

∞ -Categories and ∞ -Operads

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Abstract

Notes taken in a talk by **I. Moerdijk** at *Higher Structures in Mathematics and Physics*, Bernoulli Center, EPFL, Lausanne, Nov. 2008. Notes pretty literally reproduce what was on the board and what was said. But all mistakes are mine.

category C : a set of objects and compositions

$$C(s_0, s_1) \times C(s_1, s_2) \times \cdots \times C(s_{n-1}, s_n) \rightarrow C(s_0, s_n)$$

$$1_s \in C(s, s)$$

enriched category (in $\mathcal{E}$): each $C(s, s')$ is an object in $\mathcal{E}$, where $\mathcal{E}$ is any monoidal category

A 2-category is a category enriched in $\mathcal{E} = \text{Cat}$, the category of small categories

An $(n+1)$ -category is a category enriched in $\mathcal{E} = n\text{Cat}$.

A category with a set S of objects can be thought of as a particular algebraic structure, with a type for each pair of objects (s, s') .

These algebraic structures are governed by algebraic operads.

In other words, we can write an $\mathcal{E}$ -enriched category C as an algebra for a colored operad A_S .

$\text{Cat}_S(\mathcal{E}) = \text{Alg}_{\mathcal{E}}(A_S) = \text{Hom}(A_S, \mathcal{E})$ (S the set of objects, fixed)

we can assemble this for all S to get $\int_S \text{Cat}_S(\mathcal{E}) = \int_S \text{Hom}(A_S, \mathcal{E})$ (think of integral sign as Grothendieck construction or homotopy colimit)

so inductive definition:

$$\text{Cat}_{n+1} = \int_S \text{Hom}(A_S, \text{Cat}_n)$$

everything strict so far

Goal: give a similar inductive definition of weak higher categories

$$\text{wCat}_{n+1} = \int_S \text{Hom}(A_S, \text{wCat}_n)$$

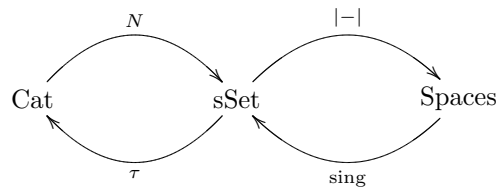
purpose of lecture: provide context for Hom and integral sign; context will be analogous to simplicial sets

how to interpret this Hom and $\int_S$?

the idea to keep in mind is that it will be analogous to the usual internal Hom and $\int$ = Grothendieck construction in categories or usual internal Hom and hocolim construction in simplicial sets

very algebraic view on categories

another way of looking at a category is as a space or a simplicial set



N is the nerve, τ the left adjoint to the nerve.
recall the horn inclusions

$$\Lambda^k[n] \hookrightarrow \Delta[n]$$

($k = 0, \dots, n$) in simplicial sets, its geometric realization is the inclusion into the standard n -simplex of all the faces except the one opposite the k th vertex

[** picture of $\Lambda^0[2]$ **]

A Kan complex is a simplicial set X for which

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{\alpha} & X \\ \downarrow & \nearrow \beta & \\ \Delta[n] & & \end{array}$$

any α extends to a β

If $X \mathcal{O} N(C)$ is the nerve of a category, it will have this extension property for $0 < k < n$.

[** picture of $\Lambda^1[2]$ **]

A simplicial set is called a weak Kan complex if it has this extension property (non-uniquely) – or has been called: was introduced by Boardman and Vogt long ago, but has had a revival lately under infinitely many other names.

These weak Kan complexes are studied under various names, namely

- inner Kan complexes;
- quasicategories (Joyal);
- ∞ -categories (Lurie);
- $(\infty, 1)$ -categories

Think of that as categories with composition which is not uniquely defined and everything holds up to homotopy. Idea: a weak higher category in which all higher cells (higher than 1-cells) are equivalences.

Joyal's claim: you can do all of category theory (limits, colimits, Grothendieck construction) with quasicategories.

This stuff is coming nicely off the ground.

Basic theory of ∞ -categories:

Theorem 1 [A. Joyal]: there is a closed model structure on $\mathbf{sSets}$ in which the cofibrations are the monos and the fibrant objects are exactly the ∞ -categories.

(question from audience: so what are the weak equivalences? they must be non-standard)

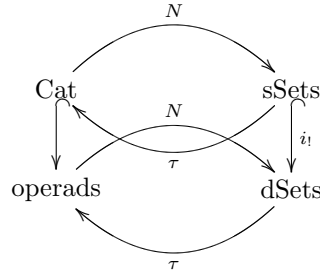
answer: A map $A \rightarrow B$ is a weak equivalence iff for any ∞ -category F the map $\tau \text{hom}(B, F) \rightarrow \tau \text{hom}(A, F)$ is an equivalence of categories for every F .

theorem 2: there are Quillen equivalences $\mathbf{sCat} \simeq \mathbf{sSet} \simeq \mathbf{sSpaces}$

in the middle: simplicial sets with the Joyal model structure; on the left categories enriched in simplicial sets (alternatively: enriched in topological spaces) with model structure due to Julia Bergner; on the right simplicial spaces (alternatively: bisimplicial sets) with Bousfield localization of Reedy model structure

proof in big book by Lurie but parts of it also by others, such as Julia Bergner;

there is a category $\mathbf{dSets}$ of dendroidal sets which includes simplicial sets, which allows an inductive definition of higher categories and which admits homotopy theory, fitting into a diagram



Properties:

1. there is a monoidal closed structure $\otimes$, hom on dSets and $i_!$ is strong monoidal functor
2. there is a closed model structure on dSets for which the fibrant objects are exactly the ∞ -operads = dendroidal weak Kan complexes
 Remark: Joyals theorem is an immediate consequence of this, because $\text{dSets}/U = \text{sSets}$ for suitable U
3. $\text{sOperads} \xrightarrow{\quad} \text{dSets} \xrightarrow{\quad} \text{dSpaces}$ on the right Quillen equivalence, on the left proof in progress: joint work with various people: Weiss, Cisinski and Clemens Berger

now definition of dendroidal sets:

dSets = presheaves on a small category Ω (just as simplicial sets are presheaves on Δ)

objects of Ω are trees: finite trees, not necessarily planar, with a specific root thought of as the output and with a number of inputs, there can be vertical with valence 0

$\Delta \subset \Omega$ by sending $[n]$ to the straight-line tree

arrows in Ω :

- there are automorphisms of trees (since these are not necessarily planar)
- there are degeneracies σ_v which collas unary vertices v
- there are internal faces ∂_e : one can contract internal edges, the face map goes from the contracted tree to the original one
- external faces ∂_w, ∂_u : from cutting the tree in two along one edge, such that one vertex is cut off (this gives a minimal set of generators)

each object T defines a representable dendroidal set $\Omega[T]$, each inner edge $e \in T$ define an inner face $\Lambda^e[T] \subset \Omega[T]$ which is the union of all faces except the one given by contracting e

if P is an operad, the nerve $N(P)$ is a dendroidal set such that $N(P)_T = T$ -shaped picture in P : label edges of T by colors of P , vertices of T by operations in P

$$\begin{array}{ccc} \Lambda^e[T] & \longrightarrow & N(P) \\ \downarrow & \nearrow \text{unique} & \\ \Omega[T] & & \end{array}$$

so nerves of operads are strict inner Kan complexes in this sense

Def: A dendroidal set is called an ∞ -operad if it has this extension property (not necessarily uniquely) tensor product of dendroidal sets from shuffles of trees enough to define on representables:

$$\Omega[S] \otimes \Omega[T] = \bigcup_i \Omega[R_i]$$

where R_i ranges over shuffles of trees (definition by “percolation”)

two arXiv articles: one with I. Weiss, one with C. Gerber (“Reedy” in the title)

questions from audience:

- Q: is this the set-version of ∞ -operads in the dg-context? A: well, maybe roughly, but not really
- Q: what is the intuition behind the definition of the weak equivalences on dendroidal sets? A: (didn’t capture reply)