TFT in two dimensions and Teichmüller spaces

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Abstract

Notes taken in a talk by **E. Getzler** at *Higher Structures in Mathematics and Physics*, Bernoulli Center, EPFL, Lausanne, Nov. 2008. Notes pretty literally reproduce what was on the board and what was said. But all mistakes are mine.

TFT in 2 dim is related to the problem of QFT in gravitational background, i.e. a pseudo-Riemannian metric on spacetime

bold changes to problem: spacetime becomes Riemannian instead of Lorentzian and it becomes 2-dimensional; second: topological theory, i.e. more interested in cohomology classes than in functions on spaces of metrics

but space of metric has group acting on it which is contractible, a special feature of 2-dimensions, namely the identity component of the diffeomorphism group on a Riemann surface, at least for negative Euler characteristic. So from point of view of topologiy it makes no difference to quotient by that, and we can quotient by conformal rescalings. The total quotient is the Teichnmueller space, with the action of the mapping class group and the cohomology of that; this is what our TFT is about:

two contractible groups act (if Euler characteristic is < 0)

$$\mathrm{Diff}_0(S); \text{conformal-rescalings}$$

quotient is $T(S)$: Teichmüller space for given surface S

already Riemann new coordinates of this space: $\dim(T(S)) = -3\chi(S)$ (recall we assume that $\chi(S)$ is negative)

the mapping class group

$$\Gamma(S) = \mathrm{Diff}_+(S)/\mathrm{Diff}_0(S) \simeq \pi_0(\mathrm{Diff}_+(S))$$

acts on $T(S)$

$$M(S) = T(S)/\Gamma(S)$$

simplest surface for which one can do this is surface of genus 2, here it comes down to looking at algebraic curves of genus 2, for which we have a very precise understanding of the Teichmueller space as a Deligne-Mumford stack, which is contractible and admits coordinates using theta-functions

today: look at compactification of $T(S)$ and show that it has a filtration which is similar to filtration of polyhedra, roughly

a way of rewriting work of Harer on triangulations of Teichmüller space

now

S is a closed oriented surface with marked points

$$S \supset (z_1, \cdots, z_n)$$

moduli space is space of hyperbolic metrics on this with cusps at marked points

there are now also examples in genus 0 and 1, as long as we have enough punctures

let $\mathcal{H}(S, Z)$ be te space of hyperbolic metrics on S with cusps at z_i

a cusp locally looks like $\frac{|d\xi|^2}{|\xi|^2(\log|\xi|)^2}$

example:

H the upper halpf plane with metric $\frac{|d\tau|}{(\mathrm{Im}\tau)}^2$: quotient by something and get a cusp at infinity

so now

$$\Gamma(S,Z) = \mathrm{Diff}_+(S,Z)/\mathrm{Diff}_0(S,Z) \simeq \pi_0(\mathrm{Diff}_+(S,Z))$$

$$T(S,Z) = \mathcal{H}(S,Z)/\mathrm{Diff}_0(S,Z)$$

$$M(S,Z) = T(S,Z)/\Gamma(S,Z) = \mathcal{H}(S,Z)/\mathrm{Diff}_+(S,Z)$$

dimension formula for n marked points (of either spaces)

$$\dim = -3\chi(S) + 2n$$

example:

$$S = \mathbb{C}P^1$$

$$Z = (z_1, \cdots, z_n)$$

cross ration gives iso

$$M(S,Z) \simeq \mathbb{C} - \{0,1\}$$

or:

$$S = \text{torus}$$

$$Z = \{z\}$$

$$M(S,Z) = H/\mathrm{SL}(2,\mathbb{Z})$$

instead of circles one should really have boundary circles: to mimic that we take angular coordinates at insertion points

$P(S,Z)$ is n-torus bundle over $M(S,Z)$

with fiber a choice of angle in the tangent space at each marked point (picking a real ray in the 2d tangent space at each marked point)

so

$$\dim P(S,Z) = 3(n - \chi(S))$$

example: for a torus with one marked point the moduli spaxce is really that of elliptic curves, with the extra information of the angle we get a 3-dimensional moduli space

now **compactify** these moduli spaces to manifolds (orbifolds if there is no marked point) with corners, such that the moduli space sits in its compactification as a homotopy equivalence

compactify to manifolds with corners P^+ such that the inclusion $P \hookrightarrow P^+$ is a homotopy equivalence (orbifold if Z is empty)

remark: all of these compactified moduli spaces assemble to form a modular operad

(introduced by Jarvis, Kimura, Stasheff, Lorjenga (spelling?))

Harvey

the universal cover of P is some extension of Teichmueller space; Harvey "borderfied" it by embedding it as open part of manifold with corners; borderification is constructed by blowing up charts each of which corresponds to a pair of pants

Harvey borderfies

$$T(S,Z)$$

and hence

$$M(S,Z)$$

and

$$P(S,Z)$$

consider surface without marked point and start drawing circles into it which are "essential", an essential circle is one such that cutting along it does not produce components with Euler characteristic zero or positive

so the process stops after cutting the surface into the same number of pairs-of-pants as given by the absolute value of the Euler characteristic.

so a pants decomposition is an isotopy class of such embedded circles; once we

once we pick a hyperbolic metric, there will be closed geodesics in each isotopy class of such circles

so hyperbolic metric gives length coordinates – namely the length of the unique closed geodesic in the isotopy classes of the closed circles.

so we have these positive numbers and they casn take all positive values;

(recall moduli space of genus 2 surface is 6-dim)

we can take boundary circle, cut there and rotate then glue again, this produces a action of mapping class group namely a Dehn twist

so we also have angle coordinate, which take values in $\mathbb{R}$ for T and in $\mathbb{R}/2\pi\mathbb{Z}$ for M.

these give a global parameterization of Teichmueller space $T(S)$, which is actually a reall ball of dim $6g-6$

when we have punctures use same techniques, but notion of pants is generalized: we still cut in pieces of Euler char -1, but now there is the pants and also the cylinder with one sups (collapsed boundary circle of pant) and similarly with two collapsed boundary circles

we have fewer parameter when the boundary circles are collapsed, because there is no longer a length to remember

now what one chart of Harvey's borderfication looks like: chart was an open subset of Euclidean space, now it will be a closed subset:

one chart of Harvey's borderfication is obtained by allowing the length coordinates to take the value 0 as well as positive values.

quick look at the elliptic curve case:

torus with one puncture, i.e. one cusp, so there is up to diffeo only one essential circle which cuts this into a cyclinder with one cusp, so parameters are a length and an angle, so it gives upper half plane with acis included;

final effect: $T^+ = T \sqcup \sqcup_{\alpha\in Q\cup\{\infty\}} mathbbR$

$$M^+ = M \sqcup \text{circle at infinity}$$

(picture of usual fundamental domain of SL2Z in upper half plane: add the boundary at top infinity)

so these Harvey-compactifications of moduli spaces we are interested in, we want to compactify them

Q: what is the relevance to physics? A: we get modular operad and TFT is just algebra for that operad; said differently: we have state space which is complex of vector spaces; tensor copy for each puncture, look at differential forms on moduli space which factor - that's it about TFTs, back to business

skeletal filtrations

people wanted to compute homotopy groups of spheres and didn't know about Postnikov tower which would be much better, so they used skeletal filtrations (40s)

invented by Blakes, developed by J.H.C Whitehead and now transformed into theory of crossed complexes

so look at spaces with filtration

$$F_0X \subset F_1X \subset \cdots \subset X$$

such that $F_kX \hookrightarrow X$ is k-connected

connected map means 0-connected map, means each connected component in target is hit.

k-connected means 0-connected , $\pi_0(F_kX) \to \pi_0(X)$ is surjective, and further $\pi_i(F_kX, x) \to \pi_i(X, x)$ is an isomorphism for $0 \leq i < k$ and an epimorphism for $i = k$.

for instance polyhedra and $F_k X$ is union of k-cells

would be lovely if our moduli spaces are polyhedra, but they are not

theorem: the skeletal filtration $F_k T^+ \subset T^+$ (compatible with operad structure) (this is not a stratification! which would have the property that interiors of strata would be equal-dimensional, which is not the case here) induces skeletal filtration on $K^+, P^+, \bar{M}$

if we are at a point of the borderfication we can choose a point which defines the chart which we are in and some of the curves will have length 0

the curves we singled out cut surface into pieces, count how many of these have genus 0, there can't be more than $|\chi(S)| +$ number of punctures, so there is a maximum number for these

so finally, here is the definition:

$$F_k := \{\text{moduli}|\text{length 0 curves which cut surface into pieces with at least } -\chi(S) - n - k \text{ having genus 0}\}$$

$$F_0 : \text{a ll curves have length 0 } = \{\text{pants decompositions}\}$$

F_1 is almost-pants-decompositions: one curve may have positive length

(** some pictures **)

recall that Teichmueller space is contractible: k-connected map into it just meeaks that homotopy groups below k vanish

so we conclude that every surface has a pants decomposition

$k = 0$: F_0 is not empty, i.e. every surface has a pants decomoposition

for $k = 1$: the theorem says that every pants deocmoposition can be joined by moves given by elements of F_1.

remark: the theorem down on moduli space is equivalent to the statement that 2dTFTs concenttrated in degree 0 (of the complexes of vector spaces assigned to marked points) are the same as commutative Frobenius algebras

in one direction easy to see : 2dTFT to Frob algebra, but reverse direction is more involved and the above tells us that the usual moves are enough to show that correlators obtained from Frob algebras are well defined.

Moore-Seiberg's prove was more by hand: classification of modular functors (kind of a categorification of 2dTFTs).

remark: that $K_k X$ is k-connected for higher k should have some significance, but not clear yet

example: disk with some points on the boundary (** pictures **), pass to the double, which is the sphere with five points on equator and orientation preserving involution fixing the equator,

consider hyperbolic metric with cusps invariant under involution and redo everything from above to get a notion of open-closed modular operad,

recover a number of theorems which have been proved using Morse theory by Moore.Seiber, Lauda-Pfeiffer and Natalson-xyz

so now filtration of moduli spaces reproduces stuff that went into constructuion of fukaya categories and Stasheff associhedra.