Periodicity conjecture via 2-Calabi-Yau categories

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Abstract

Notes taken in a talk by **B. Keller** at *Higher Structures in Mathematics and Physics*, Bernoulli Center, EPFL, Lausanne, Nov. 2008. Notes pretty literally reproduce what was on the board and what was said. But all mistakes are mine.

Contents

periodicity conjecture is statement from math. phys

A. Zamolodchikov 1991: thermodynamic Bethe ansatz to CFTs

generalized by kuniba-Nakanishi (1992)

modern form: Gliozzi-Tateo 1995

proof today based on homological algebra

study of 2-Calabi-Yau triangulated categories

overall idea of proof is that of categorification: we try to interpret the purely combinatorial statement as a combinatorial shadow of a richer categorical stetement, which is much easier to prove

in addition to this general philosophy the other main ingredient is cluster algebras invented by Fomin-Zelevinsky: interface between abstract category theory and concrete combinatorics

Plan:

1. the conjecture
2. the beginning of the proof: categorification of root systems

   (skip long middle part of proof)
3. the end of the proof: homological periodicity

0.1 the conjecture

main input are two Dynkin diagrams and their Coxeter numbers

for simplicity: restrict attention to simply laced diagrams (generalization is straightforward)

name	diagram	number vertices	coxeter number
A_n		$n \geq 1$	n+1
D_n		$n \geq 4$	2n-2
E_6		6	12
E_7		7	18
E_8		8	30

Δ, Δ' Dynkin diagrams $1, \cdots, n$ and $1, \cdots, n'$ their vertices h, h' their Coxeter numbers A, A' adjacency matrices

associated Y-system:
variables: $Y_{i,i'}$ $1 \leq i \leq n$,
$1 \leq i' \leq n'$,
$t \in \mathbb{Z}$
equations: $Y_{i,i',t-1} \cdot Y_{i,i',t+1} = \prod_{j=1}^{n}(1+Y_{j,i',t})^{a_{ij}} / \prod_{j'=1}^{n'}(1+Y_{i,j',t}^{-1})^{a_{i'j'}}$
(so these equations somehow come from the thermodynamical Bethe ansatz to certain CFTs)
Conjecture: all solutions of this equationn are periodic of period dividing $2(h+h')$
algebraic reformulation:
$K = \mathbb{Q}(Y_{i,i'} | 1 \leq i \leq n, 1 \leq i' \leq n')$
$\phi : K \overset{\simeq}{\to} K$ automorphism such that $\phi(Y_{i,i'}) = \frac{1}{Y_{i,i'}} \cdot \frac{\prod \cdots}{\prod \cdots}$
then **Conjecture**: ϕ is of finite order: $\phi^{h+h'} = \mathrm{id}_K$

proved for

- (A_n, A_1) by Frenkel-Szenes (1995) abd then by Gliozzi-Tateo (1996)
- Δ, A_1 by Fomin-Zelevinsky (2003)
- (A_n, A_m) by Volkov 2007, Szenes (2008), André Henriques (2007)
- (Δ, A_n) about to be proved by Hernandes, Leclerc (also uses categorification)

but it turns out that using 2-Calabi-Yau categories one can prove the general case:
Theorem: the conjecture holds for (Δ, Δ')

0.2 the beginning of the proof: categorification of root systems

Delta Dynkin diagram $1, \cdots, n$ its vertices, h =Coxeter number, $A = adjacencymatrix$,
e.g. $\Delta = A_2$
quadratic form: $q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{iyj} a_{ij}x_ix_j$ is positive definite

$$R = \{\text{roots}\} = \{\alpha \in \mathbb{Z}^n | q(\alpha) = 1\}$$

$\alpha \in R$. s_α = reflection at $\mathbb{R}_\alpha^\perp$ W = Weyl group

$$\alpha_1, \cdots, \alpha_n$$

root basis (i.e. such that each root is a pos. or a neg. integral lin. comb. of the α_i)

$$c = \text{Coxeter element} = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_n}$$

h =order of c= Coxeter number
idea: build triangulated category which exhibits this, such that autoequivalences of the triangulated category reproduce the time evolution system
Categories
Q a quiver (= oriented graph) with underlying graph Δ, e.g. $Q : 1 \overset{\alpha}{\to} 2$
k algebraically closed field
representation of Q =diagram of fin. dim. k-vector spaces of the shape given by Q, e.g. $V : V_1 \overset{V_\alpha}{V_2}$
$rep(Q)$= category of representations of Q
remark: this is an abelian category, we have sums, kernels and cokernels (all these are computed componentwise) and every morphism has a decomposition as a mono followed by an epi
$\ker(f : V \to W)_i = \ker(f_i : Vi \to W_i)$, $1 \leq i \leq n$
Def: $\mathcal{D}_Q$ = bounded derived category of rep(Q)
objects: bounded complexes of representations

morphisms: obtained from morphisms of complexes by formally inverting all quasi-isomorphisms
this is a nice category but almost never abelian (abelian only for $\Delta = A_1$)
but still it is a triangulated category, which is enough for our purposes

question from audience: so there is still the choice of orientation on the quiver; answer: yes, but $\mathcal{D}_Q$ is independent up to iso of this choice

remarks

- $\mathcal{D}_Q$ is abelian iff $\Delta = A_1$, but is always <u>triangulated</u>, i.e. it is additive and endowed with
  - suspension functor $\Sigma : \mathcal{D}_Q \xrightarrow{\simeq} \mathcal{D}_Q \; L \mapsto L[1]$
  - triangles: $L \to M \to N \to \Sigma L$ "induced " from short exact sequences of complexes;
  - $\mathcal{D}_Q$ has a <u>Serre functor</u>, i.e. an autoequivalence $S : \mathcal{D}_Q \xrightarrow{\simeq} \mathcal{D}_Q$ such that it makes the Serre duality formula true:

$$\mathcal{D}\mathrm{Hom}(L, M) \simeq \mathrm{Hom}(M, SL) \;\; \forall L, M \in \mathcal{D}_Q$$

    where $\mathcal{D} = \hom_k(?, k)$

Definition. $K_0(\mathcal{D}_Q)$= Grothendieck group =

$$\frac{\text{free abelian group on the isoclases } [L]}{\langle [L] - [M] + [N] \rangle}$$

where the denominator ranges over all triangles

$$S = \cdots (**\text{missed that}**)$$

theorem: (Gabriel and Happel) there exists a canonical isomorphism

$$K_0(\mathcal{D}_Q) \xrightarrow{\simeq} \mathbb{Z}^n$$

$$\{[L] | L \text{ is indecomposable (wrt direct sums)}\} \xrightarrow{\simeq} \{\text{roots}\}$$

here

$$\tau^{-1} := S^{-1} \circ \sigma$$

acts by autos on the left and c= Coxeter numver on the right, and these actions are intertwined by the above

$$\tau^{-h} \simeq \Sigma^2 \to c^h = \mathrm{Id}$$

so we have lifted the complete combinatorial picture to categories

remark: finite type cluster algebras are refinements of root systems (FZ classification theorem)
they can be categorified

Theorem (Buan-Maish-Reineke-Reiten-Todorov): The cluster algebra $\mathcal{A}_\Delta$ is "categorified" by the cluster category:

$$\mathcal{C}_Q := \text{orbit category of } \mathcal{D}_Q / (S^{-1} \circ \Sigma^2)^{\mathbb{Z}}$$

objects¿ same as those of $\mathcal{D}_Q$
morphisms $\mathrm{Hom}_{\mathcal{C}_Q}(L, M) = \oplus_{p \in \mathbb{Z}} \mathrm{Hom}(L, (S^{-1} \circ \Sigma^2)^p M)$
Q: automorphism acts freely in some sense? answer: yes

remark: $\mathcal{C}_Q$ is triangulated (which is not automatic) and it is 2-Calabi-Yau, i.e.

$$S \xrightarrow{\simeq} \Sigma^2 \, ,$$

where the "2" in the exponent is the "2" in "2-Calabi-Yau''

0.3 the end of the proof: homological periodicity statement

Δ, Δ': Dynkin diagrams

Q, Q' quivers with underlying graphs Δ and Δ'

kQ= path algebra of the quiver (spanned by paths, product is composition if defined, 0 otherwise)

mod(kQ) is equivalent to rep(Q)

$\mathcal{D}_{Q,Q'}$= bounded derived category of modules mod($kQ \otimes kQ'$)

$$\mathcal{D}_{Q,Q'}/(S^{-1} \circ \Sigma^2)^{\mathbb{Z}} \hookrightarrow \mathcal{C}_{Q,Q'}$$

left hand is not triangulated in general, but sits in smalles triangulated over-category on the right, which is still 2-Calabi-Yau: this is the *cluster category*

definition:

$\Phi := \tau^{-1} \otimes \mathrm{id}$

theorem:

1. Φ categorifies $\phi : K \xrightarrow{\simeq} K$
2. $\Phi^{h+h'} \simeq \mathrm{id}_{\mathcal{C}_{Q,Q'}}$

proof of the second point: we know

$$S \simeq \Sigma^2$$

in $\mathcal{C}_{Q,Q'}$ and then $\cdots \simeq \Sigma \otimes \Sigma$ in $\mathcal{D}_{Q,Q'}$ and $S \otimes S \simeq S$

it follows that $\tau^{-1} \otimes \tau^{-1} \simeq S^{-1}\Sigma \otimes S^{-1}\Sigma \simeq \mathrm{id}$

$$\rightarrow \Phi = \tau^{-1} \otimes \mathrm{id} \simeq \mathrm{id} \otimes \tau$$

$$\Rightarrow \Phi^{h+h'} = (\tau^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \tau)^{h'} \simeq (\Sigma^2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Sigma^{-2})$$

by Gabriel and Happel

$$= \Sigma^2 \circ \Sigma^{-2} = \mathrm{id}_{\mathcal{C}_{Q,Q'0}}$$

Q: how does this depend on the quivers being Dynkin diagrams: A: constrruction works for general quivers without cycles, but periodicity appears iff quiver is a Dynkin diagram