∞-Categories and ∞-Operads

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Abstract

Notes taken in a talk by **I. Moerdijk** at *Higher Structures in Mathematics and Physics*, Bernoulli Center, EPFL, Lausanne, Nov. 2008. Notes pretty literally reproduce what was on the board and what was said. But all mistakes are mine.

category C: a set of objects and compositions

$$C(s_0, s_1) \times C(s_1, s_2) \times \cdots C(s_{n-1}, s_n) \to C(s_0, s_n)$$

$$1_s \in C(s, s)$$

enriched category (in $\mathcal{E}$): each $C(s, s')$ is an object in $\mathcal{E}$, where $\mathcal{E}$ is any monoidal category

A 2-category is a category enriched in $\mathcal{E} = \mathrm{Cat}$, the category of small categories

An $(n+1)$-category is a category enriched in $\mathcal{E} = n\mathrm{Cat}$.

A category with a set S of objects can be thought of as a particular algebraic structure, with a type for each pair of objects (s, s').

These algebrauc structures are goverened by algebraic operads.

In other words, we can write an $\mathcal{E}$-enriched category C as an algebra for a colored operad A_S.

$\mathrm{Cat}_S(\mathcal{E}) = \mathrm{Alg}_{\mathcal{E}}(A_S) = \mathrm{Hom}(A_S, \mathcal{E})$ (S the set of objects, fixed)

we can assemble this for all S to get $\int_S \mathrm{Cat}_S(\mathcal{E}) = \int_S \mathrm{Hom}(A_s, \mathcal{E})$ (think of integral sign as Grothedieck construction or homotopy colimit)

so inductive definition:

$$\mathrm{Cat}_{n+1} = \int_S \mathrm{Hom}(A_S, \mathrm{Cat}_n)$$

everything strict so far

<u>Goal</u>: give a similar inductive definition of <u>weak</u> higher categories

$$\mathrm{wCat}_{n+1} = \int_S \mathrm{Hom}(A_S, \mathrm{wCat}_n)$$

purpose of lecture: provide context for Hom and integral sign; context will be analogous to simplicial sets

how to interpret this Hom and $\int_S$?

the idea to keep in mind is that it will be analogous to the usual internal Hom and $\int$ = rothedieck construction in categories or usual internal Hom and hocolim construction in simplicial sets

very algebraic view on categories

another way of looking at a category is as a space or a simplicial set

$$\begin{tikzcd}
\mathrm{Cat} \arrow[r, bend left, "N"] & \mathrm{sSet} \arrow[l, bend left, "\tau"] \arrow[r, bend left, "|-|"] & \mathrm{Spaces} \arrow[l, bend left, "\mathrm{sing}"]
\end{tikzcd}$$

N is the nerve, τ the left adjoint to the nerve.

recall the horn inclusions

$$\Lambda^k[n] \hookrightarrow \Delta[n]$$

($k = 0, \cdots, n$) in simplicial sets, its geometric realization is the inclusion into the standard n-simplex of all the faces except the one opposite the kth vertex

[** picture of $\Lambda^0[2]$ **]

A Kan complex is a simplicial set X for which

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "\alpha"] \arrow[d] & X \\ \Delta[n] \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

any α extends to a β

If $X0N(C)$ is the nerve of a category, it will have this extension property for $0 < k < n$.

[** picture of $\Lambda^1[2]$ **]

A simplicial set is called a weak Kan complex if it has this extension property (non-uniquely) – or has been called: was introduced by Boardman and Vogt long ago, but has had a revival lately under infinitely many other names.

These weak Kan complexes are studied under various names, namely

- inner Kan complexes;
- quasicategories (Joyal);
- ∞-categories (Lurie);
- $(\infty, 1)$-categories

Think of that as categories with composition which is not uniquely defined and everything holds up to homotopy. Idea: a weak higher category in which all higher cells (higher than 1-cells) are equivalences.

Joyal's claim: you can do all of category theory (limits, colimits, Grothendieck construction) with quasicategories.

This stuff is coming nicely off the ground.

Basic theory of ∞-categories:

Theorem 1 [A. Joyal]: there is a closed model structure on sSets in which the cofibrations are the monos and the fibrant objects are exactly the ∞-categories.

(question from audience: so what are the weak equivalences? they must be non-standard)

answer: A map $A \to B$ is a weak equivalence iff for any ∞-category F the map $\tau\mathrm{hom}(B, F) \to \tau\mathrm{hom}(A, F)$ is an equivalence of categories for every F.

theorem 2: there are Quillen equivalences sCat $\simeq$ sSet $\simeq$ sSpaces

in the middle: simplicial sets with the Joyal model structure; on the left categories enriched in simoplicial sets (alternatively: enriched in topological spaces) with model structure due to ulia Bergner; on the right simplicial spaces (alternatively: bisimplicial sets) with Bousfield localization of Reedy model structure

proof in big book by Lurie but parts of it also by others, such as Julia Bergner;

there is a category dSets of dendroidal sets which includes simplicial sets, which allows an inductive definition of higher categories and which admits homotopy theory, fitting into a diagram

$$\begin{tikzcd}
\mathrm{Cat} \arrow[r, bend left, "N"] \arrow[d, hook] & \mathrm{sSets} \arrow[l, "\tau"'] \arrow[d, hook, "i_!"] \\
\mathrm{operads} \arrow[r, "N"] & \mathrm{dSets} \arrow[l, bend left, "\tau"]
\end{tikzcd}$$

Properties:

1. there is a monoidal cllosed structure $\otimes$, hom on dSets and $i_!$ is strong monoidal functor
2. there is a closed model structure on dSets for which the fibrant objects are exactly the <u>∞-operads</u> = dendroidal weak Kan complexes

   Remark: Joyals theorem is an immediate consequence of this, because dSets$/U$ = sSets for suitable U
3. sOperads $\rightleftarrows$ dSets $\rightleftarrows$ dSpaces on the right Quillen equivalence, on the left proof in progress: joint work with various people: Weiss, Cisisnki and Clemens Berger

now definition of dendroidal sets:

dSets = presheaves on a small category Ω (just as simplicial sets are presheaves on Δ)

objects of Ω are trees: finite trees, not necessarily planar, with a specific root thought of as the output and with a number of inputs, there can be vertical with valence 0

$\Delta \subset \Omega$ by sending $[n]$ to the straight-line tree

arrows in Ω :

- there are automorphisms of trees (since these are not necessarily planar)
- there are degeneracies σ_v which collaps unary vertices v
- there are internal faces ∂_e: one can contract internal edges, the face map goes from the contracted tree to the original one
- external faces ∂_w, ∂_u: from cutting the tree in two along one edge, such that one vertex is cut off (this gives a minimal set of generators)

each object T defines a representable dendroidal set $\Omega[T]$, each inner edge $e \in T$ define an inner hor $\Lambda^e[T] \subset \Omega[T]$ which is the union of all faces except the one given by contracting e

if P is an operad, the nerve $N(P)$ is a dendroidal set such that $N(P)_T = T$-shaped picture in P: label edges of T by colors of P, vertices of T by operations in P

$$\begin{tikzcd}
\Lambda^e[T] \arrow[r] \arrow[d] & N(P) \\
\Omega[T] \arrow[ur, dashed, "\text{unique}"']
\end{tikzcd}$$

so nerves of operads are sstrict inner Kan complexes in this sense

Def: A dendroidal set is called an ∞-operad if it has this extension property (not necessarily uniquely)

tensor product of dendroidal sets from shuffles of trees

enough to define on representables:

$$\Omega[S] \otimes \Omega[T] = \bigcup_i \Omega[R_i]$$

where R_i ranges over shuffles of trees (definition by "percolation")
two arXiv articles: one with I. Weiss, one with C. Gerber ("Reedy" in the title)

questions from audience:

- Q: is this the set-version of ∞-operads in the dg-context? A: well, maybe roughly, but not really
- Q: what is the intuition behind the definition of the weak equivalences on dendroidal sets? A: (didn't capture reply)