## Module Categories and internal Bimodules

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#### Abstract

A theorem by Ostrik says that under some conditions every module category of a monoidal category C is equivalent to a category of modules internal to C. I note that the 2-category **BiMod**(C) of bimodules internal to C sits inside the 2-category  ${}_{\mathcal{C}}\mathbf{Mod}$  of module categories over C:

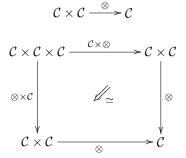
 $\mathbf{BiMod}(\mathcal{C}) \subset {}_{\mathcal{C}}\mathbf{Mod}$ 

in a certain sense. Ostrik's theorem suggests the conjecture that, when it applies, we actually have an equivalence of 2-categories.

 $\operatorname{BiMod}(\mathcal{C}) \simeq {}_{\mathcal{C}}\operatorname{Mod}.$ 

#### **Definition 1**

1. A 2-monoid or monoidal category C is a coherent monoid

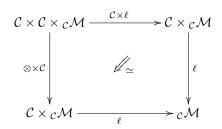


in Cat.

2. A left **2-module** or left module category  $_{\mathcal{C}}\mathcal{M}$  is a coherent left module

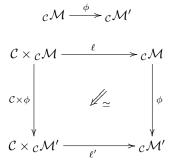
$$\mathcal{C} \times_{\mathcal{C}} \mathcal{M} \xrightarrow{\ell}_{\mathcal{C}} \mathcal{M}$$

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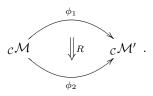
in Cat.

3. A morphism of left C-modules is a coherent morphism of left modules



in Cat, hence a functor.

4. A 2-morphism of left C-modules is a natural transformation



- 5. The 2-category of left C modules is the sub-2-category  $_{\mathcal{C}}Mod$  of Cat whose
  - objects are left C-modules
  - morphisms are morphisms of left C-modules
  - 2-morphisms are 2-morphisms of left C-modules.

#### Example 1

Let  $A \in \mathcal{C}$  be a monoid internal to the 2-monoid  $\mathcal{C}$ 

$$A \otimes A \xrightarrow{m} A$$
.

Let  $\mathbf{Mod}_A$  be the category of right A-modules internal to  $\mathcal{C}$ . For any morphism

$$\begin{array}{ccc}
N_A \\
\downarrow_f & \in & \operatorname{Mor}\left(\mathbf{Mod}_A\right) \subset \operatorname{Mor}\left(\mathcal{C}\right) \\
N'_A
\end{array}$$

and any morphism  $U \xrightarrow{g} V \in Mor(\mathcal{C})$  we get a new morphism

$$U \otimes N_A$$

$$\downarrow_{g \otimes f} \in \operatorname{Mor}(\mathbf{Mod}_A)$$

$$\bigvee_{V \otimes N'_A}$$

in a way that is clearly functorial. This makes  $\mathbf{Mod}_A$  into a left  $\mathcal C\text{-module}$ 

$$\ell$$
 :  $\mathcal{C} \times \mathbf{Mod}_A \longrightarrow \mathbf{Mod}_A$ 

$$\begin{pmatrix} U & N_A \\ \downarrow & \downarrow \\ g & \times & \downarrow \\ \psi & \psi \\ V & N'_A \end{pmatrix} \xrightarrow{U \otimes N_A} U \otimes N_A \\ \mapsto & \downarrow \\ \psi \\ V \otimes N'_A \end{pmatrix}$$

Coherence of this left action is inherited from the coherence of the associator in  $\mathcal{C}$ .

**Theorem 1 (Ostrik** [1]) Let C be a category which is

- $\bullet$  monoidal
- semisimple
- rigid
- has finitely many irreducible objects
- has an irreducible unit object.

Let  ${}_{\mathcal{C}}\mathcal{M}$  be a module category over  $\mathcal C$  which is

- $\bullet \ semisimple$
- indecomposable.

Then there exists an algebra object  $A \in \mathcal{C}$  which is

- semisimple
- $\bullet \ indecomposable$

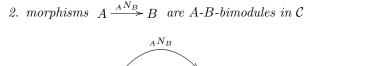
such that  ${}_{\mathcal{C}}\mathcal{M}$  is equivalent to the category  $\mathbf{Mod}_A$  of internal right A-modules:

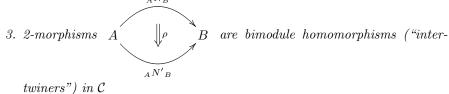
 $\exists A \in \mathcal{C} : _{\mathcal{C}} \mathcal{M} \simeq \mathbf{Mod}_A.$ 

**Remark.** Every monoidal category contains the trivial algebra object 11, the tensor unit, equipped with the trivial product  $1 \otimes 1 \longrightarrow 1$ . Every object of  $\mathcal{C}$  may be regarded as a 11-11 bimodule, and  $\otimes$  may be regarded as the tensor product over  $1 \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$ . In the same vein, every right A-module  $N_A$  in  $\mathcal{C}$  may be regarded as a 11-A-bimodule  $\mathbb{I} N_A$  internal to  $\mathcal{C}$ .

**Definition 2** Given a 2-monoid C, the (weak) 2-category of bimodules in C, BiMod(C), is the (weak) 2-category whose

1. objects are algebra objects A in C





and where horizontal composition is given by the tensor product  $\otimes_B$  of bimodules, while vertical composition is the composition of homomorphisms of bimodules.

Remark. We write

$$_{A}\mathbf{Mod}_{B} \equiv \operatorname{Hom}_{\mathbf{BiMod}(\mathcal{C})}(A, B)$$
.

In particular<sup>1</sup>

$$\mathcal{C} = \begin{tabular}{ll} & \mathcal{C} & = \begin{tabular}{ll} & \mathbf{Mod}_1 \\ & \mathbf{Mod}_A & = \begin{tabular}{ll} & \mathbf{Mod}_A \\ & & \mathbf{AMod} & = \begin{tabular}{ll} & \mathbf{Mod}_1 \\ & & \mathbf{Mod}_1 \end{tabular}. \end{tabular}$$

Horizontal composition in  $\mathbf{BiMod}(\mathcal{C})$  gives functors

 ${}_{A}\mathbf{Mod}_{B} \times {}_{B}\mathbf{Mod}_{C} \xrightarrow{\otimes_{B}} {}_{A}\mathbf{Mod}_{C} \ .$ 

The coherently weak associativity of these functors makes all  ${}_{A}\mathbf{Mod}_{A}$  into 2monoids, all categories  ${}_{A}\mathbf{Mod}_{B}$  into left  ${}_{A}\mathbf{Mod}_{A}$ -modules and all categories  ${}_{B}\mathbf{Mod}_{A}$  into right  ${}_{A}\mathbf{Mod}_{A}$ -modules, for all monoids A, B internal to C.

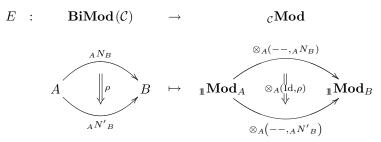
<sup>&</sup>lt;sup>1</sup>More precisely, we should write  ${}_{A}\mathbf{Mod}_{B}(\mathcal{C})$  in order to indicate the ambient 2-monoid  $\mathcal{C}$ . For our purposes however we can fix once and for all some 2-monoid  $\mathcal{C}$  and hence notationally suppress the depence of everything on this choice.

## Example 2

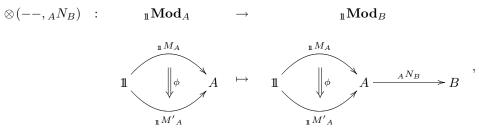
The left 2-action from example 1 can now equivalently be written as

 $\ell = \otimes_{\mathbb{1}} : {}_{\mathbb{1}}\mathbf{Mod}_{\mathbb{1}} \times {}_{\mathbb{1}}\mathbf{Mod}_A \to {}_{\mathbb{1}}\mathbf{Mod}_A \,.$ 

**Definition 3** Define the following map



Here the notation on the right is supposed to mean the following. The functor  $\otimes (--,{}_AN_B)$  acts as



and the natural transformation  $\otimes_A (\mathrm{Id}, \rho)$  is given by the map

$$\operatorname{Obj}(\mathbf{1} \operatorname{\mathbf{Mod}}_A) \ni \mathbf{1} M_A \mapsto \left( \begin{array}{c} \mathbb{1} & \overset{AN_B}{\longrightarrow} & \overset{AN_B}{\longrightarrow} \\ \mathbb{1} & \overset{AN_B}{\longrightarrow} & A & \overset{AN_B}{\longrightarrow} \\ & & & & & \\ & & & & & \\ & & & & \\ & &$$

which makes the naturality squares

$$\begin{array}{c|c} \mathbb{I} M_A \otimes_A {}_A N_B & \xrightarrow{\operatorname{Id} \otimes_A \rho} & \mathbb{I} M_A \otimes_A {}_A N'_B \\ & & & & \\ \phi \otimes_A \operatorname{Id} & & & & \\ & & & & & \\ \mathbb{I} M'_A \otimes_A {}_A N_B & \xrightarrow{} & \mathbb{I} M'_A \otimes_A {}_A N'_B \end{array}$$

commute.

### Proposition 1 E is a 2-functor.

Proof. Follows from the exchange law in  $\mathbf{BiMod}(\mathcal{C})$ .

**Remark.** The 2-functor E is clearly injective on objects as well as on 1- and 2-morphisms. Hence it "embeds" **BiMod**(C) into  $_{\mathcal{C}}$ **Mod**. So in any case we have

$$\mathbf{BiMod}(\mathcal{C}) \subset {}_{\mathcal{C}}\mathbf{Mod}$$

in some suitable sense of inclusion of 2-categories.

But Ostrik's theorem (theorem 1) says that if C is semisimple, rigid, has finitely many irreducible objects and an irreducible unit object, then E is also surjective on objects, up to equivalence. This motivates the following

**Conjecture 1** If C has all the properties listed in theorem 1, then E is an equivalence of 2-categories.

I don't yet have a proof for this. But I think one would have to follow Ostrik's proof of theorem 1 on p. 10 of [1] and use functoriality of the internal Hom.

# References

[1] V. Ostrik, Module Categories, weak Hopf Algebras and Modular Invariants, available as math.QA/0111139