The inner automorphism 3-group of a strict 2-group

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Abstract

For any group G, there is a 2-group of inner automorphisms, $\mathrm{INN}(G)$. This plays the role of the universal G-bundle. Similarly, for every 2-group $G_{(2)}$ there is a 3-group $\mathrm{INN}(G_{(2)})$ and a slightly smaller 3-group $\mathrm{INN}_0(G_{(2)})$ of inner automorphisms. We construct these for $G_{(2)}$ any strict 2-group, discuss how $\mathrm{INN}_0(G_{(2)})$ can be understood as arising from the mapping cone of the identity on $G_{(2)}$ and show that it fits into a short exact sequence

$$\mathrm{Disc}(G_{(2)}) \longrightarrow \mathrm{INN}_0(G_{(2)}) \longrightarrow \Sigma G_{(2)}$$

of strict 2-groupoids. We close by indicating how this makes $\mathrm{INN}_0(G_{(2)})$ the universal $G_{(2)}$-2-bundle.

Contents

1 Introduction

By definition, an n-group is a 1-object n-groupoid. When we forget this one object, the n-group becomes a monoidal $(n-1)$-groupoid. If this is denoted by

$$G_{(n)}$$

then the corresponding 1-object n-groupoid will be denoted

$$\Sigma G_{(n)} .$$

One source of n-groups are automorphisms: for any n-category C, the n-category

$$\mathrm{AUT}(C) := \mathrm{Aut}_{n\mathrm{Cat}}(C)$$

is an $(n+1)$-group.

In the special case that $C = \Sigma G_{(n)}$ is itself already a 1-object n-groupoid we write

$$\mathrm{AUT}(G_{(n)}) := \mathrm{Aut}_{n\mathrm{Cat}}(\Sigma G_{(n)}) .$$

Then we can speak of *inner* automorphisms: these are essentially those n-functors $C \to C$ which come from conjugating any n-morphism with a given invertible 1-morphism.

In general there may be 1-morphisms in $\Sigma G_{(n)}$ which induce the same automorphism $G_{(n)} \to G_{(n)}$ under conjugation. It is of interest to nevertheless distinguish these. We write

$$\mathrm{INN}(G_{(n)})$$

for the $(n+1)$-group whose objects are those of $G_{(n)}$ and whose higher morphisms are transformations of conjugations. We write

$$\mathrm{INN}_0(G_{(n)}) \subset \mathrm{INN}(G_{(n)})$$

for the $n+1$-group of conjugations just by objects of $G_{(n)}$.

In the present context we shall work exclusively in the 3-category

$$2\mathrm{Cat}$$

whose objects are strict 2-categories, whose morphisms are strict 2-functors, whose 2-morphisms are pseudonatural transformations and whose 3-morphisms are modifications of these: we shall study

$$\mathrm{INN}(G_{(2)})$$

for $G_{(2)}$ any strict 2-group.

(But for any n and any notion of (weak) n-category, the above notions of automorphism $(n+1)$-groups and inner automorphism $(n+1)$-groups make sense.)

The basic example is the inner automorphism 2-group of an ordinary group G. Every strict 2-group comes from a crossed module

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

of ordinary groups. $\mathrm{INN}(G)$ is the one that comes from the crossed module

$$G \xrightarrow{\mathrm{id}} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G) \ .$$

This 2-group $\mathrm{INN}(G)$ fits into the exact sequence

$$\mathrm{Disc}(G) \hookrightarrow \mathrm{INN}(G) \twoheadrightarrow \Sigma G$$

of groupoids. Here $\mathrm{Disc}(G)$ denotes the discrete category over G. This is the category whose space of objects is the space underlying G and whose only morphisms are identity morphisms.

By taking nerves and realizing geometrically, this becomes the universal G-bundle

$$G \longrightarrow EG \longrightarrow BG\,.$$

We show that and how an analogous statement holds for $\mathrm{INN}_0(G_{(2)})$, with $G_{(2)}$ any strict 2-group. Our main results are stated in 3. A brief outlook on the interpretation of $\mathrm{INN}_0(G_{(2)})$ as the universal $G_{(2)}$-2-bundle closes the discussion in 7.

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2 n-Groups in terms of groups

Sufficiently strict n-groups are equivalent to certain structures – crossed modules and generalizations theoreof – involving just collections of ordinary groups with certain structure on them.

2.1 Conventions for strict 2-groups and crossed modules

It is well known that strict 2-groups are equivalent to crossed modules of ordinary groups.

Definition 1. *A* **crossed module of groups** *is a diagram*

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

in Grp *such that*

$$\begin{tikzcd} H \arrow[rr, "\mathrm{Ad}"] \arrow[dr, "t"'] & & \mathrm{Aut}(H) \\ & G \arrow[ur, "\alpha"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} G \times H \arrow[r, "\mathrm{Id}\times t"] \arrow[d, "\alpha"'] & G \times G \arrow[d, "\mathrm{Ad}"] \\ H \arrow[r, "t"] & G \end{tikzcd} .$$

Definition 2. *A* **strict 2-group** $G_{(2)}$ *is any of the following equivalent entities*

- *a group object in* Cat
- *a category object in* Grp
- *a strict 2-groupoid with a single object*

A detailed discussion can be found in [1].
One identifies

- G is the group of objects of $G_{(2)}$.
- H is the group of morphism of $G_{(2)}$ starting at the identity object.
- $t : H \to G$ is the target homomorphism such that $h : \mathrm{Id} \to t(h)$ for all $h \in H$.

- $\alpha : G \to \mathrm{Aut}(H)$ is conjugation with identity morphisms:

$$\mathrm{Ad}_{\mathrm{Id}_g}(\ \mathrm{Id} \xrightarrow{h} t(h)\ ) = \ \mathrm{Id} \xrightarrow{\alpha(h)(h)} t(\alpha(g)(h))$$

  for all $g \in G$, $h \in H$.

We often abbreviate

$$^{g}h := \alpha(g)(h)\,.$$

Beyond that there are 2×2 choices to be made when identifying a strict 2-group $G_{(2)}$ with a crossed module of groups.

The first choice to be made is in which order to multiply elements in G. For $\bullet \xrightarrow{g_1} \bullet$ and $\bullet \xrightarrow{g_2} \bullet$ two morphisms in $\Sigma G_{(2)}$, we can either set

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \;:=\; \bullet \xrightarrow{\quad g_1 g_2 \quad} \bullet \qquad \text{(F)}$$

or

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \;:=\; \bullet \xrightarrow{\quad g_2 g_1 \quad} \bullet \qquad \text{(B)}\,.$$

The other choice to be made is how to describe arbitrary morphisms by an element in the semidirect product group $G \ltimes H$: every morphism of $G_{(2)}$ may be written as the product of one starting at the identity object with an identity morphism on some object. The choice of ordering here yields either

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, ""'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \;:=\; \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, ""'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd} \qquad \text{(R)}$$

or

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, ""'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \;:=\; \begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, ""'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \qquad \text{(L)}$$

Here we choose the convention

$$LB\,.$$

This implies

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'=t(h)g"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd}$$

for all $g \in G$, $h \in H$, as well as the following two equations for horizontal and vertical composition in $\Sigma G_{(2)}$, expressed in terms of operations in the crossed module

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "g_1"] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow h_1"] & \bullet \arrow[r, bend left=60, "g_2"] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow h_2"] & \bullet \arrow[r, phantom, "="] & \bullet \arrow[r, bend left=60, "g_2 g_1"] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow h_2{}^{g_2}h_1"] & \bullet \end{tikzcd}$$

and

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=60, "g_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow h_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow h_2"] & \bullet \arrow[r, phantom, "="] & \bullet \arrow[r, bend left=60, "g_1"] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow h_2 h_1"] & \bullet \end{tikzcd} .$$

2.2 3-Groups and 2-crossed modules

As we are considering strict models in this paper, we will assume that all 3-groups are as strict as possible. This means they will be one-object Gray-categories, or *Gray-monoids* [8]. A Gray-monoid is a (strict) 2-category $\mathcal{M}$ such that the product functor

$$\mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$$

uses the Gray tensor product, not the usual Cartesian product of 2-categories. Thus non-identity coherence morphisms only appear when we use the monoidal structure on $\mathcal{M}$.

Just as a 2-group gives rise to a crossed module, a 3-group gives rise to a 2-crossed module. Roughly, this is a complex of groups

$$L \to M \to N,$$

and a function

$$M \times M \to L \tag{1}$$

such that $L \to M$ is a crossed module, and (1) measures the failure of $M \to N$ to be a crossed module. An example is when $L = 1$, and then we have a crossed module. Now for the formal definition. See [3].

Definition 3. *A* **2-crossed module** *is a normal complex of length 2*

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

of N-groups (N acting on itself by conjugation) and an N-equivariant function

$$\{\cdot,\cdot\} : M \times M \to L\,,$$

called a Peiffer lifting, satisfying these conditions:

1. $\partial_2\{m, m'\} = (mm'm^{-1})({}^{\partial_1 m}m')^{-1}$,

2. $\{\partial_2 l, \partial_2 l'\} = [l, l'] := ll'l^{-1}l'^{-1}$

3. (a) $\{m, m'm''\} = \{m, m'\}^{mm'm^{-1}}\{m, m''\}$

   (b) $\{mm', m''\} = \{m, m'm''m'^{-1}\}^{\partial_1 m}\{m', m''\}$,

4. $\{m, \partial_2 l\} = ({}^{m}l)({}^{\partial_1 m}l)^{-1}$

5. ${}^{n}\{m, m'\} = \{{}^{n}m, {}^{n}m'\}$,

where $l, l' \in L,\ m, m', m'' \in M$ *and* $n \in N$.

Here ${}^{m}l$ denotes the action

$$\begin{array}{rcl} M \times L & \rightarrow & L \\ (m, l) & \mapsto & {}^{m}l := l\{\partial_2 l^{-1}, m\}\,. \end{array} \tag{2}$$

A *normal* complex is one in which *im* ∂ is normal in *ker* partial for all differentials.

It follows from these conditions that $\partial_2 : L \rightarrow M$ is a crossed module with the action (2).

To get from a 3-group $G_{(3)}$ to a 2-crossed module, we emulate the construction of a crossed module from a 2-group: one identitfies

- N is the group of objects of $G_{(3)}$.
- M is the group of 1-morphisms of $G_{(3)}$ starting at the identity object.
- L is the group of 2-morphisms starting at the identity 1-arrow of the identity object
- $\partial_1 : M \rightarrow N$ is the target homomorphism such that $m : \mathrm{Id} \rightarrow \partial_1(m)$ for all $m \in M$.
- $\partial_2 : L \rightarrow M$ is the target homomorphism such that $l : \mathrm{Id}_{\mathrm{Id}} \rightarrow \partial_2(l)$ for all $l \in L$.
- The various actions arise by whiskering, analogously to the case of a 2-group.

We will not go into the proof that this gives rise to a 2-crossed module for all 3-groups, but only in the case we are considering. One reason to consider 2-crossed modules is that the homotopy groups of $G_{(3)}$ can be calculated as the homology of the sequence underlying the 2-crossed module.

2.3 Mapping cones of crossed modules

Another notion equivalent to 3-groups is crossed modules internal to crossed modules (more technically known as crossed squares, [7],[5]). More generally, consider a map ϕ of crossed modules:

Definition 4 (nonabelian mapping cone [5])**.** *For*

$$\begin{tikzcd} H_2 \arrow[r, "\phi_H"] \arrow[d, "t_2"] & H_1 \arrow[d, "t_1"] \\ G_2 \arrow[r, "\phi_G"] & G_1 \end{tikzcd}$$

a 2-term complex of crossed modules ($t_i : H_i \to G_i$), we say its mapping cone is the complex of groups

$$H_2 \xrightarrow{\partial_2} G_2 \ltimes H_1 \xrightarrow{\partial_1} G_1 \ , \tag{3}$$

where

$$\partial_1 : (g_2, h_1) \mapsto t_1(h_1)\phi_G(g_2)$$

and

$$\partial_2 : h_2 \mapsto (t_2(h_2), \phi_H(h_2)^{-1}) \, .$$

Here G_2 acts on H_1 by way of the morphism $\phi_G : G_2 \to G_1$.

When no structure is imposed on ϕ, (3) is merely a complex. However, if ϕ is a crossed square, the mapping cone is a 2-crossed module (originally shown in [2], but see [3]). We will not need to define crossed squares here (the interested reader may consult [3]), but just note they come equipped with a map

$$h : G_2 \times H_1 \to H_2$$

satisfying conditions similar to the Peiffer lifting.

The only crossed square we will see in this paper is the identity map on a crossed module

$$\begin{tikzcd} H \arrow[r, "\mathrm{id}"] \arrow[d, "t"] & H \arrow[d, "t"] \\ G \arrow[r, "\mathrm{id}"] & G \end{tikzcd}$$

with the structure map

$$\begin{aligned} h : G \times H &\to H \qquad (4) \\ (g, h) &\mapsto h^g h^{-1}. \qquad (5) \end{aligned}$$

3 Main results

3.1 The exact sequence $\mathrm{Disc}(G_{(2)}) \to \mathrm{INN}_0(G_{(2)}) \to \Sigma G_{(2)}$

We construct the 3-group $\mathrm{INN}_0(G_{(2)})$ for $G_{(2)}$ any strict 2-group, and show that it plays the role of the universal principal $G_{(2)}$-bundle in that

- $\mathrm{INN}_0(G_{(2)})$ is equivalent to the trivial 3-group (hence "contractible").
- $\mathrm{INN}_0(G_{(2)})$ fits into the short exact sequence

$$\mathrm{Disc}(G_{(2)}) \hookrightarrow \mathrm{INN}_0(G_{(2)}) \twoheadrightarrow \Sigma G_{(2)}$$

  of strict 2-groupoids.

3.2 $\mathrm{INN}_0(G_{(2)})$ from a mapping cone

We show that the 3-group $\mathrm{INN}_0(G_{(2)})$ comes from a 2-crossed module

$$H \longrightarrow G \ltimes H \longrightarrow G$$

which is the mapping cone of

$$\begin{tikzcd} H \arrow[r, "\mathrm{Id}"] \arrow[d, "t"'] & H \arrow[d, "t"] \\ G \arrow[r, "\mathrm{Id}"] & G \end{tikzcd},$$

the identity map of the crossed module $(t : H \to G)$ which determines $G_{(2)}$.

Notice that this harmonizes with the analogous result for Lie 2-algebras discussed in [6].

4 Inner automorphism n-groups

An automorphism of an n-group $G_{(n)}$ is simply an automorphism of the n-category $\Sigma G_{(n)}$. We want to say that such an automorphism q is *inner* if it is equivalent to the identity automorphism

$$\begin{tikzcd} \Sigma G_{(n)} \arrow[r, bend left, "\mathrm{Id}"{name=U}] \arrow[r, bend right, "q"'{name=D}] & \Sigma G_{(n)} \arrow[Rightarrow, from=U, to=D, "\sim"] \end{tikzcd}.$$

A useful way to think of the n-groupoid of inner automorphisms is in terms of what we call "tangent categories", a slight variation of the concept of comma categories.

Tangent categories in general happen to live in interesting exact sequences. In order to be able to talk about these, we first quickly set up a our definitions for exact sequences of strict 2-groupoids.

Remember that we work entirely within the Gray category whose objects are strict 2-groupoids, whose morphisms are strict 2-functors, whose 2-morphisms are pseudonatural transformations and whose 3-morphisms are modifications of these.

4.1 Exact sequences of strict 2-groupoids

Inner automorphism n-groups turn out to live in interesting *exact sequences of* $(n+1)$*-groups*. Therefore we want to talk about generalizations of exact sequences of groups to the world of n-groupoids. Since for our purposes here only strict 2-groupoids matter, we shall be content with just using a definition applicable to that case.

Definition 5 (exact sequence of strict 2-groupoids)**.** *A collection of composable morphisms*

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} C_n$$

of strict 2-categories C_i is called an exact sequence if, as ordinary maps between spaces of 2-morphisms,

- *f_0 is injective*
- *f_n is surjective*
- *the image of f_i is the preimage under f_{i+1} of the collection of all identity 2-morphisms on identity 1-morphisms in $\mathrm{Mor}_2(C_{i+1})$, for all $1 \leq i < n$.*

In order to make this harmonize with our distinction between n-groups $G_{(n)}$ and the corresponding 1-object n-groupoids $\Sigma G_{(n)}$ we add to that

Definition 6 (exact sequences of strict 2-groups)**.** *A collection of composable morphisms*

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} G_n$$

of strict 2-groups is called an exact sequence if the corresponding chain

$$\Sigma G_0 \xrightarrow{\Sigma f_1} \Sigma G_1 \xrightarrow{\Sigma f_1} \cdots \xrightarrow{\Sigma f_n} \Sigma G_n$$

is an exact sequence of strict 2-groupoids.

Remark. Ordinary exact sequences of groups are thus precisely correspond to exact sequences of strict 2-groups all whose morphisms are identities.

4.2 Tangent 2-categories

We present a simple but useful way describe 2-categories of morphisms with coinciding source.

Definition 7 (the point). *The point is the strict 2-category*

$$\mathrm{pt} := \{\bullet\}$$

with a single object and no nontrivial morphisms. We shall carefully distinguish this from the strict 2-groupoid

$$\mathbf{pt} := \{ \bullet \xrightarrow{\sim} \circ \},$$

consisting of two objects connected by a 1-isomorphism.

The 2-groupoid **pt** might be called the "fat point". It is of course equivalent to the point – but not isomorphic. We fix one injection

$$i : \ \mathrm{pt} \hookrightarrow \mathbf{pt}$$

$$i : \bullet \mapsto \bullet$$

once and for all.

It is useful to think of morphisms

$$\mathbf{f} : \mathbf{pt} \to C$$

from the fat point to some codomain C as labeled by the corresponding image of the ordinary point

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "f"] \arrow[d, hook] & C \arrow[d, "="] \\ \mathbf{pt} \arrow[r, "\mathbf{f}"] & \mathrm{C} \end{tikzcd} .$$

Definition 8 (tangent 2-bundle). *Given any strict* 2*-category* C*, we define its tangent 2-bundle*

$$TC \subset \mathrm{Hom}_{2\mathrm{Cat}}(\mathbf{pt}, C)$$

to be that sub 2*-category of morphisms from the fat point into* C *which collapses to a 0-category when pulled back along the fixed inclusion* $i : \ \mathrm{pt} \hookrightarrow \mathbf{pt}$ *: the* *morphisms h in $T\,C$ are all those for which*

$$
\begin{tikzcd}
\mathrm{pt} \arrow[d, hook] \\
\mathbf{pt} \arrow[r, bend left=40, "\mathbf{f}"{name=U}] \arrow[r, bend right=40, "\mathbf{f}'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{pt} \arrow[d, hook] \\
\mathbf{pt} \arrow[r, "\mathbf{f}"] & C
\end{tikzcd}
\,.
$$

The tangent 2-bundle is a disjoint union

$$TC = \bigoplus_{x \in \mathrm{Obj}(C)} T_x C$$

of tangent 2*-categories at each object x of C. In this way it is a* 2*-bundle*

$$p : TC \longrightarrow \mathrm{Disc}(C)$$

over the space of objects of C.

As befits a tangent bundle, the tangent 2-bundle has a canonical section

$$e_{\mathrm{Id}} : \mathrm{Disc}(C) \to TC$$

which sends every object of C to the Identity morphism on it.

Example (slice categories). For C any 1-groupoid, i.e. a strict 2-groupoid with only identity 2-morphisms, its tangent 1-category is the comma category

$$TC = ((\mathrm{Disc}(C) \hookrightarrow C) \downarrow \mathrm{Id}_C)\,.$$

This is the disjoint union of all co-over categories on all objects of C

$$TC = \bigoplus_{a \in \mathrm{Obj}(C)} (a \downarrow C)$$

Objects of TC are morphisms $f : a \to b$ in C, and morphisms $f \xrightarrow{h} f'$ in TC are commuting triangles

$$
\begin{tikzcd}
& b \arrow[dd, "h"] \\
a \arrow[ur, bend left, "f"] \arrow[dr, bend right, "f'"'] \arrow[r, phantom, "\|"] & \\
& b'
\end{tikzcd}
$$

in C.

Example (strict tangent 2-groupoids). The example which we are mainly interested in is that where C is a strict 2-groupoid. For a any object in C, an object of T_aC is a morphism

$$a \xrightarrow{q} b\,.$$

A 1-morphism in T_aC is a filled triangle

$$\begin{tikzcd}
& b \arrow[dd, "f"] \\
a \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "\Downarrow F"] & \\
& b'
\end{tikzcd}$$

in C. Finally, a 2-morphism in T_aC looks like

$$\begin{tikzcd}
& b \arrow[dd, bend right, "f"'] \arrow[dd, bend left, "f'"] \arrow[dd, phantom, "\xRightarrow{L}"] \\
a \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "F \Downarrow\ \Downarrow F'"] & \\
& b'
\end{tikzcd}\ .$$

The composition of these 2-morphisms is the obvious one. We give a detailed description for the case the $C = \Sigma G_{(2)}$ in 5.

Proposition 1. *For any strict* 2*-category* C*, its tangent* 2*-bundle* TC *fits into an exact sequence*

$$\mathrm{Mor}(C) \hookrightarrow TC \twoheadrightarrow C$$

of strict 2-categories.

Here $\mathrm{Mor}(C) := \mathrm{Disc}(\mathrm{Mor}(C))$ is the 1-category of morphisms of C, regarded as a strict 2-category with only identity 2-morphisms.

Proof. The strict inclusion 2-functor on the left is

$$\left(g \xrightarrow{h} g' \right) \quad \mapsto \quad \begin{tikzcd}
& b \arrow[dd, bend right, "\mathrm{id}"'] \arrow[dd, bend left, "\mathrm{id}"] \arrow[dd, phantom, "\xRightarrow{\mathrm{id}}"] \\
a \arrow[ur, bend left, "g"] \arrow[dr, bend right, "g'"'] \arrow[r, phantom, "h \Downarrow\ \Downarrow h"] & \\
& b
\end{tikzcd}$$

for $g, g' : a \to b$ any two parallel morphisms in C and h any 2-morphism between them.

The strict surjection 2-functor on the right is

$$
\begin{tikzcd}
& b \arrow[dd, bend right, "f"'] \arrow[dd, bend left, "k"] \arrow[dd, phantom, "\overset{L}{\Rightarrow}"] \\
a \arrow[ur, bend left, "q"] \arrow[dr, bend right] \arrow[r, phantom, "F \Downarrow\ \overset{K}{\Downarrow}"] & \\
& b'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
b \arrow[r, bend left, "f"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow L"] & b'
\end{tikzcd} \; .
$$

The image of the injection is precisely the preimage under the surjection of the identity 2-morphism on the identity 1-morphisms . This means the sequence is exact. $\square$

4.3 Inner automorphisms

Often, for G any group, inner and outer automorphisms are regarded as sitting in a short exact sequence

$$\mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G)$$

of ordinary groups.

But we will find shortly that we ought to be regarding the conjugation automorphisms by two group elements which differ by an element in the center of the group as *different* inner automorphisms.

So adopting this point of view for ordinary groups, one gets instead the exact sequence

$$\mathrm{Z}(G) \longrightarrow \mathrm{Inn}'(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \; .$$

Of course this means setting $\mathrm{Inn}'(G) \simeq G$, which seems to make this step rather ill motivated. But it turns out that this degeneracy of concepts is a coincidence of low dimensions and will be lifted as we pass to inner automorphisms of higher groups.

First recall the standard definitions of center and automorphism of 2-groupoids:

Definition 9. *Given any strict 2-groupoid C,*

- *the automorphism 3-group*

  $$\mathrm{AUT}(C) := \mathrm{Aut}_{2\mathrm{Cat}}(C)$$

  is the 2-groupoid of equivalences on C;

- *the center of C*

  $$Z(C) := \Sigma\mathrm{AUT}(\mathrm{Id}_C)$$

  is the (suspended) automorphism 2-group of the identity on C.

Example. The automorphism 2-group of any ordinary group G (regarded as a 2-group $\mathrm{Disc}(G)$ with only identity morphisms)

$$\mathrm{AUT}(G) := \mathrm{AUT}(\Sigma G)$$

is that coming from the crossed module

$$G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G) \xrightarrow{\mathrm{Id}} \mathrm{Aut}(G)\,.$$

The center

$$Z(G) := Z(\Sigma G)$$

of any ordinary group is indeed the ordinary center of the group, regarded as a 1-object category.

To these two standard definitions, we add the following one, which is supposed to be the proper 2-categorical generalization of the concept of inner automorphisms.

Definition 10 (inner automorphisms)**.** *Given any strict* 2*-groupoid* C*, the tangent* 2*-groupoid*

$$\mathrm{INN}(C) := T_{\mathrm{Id}_C}(\mathrm{Mor}(2\mathrm{Cat}))$$

is, called the 2*-groupoid of inner automorphisms of* C*, and as such thought of as being equipped with the monoidal structure inherited from* $\mathrm{End}(C)$*.*

If the transformation starting at the identity is denoted q, it makes good sense to call the inner automorphism being the target of that transformation Ad_q:

$$\begin{tikzcd} C \arrow[r, bend left=50, "\mathrm{Id}"{name=U}] \arrow[r, bend right=50, "\mathrm{Ad}_q"'{name=D}] & C \arrow[from=U, to=D, Rightarrow, "q", "\sim"'] \end{tikzcd}\,.$$

A bigon of this form is an object in $\mathrm{INN}(C)$. The product of two such objects is the horizontal composition of these bigons in 2Cat. We shall spell this out in great detail for the case $C = \Sigma G_{(2)}$ in 5.

Proposition 2. *For* C *any strict 2-category, we have canonical morphisms*

$$\mathrm{Z}(C) \hookrightarrow \mathrm{INN}(C) \longrightarrow \mathrm{AUT}(C)$$

of strict 2-categories whose composition sends everything to the identity 2-morphism on the identity 1-morphism on the identity automorphism of C*.*

Moreover, this sits inside the exact sequence from proposition 1 as

$$\begin{tikzcd} Z(C) \arrow[r] \arrow[d, hook] & \mathrm{INN}(C) \arrow[r] \arrow[d, hook] & \mathrm{AUT}(C) \arrow[d, hook] \\ \mathrm{Mor}(\mathbf{C}) \arrow[r] & T\mathbf{C} \arrow[r] & \mathbf{C} \end{tikzcd}\,,$$

where $\mathbf{C} := \mathrm{Mor}(2\mathrm{Cat})$*.*

Proof. Recall that a morphism in $Z(C)$ is a transformation of the form

$$\begin{tikzcd} C \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Ad}_q = \mathrm{Id}"'] \arrow[r, phantom, "\sim\Downarrow q"] & C \end{tikzcd} .$$

This gives the obvious inclusion $Z(G) \hookrightarrow \mathrm{INN}(G)$. The morphism $\mathrm{INN}(G) \to \mathrm{AUT}(G)$ maps

$$\begin{tikzcd} C \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Ad}_q"'] \arrow[r, phantom, "\Downarrow q"] & C \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} C \arrow[r, "\mathrm{Ad}_q"] & C \end{tikzcd} .$$

$\square$

Remark. One would now want to define and construct the cokernel $\mathrm{OUT}(C)$ of the morphism $\mathrm{INN}(C) \to \mathrm{AUT}(C)$ and then say that

$$Z(C) \longrightarrow \mathrm{INN}(C) \longrightarrow \mathrm{AUT}(C) \longrightarrow \mathrm{OUT}(C)$$

is an exact sequence of 3-groups. But here we do not further consider this.

4.4 Inner automorphism 3-groups.

Now we apply the general concept of inner automorphisms to 2-groups. The following definition just establishes the appropriate shorthand notation.

Definition 11. *For $G_{(2)}$ a strict 2-group, we write*

$$\mathrm{INN}(G_{(2)}) := \mathrm{INN}(\Sigma G_{(2)})$$

for its 3-group of inner automorphisms.

In general this notation could be ambiguous, since one might want to consider the inner automorphisms of just the 1-groupoid underlying $G_{(2)}$. However, in the present context this will never occur and using the above definition makes a couple of expressions more manifestly appear as generalizations of familiar ones.

Example. For G an ordinary group, regarded as a discrete 2-group, one finds that

$$\mathrm{INN}(G) := T_{\mathrm{Id}_{\Sigma G}}(n\mathrm{Cat}) \simeq T_{\bullet}(\Sigma G)$$

is the codiscrete groupoid over the elements of G. Its nature as a groupoid is manifest from its realization as

$$\mathrm{INN}(G) = T_\bullet(\Sigma G)\,.$$

But it is also a (strict) 2-group. The monoidal structure is that coming from its realization as $\mathrm{INN}(G) := T_{\mathrm{Id}_{\Sigma G}}(n\mathrm{Cat})$. The crossed module corresponding to this strict 2-group is

$$G \xrightarrow{\mathrm{Id}} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G)\,.$$

The main point of interest for us is the generalization of this fact to strict 2-groups. One issue that one needs to be aware of then is that the above isomorphism $T_{\mathrm{Id}_{\Sigma G}}(n\mathrm{Cat}) \simeq T_\bullet(\Sigma G)$ becomes a mere inclusion.

Proposition 3. *For $G_{(2)}$ any strict 2-group, we have an inclusion*

$$T_\bullet \Sigma G_{(2)} \subset T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$$

of strict 2-groupoids.

This realizes $T_\bullet \Sigma G_{(2)}$ as a sub 2-groupoid of $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$.

Proof. The inclusion is essentially fixed by its action on objects: we define that an object in $T_\bullet \Sigma G_{(2)}$, which is a morphism

$$\bullet \xrightarrow{q} \bullet$$

in ΣG, is sent to the conjugation automorphism

$$\mathrm{Ad}_q \quad : \quad \Sigma G_{(2)} \quad \to \quad \Sigma G_{(2)}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, "q^{-1}"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "q"] & \bullet \end{tikzcd}$$

The transformation

$$\begin{tikzcd} \Sigma G_{(2)} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Ad}_q"'] \arrow[r, phantom, "\sim \Downarrow q"] & \Sigma G_{(2)} \end{tikzcd}\,.$$

connecting this to the identity is given by the component map

$$
\left(\bullet \xrightarrow{g} \bullet \right) \quad \mapsto \quad
\begin{tikzcd}
\bullet \arrow[rrr, "g"] \arrow[d, "q"] & & & \bullet \arrow[d, "q"] \arrow[dlll, Rightarrow, "\mathrm{Id}"'] \\
\bullet \arrow[r, "q^{-1}"'] & \bullet \arrow[r, "g"'] & \bullet \arrow[r, "q"'] & \bullet
\end{tikzcd}
.
$$

In general one could consider transformations whose component maps involve here a non-identity 2-morphism. The inclusion we are describing picks out excactly those transformations whose component map only involves identity 2-morphisms.

The crucial point to realize now is the form of the component maps of morphisms

$$
\begin{tikzcd}
 & \mathrm{Ad}_q \arrow[dd, "\mathrm{Ad}_F"] \\
\mathrm{Id}_{\Sigma G_{(2)}} \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "\Downarrow F"] & \\
 & \mathrm{Ad}_{q'}
\end{tikzcd}
$$

in $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$.

The corresponding component map equation is

$$
\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[d, "q"] \arrow[dd, bend right=60, "q'"'] & \bullet \arrow[d, "q"] \arrow[dl, Rightarrow, "\mathrm{Id}"'] \\
\bullet \arrow[r, "\mathrm{Ad}_q g"] \arrow[d, "f"] \arrow[l, phantom, "\overset{F}{\Leftarrow}"] & \bullet \arrow[d, "f"] \arrow[dl, Rightarrow, "\mathrm{Ad}_F(g)"'] \\
\bullet \arrow[r, "\mathrm{Ad}_{q'} g"] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[d, "q'"'] & \bullet \arrow[d, "q'"] \arrow[dl, Rightarrow, "\mathrm{Id}"'] \arrow[dr, "q"] & \\
\bullet \arrow[r, "\mathrm{Ad}_{q'} g"'] & \bullet & \bullet \arrow[l, "f"] \arrow[l, phantom, shift left=3, "\overset{F}{\Leftarrow}"]
\end{tikzcd}
.
$$

Solving this for Ad_F shows that this is given by conjugation

$$\mathrm{Ad}_F \quad : \quad (\bullet \xrightarrow{g} \bullet) \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[dd, "f"'] \arrow[dr, bend left, "q^{-1}"] & & & \bullet \arrow[dd, "f"] \\ & \bullet \arrow[r, "g"] & \bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "\Downarrow F"] & \\ \bullet \arrow[ur, bend right, "q'^{-1}"'] \arrow[u, phantom, "\Downarrow"] & & & \bullet \end{tikzcd}$$

with a morphism in $T_\bullet(\Sigma G_{(2)})$. And each such morphism in $T_\bullet(\Sigma G_{(2)})$ yields a morphism in $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$ this way.

Finally, 2-morphisms in $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$ between these morphisms

$$\begin{tikzcd} & \Sigma G_{(2)} \arrow[dl, dash, bend right] \arrow[dr, dash, bend left] & \\ \mathrm{Ad}_q \arrow[rr, bend left, "\mathrm{Ad}_F"] \arrow[rr, dashed, bend right, "\mathrm{Ad}_K"'] \arrow[rr, phantom, "\overset{\shortparallel}{\Downarrow} L"] \arrow[dr, bend right] & & \mathrm{Ad}_{q'} \arrow[dl, bend left] \\ & \Sigma G_{(2)} & \end{tikzcd}$$

come from component maps

$$\bullet \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "k"'] \arrow[r, phantom, "\Downarrow L"] & \bullet \end{tikzcd} \quad \in \ \mathrm{Mor}_2(\Sigma G_{(2)})\,.$$

A sufficient condition for these component maps to solve the required condition for modifications of pseudonatural transformations is that they make

$$\begin{tikzcd} & \bullet \arrow[dd, bend right, "f"'] \arrow[dd, bend left, "k"] \arrow[dd, phantom, "\Rightarrow L"] \\ \bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right] \arrow[r, phantom, "F \Downarrow \ \Downarrow K"] & \\ & \bullet \end{tikzcd}$$

2-commute. But this defines a 2-morphism in $T_\bullet \Sigma G_{(2)}$. And each such 2-morphism in $T_\bullet(\Sigma G_{(2)})$ yields a 2-morphism in $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$ this way. $\square$

The crucial point is that by the embedding

$$T_\bullet \Sigma G_{(2)} \subset T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$$

the former 2-category inherits the monoidal structure of the latter and hence becomes a 3-group in its own right. This 3-group is the object of interest here.

5 The 3-group $\mathrm{INN}_0(G_{(2)})$

Definition 12 ($\mathrm{INN}_0(G_{(2)})$)**.** *For $G_{(2)}$ any strict 2-group, the 3-group $\mathrm{INN}_0(G_{(2)})$ is, as a 2-groupoid, given by*

$$\mathrm{INN}_0(G_{(2)}) := T_\bullet \Sigma G_{(2)}$$

and equipped with the monoidal structure inherited from the embedding of proposition 3.

We now describe $\mathrm{INN}_0(G_{(2)})$ for $G_{(2)}$ coming from the crossed module

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

in more detail, in particular spelling out the monoidal structure. We extract the operations in the crossed module corresponding to the various compositions in $\mathrm{INN}_0(G_{(2)})$ and then finally identify the 2-crossed module encoded by this.

5.1 Objects

The objects of $\mathrm{INN}_0(G_{(2)})$ are exactly the objects of $G_{(2)}$, hence the elements of G:

$$\mathrm{Obj}(\mathrm{INN}(G_{(2)})) = G\,.$$

The product of two objects in $\mathrm{INN}(G_{(2)})$ is just the product in G.

5.2 Morphisms

The morphisms

$$g \to h$$

in $\mathrm{INN}(G_{(2)})$ are

$$\begin{aligned}
\mathrm{Mor}(\mathrm{INN}_0(G_{(2)})) &= \left\{ \begin{tikzcd} & \bullet \arrow[dd, "f"] \\ \bullet \arrow[ur, bend left, "g"] \arrow[dr, bend right, "h"'] \arrow[r, phantom, "\Downarrow F"] & \\ & \bullet \end{tikzcd} \;\middle|\; \begin{array}{c} f, g, h \in G,\ F \in H \\ h = t(F) f g \end{array} \right\} \\
&= \{(f, F; g) \mid f, g \in G, F \in H\}\,.
\end{aligned}$$

5.2.1 Composition

The composition of two such morphisms

$$
\begin{tikzcd}
& \bullet \arrow[d, "f_1"] \\
\bullet \arrow[ur, bend left, "q_1"] \arrow[ur, phantom, "\Downarrow F_1"] \arrow[r, "q'"] \arrow[dr, phantom, "\Downarrow F_2"] \arrow[dr, bend right, "q''"'] & \bullet \arrow[d, "f_2"] \\
& \bullet
\end{tikzcd}
.
$$

is in terms of group labels given by

$$
\begin{tikzcd}
& \bullet \arrow[d, "f_1"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[ur, phantom, "\Downarrow F_1"] \arrow[r] \arrow[dr, phantom, "\Downarrow F_2"] \arrow[dr, bend right] & \bullet \arrow[d, "f_2"] \\
& \bullet
\end{tikzcd}
=
\begin{tikzcd}
& \bullet \arrow[dd, "f_2 f_1"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[r, phantom, "\Downarrow F_2{}^{f_2}F_1"] \arrow[dr, bend right] & {} \\
& \bullet
\end{tikzcd}
.
$$

5.2.2 Product

Horizontal composition of automorphisms $\Sigma G_{(2)} \to \Sigma G_{(2)}$ gives the product in the 3-group $\mathrm{INN}(G_{(2)})$

Left whiskering of pseudonatural transformations

$$
\begin{tikzcd}
\Sigma G_{(2)} \arrow[r, "\mathrm{Ad}_g"] & \Sigma G_{(2)} \arrow[r, bend left=50, "\mathrm{Ad}_q"] \arrow[r, bend right=50, "\mathrm{Ad}_{q'}"'] \arrow[r, phantom, "\Downarrow \mathrm{Ad}_F"] & \Sigma G_{(2)}
\end{tikzcd}
$$

amounts to the operation

$$
\begin{tikzcd}
& \bullet \arrow[dd, "f"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[r, phantom, "\Downarrow F"] \arrow[dr, bend right] & {} \\
& \bullet
\end{tikzcd}
\mapsto
\begin{tikzcd}
& & \bullet \arrow[dd, "f"] \\
\bullet \arrow[r, "g"] & \bullet \arrow[ur, bend left, "q"] \arrow[r, phantom, "\Downarrow F"] \arrow[dr, bend right] & {} \\
& & \bullet
\end{tikzcd}
=
\begin{tikzcd}
& \bullet \arrow[dd, "f"] \\
\bullet \arrow[ur, bend left, "qg"] \arrow[r, phantom, "\Downarrow F"] \arrow[dr, bend right] & {} \\
& \bullet
\end{tikzcd}
$$

on the corresponding triangles.

Right whiskering of pseudonatural transformations

$$
\begin{tikzcd}
\Sigma G_{(2)} \arrow[r, bend left, "\mathrm{Ad}_q"] \arrow[r, bend right, "\mathrm{Ad}_{q'}"'] \arrow[r, phantom, "\Downarrow \mathrm{Ad}_F"] & \Sigma G_{(2)} \arrow[r, "\mathrm{Ad}_q"] & \Sigma G_{(2)}
\end{tikzcd}
$$

amounts to the operation

$$
\begin{tikzcd}
 & \bullet \arrow[dd, "f"] & & & \bullet \arrow[r, "g"] \arrow[dd, "f"] \arrow[ddr, phantom, "="] & \bullet \arrow[dd, dashed, "gfg^{-1}"] & & & \bullet \arrow[dd, "gfg^{-1}"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F"] & & \mapsto & \bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F"] & & & = & \bullet \arrow[ur, bend left, "gq"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow {}^{g}F"] & \\
 & \bullet & & & \bullet \arrow[r, "g"'] & \bullet & & & \bullet
\end{tikzcd}
$$

on the corresponding triangles.

Since 2Cat is a Gray category, the horizontal composition of pseudonatural transformations

$$
\begin{tikzcd}
\Sigma G_{(2)} \arrow[r, bend left, "\mathrm{Ad}_{q_1}"] \arrow[r, bend right, "\mathrm{Ad}_{q_1'}"'] \arrow[r, phantom, "\Downarrow \mathrm{Ad}_{F_1}"] & \Sigma G_{(2)} \arrow[r, bend left, "\mathrm{Ad}_{q_2}"] \arrow[r, bend right, "\mathrm{Ad}_{q_2'}"'] \arrow[r, phantom, "\Downarrow \mathrm{Ad}_{F_2}"] & \Sigma G_{(2)}
\end{tikzcd}
$$

is ambiguous. We shall agree to read this as

$$
\begin{tikzcd}
\Sigma G_{(2)} \arrow[r, bend left, "\mathrm{Ad}_{q_1}"] \arrow[r, bend right, "\mathrm{Ad}_{q_1'}"'] \arrow[r, phantom, "\Downarrow \mathrm{Ad}_{F_1}"] & \Sigma G_{(2)} \arrow[r, bend left, "\mathrm{Ad}_{q_2}"] & \Sigma G_{(2)} \\
\Sigma G_{(2)} \arrow[r, bend right, "\mathrm{Ad}_{q_1'}"'] & \Sigma G_{(2)} \arrow[r, bend left, "\mathrm{Ad}_{q_2}"] \arrow[r, bend right, "\mathrm{Ad}_{q_2'}"'] \arrow[r, phantom, "\Downarrow \mathrm{Ad}_{F_2}"] & \Sigma G_{(2)}
\end{tikzcd}
.
$$

The corresponding operation on triangles labelled in the crossed module is

$$
\begin{tikzcd}
& \bullet \arrow[dd, "f_1"] & \\
\bullet \arrow[ur, bend left, "q_1"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F_1"] & & \bullet \arrow[dd, "f_2"] \\
& \bullet \arrow[ur, bend left, "q_2"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F_2"] & \\
& & \bullet
\end{tikzcd}
=
\begin{tikzcd}
& & \bullet \arrow[dd, "q_2 f_1 q_2^{-1}"] \\
& \bullet \arrow[ur, bend left, "q_2"] \arrow[dd, "f_1"] \arrow[r, phantom, "="] & \\
\bullet \arrow[ur, bend left, "q_1"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F_1"] & & \bullet \arrow[dd, "f_2"] \\
& \bullet \arrow[ur, bend left, "q_2"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F_2"] & \\
& & \bullet
\end{tikzcd}
$$

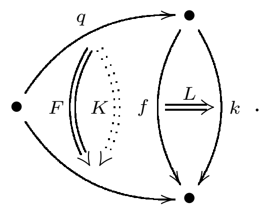
$$
=
\begin{tikzcd}
& \bullet \arrow[d, "q_2 f_1 q_2^{-1}"] \\
\bullet \arrow[ur, bend left, "q_2 q_1"] \arrow[r] \arrow[dr, bend right] \arrow[ur, phantom, "\Downarrow {}^{q_2}F_1"] \arrow[dr, phantom, "\Downarrow F_2"] & \bullet \arrow[d, "f_2"] \\
& \bullet
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
& \bullet \arrow[dd, "f_2 q_2 f_1 q_2^{-1}"] \\
\bullet \arrow[ur, bend left, "q_1"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F_2\, {}^{f_2 q_2}F_1"] & \\
& \bullet
\end{tikzcd}
$$

The non-identitcal isomorphism which relates this to the other possible way to evaluate the horizontal composition of pseudonatural transformations gives rise to the Peiffer lifting of the corresponding 2-crossed module. This is discussed in 6.4.

5.3 2-Morphisms

The 2-morphisms in $\mathrm{INN}_0(G_{(2)})$ are given by diagrams

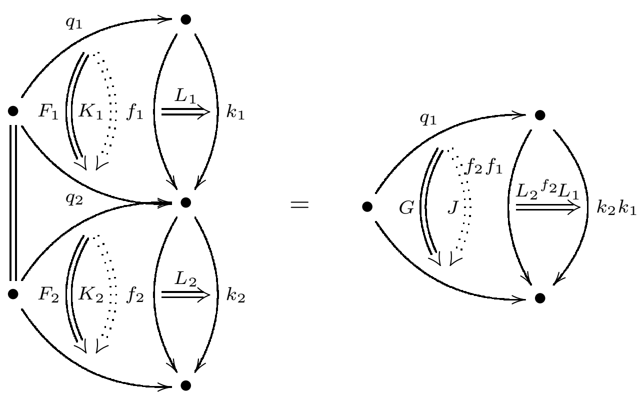


In terms of the group labels this means that $L \in H$ satisfies

$$L = K^{-1}F \,. \tag{6}$$

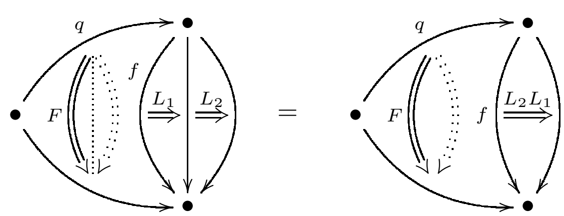
5.3.1 Composition

The horizontal composition of 2-morphisms in $\mathrm{INN}_0(G_{(2)})$ is given by



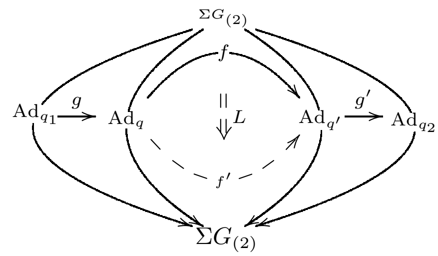
$$G = F_2{}^{f_2}F_1, \quad J = K_2{}^{k_2}K_1$$

and vertical composition by

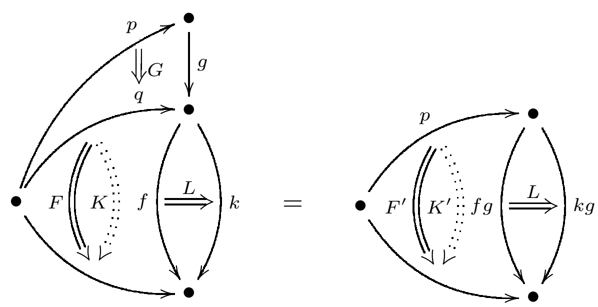


(Notice that these compositions do go horizontally and vertically, respectively, once we rotate such that the bigons have the standard orientation.)

Notice that whiskering along 1-morphisms

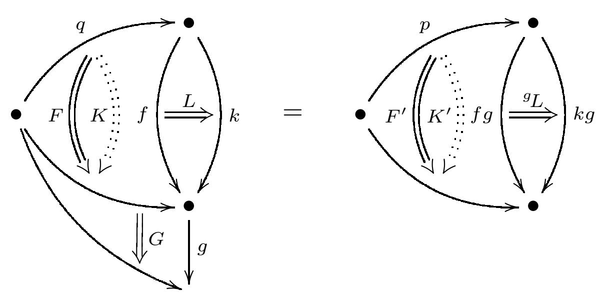


acts on the component maps as



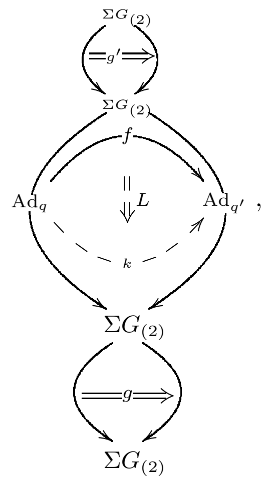
$$F' = F^f G, \quad K' = K^k G$$

and

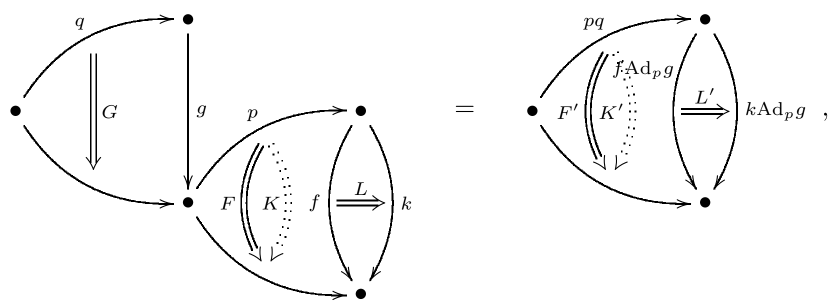


$$F' = G^g F, \quad K' = G^k K\,.$$

There is one more type of whiskering possible with 2-morphisms,



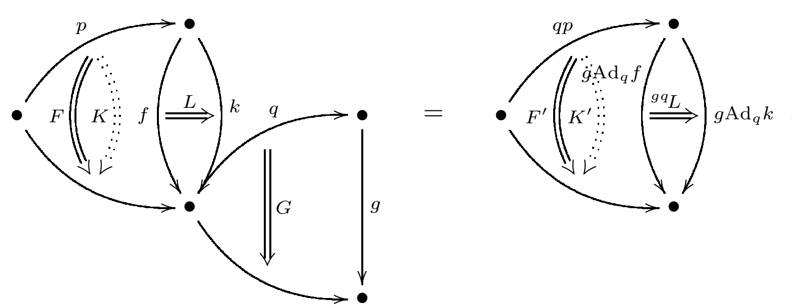
which acts in the following way on the components:



where

$$
\begin{aligned}
F' &= F^{fp}G, \\
K' &= K^{kp}G, \\
L' &= {}^{kp}G^{-1}L^{fp}G.
\end{aligned}
$$

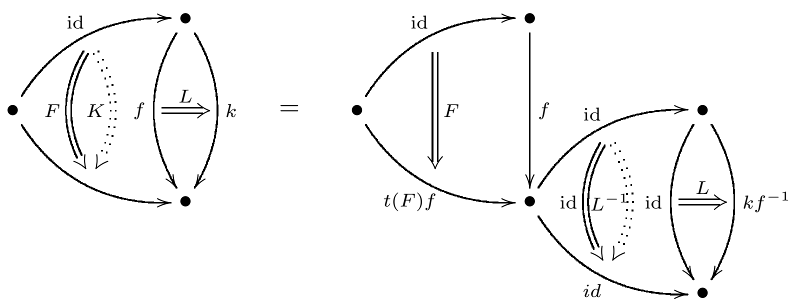
and



where

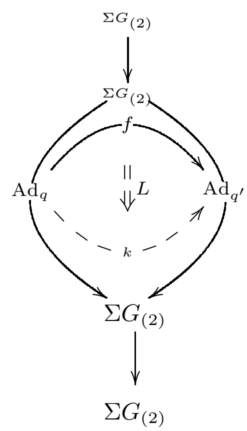
$$\begin{aligned} F' &= G^{gq}F, \\ K' &= G^{gq}K. \end{aligned}$$

An important case of this is:



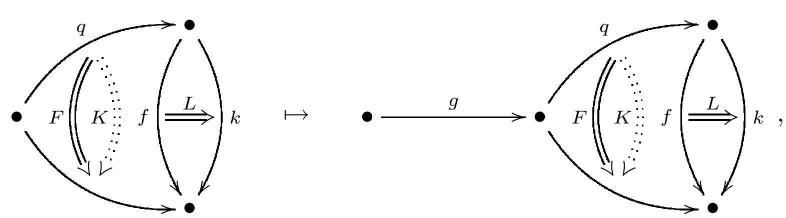
5.3.2 Product

The whiskering along objects

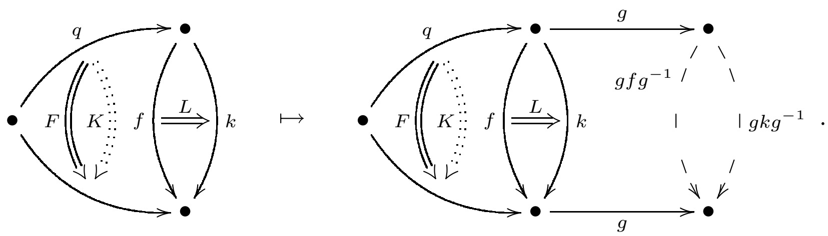


gives the product of objects with 2-morphisms in the 3-group $\mathrm{INN}(G_{(2)})$. Its action on 2-morphisms, which we have already disucssed, extends in a simple way to 3-morphisms:

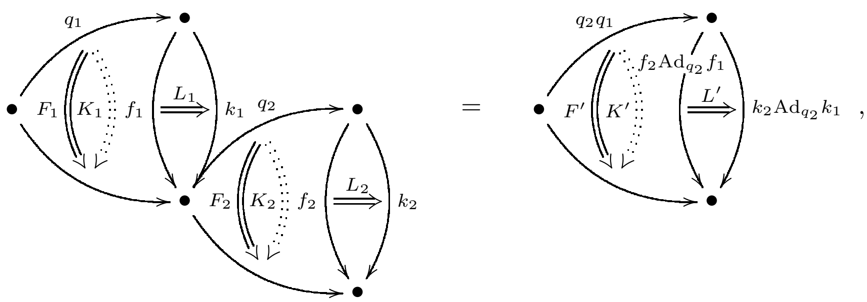
left whiskering along an object acts as



while right whiskering along an object acts as



To calculate the product of a pair of 2-morphisms, we use the fact that a 2-morphism is uniquely determined by its source and target.



with

$$
\begin{aligned}
F' &= F_2{}^{f_2 q_2} F_1 \\
K' &= K_2{}^{k_2 q_2} K_1 \\
L' &= L_2{}^{f_2 q_2} L_1
\end{aligned}
$$

6 Properties of $\mathrm{INN}_0(G_{(2)})$

6.1 Structure morphisms

We have defined $\Sigma(\mathrm{INN}_0(G_{(2)}))$ essentially as a sub 3-category of 2Cat. The latter is a Gray category, in that it is a 3-category which is strict except for the exchange law for composition of 2-morphisms. Accordingly, also $\Sigma(\mathrm{INN}_0(G_{(2)}))$ is strict except for the exchange law for 2-morphisms.

This means that as a mere 2-groupoid (forgetting the monoidal structure) $\mathrm{INN}_0(G_{(2)})$ is strict.

6.1.1 Strictness as a 2-groupoid

Proposition 4. *The underlying 2-groupoid of* $\mathrm{INN}_0(G_{(2)})$ *is strict.*

Proof. This follows from the rules for horizontal and vertical composition of 2-morphisms in $\mathrm{INN}_0(G_{(2)})$ – displayed in 5.3.1 – and the fact that $G_{(2)}$ itself is a strict 2-group, by assumption. $\square$

But the product 2-functor on $\mathrm{INN}_0(G_{(2)})$ respects horizontal composition in $\mathrm{INN}_0(G_{(2)})$ only weakly. In the language of 2-groups, this corresponds to a failure of the Peiffer identity

6.2 Trivializability

Proposition 5. *The 2-groupoid* $\mathrm{INN}_0(G_{(2)})$ *is connected,*

$$\pi_0(\mathrm{INN}(G_1)) = 1\,.$$

Proof. For any two objects q and q' there is the morphism

$$\begin{tikzcd}
 & \bullet \arrow[dd, "q'q^{-1}"] \\
\bullet \arrow[ur, bend left, "q"] \arrow[dr, bend right, "q'"'] \arrow[r, phantom, "\Downarrow \mathrm{id}"] & \\
 & \bullet
\end{tikzcd}$$

$\square$

Proposition 6. *The Hom-groupoids of the 2-category* $\mathrm{INN}_0(G_{(2)})$ *are codiscrete, meaning that they have precisely one morphism for every ordered pair of objects.*

Proof. By equation (6) there is at most one 2-morphism between any parallel pair of morphisms in $\mathrm{INN}_0(G_{(2)})$. For there to be any such 2-morphism at all, the two group elements f and k in the diagram above (6) have to satisfy $kf^{-1} \in \mathrm{im}(t)$. But by using the source-target matching condition for F and K one readily sees that this is always the case. $\square$

Theorem 1. *The 3-group* $\mathrm{INN}_0(G_{(2)})$ *is equivalent to the trivial 3-group. If* $G_{(2)}$ *is a Lie 2-group, then* $\mathrm{INN}_0(G_{(2)})$ *is equivalent to the trivial Lie 3-group even as a Lie 3-group.*

Proof. Equivalence of 3-groups $G_{(3)}$, $G'_{(3)}$ is, by definition, that of the corresponding 1-object 3-groupoids $\Sigma G_{(3)}$, $\Sigma G'_{(3)}$. For showing equivalence with the trivial 3-group, it suffices to exhibit a pseudonatural transformation of 3-functors

$$\mathrm{id}_{\Sigma(\mathrm{INN}_0(G_{(2)}))} \to I_{\Sigma(\mathrm{INN}_0(G_{(2)}))}\,,$$

where $I_{\Sigma(\mathrm{INN}_0(G_{(2)}))}$ sends everything to the identity on the single object of $\Sigma\mathrm{INN}_0(G_{(2)})$. Such a transformation is obtained by sending the single object to the identity 1-morphism on that object and sending any 1-morphism q to the 2-morphism $q \to \mathrm{id}$ from prop 5. By prop 6 this implies the existence of a unique assignment of a 3-morphism to any 2-morphism such that we do indeed obtain the component map of a pseudonatural transformation of 3-functors. By construction, this is clearly smooth when $G_{(2)}$ is Lie. $\square$

6.3 Universality

Theorem 2. *We have a short exact sequence of strict 2-groupoids*

$$\mathrm{Disc}(G_{(2)}) \hookrightarrow \mathrm{INN}_0(G_{(2)}) \twoheadrightarrow \Sigma G_{(2)} \ .$$

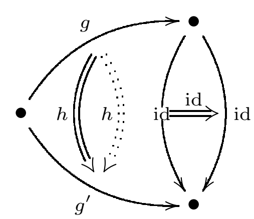
Proof. This is just proposition 1, after noticing that

$$\mathrm{Mor}(\Sigma G_{(2)}) = G_{(2)} \, .$$

$\square$

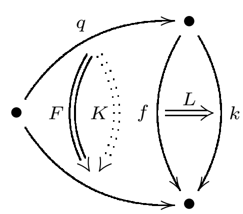
So the strict inclusion 2-functor on the left is

$$\left(g \xrightarrow{h} g' \right) \quad \mapsto$$

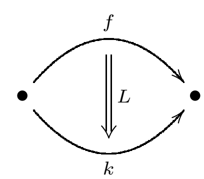


,

while the strict surjection 2-functor on the right is



$\mapsto$



.

6.4 The corresponding 2-crossed module

We now extract the structure of a 2-crossed module from $\mathrm{INN}_0(G_{(2)})$. First, let

$$\mathrm{Mor}_1^I = \mathrm{Mor}_1(\mathrm{INN}_0(G_{(2)}))|_{s^{-1}(\mathrm{Id})}$$

and

$$\mathrm{Mor}_2^I = \mathrm{Mor}_2(\mathrm{INN}_0(G_{(2)}))|_{s^{-1}(\mathrm{id}_{\mathrm{Id}})}$$

be subgroups of the 1- and 2-morphisms of $\mathrm{INN}_0(G_{(2)})$ respectively.

Proposition 7. *The group of 1-morphisms in* $\mathrm{INN}_0(G_{(2)})$ *starting at the identity object form the semidirect product group*

$$\mathrm{Mor}_1^I = G \ltimes H$$

under the identitfication

$$\begin{tikzcd}
 & \bullet \arrow[dd, "f"] \\
\bullet \arrow[ur, bend left, "\mathrm{Id}"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F"] & \\
 & \bullet
\end{tikzcd} \quad \mapsto \quad (f, F)$$

in that

$$\begin{tikzcd}
 & \bullet \arrow[dd, "f_1"] & \\
\bullet \arrow[ur, bend left, "\mathrm{Id}"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F_1"] & & \bullet \arrow[dd, "f_2"] \\
 & \bullet \arrow[ur, bend left, "\mathrm{Id}"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F_2"] & \\
 & & \bullet
\end{tikzcd} \quad = \quad \begin{tikzcd}
 & \bullet \arrow[dd, "f_2 f_1"] \\
\bullet \arrow[ur, bend left, "\mathrm{Id}"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F_2 {}^{f_2}F_1"] & \\
 & \bullet
\end{tikzcd} .$$

Proof. Use composition in $\Sigma G_{(2)}$. ◻

We have the obvious group homomorphism which is just the restriction of the target map

$$\partial_1 : \mathrm{Mor}_1^I \to \mathrm{Obj} := \mathrm{Obj}(\mathrm{INN}_0(G_{(2)}))$$

given by

$$\partial_1 \quad : \quad \begin{tikzcd}
 & \bullet \arrow[dd, "f"] \\
\bullet \arrow[ur, bend left, "\mathrm{Id}"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F"] & \\
 & \bullet
\end{tikzcd} \quad \mapsto \quad t(F)f \,.$$

This and the following constructions are to be compared with definition 4. There is an obvious action on Mor_1^I:

$$
\begin{tikzcd}
& \bullet \arrow[dd, "f"] \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F"] & \\
& \bullet
\end{tikzcd}
\mapsto
\begin{tikzcd}
& & \bullet \arrow[dd, "f"] \arrow[r, "g"] & \bullet \arrow[dd, dashed, "gfg^{-1}"] \\
\bullet \arrow[r, "g^{-1}"] & \bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow F"] & \arrow[r, phantom, "="] & \\
& & \bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
& \bullet \arrow[dd, "gfg^{-1}"] \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow {}^{g}F"] & \\
& \bullet
\end{tikzcd}
\tag{7}
$$

This action almost gives us a crossed module $\mathrm{Mor}_1^I \to \mathrm{Obj}$. But not quite, since the Peiffer identity holds only up to 3- isomorphism.

To see this, let

$$
g = \partial_1 \left(
\begin{tikzcd}
& \bullet \arrow[dd, "h"] \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & \\
& \bullet
\end{tikzcd}
\right) = t(H)h.
$$

For the Peiffer identity to hold we need the action (7) to be equal to the adjoint action of the 2-cell $(h, H; id)$. To see that this fails, first notice that the inverse of the approriate 2-cell considered as an element in the group Mor_1^I is

$$
\left(
\begin{tikzcd}
& \bullet \arrow[dd, "h"] \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "t(H)h"'] \arrow[r, phantom, "\Downarrow H"] & \\
& \bullet
\end{tikzcd}
\right)^{-1}
=
\begin{tikzcd}
& \bullet \arrow[dd, "h^{-1}"] \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "h^{-1}t(H)^{-1}"'] \arrow[r, phantom, "\Downarrow {}^{h^{-1}}H^{-1}"] & \\
& \bullet
\end{tikzcd}
.
$$

Therefore the conjugation is

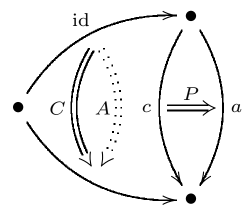
$$
\begin{tikzcd}
 & \bullet \arrow[dd, "h^{-1}"] & & & \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "g^{-1}"'] \arrow[r, phantom, "\Downarrow {}^{h^{-1}}H^{-1}"] & & \bullet \arrow[dd, "f"] & & \\
 & \bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "t(F)f"'] \arrow[r, phantom, "\Downarrow F"] & & \bullet \arrow[dd, "h"] & \\
 & & \bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & & \\
 & & & \bullet &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \bullet \arrow[dd, "h^{-1}"] & & \bullet \arrow[dd, "f"] \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "g^{-1}"'] \arrow[r, phantom, "\Downarrow {}^{h^{-1}}H^{-1}"] & & \bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dd, "f"] \arrow[r, phantom, "="] & \\
 & \bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "t(F)f"'] \arrow[r, phantom, "\Downarrow F"] & & \bullet \arrow[dd, "h"] \\
 & & \bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & \\
 & & & \bullet
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & & \bullet \arrow[dd, "h^{-1}"] \\
 & \bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dd, "h^{-1}"] \arrow[r, phantom, "="] & \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "g^{-1}"'] \arrow[r, phantom, "\Downarrow {}^{h^{-1}}H^{-1}"] & & \bullet \arrow[dd, "hf"] \\
 & \bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "gt(F)f"'] \arrow[r, phantom, "\Downarrow H\,{}^{h}F"] & \\
 & & \bullet
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & \bullet \arrow[dd, "hfh^{-1}"] \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "gt(F)fg^{-1}"'] \arrow[r, phantom, "\Downarrow H^{-1}\,{}^{h}F^{hfh^{-1}}H"] & \\
 & \bullet
\end{tikzcd}
\tag{8}
$$

$$
\neq \quad
\begin{tikzcd}
 & \bullet \arrow[dd, "gfg^{-1}"] \\
\bullet \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend right, "gt(F)fg^{-1}"'] \arrow[r, phantom, "\Downarrow {}^{g}F"] & \\
 & \bullet
\end{tikzcd}
\tag{9}
$$

Though the Peiffer identity does not hold, both actions give rise to 2-cells with the same source and target, and hence define a 3-cell P. Denote the 2-cell (8) by $(c, C; id)$ and the 2-cell (9) by $(a, A; id)$ (for $c$onjugation and $a$ction respectively).



Then

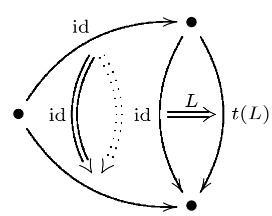
$$
\begin{aligned}
P &= A^{-1}C \\
&= {}^{g}F^{-1}\left(H^{h}F^{hfh^{-1}}H^{-1}\right) \\
&= {}^{g}F^{-1}\left({}^{t(H)h}FH^{hfh^{-1}}H^{-1}\right) \\
&= {}^{g}F^{-1}\left({}^{g}FH^{hfh^{-1}}H^{-1}\right) \\
&= H^{hfh^{-1}}H^{-1}
\end{aligned}
$$

However, what we really want is the Peiffer lifting, which will be a 3-cell with source the identity 2-cell. Hence,

Proposition 8. *The group of 2-morphisms in* $\mathrm{INN}_0(G_{(2)})$ *starting at the identity arrow on the identity object form the group*

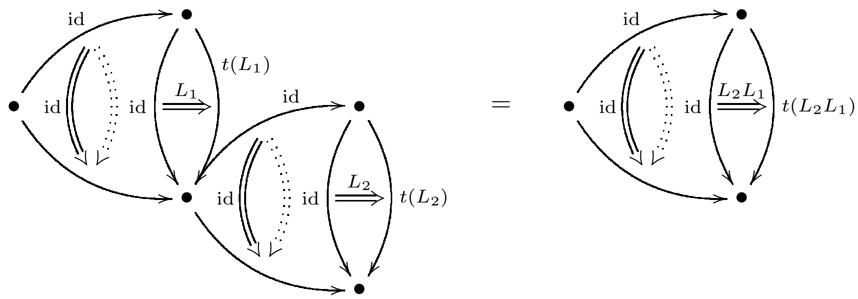
$$\mathrm{Mor}_2^I = H$$

under the identitfication



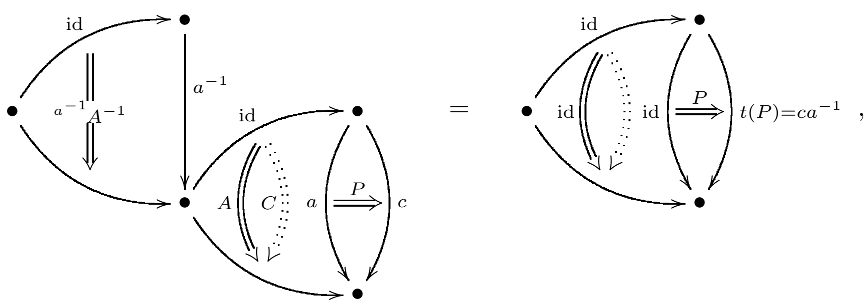
$$\mapsto \quad L$$

in that



Proof. Use the multiplication of 2-morphisms. $\square$

So, we whisker the 3-cell $(P; a, A; \mathrm{id})$ above with the inverse of $(a, A; \mathrm{id})$:

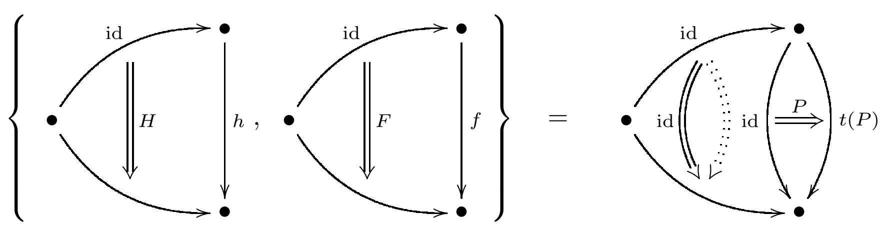


and the back face is necessarily P^{-1}.

Definition 13 (Peiffer lifting)**.** *Define the map*

$$\{\cdot,\cdot\} : \mathrm{Mor}_1^I \times \mathrm{Mor}_1^I \to \mathrm{Mor}_2^I$$

by

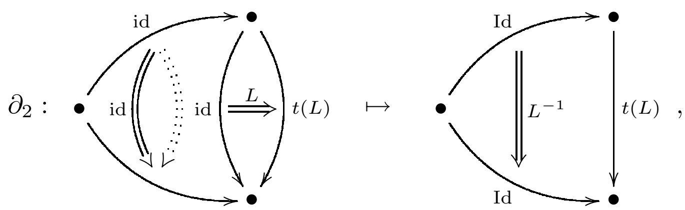


$$P = H^{hfh^{-1}}H^{-1}$$

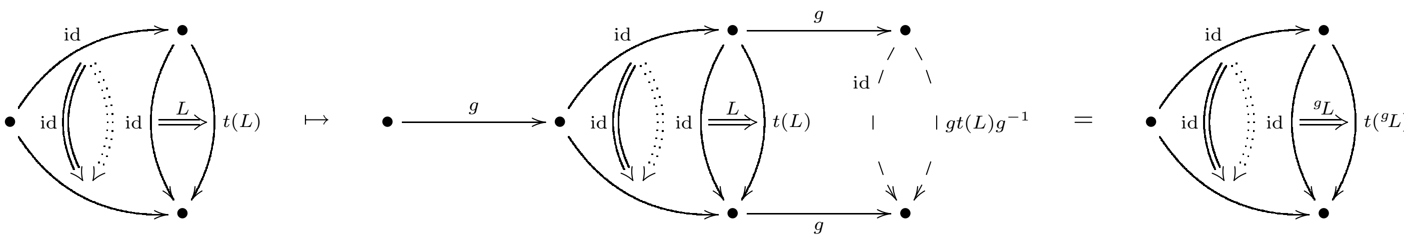
Now define the homomorphism

$$\partial_2 : \mathrm{Mor}_2^I \to \mathrm{Mor}_1^I$$

by



which is again the restriction of the target map. Note there is an action of Obj on Mor_2^I:



(10)

Clearly $\partial_2 \circ \partial_1$ is the constant map at the identity, and im ∂_2 is a normal subgroup of ker ∂_1, so

$$\mathrm{Mor}_2^I \xrightarrow{\partial_2} \mathrm{Mor}_1^I \xrightarrow{\partial_1} \mathrm{Obj} \tag{11}$$

is a sequence. We let the action of Obj on the other two groups be as described above in (7) and (10), and the maps ∂_2 and ∂_1 are clearly equivariant for this action.

Proposition 9. *The map $\{\cdot, \cdot\}$ does indeed satisfy the properties of a Peiffer lifting, and (11) is a 2- crossed module.*

Proof. The first condition holds by definition, the second and the last one are easy to check. The others are tedious. It is easy, using the crossed module properties of $H \to G$, to calculate that the actions of Mor_1^I on Mor_2^I as defined from $\mathrm{INN}_0(G_{(2)})$ and as defined via $\{\cdot, \cdot\}$ are the same. □

Since im $\partial_2 =$ ker ∂_1, ∂_2 is injective and ∂_1 is onto, this shows that (11) has trivial homology and provides us with another proof that $\mathrm{INN}_0(G_{(2)})$ is contractible.

6.4.1 Relation to the mapping cone of $H \to G$

Given a crossed square

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "u"'] & M \arrow[d, "v"] \\ N \arrow[r, "g"] & P \end{tikzcd}$$

with structure map $h : N \times M \to L$, Conduché [3]gives the Peiffer lifting of the mapping cone

$$L \longrightarrow N \ltimes M \longrightarrow P$$

as

$$\{(g,h),(k,l)\} = h(gkg^{-1}, h).$$

Recall from 2.3 that the identity map on $t : H \to G$ is a crossed square with

$$h(g,h) = h^g h^{-1},$$

so the mapping cone is a 2-crossed module

$$H \xrightarrow{\partial_2} G \ltimes H \xrightarrow{\partial_1} G,$$

where

$$d_2(h) = (t(h), h^{-1}), \qquad d_1(g,h) = t(h)g,$$

and with Peiffer lifting

$$\{(g_1,h_1),(g_2,h_2)\} = {h_1}^{g_1 g_2 g_1^{-1}} h_1^{-1}.$$

which is what we found for $\mathrm{INN}_0(G_{(2)})$.

More precisely,

Definition 14. *A morphism ψ of 2-crossed modules is a map of the underlying complexes*

$$\begin{tikzcd} L_1 \arrow[r, "\partial_2"] \arrow[d, "\psi_L"'] & M_1 \arrow[r, "\partial_1"] \arrow[d, "\psi_M"'] & N_1 \arrow[d, "\psi_N"'] \\ L_2 \arrow[r, "\partial_2"'] & M_2 \arrow[r, "\partial_1"'] & N_2 \end{tikzcd}$$

such that ψ_L, ψ_M and ψ_N are equivariant for the N- and M-actions, and

$$\{\psi_M(\cdot), \psi_M(\cdot)\}_2 = \psi_L(\{\cdot,\cdot\}_1).$$

Using propositions 7 and 8, we have a map

$$
\begin{tikzcd}
\mathrm{Mor}_2^I \arrow[r, "\partial_2"] \arrow[d, "\simeq"'] & \mathrm{Mor}_1^I \arrow[r, "\partial_1"] \arrow[d, "\simeq"] & \mathrm{Obj} \arrow[d, "\simeq"] \\
H \arrow[r, "d_2"'] & G \ltimes H \arrow[r, "d_1"'] & G
\end{tikzcd}
$$

and the actions and Peiffer lifting agree, so

Proposition 10. *The 2-crossed module associated to* $\mathrm{INN}_0(G_(2))$ *is isomorphic to the mapping cone of the identity map on the crossed module associated to* $G_{(2)}$.

———

7 Universal n-bundles

In order to put the relevance of the 3-group $\mathrm{INN}_0(G_{(2)})$ in perspective, we further illuminate our statement, 3.1, that $\mathrm{INN}_0(G_{(2)})$ *plays the role of the universal* $G_{(2)}$*-bundle*. An exhaustive discussion will be given elsewhere.

7.1 Universal 1-bundles in terms of $\mathrm{INN}(G)$

Let $\pi : Y \to X$ be a good cover of a space X and write $Y^{[2]} := Y \times_X Y$ for the corresponding groupoid.

Definition 15 (G-cocycles)**.** *A G-(1-)cocycle on X is a functor*

$$g : Y^{[2]} \to \Sigma G\,.$$

This functor can be understood as arising from a choice

$$\pi^* P \xrightarrow[\sim]{t} Y \times G$$

of trivialization of a principal right G-bundle $P \to X$ (which is essentially just a map to G) as

$$g := \pi_2^* t \circ \pi_1^* t^{-1}\,,$$

by noticing that G-equivariant isomorphisms

$$G \to G$$

are in bijection with elements of G

$$g(x,y) : h \mapsto g(x,y)h$$

acting from the left.

Observation 1 (G-bundles as morphisms of sequences of groupoids)**.** *Given a G-cocycle on X as above, its pullback along the exact sequence*

$$\mathrm{Disc}(G) \longrightarrow \mathrm{INN}(G) \longrightarrow \Sigma G \ ,$$

which we write as

$$\begin{tikzcd}
Y \times \mathrm{Disc}(G) \arrow[r] \arrow[d] & Y^{[2]} \times_g \mathrm{INN}(G) \arrow[r] \arrow[d] & Y^{[2]} \arrow[r] \arrow[d, "g"] & X \arrow[d] \\
\mathrm{Disc}(G) \arrow[r] & \mathrm{INN}(G) \arrow[r] & \Sigma G \arrow[r] & \{\bullet\}
\end{tikzcd} ,$$

produces the bundle of groupoids

$$Y^{[2]} \times_g \mathrm{INN}(G) \longrightarrow Y^{[2]}$$

which plays the role of the total space of the G-bundle classified by g.

This should be compared with the simplicial constructions described, for instance, in [4].

Remark. Using the fact that $\mathrm{INN}(G)$ is a 2-group, and using the injection $\mathrm{Disc}(G) \to \mathrm{INN}(G)$ we naturally obtain the G-action on $Y^{[2]} \times_g \mathrm{INN}(G)$.

Remark. Notice that this is closely related to the integrated Atiyah sequence

$$\mathrm{Ad}P \longrightarrow P \times_G P \longrightarrow X \times X$$

of groupoids over $X \times X$ coming from the G-principal bundle $P \to X$:

$$\mathrm{Ad}P \longrightarrow P \times_G P \longrightarrow X \times X$$

$$\begin{tikzcd}
Y \times \mathrm{Disc}(G) \arrow[r] \arrow[d] & Y^{[2]} \times_g \mathrm{INN}(G) \arrow[r] \arrow[d] & Y^{[2]} \arrow[r] \arrow[d, "g"] & X \arrow[d] \\
\mathrm{Disc}(G) \arrow[r] & \mathrm{INN}(G) \arrow[r] & \Sigma G \arrow[r] & \{\bullet\}
\end{tikzcd} .$$

We now make precise in which sense, in turn, $Y^{[2]} \times_g \mathrm{INN}(G)$ plays the role of the total space of the G-bundle characterized by the cocycle g.

To reobtain the G-bundle $P \to X$ from the groupoid $Y^{[2]} \times_g \mathrm{INN}(G)$ we form the pushout of

$$
\begin{tikzcd}
Y^{[2]} \times_g \mathrm{INN}(G) \arrow[r, "\text{target}"] \arrow[d, "\text{source}"'] & Y \times G \ . \\
Y \times G
\end{tikzcd}
\tag{12}
$$

Proposition 11. *If g is the cocycle classifying a G-bundle P on X, then the pushout of 12 is (up to isomorphism) that very G-bundle P.*

Proof. Consider the square

$$
\begin{tikzcd}
Y^{[2]} \times_g \mathrm{INN}(G) \arrow[rr, "\text{target}"] \arrow[dd, "\text{source}"'] & & Y \times G \arrow[d, "t^{-1}"] \\
& & \pi^* P \arrow[d] \\
Y \times G \arrow[r, "t^{-1}"] & \pi^* P \arrow[r] & P
\end{tikzcd}
,
$$

where $t : \ \pi^* P \xrightarrow{\sim} Y \times G$ is the local trivialization of P which gives rise to the transition function g. Then the diagram commutes by the very definition of g. Since t is an isomorphism and since $\pi^* P \to P$ is locally an isomorphism, it follows that this is the universal pushout. $\square$

7.2 Universal 2-bundles in terms of $\mathrm{INN}_0(G_{(2)})$

Now let $G_{(2)}$ be any strict 2-group. Let $Y^{[3]}$ be the 2-groupoid whose 2-morphisms are triples of lifts to Y of points in X. A principal $G_{(2)}$-2-bundle [9, 10] has local trivializations characterized by 2-functors

$$g : Y^{[3]} \to \Sigma G_{(2)} \, .$$

Definition 16 ($G_{(2)}$-cocycles)**.** *A $G_{(2)}$-(2-)cocycle on X is a 2-functor*

$$g : Y^{[3]} \to \Sigma G_{(2)} \, .$$

(Instead of 2-functors on $Y^{[3]}$ one could use pseudo functors on $Y^{[2]}$.)

As before, we can pull these back along our exact sequence of 2-groupoids 3.1

$$\mathrm{Disc}(G_{(2)}) \longrightarrow \mathrm{INN}(G_{(2)}) \longrightarrow \Sigma G_{(2)}$$

to obtain

$$
\begin{tikzcd}
Y \times \mathrm{Disc}(G_{(2)}) \arrow[r] \arrow[d] & Y^{[3]} \times_g \mathrm{INN}(G_{(2)}) \arrow[r] \arrow[d] & Y^{[3]} \arrow[r] \arrow[d, "g"] & X \arrow[d] \\
\mathrm{Disc}(G_{(2)}) \arrow[r] & \mathrm{INN}(G_{(2)}) \arrow[r] & \Sigma G_{(2)} \arrow[r] & \{\bullet\}
\end{tikzcd}
.
$$

We reconstruct the total 2-space of the 2-bundle by forming the weak pushout of

$$
\begin{tikzcd}
Y^{[3]} \times_g \mathrm{INN}(G_{(2)}) \arrow[r, "\text{target}"] \arrow[d, "\text{source}"'] & Y \times G_{(2)} \\
Y \times G_{(2)}
\end{tikzcd}
. \tag{13}
$$

Here "source" and "target" are defined relative to the inclusion

$$
Y \times \mathrm{Disc}(G_{(2)}) \hookrightarrow Y^{[2]} \times_g \mathrm{INN}(G_{(2)})\,.
$$

This means that for a given 1-morphism

$$
\begin{tikzcd}
 & \bullet \arrow[dd, "f:=g(x,y)"] & x \arrow[dd] \\
\bullet \arrow[ur, bend left, "q_1"] \arrow[dr, bend right, "q_2"'] \arrow[r, phantom, "\Downarrow F"] & & \\
 & \bullet & y
\end{tikzcd}
$$

in $Y^{[3]} \times_g \mathrm{INN}(G_{(2)})$ (for any $x, y \in Y$ with $\pi(x) = \pi(y)$ and for $g(x,y)$ the corresponding component of the given 2-cocycle) which we may equivalently rewrite as

$$
\begin{tikzcd}
 & \bullet \arrow[dd, "f"] & x \arrow[dd] \\
\bullet \arrow[ur, bend left=40, "q_1"] \arrow[ur, bend right=20, "f^{-1}q_2"'] \arrow[ur, phantom, "\Downarrow f^{-1}F"] & & \\
 & \bullet & y
\end{tikzcd}
$$

the source in this sense is

$$
\begin{tikzcd}
 & \bullet & x \\
\bullet \arrow[ur, bend left, "q_1"] \arrow[ur, bend right, "f^{-1}q_2"'] \arrow[ur, phantom, "\Downarrow f^{-1}F"] & &
\end{tikzcd}
,
$$

regarded as a morphism in $Y \times G_{(2)}$, while the target is

$$
\begin{tikzcd}
\bullet \arrow[dr, bend left, "fq_1"] \arrow[dr, bend right, "q_2"'] \arrow[dr, phantom, "\Downarrow F"] & & \\
 & \bullet & y
\end{tikzcd}
$$

regarded as a morphism in $Y \times G_{(2)}$.

This way the transition function $g(x, y)$ acts on the copies of $G_{(2)}$ which appear as the trivialized fibers of the $G_{(2)}$-bundle.

Bartels [9][proof of prop. 22] gives a reconstruction of total space of principal 2-bundle from their 2-cocycles which is closely related to $Y^{[3]} \times_g \mathrm{INN}(G_{(2)})$.

We end by saying that $\mathrm{INN}(G_{(2)})$ is a special case of a much more general construction in 2-bundles [11], and that the results of this paper will, in a future paper, be connected to similar results using simplicial groups and simplicial universal bundles.

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