notes taken in

A. Jaffe: Infinite dimensimal geometry and C(constructive/conformal)QFT
, July 02, 2008, Max Planck institute for Math in Bonn, conference *The manifold geometries of quantum field theory*

some methods in QFT can be used in inf. dim geometry
Connes: Journal of K-theory 1988
Jaffe 1999
new work with Jaeckel and Moser

Contents

0.1 cyclic cohomology

cohomology suited for infinite dimensional spaces

cochains are sequences of functions on tensor powers of an algebra $\mathcal{A}$ that map

$$f_n : \mathcal{A}^{n+1} \times G \to \mathbb{C}$$

they are required to be $(n+1)$-multilinear in $a \in \mathcal{A}$ and continuous in $\mathcal{A}$ and G

pick a representation $V : G \to U(\mathcal{A})$ of G on the unitary elements in $\mathcal{A}$

The cochains are required to have the property

1. that
$$f_n(a_0, \ldots, a_n, g) = 0$$
if $a_j = I$, $j \neq 0$

2. there is a norm in the game such that
$$n^{1/2} \|f_n\|^{1/n} \to 0$$
as $n \to \infty$ [I *think*]

3. the cochains are G-invariant in that for $a \mapsto a^g = V(g)aV(g)^*$ we have
$$f(a_0^g, \ldots, a_n^g, g) = f_n(a_0, \ldots, a_n, g)$$

Write $\mathcal{C}$ for the space of cochains. On that we act with the coboundary operator

$$\partial : \mathcal{C} \to \mathcal{C}$$

given by

$$\partial = b + B$$

(B is the Connes operator) defined as follows:

$$(bf_n)(a_0, \cdots, a_{n+1}) = \sum_{j=0} (-1)^j f_n(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}, g) + (-1)^{j+1} f_n(a_{n+1}^{g^{-1}}, a_0, \cdots, a_n, g)$$

$$(Bf_n)(a_0, \ldots, a_{n-1}) = Af_n(I, a_0, \cdots, a_{n-1}, g)$$

where "A" is antisymmetrization in the a_i

cocycle is a cochain τ such that $\partial \tau = 0$

τ is called even if $\tau_{2n+1} = 0$

we want to pair cocycles with idempotents a in the algebra

$$a \in \mathcal{A}, a^2 = I$$

assume that τ is even

the pairing should have the property that

$$\langle \tau_1, a \rangle + \langle \tau_2, a \rangle = \langle \tau_1 + \tau_2, a \rangle$$

$$\langle \partial G, a \rangle = 0$$

we construct such a pairing from the "generating function" of a cocycle

$$J(t, a) = \sum_{n=0}^{\infty} (it)^n \tau_n(a, a, a, \cdots, a, g)$$

by setting

$$\langle \tau, a \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} J(\tau, a) e^{-t^2}$$

0.2 QFT

SUSY QM:

Hilbert space $\mathcal{H}$ with rep of $\mathcal{A}$, $V(g)$ unitary rep

H, Q operators on $\mathcal{A}$, the Hamiltonian and its supercharge

$$H = Q^2$$

unitary Γ

$$\Gamma^2 = 1$$

induces grading on $\mathcal{H}$, write

$$A^\Gamma := \Gamma A \Gamma$$

for any operator A. Then

$$Q^\Gamma = Q$$

$$a^g = V(g) a V(g)^{-1}$$

and

$$a^\Gamma = a$$

for $a \in \mathcal{A}^{\text{even}}$

[at this point the speaker gave a long speach on some ideas about approaches to infinite dimensional analysis. here are some keywords:]

infinite diemsnional theory: approximate by finite things and then take the limit, trying to make sure that not all necessary properties break

some comments on how one wants finite number of degrees of freedom within finite regions of phase space $\to$

"phase cell localization methods" : large list of applications in AQFT

[end of speech, on with the main content (for appreciating this speech fully my impression was that knowledge of the upcoming work with Jaeckel and Moser would have been helpful...)]

how does SUSY QFT connect to the previous section?

there is a cocycle

$$\partial \tau^{\text{JLO}} = 0$$

defined as follows:

$$\tau_n^{\text{JLO}}(a_0, \ldots, a_n, g) = \int_{\{\beta_j > 0\}} \text{Tr}_{\text{super}}(V(g) e^{-\beta_0 Q^2} da_1 e^{-\beta_1 Q^2} \cdots da_n e^{-\beta_n Q^2}) \delta(1 - \sum_{j=0}^{n} \beta_j) d\beta_1 \cdots d\beta_n$$

where

$$da = [Q, a] = Qa - a^\Gamma Q$$

Theorem 1

$$\partial \tau^{\text{JLO}} = 0$$

[Jaffe didn't say so explicitly, but this is of course a correlator in susy QM in the "g-twisted sector", then usually written as $\langle da_1(\beta_0) \cdots d_n(\sum_j \beta_j) \rangle_g$]

for odd n this τ vanishes, so it is an even cocycle

here e^{-Q^2} has to be trace class

this formula simplifies when we look at the pairing, which is just

$$\langle \tau^{\text{JLO}}, a \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \text{Tr}_S(U \exp(-Q^2 + itda - t^2)) dt$$

if $a = I$ then this reduces to the formula originally considered by Witten

$$\langle \tau^{\text{JLO}}, I \rangle = \text{Index}(Q_+)$$

where

$$Q_+ = \frac{1}{2}(Q + \Gamma Q)$$

is one of the two off diagonal blocks of Q in the basis in which Γ is block diagonal

interlude: vertex calculus

$$X_n = \{x_0, \cdots, x_n\}$$

$$= \int_{\{\beta_j>0\}} x_0 e^{-\beta_0 Q^2} \cdots x_n e^{-\beta_n^2 Q^2} \delta(1 - \sum \beta_j) d\beta_1 \cdots d\beta_j$$

Theorem 2

$$X_n = \{x_0, I, \cdots, x_n\} + \sum_{j=1}^{n+1} \{x_0, \ldots, x_j, I, \cdots, x_n\}$$

$$\langle X_n \rangle = \mathrm{Tr}_S(U X_n)$$

$$\langle dX_n \rangle = 0$$

proof using these identities

Write

$$e^{-H} = e^{-Q^2}$$

as a Euclidean field theory and use heat kernel expansion.

different point of view:

enlarge our Hilbert space $\mathcal{H}$ with one more fermionic degree of freedom to get $\hat{\mathcal{H}}$

$$\{b, b^*\} = 1$$

$$bQ + Qb = 0$$

$$\eta = b + b^*$$

on $\hat{\mathcal{H}}$ new Dirac operator

$$q = Q + ta\eta$$

$t \in \mathbb{R}$
$a^2 = I$

$$q^2 = Q^2 + t^2 - \eta da$$

$$\langle \tau^{\mathrm{JLO}}, a \rangle = \frac{1}{\sqrt{t\pi}} \int_{-\infty}^{\infty} \mathrm{Tr}_{\hat{\mathcal{H}}}((-1)^{N_b} a U(g) e^{-q^2} dt)$$

$$= \langle \langle (-1)^{N_b} a \rangle \rangle$$

outer bracket is expectation value on $\hat{\mathcal{H}}$

Theorem 3 *if*

$$\tau_\lambda^{\mathrm{JLO}}$$

depends on λ *through* $Q = Q(\lambda)$ *in a differential way, then*

$$e^{-Q^2(\lambda)}$$

is differentiable in trace class operators then

$$\frac{d}{d\lambda}\tau_\lambda^{\mathrm{JLO}} = \partial G_\lambda$$

so this leads to homotopies between cocycles

[the following went along with another long speech. I am not sure what to make of it. I even asked, but am not sure if I did receive an answer – it is somehow about applications to 2-dimensional AQFT]

$$V(z), z \in C^N$$

$$V(z) = \sum_{j=1}^{N} \Omega_j \frac{\partial V}{\partial z_j} z_j$$

$$0 < \Omega_j$$

uniform ellipticity condition

$$M + |\nabla V|^2 \geq \epsilon |z|^2$$

twisted fields

$$V(\theta)\phi_j(x)V(\theta)^* = e^{i\theta\Omega_j}\phi_j(x)$$