TFTs with defects and higher categories

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Abstract

Notes taken in a talk by **Lazaroiu** at *Higher Structures in Mathematics and Physics*, Bernoulli Center, EPFL, Lausanne, Nov. 2008. Notes pretty literally reproduce what was on the board and what was said. But all mistakes are mine.

TFTDs

enrichments of 2d TFTs on oriented Riemann surfaces, bulk theireiews,without bundary, in the sense considered by Atiyah and Segal,

2-dimensional surface, oriented (unoriented possible) punctures, i.e. boundary circles, but no boundaries (but can be done, too, of course)

2d TFT (Atiyah-Segal)

consider categeory of Cob cobordisms, 2d oriented, forms a symmetric monoidal category

TFT is strict monoidal functor

$$F : \mathrm{Cob} \to \mathcal{V}$$

for $(\mathcal{V}, \otimes)$ some other strict symmetric monoidal category.

$(\mathrm{Cob}, \sqcup)$ where $\sqcup$ is essentially disjoint union (but beware of collars)

so

$$\mathrm{Obj(Cob)} = \text{finite disjoint unions of circles}$$

$$\mathrm{Mor(Cob)} = \{\text{Riemann surfaces}\}$$

everything up to orientation preserving diffeomorphisms

for instance $\mathcal{V} = \mathrm{FinVect}_k$.

generalize this by allowing more general classes of objects and morphisms, by decorating these and introducing labels.

aim: obtain protocategory, to be explained

0.1 TFTD

Cobd

category of 2d cobordsims with defects

$$\mathrm{Mor(Cobd)} = \{ \text{ cob } \Sigma \text{ as before but with smoothly embedded oriented curve } \Gamma, \text{ not nec. connected. } \partial\Sigma \subset \partial\Sigma \ \}$$

up to orientation preserving diffeos of Σ and isotopies of Γ in Σ rel $\partial\Sigma$

$\forall \gamma \in \Pi_0(\Gamma) : (\Sigma, -\gamma, \Gamma - \{\gamma\}) := (\Sigma, +\gamma, \Gamma - \{\gamma\})$

we always require that γ meets Σ transversely

$\pi_0(\Gamma) = \text{finite}$

Γ is called the <u>defect</u>.

$$\mathrm{Obj(Cobd)} = \text{ finite disjoint unions of marked circles up to orientation peserving diffeos}$$

(** pictures **)
$(\mathrm{Cobd}, \sqcup)$ is strict monoidal category
a TFT with defects, TFTD, is a strict monoidal functor

$$F : \mathrm{Cobd} \to \mathcal{V}$$

there is a more complicated operation called fusion of defects.
<u>decorations</u>
$(g, -)$ = graph *with involution* $\bar{(\cdot)}$

$$g_1 \overset{s}{\underset{t}{\rightrightarrows}} g_0$$

write $g_1(a, b)$ for the set of arrows from a to b
let Cobd_g be cobordisms with defects as before
color elements of $\pi_0(\Sigma - \Gamma)$ with elemnts of g_0 and decorate elements of $\pi_0(\Gamma)$ by chosen elements of g_1
+
let

$$(\gamma, f) = (-\gamma, \bar{f})$$

(recall bar denotes involution in the graph)
(** pictures **)
so symmetric monoidal functor on that is TFTD with fusion over g

$$\gamma_1, \gamma_2 \in \pi_0(\Gamma_v)$$

$$(\gamma_1, f_1) \overset{\text{isotopy}}{\sim} (\gamma_2, f_2)$$

so fusion means merging locally two parallel defect lines
definition: a <u>protocategory</u> is a pair $E = (g, \circ)$ where g is a graph and $\circ$ a composition operation on its arrows
such that for every object there is a singled out endomorphism called the <u>protounit</u> of that object
(so we do not require ordinary units!)
now $(\mathrm{Cobd}_E, \sqcup)$ defect cobordisms with fusion over E
so TFT with defects and fusion is a symm. monoidal functor on that.
notice that we have a map

$$\mathrm{Cobd}_g \to \mathrm{Cobd}_E$$

0.2 operadic description

scholium
follow Australian school and say:
colored operad = multicategory
classical operad = multiateghory with single object
morphisms of of (colored) operad = morphisms of multicats = multifunctor
for all $a_1, \cdots, a_n, a \in \mathrm{Obj}(O)$
$\mathrm{Hom}(a_1, \cdots, a_n, a)$ is the usual Hom space of colored operads
(** usual associativity condition etc. pp. **)
notice that we have/require a unit operation on every object.
(** pictures **)
representable multicategories
$(\mathcal{V}, \otimes)$ = monoidal $\Rightarrow$ mult(ν)
$\mathrm{Obj}(\mathrm{Mult}(\mathcal{V})) = \mathrm{Ob}(\mathcal{V})$

$$\mathrm{Hom}(a_1, \cdots, a_m, a) = \mathrm{Hom}_{\mathrm{Mult}(\mathcal{V})}(a_1 \otimes \cdots a_m, a)$$

def: a • $\mathcal{V}$-valued algebra over a (colored) operad O is a multifunctor $F : O \to \mathrm{Mult}(\mathcal{V})$

$$\{F : O \to \mathrm{Mult}(\mathcal{V})\} =: A(O, \mathcal{V})$$

restrict attention to multipants (topologically spheres with many incoming boundary circles cut out)
(** pictures **)
"enrichment of little disk operad"
gives colored operads $P_g \to P_E$
= defects of operads over g, E
this is vast generalization of the planar operads of Vaughan Jones related to
defect algebras
without fusion over g, $F : P_g \to \mathrm{Mult}(\mathcal{V})$ with fusion over g, $F : P_E \to \mathrm{Mult}(\mathcal{V})$, this gives $A(E, \mathcal{V})$
with multi-natural transformations these $A(\cdot, \cdot)$ are categories

Q: what's the difference to using cobordisms categories?
A: it's simpler, since with operads we allow only one output now

fusion representability
remark: there are multibicategories (categories weakly enriched over multi categories)
$mbicat_g$ and $mbicat_E$
these have objects a, b
morphisms (1-cells) $a \to b$
multi-2-cells

Def.: multibicategories are representable if of the form

$$\mathrm{Mult}(\mathcal{D})$$

for $\mathcal{D}$ = bicategory
there is a representability criterion by Claudio Hermida
there exist usual categoiries $mbcat_g$, $mbcat_E$
there exist equivalence of categories

$$A(g, \mathcal{V}) \simeq \mathcal{V} - \mathrm{mbicat}_g^{\mathrm{piv}}$$

$$A(E, \mathcal{V}) \simeq \mathcal{V} - \mathrm{bicat}_E^{\mathrm{piv}}$$

where piv means *pivotal*, which arises because we singled out basepoints on boundary circles

0.3 application

if representable ⇒ bicategories with duality a la May-Sigurdsson
Applications:

- topological Landau-Ginzburg models TLGB

  these are top. theores with fusion

- Caldraruu-Willerton B-twisted TSM (in Calabi-Yau) are top TFTs with defects and fusion