Note on Lax Functors and Bimodules

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Abstract

This is a private note on how lax functors into a tensor category $\mathcal{C}$ give rise to a subcategory of $\mathbf{BiMod}(\mathcal{C})$. Using the FRS theorem it hence makes good sense to define a (rational) string background to be a lax functor into the suspension of a modular category. I notice how this is very similar to how quiver representations define B-model backgrounds.

1 Introduction

In the following I try to sketch the observation

1. that giving a lax functor from a graph into a tensor category is the same thing as
   - assigning an algebra A_a in $\mathcal{C}$ to each vertex a.
   - assigning an A_a-A_b bimodule to each edge; $a \xrightarrow{r} b$
2. that, according to FRS, the data of a full RCFT with chiral data $\mathcal{C}$ is the same as
   - finding an algebra $A_a \in \mathcal{C}$ of open a-a strings for each D-brane a,
   - finding A_a-A_b bimodules from spaces of states of open a-b strings;
3. that defining a background for B-model strings on $\mathbb{C}^3/G$ is the same as
   - assigning fractional D-branes to vertices a
   - assigning "string condensates" stretching between D-branes to edges $a \longrightarrow b$ of some graph (the "quiver" associated to G).

If we distinguish a full (R)CFT with all its possible boundary conditions from a particular background it defines, which may contain just a subset of all possible D-brane types, then, using the FRS theorem, it makes good sense to define a (rational) string background to be a lax functor

$$\Gamma \to \Sigma(\mathcal{C}) \; .$$

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2 Lax Functors and Bimodules

(The content of this section must be something well known, but I wasn't able to find a reference.)

Let C be a category and D be a bicategory. (I'll call D a weak 2-category, or even just a 2-category.) Thinking of C as a bicategory with only identity 2-morphisms we define

Definition 1 *A* **lax functor**

$$F : C \to D$$

is like an ordinary functor, only that it respects units and composition only up to coherent 2-morphisms:

$$\begin{tikzcd} a \arrow[r, "\mathrm{Id}"] & a \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F(a) \arrow[rr, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "F(\mathrm{Id}_a)"'] \arrow[rr, phantom, "\Downarrow i_a"] & & F(a) \end{tikzcd}$$

and

$$\begin{tikzcd} a \arrow[r, "r"] & b \arrow[r, "s"] & c \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & F(b) \arrow[dr, "F(s)"] \arrow[d, phantom, "\Downarrow m_{r,s}"] & \\ F(a) \arrow[ur, "F(r)"] \arrow[rr, "F(s \circ r)"'] & {} & F(a) \end{tikzcd} .$$

The coherence conditions are such that $m_{\cdot,\cdot}$ behaves like an associative product with $i_\cdot$ as its units.

Now let Γ be the free category of some finite graph. Let $\mathcal{C}$ be a monidal category. Denote by $\Sigma(\mathcal{C})$ its suspension, i.e. the bicategory with a single object and $\mathrm{Mor}_{\Sigma(\mathcal{C})}(\bullet, \bullet) = \mathcal{C}$. Denote by $\mathbf{BiMod}(\mathcal{C})$ the bicategors of bimodules internal to $\mathcal{C}$.

Observation 1 *Every proper functor*

$$\tilde{F} : \Gamma \to \mathbf{BiMod}(\mathcal{C})$$

defines a lax functor

$$F : \Gamma \to \Sigma(\mathcal{C}) \; .$$

Proof. The image under F of each identity morphism $a \xrightarrow{\mathrm{Id}} a$ in Γ is clearly an associative algebra

$$A_a \equiv F(\mathrm{Id}_a)$$

internal to $\mathcal{C}$. For any morphism $a \xrightarrow{r} b$ the 2-morphism m_{r,Id_b} equips $F(r)$ with a right module action for A_b and 2-morphism $m_{\mathrm{Id}_a,r}$ equips $F(r)$ with a left module action for A_a. Both actions commute and make $F(r)$ an A_a-A_b-bimodule.

Furthermore, the coherence laws enforce that the product $m_{r,s}$ of two bimodules satisfies the property of a product over A_b:

$$\begin{tikzcd}
F(r) \arrow[ddr] & & A_b \arrow[d] & F(s) \arrow[dl] \\
& & \boxed{m_{\mathrm{Id}_b,s}} \arrow[dl] & \\
& \boxed{m_{r,s}} \arrow[d] & & \\
& F(s \circ r) & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F(r) \arrow[dr] & & A_b \arrow[dl] & F(s) \arrow[ddl] \\
& \boxed{m_{r,\mathrm{Id}_b}} \arrow[dr] & & \\
& & \boxed{m_{r,s}} \arrow[d] & \\
& & F(s \circ r) &
\end{tikzcd}$$

$\square$

Remark. The lax functor property does not seem to ensure that $F(s \circ r)$ is the *universal* product over A_b of $F(r)$ with $F(s)$. In other words, it could be just a subobject of $F(r) \otimes_A F(s)$. (?)

Definition 2 *A* **morphism of lax functors**

$$\begin{tikzcd}
C \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=L}] & D
\arrow[from=U, to=L, Rightarrow, "\phi"]
\end{tikzcd}$$

*is an assignment of 2-*iso*morphisms*

$$\mathrm{Mor}_1(C) \ni \quad a \xrightarrow{r} b \quad \mapsto \quad
\begin{tikzcd}
F(a) \arrow[r, "F(r)"] \arrow[d, "\phi(a)"'] \arrow[dr, phantom, "\swarrow \phi(r)"] & F(b) \arrow[d, "\phi(b)"] \\
F'(a) \arrow[r, "F'(r)"'] & F'(b)
\end{tikzcd}
\quad \in \mathrm{Mor}_2(D)$$

which respects the lax structure in that

$$
\begin{tikzcd}
F(a) \arrow[r, "F(r)"] \arrow[d, "\phi(a)"'] \arrow[dr, phantom, "\Swarrow \phi(r)"] & F(b) \arrow[r, "F(s)"] \arrow[d, "\phi(b)"] \arrow[dr, phantom, "\Swarrow \phi(s)"] & F(c) \arrow[d, "\phi(c)"] \\
F'(a) \arrow[r, "F'(r)"] \arrow[rr, bend right=50, "F'(s \circ r)"'] & F'(b) \arrow[r, "F'(s)"] \arrow[d, phantom, "\Downarrow m'_{r,s}"] & F'(c) \\
& &
\end{tikzcd}
=
\begin{tikzcd}
& F(b) \arrow[dr, "F(s)"] \arrow[d, phantom, "\Downarrow m_{r,s}"] & \\
F(a) \arrow[ur, "F(r)"] \arrow[rr, "F(s \circ r)"] \arrow[d, "\phi(a)"'] & \arrow[d, phantom, "\Swarrow \phi(s \circ r)"] & F(c) \arrow[d, "\phi(b)"] \\
F'(a) \arrow[rr, "F'(s \circ r)"'] & & F'(b)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
F(a) \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, phantom, bend left=25, "\Downarrow i_a"] \arrow[r, "F(\mathrm{Id}_a)"] \arrow[d, "\phi(a)"'] \arrow[dr, phantom, "\Swarrow \phi(\mathrm{Id}_a)"] & F(a) \arrow[d, "\phi(a)"] \\
F'(a) \arrow[r, "F'(\mathrm{Id}_a)"'] & F'(a)
\end{tikzcd}
=
\begin{tikzcd}
F(a) \arrow[r, "\mathrm{Id}"] \arrow[d, "\phi(a)"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & F(a) \arrow[d, "\phi(a)"] \\
F'(a) \arrow[r, "\mathrm{Id}"] \arrow[r, bend right=50, "F'(\mathrm{Id}_a)"'] \arrow[r, phantom, bend right=25, "\Downarrow i'_a"] & F'(a)
\end{tikzcd}
.
$$

Observation 2 *For lax functors $F, F' : \Gamma \to \Sigma(C)$ as before, a morphism $F \to F'$ induces a collection of bimodule isomorphism of induced bimodules.*

Like this: Consider the above diagram evaluated for $b = c$ and $s = \mathrm{Id}_b$. This yields an equation like

$$
\begin{tikzcd}
\bullet \arrow[r, "N"] \arrow[d, "U"'] \arrow[dr, phantom, "\swarrow\lambda"] & \bullet \arrow[r, "A"] \arrow[d, "V"] \arrow[dr, phantom, "\swarrow\phi"] & \bullet \arrow[d, "V"] \\
\bullet \arrow[r, "N'"] \arrow[rr, bend right=60, "N'"', ""{name=B}] & \bullet \arrow[r, "A'"] \arrow[d, phantom, "\Downarrow m'_{N',A'}"] & \bullet \\
& &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \bullet \arrow[dr, "A"] \arrow[d, phantom, "\Downarrow m_{N,A}"] & \\
\bullet \arrow[ur, "N"] \arrow[rr, "N"] \arrow[d, "U"'] \arrow[drr, phantom, "\swarrow\lambda"] & & \bullet \arrow[d, "V"] \\
\bullet \arrow[rr, "N'"'] & & \bullet
\end{tikzcd}
.
$$

This equation says that

$$
\begin{tikzcd}
N \otimes V \arrow[d, "\lambda"'] \\
U \otimes N'
\end{tikzcd}
$$

is an intertwiner from the right A-module $N \otimes V$ (with the right A action induced from passing A through V by means of ϕ^{-1}) to the right A'-module $U \otimes N'$ (with the obvious A' action) $\square$

Since there is no requirement on the invertibility of the 2-morphisms $i_\cdot$ and $m_{\cdot,\cdot}$ in the definition of a lax functor, a lax functor is much less rigid than a pseudofunctor, for which $i_\cdot$ and $m_{\cdot,\cdot}$ would all be required to be 2-isomorphisms. It it natural to define a notion of functor in between lax and pseudo.

Definition 3 *An* **ambidextrous lax functor** *is a lax functor with co-versions of the structure 2-morphisms, namely*

$$
a \xrightarrow{\mathrm{Id}} a \quad \mapsto \quad
\begin{tikzcd}[column sep=large]
F(a) \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "F(\mathrm{Id}_a)"'] \arrow[r, phantom, "\Uparrow e_a"] & F(a)
\end{tikzcd}
$$

and

$$
a \xrightarrow{r} b \xrightarrow{s} c \quad \mapsto \quad
\begin{tikzcd}
& F(b) \arrow[dr, "F(s)"] \arrow[d, phantom, "\Uparrow \Delta_{r,s}"] & \\
F(a) \arrow[ur, "F(r)"] \arrow[rr, "F(s \circ r)"'] & & F(a)
\end{tikzcd}
,
$$

such that some obvious compatibility conditons hold.

Observation 3 *The statement in observation 1 (p. 1) applies to ambidextrous lax functors with algebras replaced by bialgebras everywhere. If we choose the above mentioned compatibility conditions nicely, these bialgebras will be special Frobenius algebras.*

Remark. The sequence

$$\text{lax functors} \supset \text{ambi lax functors} \supset \text{pseudofunctors}$$

reflects the sequence

$$\text{adjunctions} \supset \text{ambijunctions} \supset \text{equivalences}\,.$$

3 Lax Quiver Representations and RCFT

FRS show that an RCFT is the same as a special symmetric Frobenius algebra A (in the oriented case, or Jandl algebra in the unoriented case) internal to a modular tensor category $\mathcal{C}$.

More in detail, every type a of D-brane in the RCFT corresponds to such an internal algebra $A_a \in \mathcal{C}$, which is the algebra of open a-a strings. The space of open a-b strings furnishes a left A_a and a right A_b bimodule. Regarding these as just left A_a-modules and restricting to the simple subobjects yields the simple boundary conditions of the theory.

Instead of considering RCFTs with all of their possible boundary conditions, let us consider a (rational) "string background" to be a *choice* of (Morita equivalent) algebras $A_a \in \mathcal{C}$ together with a choice of subcollection of their bimodules. (Heuristically, this is a choice of background together with a choice of D-branes in that background. The point is that not all types of allowed D-branes need to be present. Hence the distinction between the full RCFT and one of the backgrounds defined by it.)

Using observation 1 (p. 2) we can succinctly express this as follows.

Observation 4 *A (rational)* **string background** *B is a (ambidextrous) lax functor (def. 3)*

$$B : \Gamma \to \Sigma(\mathcal{C})$$

from some (free category over a) finite graph Γ to the suspension of a modular tensor category $\mathcal{C}$.

Notice how this is rather similar to the well known conception of a string background for topological (B-model) strings: for the subclass of such backgrounds which appear as global quotients of $\mathbb{C}^3$ by a finite subgroup $G \subset SU(3)$, such a background is specified by a **quiver representation**, i.e. by a functor

$$\Gamma \to \mathbf{Vect}$$

from some (free category over a) finite graph to the category of vector spaces.

In this context, too, the vertices of Γ map to types of branes (called the "fractional branes"), while the edges map to open string (condensates, in this case) states, stretching between these branes.

In order to understand if there is more to this similarity, one should think about the following

Exercise 1 *Formulate B-model strings on* $\mathbb{C}^3/G$ *algebraically in the FHK limit of FRS. Find the boundary conditions and identitfy the "fractional branes". Check if quiver representations* $\Gamma \to \mathbf{Vect}$ *(for the quiver* Γ *obtained from* G*) correspond to string backgrounds in the sense of obs. 4.*

To be more precise, a background for the B-model string on $\mathbb{C}^3/G$ is not just a quiver representation, but possibly an entire complex of these - an object in the derived category of quiver representations.

Hence

Exercise 2 *Define and study the derived category of lax functors*

$$\Gamma \to \Sigma(\mathcal{C})$$

and repeat exercise 1 in this context.