Nonabelian Bundle Gerbes from 2-Transport
Part I: Synthetic Bibundles

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Abstract

It is shown that 1-morphisms between 2-functors from 2-paths to the category $\mathbf{BiTor}(H)$ define bibundles with bibundle connection the way they appear in nonabelian bundle gerbes. An arrow theoretic description of bibundle connections is given using synthetic differential geometry. In a sequel to this paper this will be used to show that pre-trivializations of 2-functors to $\mathbf{BiTor}(H)$ are in bijection with fake flat nonabelian bundle gerbes.

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1 Introduction

A fiber bundle with connection is essentially a parallel transport functor from paths in base space to the transport groupoid of the bundle.

As a categorification of this fact, locally trivialized 2-transport 2-functors from 2-paths (surfaces) to certain 2-groupoids were shown in [9, 10] to encode the same cocycle data as fake flat nonabelian gerbes with connection and curving [7].

It turns out that a 2-transport 2-functor can be locally trivialized in two steps. Performing only one of these steps trivializes the 2-functor locally, but does not trivialize the 2-functor 1-morphisms which give the transitions between the local trivializations. This first step shall hence be called a **pre-trivialization** of the 2-transport 2-functor.

It was shown in [3] that pre-trivializations of 2-functors with values in $\mathbf{Vect}_1$, the monoidal category of 1-dimensional vector spaces (when regarded as a 2-category with a single object), are in bijection with *abelian bundle gerbes* [1] with connection and curving. A bundle gerbe with connection is a realization of the information contained in a proper gerbe with connection in terms of trivializable transition bundles with connection on double intersections of a good covering of base space. These transition bundles with connection are in bijection with pseudonatural transformations of local trivial transport 2-functors.

Here this result is generalized to *nonabelian bundle gerbes* (NABGs), which were defined in [2] as a generalization of the concept of an abelian bundle gerbe.

While it is straightforward to define a nonabelian bundle gerbe *without* connection and curving, the proper notion of connection in an NABG turns out to be non-obvious. The solution found in [2] justifies itself mainly in that it leads to the same cocylce data as found in [7].

We show that 2-transport 2-functors from 2-paths to the monoidal category $\mathbf{BiTor}(H)$ of bitorsors of a group H (when regarded as a 2-category with a single object), have pre-trivializations that are in bijection with "fake flat" NABGs with connection and curving.

In particular, it is shown that 2-functor 1-morphisms between local trivial 2-transport 2-functors are in bijection with H-bibundles that are equipped with a generalized notion of connection which reproduces precisely the definition found in [2].

Hence we find and investigate a purely arrow-theoretic description of bibundle connections on H-bibundles. All the curious properties which distinguish a bibundle connection from an ordinary connection are shown to be results of the fact that bibundle connections are not related to transport functors (like ordinary connections are) but to pseudonatural transformations between 2-transport 2-functors. A pseudonatural transformation has properties similar to but different from those of a proper functor.

The translation between the arrow-theory which we use and the differential-form description of bibundle connections used in [2] is done using synthetic differential calculus as described in [4, 5].

The main part of this paper is §2, which investigates the arrow-theoretic description of bibundle connections.

In order to introduce our language, ordinary bundles with connection are described in that fashion in §2.1, mainly reviewing material presented by A. Kock. Bibundle connections are then defined arrow-theoretically in §2.2 and it is shown how their synthetic differential version is equivalent to the structures defined in [2].

The remaining part §3 introduces 2-transport 2-functors with values in $\mathbf{BiTor}(H)$ and establishes the theorem which relates their 1-morphsism to bibundles with bibundle connection. In a sequel to this paper pre-trivializations of such 2-functors will be defined and will be shown to be in bijection with fake flat nonabelian bundle gerbes.

§3 makes essential use of pseudonatural transformations between 2-functors, the details of whose definition can be found in the appendix of [3].

2 Connections on Bi-Bundles, Synthetically

2.1 Connections on Bundles, Synthetically

The content of this subsection is a summary of material presented in a series of papers by A. Kock [4, 5]. We try to emphasize the graphical (arrow-theoretic) point of view even more than already done by Kock himself. On the one hand this is because we feel it makes his approach even more enjoyable. On the other hand, the emphasis on the functorial aspects of synthetic differential geometry make its categorification and hence its application to parallel transport of string rather transparent. In particular, some intricate relations concerning connection 1-forms in bibundles become obvious tautologies when expressed in graphical calculus.

2.1.1 Graphical Notation for Torsors

Given a group H, we think of it as a category with a single object "$\bullet$" and all morphisms invertible. Hence we denote a group element $h \in H$ by an arrow

$$\bullet \xrightarrow{h} \bullet \ .$$

Let (T, ℓ) be a left H-torsor[1] with left action ℓ. In order to adapt the above notation to the presence of left H-torsors, we denote an element $\rho \in T$ as

$$\bullet \xrightarrow{\rho} T \ .$$

The result of acting with $h \in H$ from the left on $\rho \in T$ is then denoted

$$\bullet \xrightarrow{\ell(h,\rho)} T \ \equiv \ \bullet \xrightarrow{h} \bullet \xrightarrow{\rho} T \ .$$

By construction, any two torsor elements ρ_1, ρ_2 differ by the left action of a unique element of H, usually called $\rho_1\bar{\rho}_2$. We write

$$\bullet \xrightarrow{\rho_1\bar{\rho}_2} \bullet \ \equiv \ \bullet \xrightarrow{\rho_1} T \xrightarrow{\bar{\rho}_2} \bullet$$

so that manifestly

$$\begin{aligned} \bullet \xrightarrow{\ell(\rho_1\bar{\rho}_2,\rho_2)} T \quad &= \quad \bullet \xrightarrow{\rho_1} T \xrightarrow{\bar{\rho}_2} \bullet \xrightarrow{\rho_2} T \\ &= \quad \bullet \xrightarrow{\rho_1} T \ . \end{aligned}$$

Now consider another left H-torsor (T', ℓ'). Any torsor morphism $T \xrightarrow{\phi} T'$ is specified (non-uniquely) by a pair $(\rho, \rho') \in T \times T'$ with $\rho' = \phi(\rho)$. It is consistent with the above to write

$$T \xrightarrow{\phi} T' \ \equiv \ T \xrightarrow{\bar{\rho}} \bullet \xrightarrow{\rho'} T'$$

[1] For us, a (left) H-torsor is a a (left) H-space which is isomorphic to H as a (left) H space. Sometimes "H-torsor" is used instead to mean more generally any H-space on which H acts freely, like a principal H-bundle.

such that we can graphically compute as follows:

$$\begin{aligned}
\bullet \xrightarrow{\phi(\rho)} T' &= \bullet \xrightarrow{\rho} T \xrightarrow{\phi} T' \\
&= \bullet \xrightarrow{\rho} T \xrightarrow{\bar{\rho}} \bullet \xrightarrow{\rho'} T' \\
&= \bullet \xrightarrow{\rho'} T'
\end{aligned}$$

All this suggests to think of elements $\bullet \xrightarrow{h} \bullet$ of H together with all left H-torsor elements $\bullet \xrightarrow{\rho} T$ as well as their formal inverses $T \xrightarrow{\bar{\rho}} \bullet$ (in the above sense) as morphisms of a groupoid whose objects are all left H-torsors (T, ℓ) together with the object $\bullet$.

In fact, if we identify $\bullet = H$ and think of H as a left H-torsor over itself, this groupoid is simply isomorphic to the category of left H-torsors. All we have done above is essentially to note that torsor morphisms $H \longrightarrow (T, \ell)$ are in bijection with elements of T in a way that makes the above identifications viable.

Still, the above way of thinking proves to be very convenient in the following. We shall review below how A. Kock uses this approach [4] to describe (synthetically) fiber bundles with connection. After that we extent his discussion to bitorsors and apply it to connections on bibundles.

2.1.2 The Comprehensive Groupoid of a Principal Bundle

Let $\begin{array}{c} E \\ \downarrow{\scriptstyle \pi} \\ M \end{array}$ be a principal left H-bundle. This means each fiber E_x is a left H-torsor. (Compare the comment in the footnote above.) In the notation of §2.1.1 this means that E is a collection of *morphisms*

$$E = \left\{ \bullet \xrightarrow{\rho} E_x \right\} .$$

In the spirit of the above discussion it makes good sense to consider the collection of formal inverses of all these morphisms:

$$E^{-1} \equiv \left\{ E_x \xrightarrow{\bar{\rho}} \bullet \right\} .$$

The groupoid

$$\mathrm{Trans}\,(E) \equiv EE^{-1} \equiv \left\{ E_x \xrightarrow{\bar{\rho}_1} \bullet \xrightarrow{\rho_2} E_y \right\}$$

whose objects are all the fibers of E and whose morphisms are all the torsor morphisms between these, shall be called the **transport groupoid** of E. Similarly we can form the groupoid

$$H \simeq E^{-1}E \equiv \left\{ \bullet \xrightarrow{\rho_1} E_x \xrightarrow{\bar{\rho}_2} \bullet \right\} ,$$

which is nothing but the structure group H itself.

Definition 1 *The union of these four categories*

$$E \oplus E^{-1} \oplus EE^{-1} \oplus E^{-1}E$$

is a groupoid associated with the bundle $E \xrightarrow{\pi} M$ *which is called in [4] the* **comprehensive groupoid** *of* E*.*

2.1.3 Connections and Connection Forms

Let $\mathcal{P}_1(M)$ be the groupoid of thin-homotopy classes of paths in M.

Consider a functor $\mathrm{tra}_\nabla : \mathcal{P}_1(M) \to \mathrm{Trans}(E)$. which associates fibers of the bundle with points in M and "parallel transport" torsor-morphisms with paths in M.

$$\mathrm{tra}_\nabla \left(x \xrightarrow{\gamma} y \right) = E_x \xrightarrow{\mathrm{tra}_\nabla(\gamma)} E_y$$

Using synthetic differential geometry as in [5] we can "differentiate" such a parallel transport functor to obtain something that is no longer a functor, but just a graph map.

Definition 2 *Given a transport functor* tra_∇*, its restriction to infinitesimnal paths yields a graph map denoted*

$$\nabla : M_{(1)} \longrightarrow \mathrm{Trans}(E)$$

from pairs of infinitesimally close points in M *to* $\mathrm{Trans}(E)$*. This we call the* **(principal) connection** *of the parallel transport* tra_∇*.*

We write

$$x \rightsquigarrow y$$

for an infinitesimal path from x to y, where x and y are neighbours in the *first neighborhood of the diagonal* of M [5]. The result of acting with ∇ on such an infinitesimal path is

$$\nabla\left(x \rightsquigarrow y \right) \equiv E_x \xrightarrow{\nabla(x,y)} E_y \ .$$

Here the arrow on the right is an infinitesimal arrow in the transport groupoid.

Definition 3 *Given a connection* ∇ *as in def. 2, we get a group valued 1-form, the* **connection 1-form***, on the total space of the bundle. This is a graph map*

$$\omega : E_{(1)} \longrightarrow H$$

defined by

$$\omega : \left(e_x \rightsquigarrow e_y \right) \mapsto \left(\bullet \xrightarrow{e_x} E_x \xrightarrow{\nabla(x,y)} E_y \xrightarrow{\bar{e}_y} \bullet \right)$$

for all $E \ni e_x \overset{\pi}{\longmapsto} x$ *and* $E \ni e_y \overset{\pi}{\longmapsto} y$.

Note that ω is functorial in the sense that

$$\begin{aligned} \omega : \quad & \left(e_x \rightsquigarrow e_y \rightsquigarrow e_z \right) \\ \mapsto \quad & \left(\bullet \xrightarrow{e_x} E_x \xrightarrow{\nabla(x,y)} E_y \xrightarrow{\bar{e}_y} \bullet \xrightarrow{e_y} E_y \xrightarrow{\nabla(y,z)} E_z \xrightarrow{\bar{e}_z} \bullet \right) \\ = \quad & \left(\bullet \xrightarrow{e_x} E_x \xrightarrow{\nabla(x,y)\circ\nabla(y,z)} E_z \xrightarrow{\bar{e}_y} \bullet \right) \end{aligned}$$

The following crucial properties of the connection 1-form are trivial tautologies in the arrow-theoretic calculus:

Proposition 1 ([4])

1. *For all* $h \in H$ *such that* $e_x \sim e_y \Rightarrow e_x \sim \ell(h, e_y)$ *we have*
$$\omega(e_x, \ell(h, e_y)) = \omega(e_x, e_y)\; h^{-1} .$$
2. *For all* $h \in H$ *we have*
$$\omega(\ell(h, e_x), \ell(h, e_y)) = h\, \omega(e_x, e_y)\; h^{-1} .$$

We give the proof just in order to emphasize that in arrow notation there is essentially nothing to prove.

Proof.

1.
$$\omega(\ell(h, e_x), e_y) = \bullet \xrightarrow{e_x} E_x \xrightarrow{\nabla(x,y)} E_y \xrightarrow{\bar{e}_y} \bullet \xrightarrow{h^{-1}} \bullet$$

2.
$$\omega(\ell(h, e_x), \ell(h, e_y)) = \bullet \xrightarrow{h} \bullet \xrightarrow{e_x} E_x \xrightarrow{\nabla(x,y)} E_y \xrightarrow{\bar{e}_y} \bullet \xrightarrow{h^{-1}} \bullet$$

$\square$

These two properties are nothing but the synthetic version of the familiar properties of a connection 1-form. Recall

Definition 4 (e.g. [6]) *A connection 1-form on a principal* H*-bundle* E *is a 1-form*

$$c \in \Omega^1(E, \mathrm{Lie}(H))$$

satisfying the following two conditions. Let

$$\begin{array}{rcl} \mathbb{R} \supset (-\epsilon, \epsilon) & \to & H \\ t & \mapsto & h(t) \end{array}$$

and denote tangent vectors to curves by square brackets, then

1.

$$c\left([\ell(h(t)\,,e_x)]\right) = -[h(t)]$$

2.

$$\ell\left(h,\cdot\right)^* c = \mathrm{Ad}_h c\,.$$

2.1.4 Curvature

Definition 5

$$E_x \xrightarrow{\exp(F_\nabla)(x,y,z)} E_x \;\equiv\; E_x \xrightarrow{\quad\nabla(x,y)\quad} E_y \xrightarrow{\quad\nabla(y,z)\quad} E_z \xrightarrow{\quad\nabla(z,x)\quad} E_x$$

Definition 6

$$\begin{aligned}
\mathbf{d}\omega\left(e_x,e_y,e_z\right) &= \omega\left(e_x,e_y\right)\omega\left(e_y,e_z\right)\omega\left(e_z,e_x\right) \\
&= \bullet \xrightarrow{e_x} E_x \xrightarrow{\quad\nabla(x,y)\quad} E_y \xrightarrow{\quad\nabla(y,z)\quad} E_z \xrightarrow{\quad\nabla(z,x)\quad} E_x \xrightarrow{\bar{e}_x} \bullet \\
&= \bullet \xrightarrow{e_x} E_x \xrightarrow{\exp(F_\nabla)(x,y,z)} E_x \xrightarrow{\bar{e}_x} \bullet
\end{aligned}$$

2.2 Bibundle Connections

In our terminology, a bibundle is a bundle whose fibers are bitorsors. (Compare the footnote above). In §2.2.1 we recall some elementary facts about bitorsors, mainly in order to establish our notation. The category $\mathbf{BiTor}(H)$ of H-bitorsors is equivalent to the 2-group associated to the crossed module $(\, H \xrightarrow{t} \mathrm{Aut}(H)\,)$ [8]. This explains the appearance of crossed module structures in computations with bitorsors. It is due to this relation of bitorsors of an ordinary group to the structure of a 2-group that bibundles are sort of half-way between ordinary bundles and 2-bundles.

2.2.1 Bitorsors

An H-bitorsor (T, ℓ, r) is a left and right H-torsor such that the left action commutes with the right action. An H-bitorsor morphism is a map between bitorsors that commutes with both these actions. The category whose objects are H-bitorsors and whose morphisms are bitorsor morphisms shall be called $\mathbf{BiTor}(H)$.

The group H itself is a bitorsor, with the obvious left and right action on itself. Regarded as a bitorsor in this sense, we write (H, Id) or $\bullet_{\mathrm{Id}}$ for H. The "Id" indicates that there are more general bitorsor structures on H. Let G be any group and

$$\alpha : \; G \longrightarrow \mathrm{Aut}(H)$$

a group homomorphism from G to the automorphism group of H. Then (H, ℓ, r_g) with

$$\begin{aligned} \ell(h, h_0) &\equiv h\, h_0 \\ r_g(h, h_0) &\equiv h_0\, \alpha(g)(h) \end{aligned}$$

defines a bitorsor structure on H for every $g \in G$.

Definition 7 *H equipped with this bitorsor structure will be denoted (H, g) or simply $\bullet_g$.*

A bitorsor morphism between bitorsors coming from H itself

$$\phi : \; \bullet_g \longrightarrow \bullet_{g'}$$

is clearly fixed by specifying the image of the neutral element, $h \equiv \phi(\mathrm{Id})$. Hence we write

$$\bullet_g \xrightarrow{h} \bullet_{g'} \; .$$

One easily checks that this can respect both left and right action if and only if

$$g \;=\; t(h)\, g' \tag{1}$$

where

$$t : \; H \longrightarrow G$$

is some group homomorphism such that

$$\alpha\,(t\,(h))\,(h_0) = h\,h_0\,h^{-1}\,.$$

Given two bitorsors T, T', their **tensor product** is the set of equivalence classes

$$T \otimes T' = \{(\rho, \rho') \mid (r\,(h, \rho)\,, \rho') \sim (\rho, \ell\,(h, \rho'))\}$$

with the obvious left and right action inherited from T and T':

$$\begin{aligned} \ell\,(h, (\rho, \rho')) &\equiv (\ell\,(h, \rho)\,, \rho') \\ r\,(h, (\rho, \rho')) &\equiv (\rho, r\,(h, \rho'))\ . \end{aligned}$$

One checks that under this tensor product we have

$$\begin{aligned} \bullet_g \otimes \bullet_{g'} &\simeq \bullet_{gg'} \\ (h, 1) &\simeq h \end{aligned}$$

and

$$\begin{tikzcd} \bullet_{g_1} \arrow[dd, "h_1"'] & \otimes & \bullet_{g_2} \arrow[dd, "h_2"] & & \bullet_{g_1 g_2} \arrow[dd, "h_1\alpha(g_1)(h_2)"] \\ & & & = & \\ \bullet_{g'_1} & \otimes & \bullet_{g'_2} & & \bullet_{g'_1 g'_2} \end{tikzcd}\ . \tag{2}$$

Hence $\mathbf{BiTor}\,(H)$ is in fact a *monoidal categegory.* Just like every monoid can be regarded as a category with a single object, every monoidal category can be regarded as a (weak) 2-category with a single object.

Definition 8 *As a weak 2-category,* $\mathbf{BiTor}\,(H)$ *has*

- *a single object* $\bullet$
- *a morphism* $\begin{tikzcd} \bullet \arrow[r, bend left, "T"] & \bullet \end{tikzcd}$ *for every H-bitorsor T,*
- *a 2-morphism* $\begin{tikzcd} \bullet \arrow[r, bend left=50, "T"] \arrow[r, bend right=50, "T'"'] \arrow[r, phantom, "\Downarrow\phi"] & \bullet \end{tikzcd}$ *for every bitorsor morphism* $T \xrightarrow{\phi} T'$ *.*

Horizontal composition of 2-morphisms is the tensor product described above. Vertical composition of 2-morphisms is simply composition of the corresponding bitorsor morphisms.

Definition 9 *Since the left and right H-action on a bitorsor are both free and transitive, there is, for every $\rho \in (T, \ell, r)$ a bijection*

$$\phi_\rho^T : H \to H$$

such that

$$\ell(h,\rho) = r\left(\phi_\rho^T(h), \rho\right) .$$

By the associativity of the group action on the torsor this is clearly an automorphism

$$\phi_\rho^T \in \mathrm{Aut}(H) .$$

Proposition 2

1. *For any $\phi = \phi^T$ we have*

$$\phi_{\ell(h,\rho)} = \phi_\rho \, t(h)^{-1} .$$

(Here the product on the right is in $\mathrm{Aut}(H)$.)

2. *For $\phi = \phi^{(H,g)}$ we have*

$$\phi_h = \alpha(t(h)\ g)^{-1} .$$

3. *For $\phi = \phi^{T\otimes T'}$ we have*

$$\phi_{(\rho,\rho')} = \phi'_{\rho'}\phi_\rho .$$

4. *For $T \xrightarrow{f} T'$ we have*

$$\phi'_{f(\rho)} = \phi_\rho .$$

Definition 10 *An H-bibundle $\begin{array}{c} E \\ \downarrow{\scriptstyle\pi} \\ M \end{array}$ is a principal left H-bundle which is also a principal right H-bundle such that left and right action commute.*

Hence the fibers E_x of a principal H-bibundle are H-bitorsors as defined above.

Definition 11 *According to def. 9 every H-bibundle E comes with a map*

$$\begin{array}{rcl} \phi : E & \to & \mathrm{Aut}(H) \\ \rho_x & \mapsto & \phi_{\rho_x}^{E_x} \end{array}$$

that maps each point in the total space to the H-automorphism that relates the left and right action of H at that point.

2.2.2 Connections and Connection Forms on Bibundles

It seems tempting to define parallel transport on a bibundle E to be a functor $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow \mathrm{Trans}_{\mathrm{bi}}(E)$. However, this is different from a transport functor on the same bundle regarded as just a left (or right) principal bundle, since the definition of the transport groupoid differs in both cases. For a bibundle the morphisms in the groupoid are bitorsor morphisms, which are more restricted than mere left (or right) torsor morphisms.

One consequence of this is that the holonomy of such a naive parallel transport (and hence the curvature of the corresponding connnection) would be restricted to lie in the center of H. Namely consider any loop $x \xrightarrow{\gamma} x$ in M, and assume without restriction of generality that the fiber over x has been identified with H itself, $E_x = \bullet_g$. Then $\bullet_g \xrightarrow{\mathrm{tra}(\gamma)} \bullet_g = \bullet_g \xrightarrow{h} \bullet_g$ for some $h \in H$. But by (1) it follows then that $t(h) = \mathrm{Id}$, which implies that h is in the center of H.

A good generalization of the concept of connection applicable to bibundles, which evades the above restrictions, has been introduced and studied in [2]. In this section we give an arrow-theoretic formulation of these bibundle connections.

It turns out that the naive notion of parallel transport on a bibundle differs from the one we shall study in that the latter is not a functor, but a *pseudo-natural transformation* between certain 2-functors. All this will be clarified in §3, where we *derive* the concept of a bibundle connection from certain automorphisms of 2-functors with values in $\mathbf{BiTor}(H)$. If without this understanding of the origin of the concept of bibundle connections their definition below appears ad hoc, the reader is invited to skip to §3 and come back to this point afterwards.

Definition 12 *A* **bibundle parallel transport** *on an H-bibundle $E \xrightarrow{\pi} M$ is*

1. *two functors*

$$\begin{aligned} \mathrm{tra}_A &: \mathcal{P}_1(M) \longrightarrow \mathrm{Aut}(H) \\ \mathrm{tra}_{A'} &: \mathcal{P}_1(M) \longrightarrow \mathrm{Aut}(H) \end{aligned}$$

   and

2. *a* map

$$\mathrm{tra}_{\mathrm{bi}} : \mathcal{P}_1(M)^{\mathrm{op}} \longrightarrow \mathbf{BiTor}(E)$$

such that

$$\mathrm{tra}_{\mathrm{bi}}\left(\begin{tikzcd} y \arrow[d, "\gamma^{\mathrm{op}}"'] \\ x \end{tikzcd}\right) \quad = \quad \begin{tikzcd} \bullet_{\mathrm{tra}_A(\gamma)} \otimes E_y \arrow[d, "\mathrm{tra}_{\mathrm{bi}}(\gamma)"] \\ E_x \otimes \bullet_{\mathrm{tra}_{A'}(\gamma)} \end{tikzcd}$$

for all $\gamma \in \mathcal{P}_1(M)$ *and*

$$\mathrm{tra}_{\mathrm{bi}}\left(\begin{tikzcd} z \arrow[d, "\gamma_2^{\mathrm{op}}"'] \\ y \arrow[d, "\gamma_1^{\mathrm{op}}"'] \\ x \end{tikzcd}\right) \quad = \quad \begin{tikzcd}[column sep=tiny] \bullet_{\mathrm{tra}_A(\gamma_1)} \otimes \bullet_{\mathrm{tra}_A(\gamma_2)} \otimes E_z \arrow[d, "\mathrm{Id}"'] \arrow[d, shift left=12ex, "\mathrm{tra}_{\mathrm{bi}}(\gamma_2)"] \\ \bullet_{\mathrm{tra}_A(\gamma_1)} \otimes E_y \otimes \bullet_{\mathrm{tra}_{A'}(\gamma_2)} \arrow[d, "\mathrm{tra}_{\mathrm{bi}}(\gamma_1)"'] \arrow[d, shift left=12ex, "\mathrm{Id}"] \\ E_x \otimes \bullet_{\mathrm{tra}_{A'}(\gamma_1)} \otimes \bullet_{\mathrm{tra}_{A'}(\gamma_2)} \end{tikzcd}$$

for all composable $\gamma_1, \gamma_2 \in \mathcal{P}_1(M)$

Remarks.

1. Obviously $\mathrm{tra}_{\mathrm{bi}}$ is not a functor, though not totally unrelated to a (contravariant) functor.

2. Recall from §2.2.1 that $\bullet_{\mathrm{tra}_A(\gamma)}$ denotes the group H regarded as a bitorsor over itself with the obvious action from the left and with the action from the right twisted by the automorphism $\mathrm{tra}_A(\gamma)$.

3. According to prop. 2 tra_A and $\mathrm{tra}_{A'}$ are not independent. Given $\mathrm{tra}_{\mathrm{bi}}$ and tra_A the functor $\mathrm{tra}_{A'}$ is fixed. The precise relation is the content of prop. 4 below.

As before for connections in ordinary bundles, the bibundle parallel transport gives rise to a graph map

$$\nabla_{\mathrm{bi}} : \; M_{(1)} \longrightarrow \mathbf{BiTor}(H) \;,$$

the **bibundle connection**. Now there are in addition graph maps

$$A : \; M_{(1)} \longrightarrow H$$

and

$$A' : \; M_{(1)} \longrightarrow H$$

coming from tra_A and $\mathrm{tra}_{A'}$, respectively. These are synthetic differential 1-forms with values in H [5].

Definition 13 *Given a bibundle connection ∇_{bi} as above, we get an H-valued 1-form, the* **bibundle connection 1-form***, on the total space of the bundle. This is a graph map*

$$\omega_{\mathrm{bi}} : \; E_{(1)} \longrightarrow H$$

defined by

$$\omega_{\mathrm{bi}}\,(e_y, e_x) \quad \equiv \quad \omega_{\mathrm{bi}} \left(\begin{array}{c} e_y \\ \rightsquigarrow \\ e_x \end{array} \right) \quad \equiv \quad \begin{array}{c} \bullet_{A(x,y)} \otimes \bullet_g \\ \mathrm{Id} \downarrow \quad\; \downarrow e_y \\ \bullet_{A(x,y)} \otimes E_y \\ \downarrow \nabla_{\mathrm{bi}}(y,x) \\ E_x \otimes \bullet_{A'(x,y)} \\ \bar{e}_x \downarrow \quad\; \downarrow \mathrm{Id} \\ \bullet_{g'} \otimes \bullet_{A'(x,y)} \end{array}$$

for all $E \ni \; e_x \stackrel{\pi}{\longmapsto} x$ and $E \ni \; e_y \stackrel{\pi}{\longmapsto} y$.

The construction of the bibundle connection 1-form ω_{bi} is completely analogous to that of the ordinary connection 1-form in def. 3. Accordingly, it essentially satisfies the same two relations as those of prop. 1, except for a certain twist by tra_A induced by the peculiar nature of parallel transport in bibundles.

Proposition 3 *Let ω_{bi} be a bibundle connection 1-form as a above.*

1. *For all $h \in H$ such that $e_y \sim e_x \;\Rightarrow\; e_y \sim \ell(h, e_x)$ we have*

$$\omega\,(e_y, \ell\,(h, e_x)) = \omega\,(e_y, e_x)\;\; h^{-1}\,.$$

2. *For all $h \in H$ we have*

$$\omega\,(\ell\,(h, e_y)\,, \ell\,(h, e_x)) = \alpha\,(A\,(y, x))\,(h)\;\; \omega\,(e_y, e_x)\;\; h^{-1}\,.$$

Proof. The second property arises as follows:

$$
\omega\left(\ell\left(h,e_y\right),\ell\left(h,e_x\right)\right) \quad = \quad
\begin{tikzcd}
\bullet_{A(x,y)} \arrow[d, "\mathrm{Id}"'] & \otimes & \bullet_{t(h)g} \arrow[d, "h"] \\
\bullet_{A(x,y)} \arrow[d, "\mathrm{Id}"'] & \otimes & \bullet_{g} \arrow[d, "e_y"] \\
\bullet_{A(x,y)} & \otimes \arrow[d, "\nabla(y,x)"] & E_y \\
E_x \arrow[d, "\bar{e}_x"'] & \otimes & \bullet_{A'(x,y)} \arrow[d, "\mathrm{Id}"] \\
\bullet_{g'} \arrow[d, "h^{-1}"'] & \otimes & \bullet_{A'(x,y)} \arrow[d, "\mathrm{Id}"] \\
\bullet_{t(h)g'} & \otimes & \bullet_{A'(x,y)}
\end{tikzcd}
\quad \overset{(2)}{=} \quad
\begin{tikzcd}
\bullet_{A(x,y)} & \otimes \arrow[d, "\alpha(A(x,y))(h)"] & \bullet_{g} \\
\bullet_{A(x,y)} \arrow[d, "\mathrm{Id}"'] & \otimes & \bullet_{g} \arrow[d, "e_y"] \\
\bullet_{A(x,y)} & \otimes \arrow[d, "\nabla(y,x)"] & E_y \\
E_x \arrow[d, "\bar{e}_x"'] & \otimes & \bullet_{A'(x,y)} \arrow[d, "\mathrm{Id}"] \\
\bullet_{g'} & \otimes \arrow[d, "h^{-1}"'] & \bullet_{A'(x,y)} \\
\bullet_{t(h)g'} & \otimes & \bullet_{A'(x,y)}
\end{tikzcd}
$$

The proof of the first property follows from the same diagrams but with the topmost morphisms discarded. □

These two properties are nothing but the synthetic version of the properties of bibundle connection 1-forms as discussed in [2]. There the pair $(\omega_{\mathrm{bi}}, A)$ is called a "2-connection" (def. 8).

Definition 14 ([2]) *A bibundle connection 1-form on an H-bibundle E is a 1-form*

$$c \in \Omega^1\left(E, \mathrm{Lie}\left(H\right)\right)$$

together with a 1-form

$$A \in \Omega^1\left(M, \mathrm{Lie}\left(\mathrm{Aut}\left(H\right)\right)\right)$$

satisfying the following two conditions. Let

$$\begin{array}{rcl} \mathbb{R} \supset (-\epsilon, \epsilon) & \to & H \\ t & \mapsto & h\left(t\right) \end{array}$$

and denote tangent vectors to curves by square brackets, then

1.
$$c\left(\left[\ell(h\left(t\right), e_x)\right]\right) = -[h\left(t\right)]$$

2.
$$\ell\left(h, \cdot\right)^* c = \mathrm{Ad}_h c - \pi^*\left(h\ (d\alpha)\left(A\right)\left(h^{-1}\right)\right)\ .$$

There is only one $\mathrm{Lie}(\mathrm{Aut}(H))$-valued 1-form appearing here, while in def. 13 there were two of them, A and A'. As remarked above, A' is fixed once (ω_{bi},A) is:

Proposition 4

$$A'(x,y) = \phi(e_x)\, t(\omega_{\mathrm{bi}}(e_y,e_x))\, A(x,y)\, \phi(e_y)^{-1}$$

Proof. Consider a transport of the form

$$\bullet_{A(x,y)} \otimes E_y \ni\ (\mathrm{Id}, e_y) \xmapsto{\nabla_{\mathrm{bi}}(y,x)} (e_x, \mathrm{Id})\ \in E_x \otimes \bullet_{A'(x,y)}\ .$$

The source and target elements can always be brought into the given form by using the property of the tensor product of bitorsors. Since $\nabla_{\mathrm{bi}}(y,x)$ is a bitorsor morphism we have, according to prop. 2, the relation

$$\phi^{\bullet_{A(x,y)}\otimes E_y}_{(\mathrm{Id},e_y)} = \phi^{E_x\otimes \bullet_{A'(x,y)}}_{(e_x,\mathrm{Id})} = \phi^{E_x\otimes \bullet_{A'(x,y)}}_{\nabla_{\mathrm{bi}}(y,x)(\mathrm{Id},e_y)}\,.$$

This looks complicated, but simplifies using a little graphical calculus. First use the following

Lemma 1

$$\nabla_{\mathrm{bi}}(y,x)(\mathrm{Id},e_y) = \ell(\omega_{\mathrm{bi}}(y,x)\,,\,(e_x,\mathrm{Id}))\ .$$

Proof. This is essentially the definition of ω_{bi}:

$$\nabla_{\mathrm{bi}}(y,x)(\mathrm{Id},e_y) = \begin{tikzcd}
\bullet_{A(x,y)} \arrow[d, "\mathrm{Id}"'] & \otimes & \bullet_g \arrow[d, "e_y"] \\
\bullet_{A(x,y)} & \otimes \arrow[d, "\nabla_{\mathrm{bi}}(y,x)"] & E_y \\
E_x & \otimes & \bullet_{A'(x,y)}
\end{tikzcd}
=
\begin{tikzcd}
\bullet_{A(x,y)} \arrow[d, "\mathrm{Id}"'] & \otimes & \bullet_g \arrow[d, "e_y"] \\
\bullet_{A(x,y)} & \otimes \arrow[d, "\nabla_{\mathrm{bi}}(y,x)"] & E_y \\
E_x \arrow[d, "\bar{e}_x"'] & \otimes & \bullet_{A'(x,y)} \arrow[d, "\mathrm{Id}"] \\
\bullet_{g'} \arrow[d, "e_x"'] & \otimes & \bullet_{A'(x,y)} \arrow[d, "\mathrm{Id}"] \\
\bullet_{g'} & \otimes & \bullet_{A'(x,y)}
\end{tikzcd}
= \ell(\omega_{\mathrm{bi}}(y,x)\,,\,(e_x,\mathrm{Id}))$$

□

Hence we are left with

$$\begin{aligned}
\phi^{\bullet_{A(x,y)}\otimes E_y}_{(\mathrm{Id},e_y)} &= \phi^{E_x\otimes \bullet_{A'(x,y)}}_{\ell(\omega_{\mathrm{bi}}(e_y,e_x),(e_x,\mathrm{Id}))} \\
&\overset{(2)}{=} \phi^{E_x\otimes \bullet_{A'(x,y)}}_{(e_x,\mathrm{Id})}\, t\left(\omega_{\mathrm{bi}}(e_y,e_x)^{-1}\right)\ .
\end{aligned}$$

Using prop. 2 once again, the tensor products in this expression can be taken apart to get

$$\phi_{e_y}^{E_y} \phi_{\mathrm{Id}}^{\bullet A(x,y)} = \phi_{\mathrm{Id}}^{\bullet A'(x,y)} \phi_{e_x}^{E_x} t \left(\omega_{\mathrm{bi}} \left(e_y, e_x \right)^{-1} \right) .$$

With yet another application of prop. 2 we can replace

$$\phi_{\mathrm{Id}}^{\bullet A(x,y)} = A(x,y)^{-1}$$

and

$$\phi_{\mathrm{Id}}^{\bullet A'(y,x)} = A'(x,y)^{-1} ,$$

and with the notation introduced in def. 11 the above expression finally turns into a more agreeable form

$$\phi(e_y)\, A(x,y)^{-1} = A'(x,y)^{-1}\, \phi(e_x)\, t(\omega_{\mathrm{bi}}(e_y, e_x))^{-1} .$$

This is the equation to be proven, up to some trivial manipulations. □

Remark. Using the synthetic exterior derivative for group-valued differential 0-forms, $\mathbf{d}\phi(e_x, e_y) \equiv \phi(e_x)^{-1} \phi(e_y)$, the statement of proposition 4 can be equivalently rewritten as

$$A'(x,y) = \phi(e_x)\, t(\omega_{\mathrm{bi}}(e_y, e_x))\, \phi(e_x)^{-1} \;\; \phi(e_x)\, A(x,y)\, \phi(e_x)^{-1} \;\; \mathbf{d}\phi(e_x, e_y)^{-1} .$$

This is manifestly the synthetic version of equation (53) in [2].

This relation between A and A' is crucial for understanding the tensor product of parallel transport in two bibundles. This is the content of the next subsection.

2.2.3 Tensor Product of Bibundles with Bibundle Connection

Definition 15 *Given H-bibundles $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$, their* **product bibundle** $E_1 \otimes_M E_2 \xrightarrow{\pi} M$ *is the H-bibundle with fibers*

$$(E_1 \otimes_M E_2)_x = (E_1)_x \otimes (E_2)_x .$$

Definition 16 *Given H-bibundles $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$ with bibundle parallel transport $(\mathrm{tra}_{\mathrm{bi}1}, \mathrm{tra}_{A_1}, \mathrm{tra}_{A'_1})$ and $(\mathrm{tra}_{\mathrm{bi}1}, \mathrm{tra}_{A_1}, \mathrm{tra}_{A'_1})$, respectively such that the matching condition*

$$\mathrm{tra}_{A'_1} = \mathrm{tra}_{A_2}$$

if fulfilled, then the **product bibundle transport** $(\mathrm{tra}_{\mathrm{bi}\,12}, \mathrm{tra}_{A_1}, \mathrm{tra}_{A_2'})$ *on* $E_1 \otimes_M E_2$ *is defined as*

$$\mathrm{tra}_{\mathrm{bi}\,12}\left(\begin{tikzcd} y \arrow[d, "\gamma^{\mathrm{op}}"'] \\ x \end{tikzcd}\right) = \begin{tikzcd}[column sep=tiny] \bullet_{\mathrm{tra}_{A_1}(\gamma)} \otimes (E_1)_y \arrow[d, "\mathrm{tra}_{\mathrm{bi}\,1}(\gamma)"] & \otimes & (E_2)_y \arrow[d, "\mathrm{Id}"] \\ (E_1)_x \otimes \bullet_{\mathrm{tra}_{A_1'}(\gamma)} \arrow[d, equal] & \otimes & (E_2)_x \\ (E_1)_x \arrow[d, "\mathrm{Id}"] & \otimes & \bullet_{\mathrm{tra}_{A_2}(\gamma)} \otimes (E_2)_y \arrow[d, "\mathrm{tra}_{\mathrm{bi}\,2}(\gamma)"] \\ (E_1)_x & \otimes & (E_2)_x \otimes \bullet_{\mathrm{tra}_{A_2'}(\gamma)} \end{tikzcd} .$$

In §3 it is shown that this is nothing but the composition of pseudonatural transformations of 2-functors with values in $\mathbf{BiTor}(H)$.

2.2.4 Bibundle Curvature

$$\begin{tikzcd} \bullet_{A(z,x)A(x,y)A(y,z)} \otimes E_z \arrow[d, "\exp\left(F_{\nabla_{\mathrm{bi}}}\right)(z,y,x)"] \\ E_z \otimes \bullet_{A'(z,x)A'(x,y)A'(y,z)} \end{tikzcd} \equiv \begin{tikzcd} \bullet_{A(z,x)A(x,y)A(y,z)} \otimes E_z \arrow[d, "\nabla(z,y)"] \\ \bullet_{A(z,x)A(x,y)} \otimes E_y \otimes \bullet_{A'(y,z)} \arrow[d, "\nabla(y,x)"] \\ \bullet_{A(z,x)} \otimes E_x \otimes \bullet_{A'(x,y)A'(y,z)} \arrow[d, "\nabla(x,z)"] \\ E_z \otimes \bullet_{A'(z,x)A'(x,y)A'(y,z)} \end{tikzcd}$$

[...]

3 2-Transport with Values in $\mathbf{BiTor}(H)$

Definition 17 *A* **2-bundle with 2-transport with values in $\mathbf{BiTor}(H)$** *is a 2-functor*

$$\mathrm{tra}: \mathcal{P}_2(M) \longrightarrow \mathbf{BiTor}(H) .$$

Here $\mathbf{BiTor}(H)$ is regarded as a 2-category, as described in def. 8 (p. 10).

Hence a 2-bundle with 2-transport with values in $\mathbf{BiTor}(H)$ sends every surface $S \in \mathrm{Mor}_2(\mathcal{P}_2(M))$ to a pair T, T' of H-bitorsors associated to the boundary of that surface, together with a bitorsor morphism $T \xrightarrow{\phi} T'$ between these associated to the interior of the surface:

$$\mathrm{tra}\left(\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow [S]"] & y \end{tikzcd}\right) \equiv \begin{tikzcd} \bullet \arrow[r, bend left, "(T,\ell,r)_{\gamma_1}"] \arrow[r, bend right, "(T',\ell',r')_{\gamma_2}"'] \arrow[r, phantom, "\Downarrow \phi_S"] & \bullet \end{tikzcd} \in \mathrm{Mor}_2(\mathbf{BiTor}(H)) ,$$

For establishing the relation with (nonabelian) bundle gerbes, the concept of a *trivial* 2-transport with values in $\mathbf{BiTor}(H)$ plays an important role. Trivialization here corresponds to identifying all H-torsors with H itself. But since we are dealing with *bi*torsors such an identification involves in addition a choice of an automorphism of H, which twists the right action of H on itself (see §2.2.1). Hence for a 2-transport with values in $\mathbf{BiTor}(H)$ to be trivial, all the H-bitorsors (T, ℓ, r) it associates with 1-paths have to be to of the form $(T, \ell, r) = (H, g)$, as in def. 7. The assignment of g has to be functorial.

Definition 18 *A 2-bundle with 2-transport with values in* $\mathbf{BiTor}(H)$ *is called* **trivial** *precisely if the H-bitorsors it assigns to paths in M are H itself equipped with a twisted right action (def. 7), i.e. precisely if there exists a functor*

$$\mathrm{tra}_A : \mathcal{P}_1(M) \to \mathrm{Aut}(H)$$

for all $S \in \mathrm{Mor}_2(\mathcal{P}(M))$ such that

$$\mathrm{tra}\left(\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow [S]"] & y \end{tikzcd}\right) \equiv \begin{tikzcd} \bullet \arrow[r, bend left, "(H,\mathrm{tra}_A(\gamma_1))"] \arrow[r, bend right, "(H,\mathrm{tra}_A(\gamma_2))"'] \arrow[r, phantom, "\Downarrow h(S)\equiv\mathrm{tra}(S)"] & \bullet \end{tikzcd} \in \mathrm{Mor}_2(\mathbf{BiTor}(H)) ,$$

1-Morphisms of bitor-2-bundles with 2-transport are pseudonatural transformations of the respective 2-functors. 2-morphisms are modifications of these.

We now state our main result, which relates 1-morphisms of trivial 2-transport with values in $\mathbf{BiTor}(H)$ to bibundles with bibundle connections.

Proposition 5

1. *1-morphisms* $\mathrm{tra} \xrightarrow{\phi} \mathrm{tra}'$ *of trivial 2-bundles with* $\mathbf{BiTor}(H)$*-2-transport are in bijection with* H*-bibundles with bibundle parallel transport (def. 12) on* M.

2. *The composition* $\mathrm{tra}_1 \xrightarrow{\phi_1} \mathrm{tra}_2 \xrightarrow{\phi_2} \mathrm{tra}_3$ *of 1-morphisms of trivial 2-bundle with bitor-2-transport corresponds to the tensor product of the corresponding bibundles with bibundle transport (def. 16).*

3. *2-morphisms*
$$\begin{tikzcd} \mathrm{tra} \arrow[r, bend left=50, "\phi_1"{name=U}] \arrow[r, bend right=50, "\phi_2"'{name=D}] & \mathrm{tra}' \arrow[Rightarrow, from=U, to=D, "\mathcal{A}"] \end{tikzcd}$$
of trivial bitor-2-transport 2-functors are in bijection with bibundle isomorphisms of the bibundles associated with ϕ_1 *and* ϕ_2.

(Compare this with the respective proposition on 2-transport with values in $\mathbf{Vect}_1$ stated in [3].)

Proof.

1. The 1-morphism ϕ, being a pseudonatural transformation (see the appendix of [3] for details), is a map
$$\mathrm{Mor}_1(\mathcal{P}_1(M)) \ni \quad x \xrightarrow{\gamma} y \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, "\mathrm{tra}_A(\gamma)"] \arrow[d, "\phi(x)"'] \arrow[dr, phantom, "\swarrow \phi(\gamma)"] & \bullet \arrow[d, "\phi(y)"] \\ \bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma)"'] & \bullet \end{tikzcd} \in \mathrm{Mor}_2(\mathbf{BiTor}(H))$$
such that
$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{tra}_A(\gamma_1)"] \arrow[d, "\phi(x)"'] \arrow[dr, phantom, "\swarrow \phi(\gamma_1)"] & \bullet \arrow[r, "\mathrm{tra}_A(\gamma_2)"] \arrow[d, "\phi(y)"] \arrow[dr, phantom, "\swarrow \phi(\gamma_2)"] & \bullet \arrow[d, "\phi(z)"] \\ \bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma_1)"'] & \bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma_2)"'] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, "\mathrm{tra}_A(\gamma_1 \cdot \gamma_2)"] \arrow[d, "\phi(x)"'] \arrow[dr, phantom, "\swarrow \phi(\gamma_1 \cdot \gamma_2)"] & \bullet \arrow[d, "\phi(z)"] \\ \bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma_1 \cdot \gamma_2)"'] & \bullet \end{tikzcd} .$$
Using the translation between the interpretation of $\mathbf{BiTor}(H)$ as a monoidal 1-category and its interpretation as a weak 2-category with a single object given in def. 8 one sees that this are precisely the two conditions for a bibundle with bibundle parallel transport as defined in def. 12.

2. The composition of pseudonatural transformations

$$\mathrm{tra}_1 \xrightarrow{\phi_1} \mathrm{tra}_2 \xrightarrow{\phi_2} \mathrm{tra}_3$$

is by definition given by the map

$$\mathrm{Mor}_1(\mathcal{P}_1(M)) \ni \; x \xrightarrow{\gamma} y \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, "\mathrm{tra}_{A_1}(\gamma)"] \arrow[d, "\phi(x)"'] \arrow[dr, phantom, "\swarrow \phi(\gamma)"] & \bullet \arrow[d, "\phi(y)"] \\ \bullet \arrow[r, "\mathrm{tra}_{A'_2}(\gamma)"'] & \bullet \end{tikzcd} \quad \equiv \quad \begin{tikzcd} \bullet \arrow[r, "\mathrm{tra}_{A_1}(\gamma)"] \arrow[d, "\phi_1(x)"'] \arrow[dr, phantom, "\swarrow \phi_1(\gamma)"] & \bullet \arrow[d, "\phi_1(y)"] \\ \bullet \arrow[r, "\mathrm{tra}_{A'_1}(\gamma)", "\mathrm{tra}_{A_2}(\gamma)"'] \arrow[d, "\phi_2(x)"'] \arrow[dr, phantom, "\swarrow \phi_2(\gamma)"] & \bullet \arrow[d, "\phi_2(y)"] \\ \bullet \arrow[r, "\mathrm{tra}_{A'_2}(\gamma)", "\mathrm{tra}_{A_3}(\gamma)"'] & \bullet \end{tikzcd}$$

This is precisely the definition of the tensor product of bibundle parallel transports ϕ_1 and ϕ_2 given in def. 16. □

Hence the curious definition of parallel transport in a bibundle is seen to be due to the fact that it is not a functor, but a pseudonatural transformation.

Proposition 6 *(relation of curving with bibundle curvature)*

Proof.

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{tra}(\gamma_1)"] \arrow[d, "\phi(x)"'] \arrow[dr, phantom, "\swarrow \phi(\gamma_1)"] & \bullet \arrow[d, "\phi(y)"] \\ \bullet \arrow[r, "\mathrm{tra}(\gamma_1)"{name=U}] \arrow[r, bend right=60, "\mathrm{tra}(\gamma_2)"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "\mathrm{tra}(S)"] \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, bend left=60, "\mathrm{tra}(\gamma_1)"{name=U}] \arrow[r, "\mathrm{tra}(\gamma_2)"'{name=D}] \arrow[d, "\phi(x)"'] \arrow[dr, phantom, "\swarrow \phi(\gamma_2)"] & \bullet \arrow[d, "\phi(y)"] \arrow[from=U, to=D, Rightarrow, "\mathrm{tra}(S)"] \\ \bullet \arrow[r, "\mathrm{tra}(\gamma_2)"] & \bullet \end{tikzcd} \; .$$

□

[. . .]

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