 2-transport

$$\text{tra} : \mathcal{P}_2(X) \rightarrow \text{Tor}_{G_2}$$

we are looking for the associated 2-transport

$$\text{tra}_\rho : \mathcal{P}_2(X) \rightarrow \text{Tor}_{G_2} \rightarrow \text{Mod}_C$$

which is uniquely determined by the right face of a tin can diagram of the following form

So we'd better use a version of coequalizers in  $\text{Cat}$  which makes this true. . .

This is discussed in section 1.5.4. The above tin can is an instance of (4), given there.

## 1.5 Colimits in $\text{Cat}$

### 1.5.1 Introduction.

Associating 2-vector bundles crucially depends on a notion of coequalizer in  $\text{Cat}$ . Strict coequalizers in  $\text{Cat}$  are explicitly constructed in [1]. It is not a priori clear, though, that strict coequalizers are sufficient for our needs.

We want to internalize the notion of limit and colimit in  $\text{Cat}$ . Certainly some Australian category theorists know all about this. But I don't. So here I give some notes on how I would try to approach this.

### 1.5.2 Internal limits and colimits.

Let  $C$  be any category. A **diagram** in  $C$  is a functor

$$D : S \rightarrow C$$

from any small category  $S$  (a “**shape**”) to  $C$ . Let

$$D_t : 1 \rightarrow C$$

be the diagram consisting of a single object  $t \in \text{Obj}(C)$ . A **cone**  $c$  over a diagram  $D$  with **tip**  $t$  is a natural transformation

$$\begin{array}{ccc} & 1 & \\ \nearrow & \Downarrow c & \searrow^{D_c} \\ S & \xrightarrow{D} & C \end{array} .$$

Given two cones

$$\begin{array}{ccc} & 1 & \\ \nearrow & \Downarrow c & \searrow^{D_t} \\ S & \xrightarrow{D} & C \end{array} , \quad \begin{array}{ccc} & 1 & \\ \nearrow & \Downarrow c' & \searrow^{D_{t'}} \\ S & \xrightarrow{D} & C \end{array}$$

over the same diagram, a **morphism of cones**

$$c \xrightarrow{f} c'$$

is a natural transformation

$$\begin{array}{ccc} & D_t & \\ \curvearrowright & \Downarrow f & \curvearrowleft \\ 1 & & C \\ & D_{t'} & \end{array}$$

(hence nothing but a morphism  $t \rightarrow t'$  between the tips in  $C$ ) such that

$$\begin{array}{ccc} & 1 & \\ \nearrow & \Downarrow c & \searrow^{D_t} \\ S & \xrightarrow{D} & C \end{array} = \begin{array}{ccc} & 1 & \\ \nearrow & \Downarrow c' & \searrow^{D_{t'}} \\ S & \xrightarrow{D} & C \end{array} \begin{array}{ccc} & \curvearrowleft^{D_t} & \\ & \Downarrow f & \\ & \curvearrowright_{D_{t'}} & \end{array} .$$

We get a **category of cones** over the diagram  $D$  this way. The **limit** over  $D$  is (if it exists) the terminal object in that category.

Similarly for cocones and colimits.

So a cocone over  $D$  with tip  $t$  is a natural transformation

$$\begin{array}{ccc} S & \xrightarrow{D} & C \\ & \searrow & \nearrow \\ & 1 & \end{array} \begin{array}{c} c \Downarrow \\ D_t \end{array} ,$$

and a morphism of cocones

$$c \xrightarrow{f} c'$$

is a natural transformation

$$\begin{array}{ccc} & D_t & \\ & \curvearrowright & \\ 1 & \Downarrow f & C \\ & \curvearrowleft & \\ & D_{t'} & \end{array}$$

such that

$$\begin{array}{ccc} S & \xrightarrow{D} & C \\ & \searrow & \nearrow \\ & 1 & \end{array} \begin{array}{c} c \Downarrow \\ D_t \\ f \\ D_{t'} \end{array} = \begin{array}{ccc} S & \xrightarrow{D} & C \\ & \searrow & \nearrow \\ & 1 & \end{array} \begin{array}{c} c' \Downarrow \\ D_{t'} \end{array} .$$

We get a **category of cocones** this way. The **colimit** over  $D$  (if it exists) is the initial object in this category.

### 1.5.3 Limits and colimits in 2Cat

Where we had natural transformation before we now have pseudonatural transformations. These have modifications going between them.

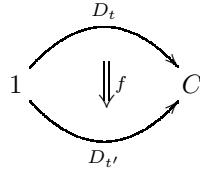
So a cocone is now a pseudonatural transformation

$$\begin{array}{ccc} S & \xrightarrow{D} & C \\ & \searrow & \nearrow \\ & 1 & \end{array} \begin{array}{c} c \Downarrow \\ D_t \end{array} .$$

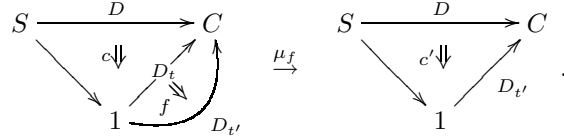
Two of these may be related by a modification

$$\begin{array}{ccc} S & \xrightarrow{D} & C \\ & \searrow & \nearrow \\ & 1 & \end{array} \begin{array}{c} c \Downarrow \\ D_t \end{array} \xrightarrow{\mu} \begin{array}{ccc} S & \xrightarrow{D} & C \\ & \searrow & \nearrow \\ & 1 & \end{array} \begin{array}{c} c' \Downarrow \\ D_t \end{array} .$$

A morphism of cocones is now a pseudonatural transformation



together with a specified isomodification



We require this assignment to be natural with respect to modifications of  $c$  in the sense that

$$\begin{array}{ccc}
 \begin{array}{c} S \xrightarrow{D} C \\ \searrow \quad \nearrow \\ 1 \end{array} \begin{array}{c} \xrightarrow{D_t} \\ \downarrow f \\ \xrightarrow{D_{t'}} \end{array} & \xrightarrow{\mu_f} & \begin{array}{c} S \xrightarrow{D} C \\ \searrow \quad \nearrow \\ 1 \end{array} \begin{array}{c} \xrightarrow{D_t} \\ \downarrow f \\ \xrightarrow{D_{t'}} \end{array} \\
 \nu_f \downarrow & & \downarrow \nu \\
 \begin{array}{c} S \xrightarrow{D} C \\ \searrow \quad \nearrow \\ 1 \end{array} \begin{array}{c} \xrightarrow{D_t} \\ \downarrow f \\ \xrightarrow{D_{t'}} \end{array} & \xrightarrow{\mu_{\tilde{f}}} & \begin{array}{c} S \xrightarrow{D} C \\ \searrow \quad \nearrow \\ 1 \end{array} \begin{array}{c} \xrightarrow{D_t} \\ \downarrow f \\ \xrightarrow{D_{t'}} \end{array}
 \end{array} \quad . \quad (3)$$

Cocones and morphisms between them form a category and the colimit (if it exists) is the initial object of that category.

### 1.5.4 Coequalizers in 2Cat

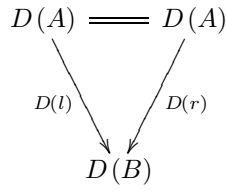
We now apply the above to coequalizers.

In this case, the shape in question is the small category

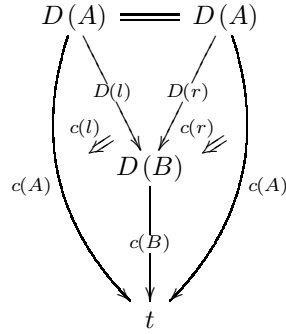
$$S = \left\{ \begin{array}{c} A = A \\ \swarrow \quad \searrow \\ l \quad r \\ \downarrow \\ B \end{array} \right\} ,$$

regarded as a 2-category with only identity 2-morphisms. A strict coequalizer would be a 2-functor on this shape which is just an ordinary functor, regarded

as a 2-functor

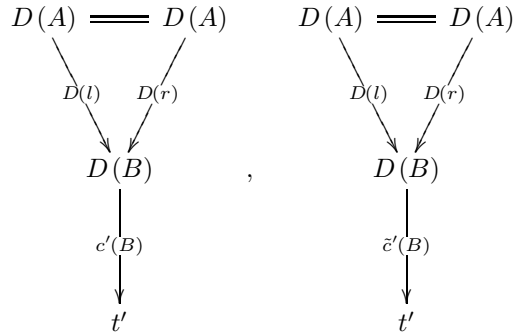


A cocone over this looks like

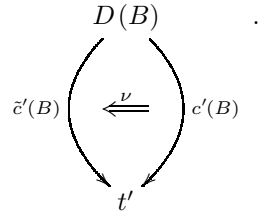


**Strict coequalizers in 2Cat** For the moment, restrict attention to the case where the 2-morphisms  $c(l)$  and  $c(r)$  above are taken to be identity 2-cells.

Then two cocones with the same tip look simply like



and a modification of cocones  $c' \xrightarrow{\nu} \tilde{c}'$  is nothing but a 2-morphism





By assumption (3), this comes with modifications  $\mu_f$  and  $\mu_{\tilde{f}}$  such that

$$\begin{array}{ccc}
 D(B) & \xrightarrow{c(B)} & t \\
 \downarrow \nu & & \downarrow \nu_f \\
 t' & \xrightarrow{\text{Id}} & t' \\
 \uparrow c'(B) & & \uparrow f(B) \\
 & & \mu_{\tilde{f}} \\
 & & \tilde{f}(B)
 \end{array}
 \quad (4)$$

2-commutes.

### 1.5.5 Enriched categories in terms of lax functors

Here I say a couple of words about how to think of enriched categories in terms of lax functors. Not directly relevant for the rest of these notes.

Let  $\mathcal{V}$  be any bicategory (usually the suspension of a monoidal category). Let  $C$  be any  $\mathcal{V}$ -enriched category. This is the same as a lax functor

$$F_C : PC \rightarrow \mathcal{V},$$

where  $PC$  is the pair groupoid of the set of objects of  $C$ .

More generally, let  $S$  be any category, then we can say that enriching  $S$  over  $\mathcal{V}$  is specifying a lax functor

$$C : S \rightarrow \mathcal{V}.$$

These lax functors naturally live in a slice-2-category, where the 1-morphisms

$$C \xrightarrow{f} C'$$

are given by 2-cells

$$\begin{array}{ccc}
 S & \xrightarrow{f_1} & S' \\
 \searrow C & \Downarrow f_2 & \swarrow C' \\
 & \mathcal{V} &
 \end{array}$$

with horizontal composition

$$C \xrightarrow{f} C' \xrightarrow{g} C''$$

given by

$$\begin{array}{ccccc}
 S & \xrightarrow{f_1} & S' & \xrightarrow{g_1} & S'' \\
 \searrow C & \Downarrow f_2 & \downarrow C' & \Downarrow g_2 & \swarrow C'' \\
 & & \mathcal{V} & &
 \end{array}$$

We find that  $f_2$  is given by 2-cells

$$\begin{array}{ccc}
 C(s) & \xrightarrow{C(s,s')} & C(s') \\
 \downarrow f_2(s) & \Downarrow f_2(s,s') & \downarrow f_2(s') \\
 C'(f_1(s)) & \xrightarrow{C'(f_1(s),f_1(s'))} & C'(f_1(s'))
 \end{array}$$

in  $\mathcal{V}$  satisfying a couple of tin can equations expressing the compatibility with composition and units.

For the special case that the vertical 1-morphisms in the above are taken to be identities, this reproduces the standard definition of functors of enriched categories.

[Next I need to define 2-morphisms in the slice category and show that they reproduce natural transformations in the enriched setup.]

## References

- [1] M. Bednarczyk, A. Borzyszkowski, W. Pawłowski, *Generalized Congruences – Epimorphisms in Cat*, TAC, **5** 11 (1999) 266-280