FRS Formalism from 2-Transport

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Abstract

Fuchs, Runkel, Schweigert et al. ("FRS") have developed a detailed formalism for studying CFT in terms of (Wilson-)graphs decorated in modular tensor categories. Here we discuss aspects of how the Poincaré dual of their prescription can be understood in terms of locally trivialized 2-functors from surface elements to bimodules, similar to how Stolz and Teichner describe enriched elliptic objects.

Even though this might begin to look like a paper, the following are unfinished private notes.

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1 Introduction

1.1 CFT in (2-)Functorial Language

There are two apparently different approaches to a rigorous formulation of 2-dimensional conformal field theory (CFT).

- Segal [1, 2] defined a CFT to be a certain functor

  $$\mathrm{CFT} : \mathbf{RCob} \to \mathbf{Hilb}$$

  from the category of 2-dimensional Riemannian cobordisms to the category of Hilbert spaces.

  A certain locality requirement (known as *excision* in the context of elliptic cohomology) forces one to refine this definition. Stolz and Teichner [3] refine the source category RCob to a 2-category $\mathcal{P}_2$ obtained by decomposing Riemannian cobordisms

  $$(S^1)^m \xrightarrow{\Sigma} (S^1)^n$$

  into Riemannian surface elements

  $$x \underset{\gamma_2}{\overset{\gamma_1}{\Downarrow S}} y \qquad \in \mathrm{Mor}_2\left(\mathcal{P}_2\right)\,.$$

  Moreover, they refine **Hilb** to a weak 2-category (bicategory) $\mathbf{BiMod}_{\mathrm{vN}}$ by decomposing Hilbert space morphisms

  $$H_1 \xrightarrow{\phi} H_2$$

  into morphisms of (Hilbert-)bimodules of von Neumann algebras

  $$A \underset{{}_A N'_B}{\overset{{}_A N_B}{\Downarrow f}} B \qquad \in \mathrm{Mor}_2\left(\mathbf{BiMod}_{\mathrm{vN}}\right)\,.$$

  As a result, in the refined formulation Segal's functor becomes a 2-functor

  $$\mathrm{CFT} : \mathcal{P}_2 \to \mathbf{BiMod}_{\mathrm{vN}}\,.$$

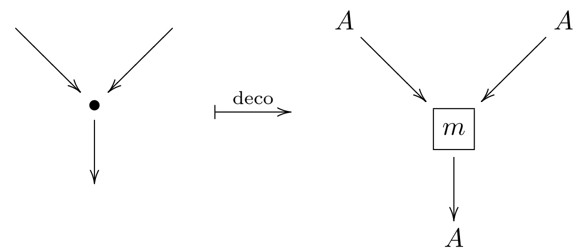
- Fuchs, Runkel, Schweigert et al. [4] construct CFTs in terms of graphs drawn on Riemannian surfaces which are decorated by objects and morphisms in a modular tensor category $\mathcal{C}$. This procedure is a (vast) generalization of the description of 2-dimensional *topological* field theory introduced by Fukuma, Hosono and Kawai [5, 6].[1]

In more functorial terms, it amounts roughly to studying a certain functor

$$\mathrm{deco} : \mathbf{Cob}^1(\Sigma) \to \mathcal{C}$$

from enriched 1-dimensional cobordisms (embedded into a Riemannian surface Σ and allowed to join and split) to $\mathcal{C}$. In particular, this functor acts as

$\mathrm{Mor}\left(\mathbf{Cob}^1(\Sigma)\right) \ni$



$\in \mathrm{Mor}(\mathcal{C})$,

where A is a Frobenius algebra object internal to $\mathcal{C}$ and $A \otimes A \xrightarrow{m} A$ is the respective product morphism.

The full definition of the decoration functor takes more structure into acoount. Modules and bimodules of A encode boundary conditions and defect lines in CFT, and field insertions manifest themselves as certain (bi-)module homomorphisms. Hence it was eventually recognized [7, 8], that $\mathcal{C}$ has to be regarded as sitting inside the weak 2-category of bimodules of (Frobenius-)algebras internal to $\mathcal{C}$:

$$\mathcal{C} \simeq \mathrm{Hom}(\mathbb{1}, \mathbb{1}) \subset \mathbf{BiMod}_{\mathrm{Frob}}(\mathcal{C}) \; .$$

In this language, the algebra object A is regarded as a bimodule for the trivial algebra object given by the tensor unit $\mathbb{1}$

$$A \simeq \; \mathbb{1} \xrightarrow{{}_{\mathbb{1}}A_{\mathbb{1}}} \mathbb{1} \; ,$$

and the tensor product in $\mathcal{C}$ is identified with the bimodule tensor product over $\mathbb{1}$:

$$A \otimes A \simeq \; \mathbb{1} \xrightarrow{{}_{\mathbb{1}}A_{\mathbb{1}}} \mathbb{1} \xrightarrow{{}_{\mathbb{1}}A_{\mathbb{1}}} \mathbb{1}$$

Clearly, both these formulations of CFT, while superficially looking different, share some similarities. They ought to be related in a conceptual manner. The similarities between both frameworks are drastically amplified once we employ a trivial change of perspective concerning the diagrams depicted above.

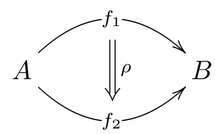
[1] In fact, in this approach the decorated Riemannian surface is ultimately to be embedded into a 3-manifold which is to be thought of as a 3-dimensional cobordism of a 3-dimensional topological field theory. However, for the present discussion this higher dimensional aspect shall not concern us. The details are described in the series of papers [17, 18, **?**, 19, **?**, **?**].

Any 2-morphism

$$(\; A \xrightarrow{f_1} B \;) \stackrel{\rho}{\Rightarrow} (\; A \xrightarrow{f_2} B \;)$$

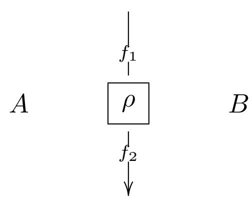
in a 2-category may be represented graphically in two different ways,

- either by a <u>globular diagram</u>



  where objects are depicted by points, morphisms by arcs and 2-morphisms by surfaces,

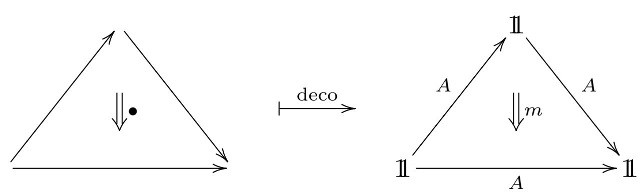
- or by a <u>string diagram</u>



  where objects are depicted by surfaces, morphisms by arcs and 2-morphisms by points.

These two descriptions are mutually Poincaré dual.

When we regard $\mathcal{C}$ as a 2-category with the single object $\mathbb{1}$, and when we use globular diagrams, the FRS decoration functor becomes a 2-functor, and its action on trivalent vertices looks like



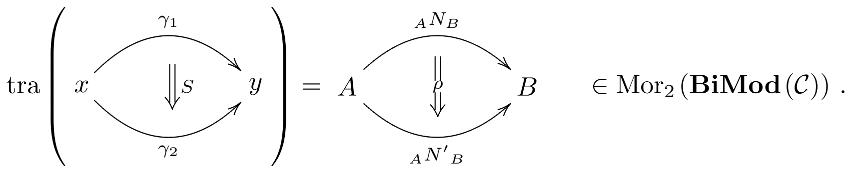
While not directly interpretable as coming from the approach by Segal, Stolz & Teichner, this structure strongly suggests that along these lines a relation between the two approaches could be established. The purpose of the present work is to elaborate on this.

Our main result is that 2-functors from a geometric 2-category of surface elements to $\mathbf{BiMod}_{\mathrm{Frob}}(\mathcal{C})$ give rise to at least some aspects of the FRS formulation when *locally trivialized.*

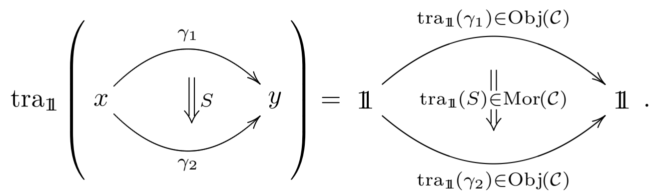
The notion of local trivialization of a 2-functor that we use is a special case of the general theory of **2-transport** [9].

1.2 Outline

- The key concept of the present discussion is that of a trivialization of a **2-transport** 2-functor tra with values in bimodules.

  We define a **trivial** 2-transport functor $\mathrm{tra}_{\mathbb{1}}$ to be one which assigns only the trivial algebra object $\mathbb{1}$:

  A **trivialization** of tra is defined to be a **trivialization morphism**

  $$\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathbb{1}}$$

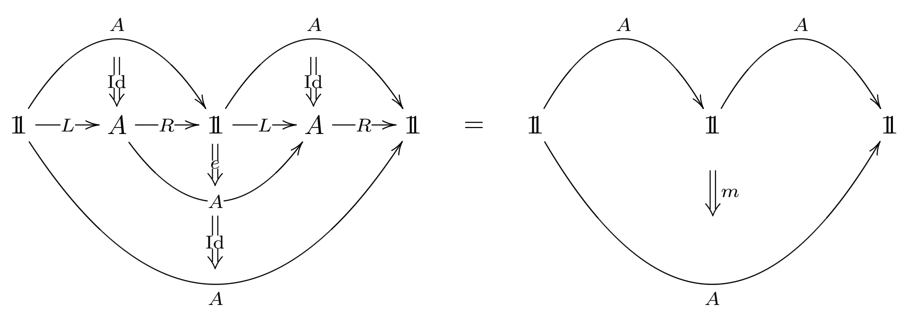
  of 2-functors (a pseudonatural transformation, see §A.2) such that t has an ("ambidextrous") adjoint

  $$\mathrm{tra}_{\mathbb{1}} \xrightarrow{\bar{t}} \mathrm{tra} \ .$$

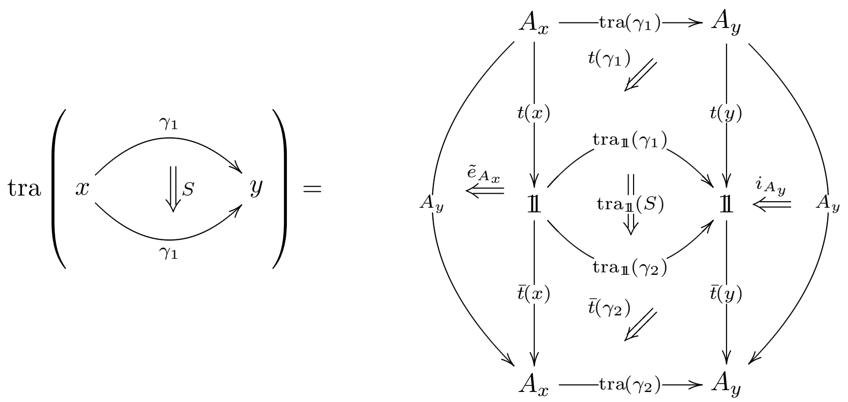
  These definitions are given in §3.1.

- Working out the details of how a trivialization of a 2-transport 2-functor looks like requires some technical facts about internal bimodules and in particular about left-induced bimodules. These are discussed in §2.3.

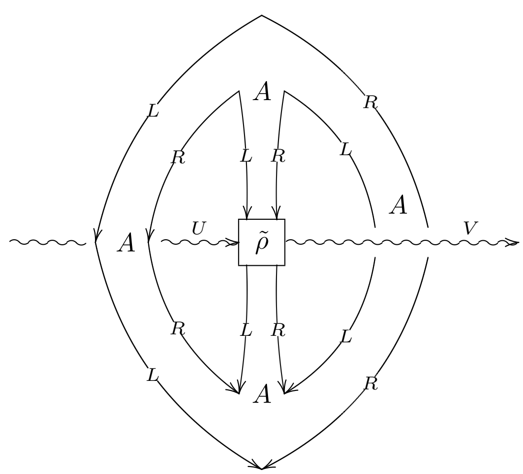
  In particular, the existence of trivializations of 2-transport relies on the fact, recently discussed in [11] (extending an older result in [13]), that *every Frobenius algebra object in $\mathcal{C}$ arises from an ambidextrous adjunction* in $\mathbf{BiMod}(\mathcal{C})$. We spell out how this works in detail in §2.4.



- Using these facts on bimodules and Frobenius algebras, we derive two crucial theorems:
  - Theorem 1 (p. 44) states that a certain class of 2-transport 2-functors is trivializable. The proof, given in §3.2 amounts to constructing the respective pseudonatural transformations and their modifications.[2] These will be essential for the following.
  - Propositon 14 (p. 51) states that the image of any trivializable 2-transport 2-functor can be re-expressed entirely in terms of trivial 2-transport and 2-morphisms coming from the trivialization. More precisely, the image of a 2-transport tra trivialized by $\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathbb{1}}$ is



- With an understanding of (locally) trivialized 2-transport in hand, we can study examples. As a first step, §4.3 shows how the dual triangulation and its decoration prescription describing the disk with one bulk insertion in FRS formalism



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[2]I do show that a certain class of 2-transport 2-functors is trivializable. I suspect that, moreover, every trivializable 2-transport is in this class. But this I cannot show yet.

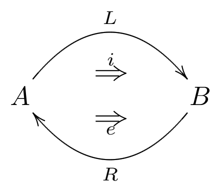
arises from locally trivializing a certain 2-transport.

2 Frobenius Algebras and Adjunctions

2.1 Adjunctions

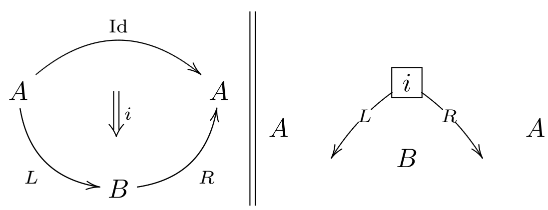
In a 2-categorical context invertibility of morphisms is in general replaced by *equivalence*, i.e. by invertibility up to 2-isomorphism. In some situations however, even the notion of equivalence is too strong, and one is left merely with *adjunctions*. For applications as those to be presented in the following, an *ambidextrous adjunction* which satisfies a *bubble move equation* will be seen to provide sufficiently many features of a true equivalence to admit the inversion operations needed here.

Definition 1 *An* **adjunction**



in a 2-category $\mathcal{K}$ is a collection of

1. *1-morphisms* $A \xrightarrow{L} B$ *and* $B \xrightarrow{R} A$ *in* $\mathrm{Mor}_1(\mathcal{K})$
2. *2-morphisms*



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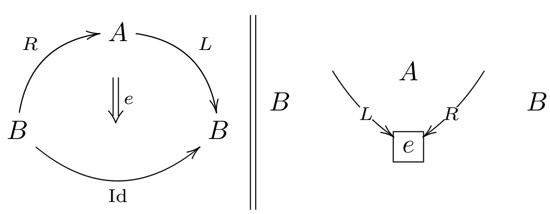
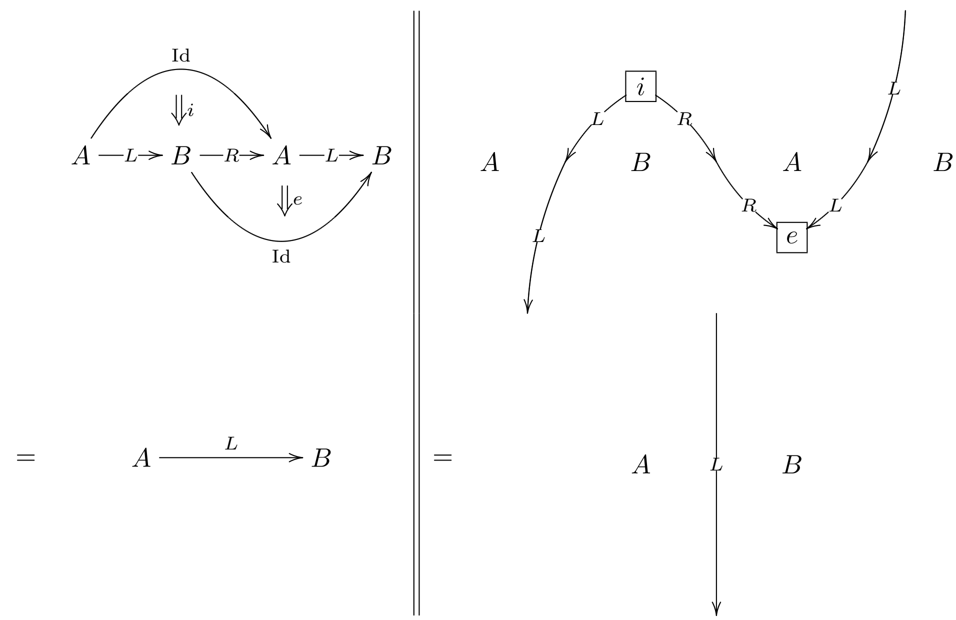
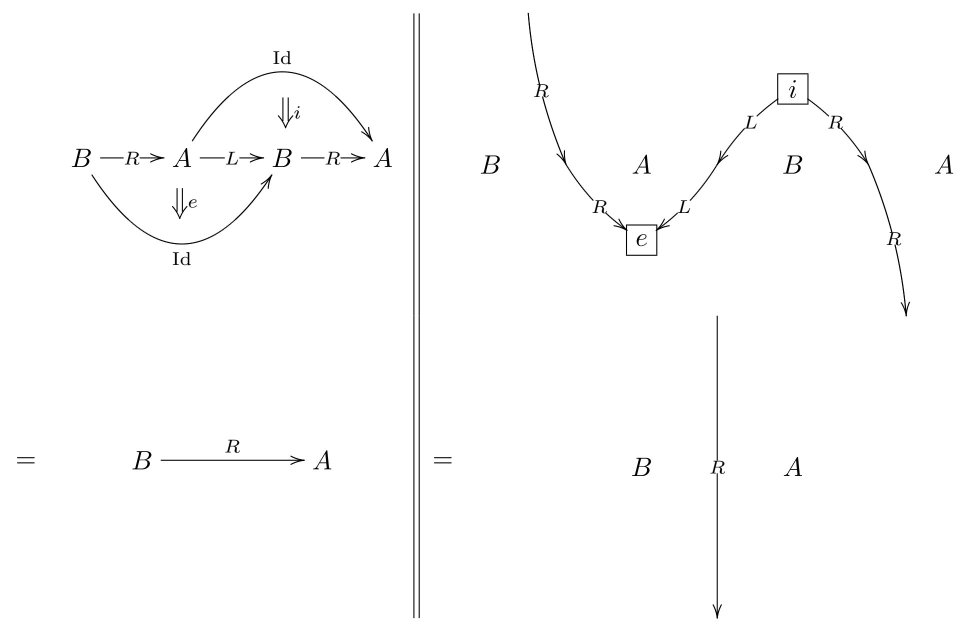


in $\mathrm{Mor}_2(\mathcal{K})$

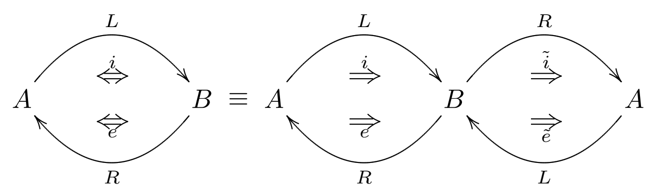
satisfying the **zig-zag idenities**, *which look like*



and

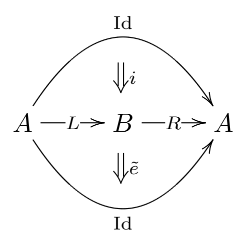


Definition 2 ([11, 12]) *A pair of adjunctions*



is called an **ambidextrous adjunction***.*

Definition 3 *We call an ambidextrous adjunction* **special** *iff*



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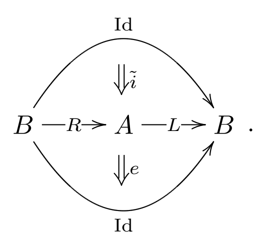
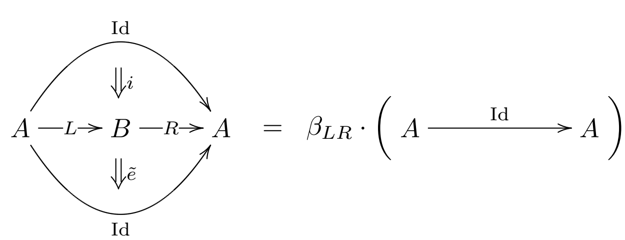


both are 2-*iso*morphisms.

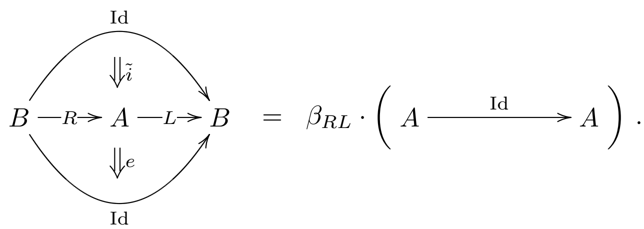
A special ambidextrous adjunction is hence similar to an equivalence, but weaker.

We will mostly be interested in special cases where all the 1-morphisms sets are vector spaces. (More precisely, we will be intersted in the case where our 2-category $\mathcal{K}$ is the 2-category of biomodules of a modular tensor category $\mathcal{C}$.) In this case we shall be more specific about the precise nature of the above 2-isomorphisms.

Definition 4 *When the 1-morphism sets of $\mathcal{K}$ are vector spaces, we call an ambidextrous adjunction in $\mathcal{K}$* **special** *iff*



and



for β_{LR} and β_{RL} elements of the ground field.

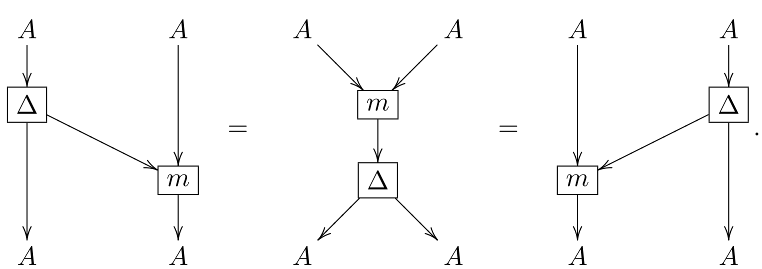
Remark. Below we will see (prop. 10) how speciality of ambidextrous adjunctions translates into speciality of the Frobenius algebras that they give rise to. Speciality for Frobenius algebras is an established concept (def. 6 below) which hence motivated our choice of the term *special* for the above property of ambidextrous adjunctions.

2.2 Frobenius Algebras

Definition 5 *A* **Frobenius algebra** *in a monoidal category $\mathcal{C}$ is an object $A \in \mathrm{Obj}(\mathcal{C})$ togther with morphisms*

1. *product*
$$A \otimes A \xrightarrow{m} A$$
2. *unit*
$$\mathbb{1} \xrightarrow{i} A$$
3. *coproduct*
$$A \xrightarrow{\Delta} A \otimes A$$
4. *counit*
$$A \xrightarrow{e} \mathbb{1}$$

such that (m, i) is an algebra, (Δ, e) is a coalgebra and such that product and coproduct satisfy the **Frobenius property**



Remark. For manipulations of diagrams as in the following it is often helpful to think of the Frobenius property as saying that, with A regarded as a bimodule over itself, the coproduct is a bimodule homomorphism form ${}_A(A)_A$ to ${}_A(A \otimes A)_A$

We will be interested in Frobenius algebras with additional properties. The Frobenius algebras of relevance here are

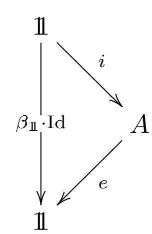
- special (def. 6)
- symmetric (def. 7) .

Unfortunately, while standard, the terms "special" and "symmetric" are rather unsuggestive of the phenomena they are suppposed to describe.

1. Seciality says that the two "bubble diagrams" in a Frobenius algebra are proportional to identity morphisms.
2. Symmetry of a Frobenius algebra says that the two obvious isomorphisms of A with its dual object $A^\vee$ are equal.

The reader should in particular be warned that symmetry, in this sense, of a Frobenius algebra is not directly related to whether or not that algebra is (braided) *commutative*. (But in modular tensor categories braided commutativity together with triviality of the twist implies symmetry.)

Definition 6 ([17], def. 3.4) *Let A be a Frobenius algebra object in an abelian tensor category. A is* **special** *precisely if*



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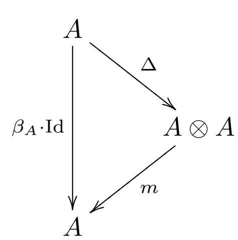
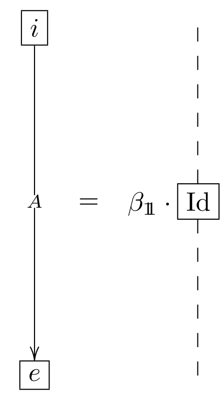


for some constants $\beta_{\mathbb{1}}$ and β_A

In terms of string diagrams in the suspension of $\mathcal{C}$ these two conditions look like



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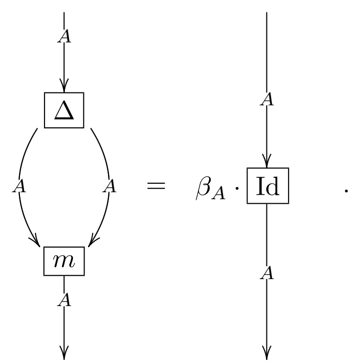
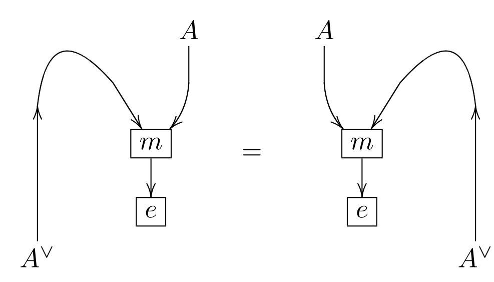


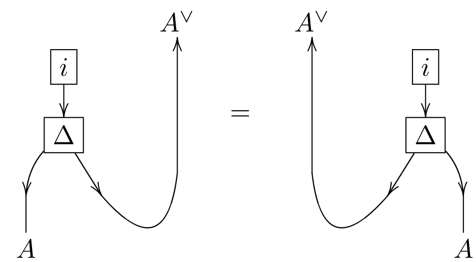
Speciality of Frobenius algebras will be related to speciality of ambidextrous adjunctions in prop. 10 on p. 10.

Definition 7 ([17], (3.33)) *A Frobenius algebra is called* **symmetric** *if the following two isomorphisms of A with its dual, $A^\vee$, are equal:*



Proposition 1 ([17], (3.35)) *The morphisms in (7) are indeed isomorphisms.*

Proof. Using the Frobenius property, one checks that the inverse morphisms are



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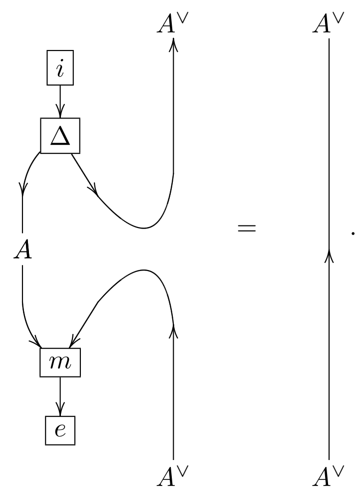
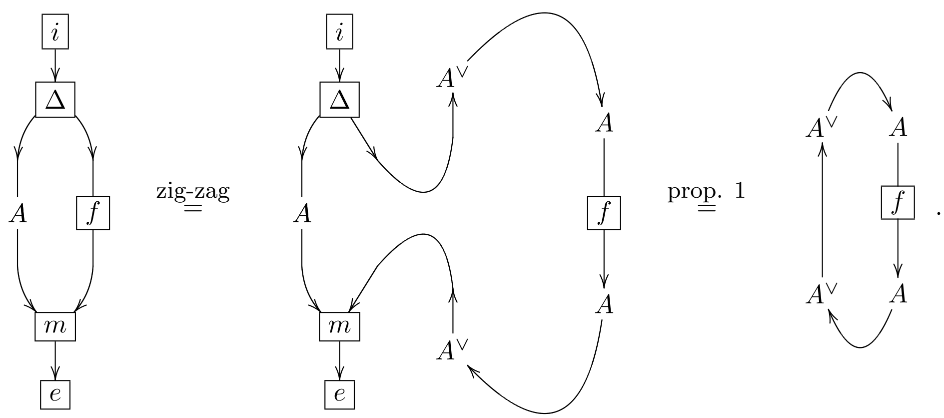
Hence we have in particular



Using relations like this we frequently pass back and forth between diagrams with or without occurence of the dual $A^\vee$ of A.

Example 1

When dealing with FRS diagrams for disk correlators (§4.3) one encounters the following situation. Let $A \xrightarrow{f} A$ be any morphism. Then



2.3 Bimodules

We can define bimodules in any abelian monoidal category.

Definition 8 *An* **abelian category** $\mathcal{C}$ *is a category with the following properties:*

1. *The* **hom-spaces** $\mathrm{Hom}\,(a, b)$ **are abelian groups** *for all* $a, b \in \mathrm{Obj}\,(\mathcal{C})$.

   The abelian group operation **'+' distributes over composition** *of morphisms. This means that for every diagram*

   $$a \xrightarrow{f} b \underset{g_2}{\overset{g_1}{\rightrightarrows}} c \xrightarrow{h} d$$

   we have

   $$a \xrightarrow{f} b \circ \left(\begin{array}{c} b \xrightarrow{g_1} c \\ + \\ b \xrightarrow[g_2]{} c \end{array} \right) \circ c \xrightarrow{h} d \;=\; \begin{array}{c} a \xrightarrow{f} b \xrightarrow{g_1} c \xrightarrow{h} d \\ + \\ a \xrightarrow[f]{} b \xrightarrow[g_2]{} c \xrightarrow[h]{} d \end{array} .$$

2. $\mathcal{C}$ **contains a zero-object** 0, *(an object which is both initial and terminal).*

3. *For all* $a, b \in \mathrm{Obj}\,(\mathcal{C})$ *the* **direct product** $a \times b$ **exists.**

4. *Every morphism* f *in* $\mathcal{C}$ *has* **kernel and cokernel** $\ker(f)$, $\mathrm{coker}\,(f)$ *in* $\mathcal{C}$.

5. $\mathrm{coker}\,(\ker(f)) = f$ *for every* $f \in \mathrm{Mor}\,(\mathcal{C})$

6. $\ker(\operatorname{coker}(f)) = f$ *for every* $f \in \operatorname{Mor}(\mathcal{C})$

Definition 9 *Let $\mathcal{C}$ be any monoidal category and let A and B be algebra objects in $\mathcal{C}$. An A-B* **bimodule** *in $\mathcal{C}$ is an object ${}_AN_B \in \operatorname{Obj}(\mathcal{C})$ together with left and right action morphisms*

$$A \otimes {}_AN_B \xrightarrow{\ell} A$$

and

$$ {}_AN_B \otimes B \xrightarrow{r} A$$

satisfying

1. <u>*compatibility with the product*</u>

$$\begin{array}{ccc} A \otimes A \otimes {}_AN_B & \xrightarrow{m \otimes {}_AN_B} & A \otimes {}_AN_B \\ {\scriptstyle A\otimes\ell}\downarrow & & \downarrow{\scriptstyle \ell} \\ A \otimes {}_AN_B & \xrightarrow[\ell]{} & {}_AN_B \end{array}$$

$$\begin{array}{ccc} {}_AN_B \otimes B \otimes B & \xrightarrow{{}_AN_B \otimes m} & {}_AN_B \otimes B \\ {\scriptstyle r\otimes B}\downarrow & & \downarrow{\scriptstyle r} \\ {}_AN_B \otimes B & \xrightarrow[r]{} & {}_AN_B \end{array}$$

2. <u>*compatibility with the unit*</u>

$$\begin{array}{ccc} \mathbb{1} \otimes {}_AN_B & \longrightarrow & {}_AN_B \\ & {\scriptstyle i_A \otimes {}_AN_B}\searrow \quad \nearrow{\scriptstyle \ell} & \\ & A \otimes {}_AN_B & \end{array}$$

$$\begin{array}{ccc} {}_AN_B \otimes \mathbb{1} & \longrightarrow & {}_AN_B \\ & {\scriptstyle {}_AN_B \otimes i_B}\searrow \quad \nearrow{\scriptstyle r} & \\ & A \otimes {}_AN_B & \end{array}$$

Definition 10 *Let $\mathcal{C}$ be an abelian monoidal category. Let ${}_AM_B$ and ${}_BN_C$ be bimodules in $\mathcal{C}$. Then the* **bimodule tensor product** *is the cokernel*

$$ {}_AM_B \otimes {}_BN_C \xrightarrow{\otimes_B(M,N)} {}_AM_B \otimes_B {}_BN_C \equiv \operatorname{coker}\left({}_AM_B \otimes B \otimes {}_BN_C \xrightarrow{r\otimes N - M\otimes\ell} {}_AM_B \otimes {}_BN_C \right).$$

Example 2

Let $A \in \mathrm{Obj}(\mathcal{C})$ be a special Frobenius algebra. Let ${}_AA_A$ be A but regarded as a bimodule over itself with $\ell = m = r$.

Set

$$ {}_AA_A \otimes {}_AA_A \xrightarrow{\otimes_B(M,N)} {}_AA_A \otimes_A {}_AA_A = {}_AA_A \otimes {}_AA_A \xrightarrow{m} {}_AA_A \ . $$

We have that

$$ {}_AA_A \otimes A \otimes {}_AA_A \xrightarrow{m\otimes A - A\otimes m} {}_AA_A \otimes {}_AA_A \xrightarrow{m} {}_AA_A $$

is the 0-arrow, due to the associativity of m. Given any other arrow ϕ with this property we have

$$ \begin{array}{ccccc} {}_AA_A \otimes A \otimes {}_AA_A & \xrightarrow{m\otimes A - A\otimes m} & {}_AA_A \otimes {}_AA_A & \xrightarrow{m} & {}_AA_A \ . \\ & & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \Delta\circ\phi} & \\ & & {}_AA_A & & \end{array} $$

The fact that the morphism $\Delta \circ \phi$ makes this diagram commute depends on the special Frobenius property of A as well as on the fact that $(m\otimes A - A\otimes m)\circ\phi = 0$.

Definition 11 *Let $\mathcal{C}$ be any monoidal category. The* **2-category of (Frobenius) algebra bi-modules internal to** $\mathcal{C}$, *denoted* $\mathbf{BiMod}(\mathcal{C})$, *is defined as follows:*

1. *objects are all (Frobenius) algebras A internal to $\mathcal{C}$*
2. *1-morphisms $A \xrightarrow{{}_AM_B} B$ are all internal $A - B$ bimodules ${}_AM_B$*
3. *2-morphisms*

$$ A \underset{{}_AN_B}{\overset{{}_AM_B}{\rightrightarrows}} B \quad (\Downarrow \phi) $$

are all internal bimodule homomorphisms (intertwiners) ${}_AM_B \xrightarrow{\phi} {}_AN_B$.

Horizontal composition in $\mathbf{BiMod}(\mathcal{C})$ *is the tensor product of bimodules. Vertical composition is the composition of bimodule homomorphisms.*

Remark.

1. $\mathbf{BiMod}(\mathcal{C})$ is really a *weak* 2-category (a bicategory) with nontrivial associator. As usual, we here consider its strictification and suppress all appearances of the associator.

2. The tensor unit $\mathbb{1} \in \mathcal{C}$ equipped with the trivial (co)product is always a (Frobenius) algebra internal to $\mathcal{C}$. The sub-2-category $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ of $\mathbf{BiMod}(\mathcal{C})$ is $\mathcal{C}$ itself:

$$\mathrm{Hom}_{\mathbf{BiMod}(\mathcal{C})}(\mathbb{1}, \mathbb{1}) \simeq \mathcal{C}\,.$$

2.3.1 Left-induced Bimodules

A particularly important role for our construction is played by left-induced bimodules.

Definition 12 *A* **left-induced bimodule** *in* $\mathbf{BiMod}(\mathcal{C})$ *is a bimodule of the form*

$$ {}_A N_B \equiv \; A \overset{m}{\dashrightarrow} A \otimes V \overset{\phi \circ m}{\dashleftarrow} B$$

for $V \in \mathrm{Obj}(\mathcal{C})$, where the left action by A comes from the action of A on itself and where the right action by B comes from composing the morphism

$$V \otimes B \xrightarrow{\phi} A \otimes V \;\; \in \mathrm{Mor}_1(\mathcal{C})$$

with the right action of A on itself. ϕ is required to make the following diagrams commute:

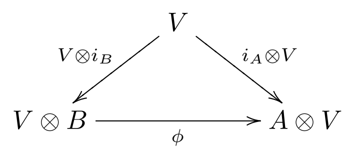
1. <u>*compatibility with the product*</u>

$$\begin{array}{ccccc}
V \otimes B \otimes B & \xrightarrow{\phi \otimes B} & A \otimes V \otimes B & \xrightarrow{A \otimes \phi} & A \otimes A \otimes V \\
{\scriptstyle V \otimes m}\downarrow & & & & \downarrow{\scriptstyle m \otimes V} \\
V \otimes B & & \xrightarrow[\phi]{\hspace{6em}} & & A \otimes V
\end{array}$$

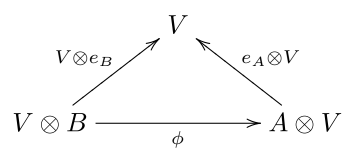
2. <u>*compatibility with the coproduct*</u>

$$\begin{array}{ccccc}
V \otimes B \otimes B & \xrightarrow{\phi \otimes B} & A \otimes V \otimes B & \xrightarrow{A \otimes \phi} & A \otimes A \otimes V \\
{\scriptstyle V \otimes \Delta}\uparrow & & & & \uparrow{\scriptstyle \Delta \otimes V} \\
V \otimes B & & \xrightarrow[\phi]{\hspace{6em}} & & A \otimes V
\end{array}$$

3. *compatibility with the unit*



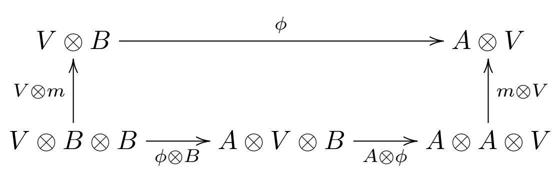
4. *compatibility with the counit*



Proposition 2 *For* special *Frobenius algebras the four conditions in def 12 may not be independent. For A and B special Frobenius algebras*

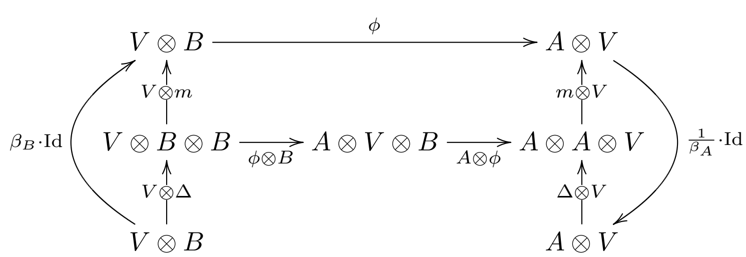
- *compatibility with the coproduct is* implied *by compatibility with the product if $\beta_A = \beta_B$*
- *compatibility with the counit is* implied *by compatibility with the product if the constants $\beta_{\mathbb{1}}$ agree.*

Proof. From the commuting diagram describing the compatibility with the product



$V \otimes B$

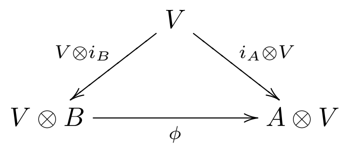
we obtain, by definition 6, the commuting diagram



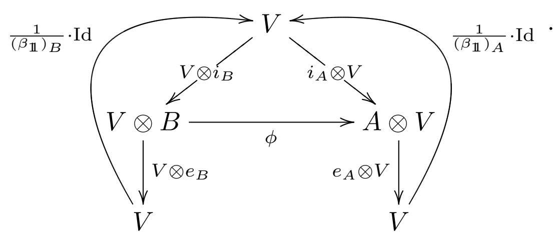
.

This immediately implies the diagram which expresses compatibility with the coproduct iff $\beta_A = \beta_B$.

Similarly, from the commuting diagram describing the compatibility with the unit



we obtain, by definition 6, the commuting diagram



This immediately implies the diagram which expresses compatibility with the counit if $(\beta_{\mathbb{1}})_A = (\beta_{\mathbb{1}})_B$.

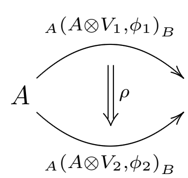
□

Example 3 *We get a left-induced $A - A$ bimodule $(A \otimes V, \phi = c^{\pm}_{A,V})$, where*

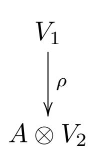
$$V \otimes A \xrightarrow{\phi} A \otimes V \;=\; V \otimes A \xrightarrow{c^{\pm}_{A,V}} A \otimes V$$

is the left or right **braiding** *in $\mathcal{C}$. This is the crucial example for the application to FRS formalism, where modules of this form describe* **field insertions** *with V being interpreted as the chiral data of the field.*

Proposition 3 *A* **morphism of left-induced bimodules**



is specified by a morphism



as

$$\begin{array}{c} A \otimes V_1 \\ \Big\downarrow {\scriptstyle A\otimes\rho} \\ A \otimes A \otimes V_2 \\ \Big\downarrow {\scriptstyle m\otimes V_2} \\ A \otimes V_2 \end{array} \quad .$$

This ρ has to make the diagrams

$$\begin{array}{ccc} V_1 \otimes B & \xrightarrow{\phi_1} & A \otimes V_1 \\ {\scriptstyle \rho\otimes B}\Big\downarrow & & \Big\downarrow{\scriptstyle A\otimes\rho} \\ A \otimes V_2 \otimes B & \xrightarrow{A\otimes\phi_2} & A \otimes A \otimes V_2 \end{array}$$

commute.

Remark. Note that, in general, ρ is not unique.

Definition 13 *We denote the sub-2-category of left-induced bimodules by*

$$\mathbf{LFBiMod}(\mathcal{C}) \subset \mathbf{BiMod}(\mathcal{C}) \ .$$

Proposition 4 *The bimodule tensor product* $A \xrightarrow{{}_A N_B} B \xrightarrow{{}_B N'_C} C$ *of two left-induced bimodules is the left-induced bimodule*

$${}_A N_B \otimes_B {}_B N'_C \equiv \ A \overset{m}{\dashrightarrow} A \otimes V \otimes V' \overset{\phi'\circ\phi\circ m}{\dashleftarrow} C \ .$$

Proof.

We claim that the map

$$\begin{array}{ccc} A \otimes V \ \otimes \ B \otimes V' & \xrightarrow{\quad f \quad} & A \otimes V \otimes V' \\ & {\scriptstyle A\otimes\phi\otimes V'} \searrow \qquad \nearrow {\scriptstyle m_A\otimes V\otimes V'} & \\ & A \otimes A \otimes V \otimes V' & \end{array}$$

is a cokernel for

$$A \otimes V \ \otimes \ B \ \otimes \ B \otimes V' \xrightarrow{r\otimes(B\otimes V')-(A\otimes V)\otimes\ell} A \otimes V \ \otimes \ B \otimes V' \ .$$

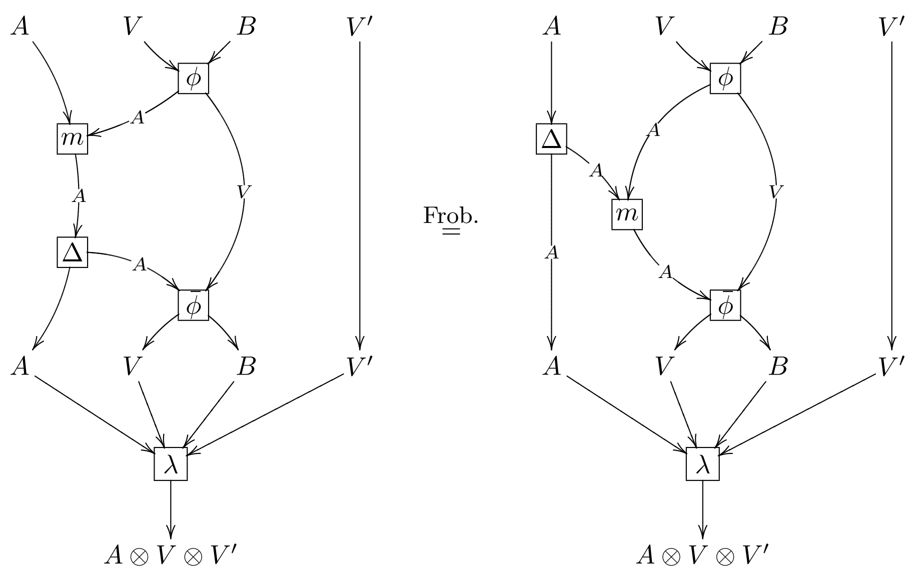
Consider the sequence

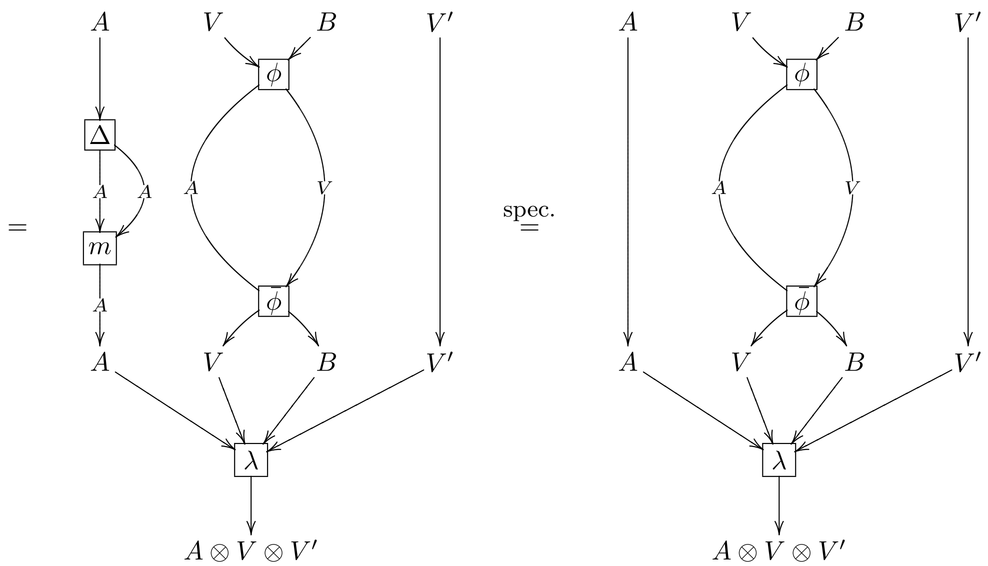
$$
\begin{array}{ccccc}
A \otimes V \ \otimes B \ \otimes B \otimes V' & \xrightarrow{r\otimes(B\otimes V') - (A\otimes V)\otimes \ell} & (A \otimes V) \otimes (B \otimes V') & \xrightarrow{f} & (A \otimes V \otimes V') \ , \\
 & & \downarrow{\scriptstyle \lambda} & \swarrow{\scriptstyle g} & \\
 & & A \otimes V \otimes V' & &
\end{array}
$$

and set

$$
\begin{array}{ccc}
A \otimes V \otimes V' & \xrightarrow{g} & A \otimes V \otimes V' \\
{\scriptstyle \Delta\otimes V\otimes V'}\downarrow & & \uparrow{\scriptstyle \lambda} \\
A \otimes A \otimes V \otimes V' & \xrightarrow[A\otimes\bar{\phi}\otimes V']{} & (A \otimes V) \otimes (B \otimes V')
\end{array} \ .
$$

One sees that g really makes the diagram commute by the following computation.





In the first step we have used the Frobenius property of A, in the second the compatibility of ϕ with the product and of λ with the left and right action. Finally, in the third step we have used speciality of A, assuming that $\beta_A = 1$. The resulting morphism is clearly equal to λ.

(UNIQUENESS OF g REMAINS TO BE SHOWN) □

Definition 14 *Every algebra homomorphism* $B \xrightarrow{\rho} A$ *defines a left-induced bimodule*

$$ {}_A\rho_B \equiv A \overset{m}{\dashrightarrow} A \otimes \mathbb{1} \overset{\rho \circ m}{\dashleftarrow} B \ . $$

Proposition 5 *The bimodule tensor product of bimodules coming from algebra homomorphisms corresponds to the composition of the respective morphisms. More precisely, given algebra homomorphisms* $C \xrightarrow{\rho'} B \xrightarrow{\rho} A$ *we have*

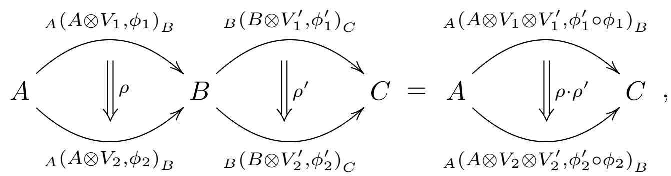
$$ {}_A\rho_B \otimes_B {}_B\rho'_C = A \overset{m}{\dashrightarrow} A \otimes \mathbb{1} \overset{\rho' \circ \rho \circ m}{\dashleftarrow} C \ . $$

Proposition 6

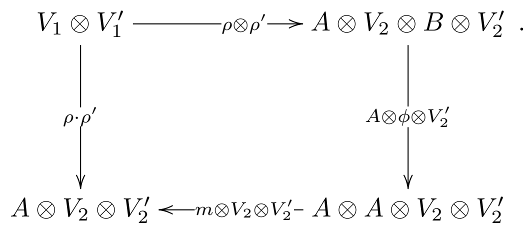
1. *The bimodule tensor product of left-induced bimodules* ${}_A(A \otimes V_1, \phi_1)_B$ *and* ${}_B(C \otimes V_2, \phi_2)_C$ *is*

$$ {}_A(A \otimes V_1, \phi_1)_B \otimes_B {}_B(B \otimes V_2, \phi_2)_C = {}_A(A \otimes V_1 \otimes V_2, \phi_2 \circ \phi_1)_C $$

2. *The horizontal product in* **LFBiMod**$(\mathcal{C})$ *is given by the following expression:*



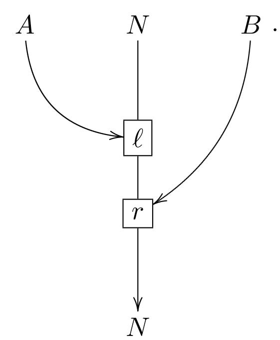
where



2.3.2 Conjugation of Bimodules

There are three kinds of conjugation operations on bimodules.

Proposition 7 ([18], prop. 2.10) *Let* $_AN_B$ *be a bimodule with action*



1.

2.4 Expressing Frobenius Algebras in Terms of Adjunctions

Every Frobenius algebra object in $\mathcal{C}$ can be expressed in terms of an adjunction in **BiMod**$(\mathcal{C})$.

In the literature one can find (at least) two slightly different realizations of this fact.

- From the general perspective of Eilenberg-Moore objects (and actually in more generality than we need here) in [11] (extending a similar construction in [13]) a construction using left-induced bimodules is given, where the two units and and counits of the ambijunction are built directly from the action of the Frobenius algebra's (co)product and (co)unit.

- In def. 2.12 of [18] a construction in terms of left A modules and their duals is given, where, implicitly, the units and counits of the ambijunction are constructed from the unit and counit of the *duality* on objects, composed with a projection operation.

Both these sources do not make all the details explicit that we will need. For instance [18] does not mention adjunctions at all. Therefore we spell out the details in the following two subsections.

2.4.1 Adjunctions using left-induced Bimodules

The algebra A may trivially be taken as a left, right or a bimodule over itself. We write ${}_A A_{\mathbb{1}}$, ${}_{\mathbb{1}} A_A$ and ${}_A A_A$, respectively, for the object A equipped with an A-module structure this way.

All three of these are left-induced bimodules. In order to be able to make full use of the rules for tensor products of left-induced bimodules, the following definition spells out the left-induced bimodule structure on

$$ {}_A L_{\mathbb{1}} \equiv {}_A A_{\mathbb{1}} $$

and

$$ {}_{\mathbb{1}} R_A \equiv {}_{\mathbb{1}} A_A $$

according to def. 12.

Definition 15 *Given a Frobenius algebra A in $\mathcal{C}$, we define the following left-induced bimodules.*

1.

$$ {}_{\mathbb{1}} L_A \equiv {}_{\mathbb{1}}(\mathbb{1} \otimes A, \phi)_A \equiv \; \mathbb{1} \overset{m}{\dashrightarrow} \mathbb{1} \otimes A \overset{\phi\circ m}{\dashleftarrow} A $$

with

$$ \begin{array}{ccc} A \otimes A & \xrightarrow{\;\phi\;} & \mathbb{1} \otimes A \\ & {\scriptstyle m}\searrow \qquad \nearrow & \\ & A & \end{array} $$

2.

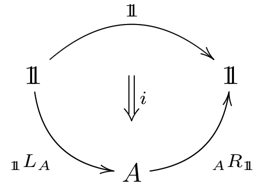
$$ {}_A R_{\mathbb{1}} \equiv {}_A(A \otimes \mathbb{1}, \phi)_{\mathbb{1}} \equiv \; A \overset{m}{\dashrightarrow} A \otimes \mathbb{1} \overset{\phi\circ m}{\dashleftarrow} \mathbb{1} $$

with

$$\begin{array}{ccc} \mathbb{1}\otimes\mathbb{1} & \xrightarrow{\phi} & A\otimes\mathbb{1} \\ \downarrow & & \uparrow \\ \mathbb{1} & \xrightarrow[i]{} & A \end{array}$$

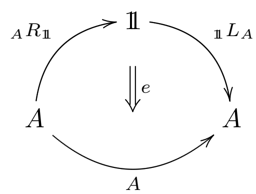
and the following bimodule morphisms

1. <u>*left unit*</u>



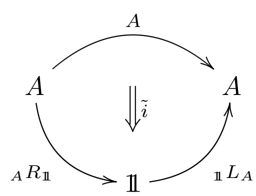
given by $\mathbb{1} \xrightarrow{i} \mathbb{1}\otimes A$ *and hence inducing* $\mathbb{1} \xrightarrow{i} A$

2. <u>*left counit*</u>



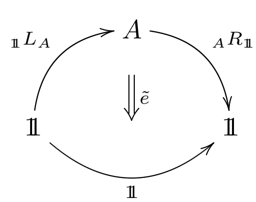
given by $A \longrightarrow A\otimes\mathbb{1}$ *and hence inducing* $A\otimes A \xrightarrow{m} A$

3. <u>*right unit*</u>



given by $\mathbb{1} \xrightarrow{i} A \xrightarrow{\Delta} A\otimes A$ *and hence inducing* $A \xrightarrow{\Delta} A\otimes A$

4. <u>*right counit*</u>



given by $A \xrightarrow{e} \mathbb{1}\otimes\mathbb{1}$ *and hence inducing* $A \xrightarrow{e} \mathbb{1}$

Proposition 8 *For A a special Frobenius algebra, this defines a special ambidextrous adjunction* $\mathrm{Adj}(A)$.

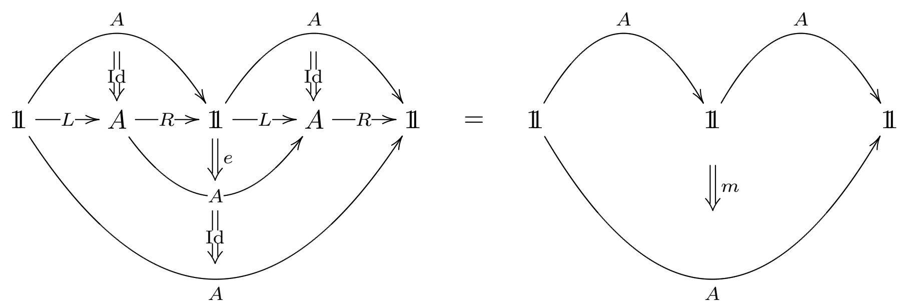
Proof. Using the rules for horizontal and vertical composition of left-induced bimodules given in §2.3.1 one straightforwardly checks the required zig-zag identities as well as the specialty property. □

Proposition 9 *The Frobenius algebra* Frob(Adj(A)) *obtained from this ambidextrous adjunction is* A *itself*

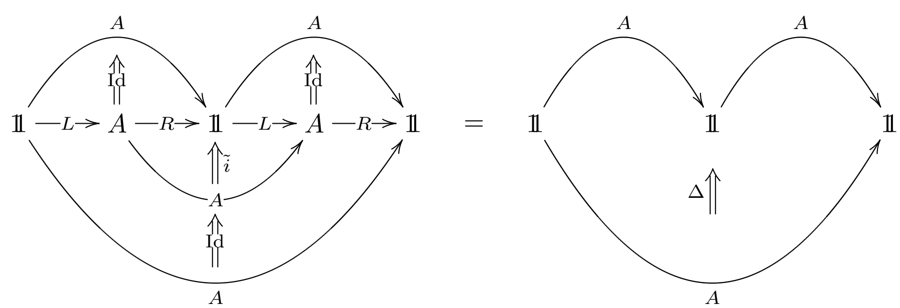
$$\text{Frob}(\text{Adj}(A)) = A\,.$$

Proof. Applying the rules for horizontal and vertical composition of left-induced bimodules yields the following identities.

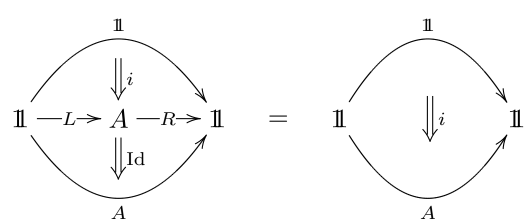
1. <u>product</u>



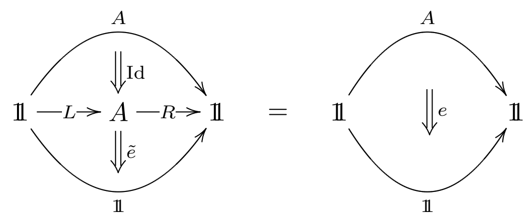
2. <u>coproduct</u>



3. <u>unit</u>



4. <u>counit</u>

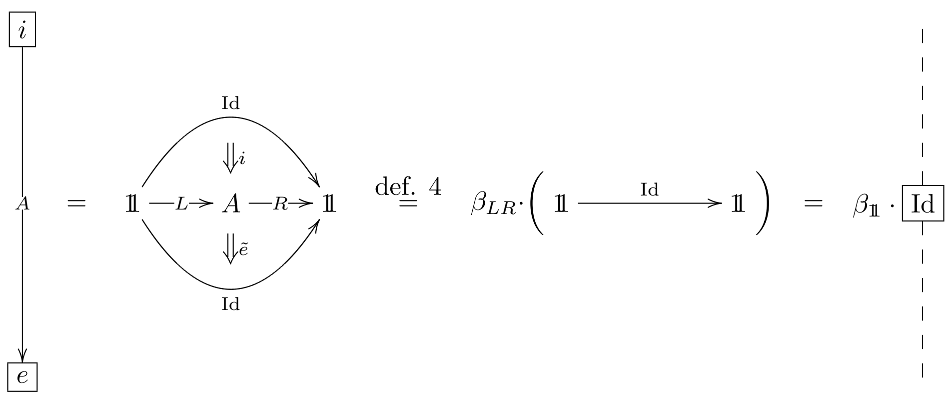


□

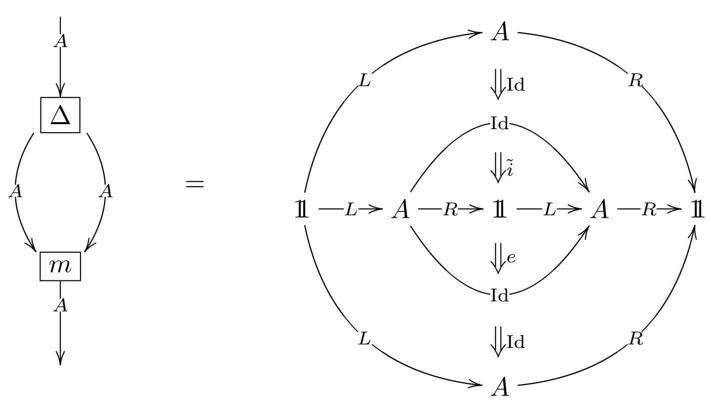
Proposition 10 *Under the relation of Frobenius algebras with ambidextrous adjunctions (prop. 8 and 9)* special *Frobenius algebras (def. 6) correspond to* special *ambijunctions (def. 4). The constants are related by*

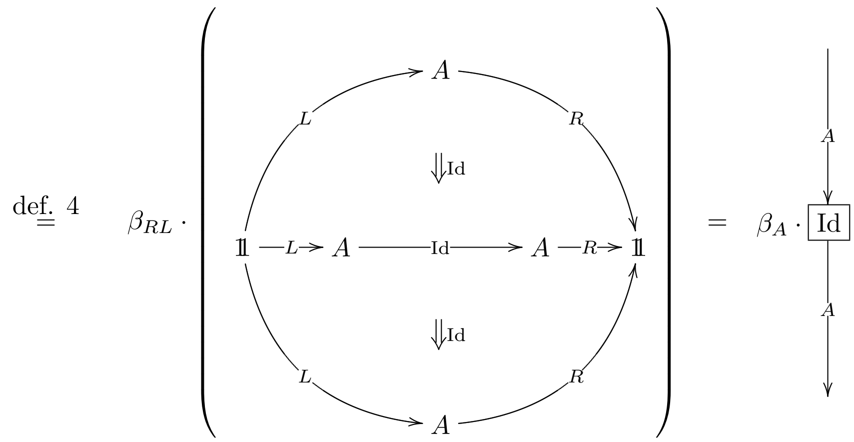
$$\begin{aligned} \beta_{\mathbb{1}} &= \beta_{LR} \\ \beta_A &= \beta_{RL}\,. \end{aligned}$$

Proof. We have



and





□

2.4.2 Adjunctions using Duality and Projection

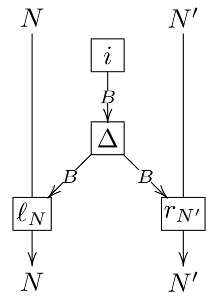
Due to the extra structure present on Frobenius algebras, bimodules for Frobenius algebras are easier to handle than general bimodules. Using the coproduct one can built the following projector, which sends objects in the ordinary tensor product of two bimodules to their images in the bimodule tensor product.

Definition 16 ([18], (2.45)) *Let A, B, C be special, symmetric Frobenius algebras in the rigid (meaning that all duals exist) monoidal category $\mathcal{C}$. For every pair ${}_AN_B$, ${}_BN'_C$ of bimodules, let* $N\otimes N' \xrightarrow{P_{N,N'}} N\otimes N'$ *be given by*

$$\begin{array}{ccc} N\otimes N' & \xrightarrow{P_{N,N'}} & N\otimes N' \\ {\scriptstyle N\otimes i\otimes N'}\downarrow & & \uparrow{\scriptstyle \ell_N\otimes r_{N'}} \\ N\otimes B\otimes N' & \xrightarrow[\Delta]{} & N\otimes B\otimes B\otimes N' \end{array} \quad .$$

Note that the tensor product '$\otimes$' is that of $\mathcal{C}$. The bimodule tensor product (over B, say), will be denoted $\otimes_B$.

In string diagrams $P_{N,N'}$ looks like



.

Using associativity and the Frobenius property, one readily checks that $P_{N,N'}$ is a projector and, when $\mathcal{C}$ is abelian, that it annihilates elements that vanish in the bimodule tensor product.

Hence the image of $P_{N,N'}$ is indeed the bimodule tensor product of N with N',

$$N \otimes_B N' = \mathrm{im}\,(P_{N,N'}) \; .$$

More precisely, we have the following general definition of images

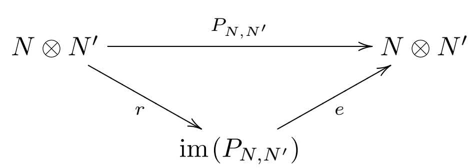
Definition 17 (compare [18], def. 2.12) *The object* $\mathrm{im}\,(P_{N,N'}) \in \mathrm{Obj}\,(\mathcal{C})$ *is called the* **image** *of* $N \otimes N' \xrightarrow{P_{N,N'}} N \otimes N'$ *if there are morphisms*

$$\mathrm{im}\,(P_{N,N'}) \xrightarrow{e} N \otimes N'$$

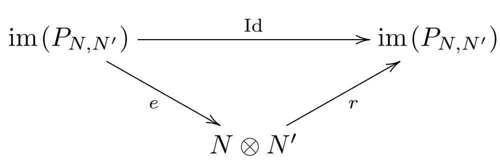
(the **injection of the image into the domain***) and*

$$N \otimes N' \xrightarrow{r} \mathrm{im}\,(P_{N,N'})$$

(the **projection of the domain onto the image***) such that*



and

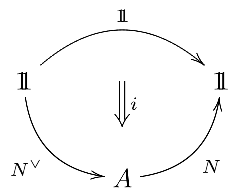


.

In the cases of interest here, where $\mathcal{C}$ is abelian and semisimple, such morphisms e and r do exist for every projector.

Next we construct ambijunctions, using left A-modules N and the projector $P_{N^\vee,N}$, along the lines of [18], prop. 2.13

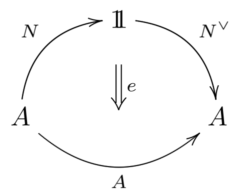
1. left unit



given by

$$\begin{array}{c} \mathbb{1} \\ \downarrow \\ N^\vee \otimes N \\ \downarrow{\scriptstyle r} \\ N^\vee \otimes_A N \end{array}$$

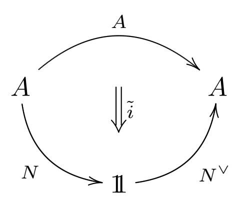
2. left counit



given by

$$\begin{array}{c} N \otimes N^\vee \\ \downarrow \\ A \otimes_A N \otimes N^\vee \otimes_A A \\ \downarrow{\scriptstyle e\otimes e} \\ A \otimes N \otimes N^\vee \otimes A \\ \downarrow{\scriptstyle A\otimes b\otimes A} \\ \mathbb{1} \end{array}$$

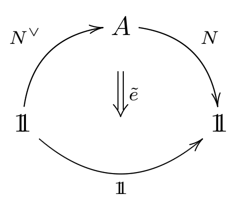
3. right unit



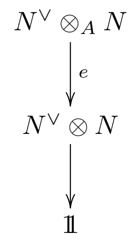
given by

$$\begin{array}{c} A \\ \downarrow \\ A \otimes_A A \\ \downarrow{\scriptstyle e} \\ A \otimes A \\ \downarrow \\ A \otimes N \otimes N^\vee \otimes A \\ \downarrow{\scriptstyle r\otimes r} \\ N \otimes N^\vee \\ \downarrow{\scriptstyle r} \\ N \otimes_A N^\vee \end{array}$$

4. right counit



given by



[...]

2.4.3 Relation between the two Constructions

The two constructions described above are closely related whenever the left A module N is A itself, regarded as a left module over itself, i.e. whenever

$$ {}_A N_{\mathbb{1}} = {}_A R_{\mathbb{1}} \,, $$

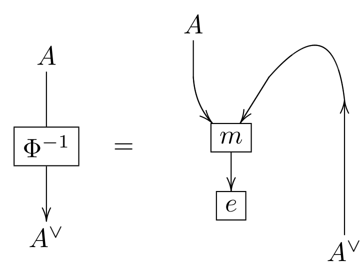
where ${}_A R_{\mathbb{1}}$ was defined in def. 15.

Here we will relate the construction of special ambidextrous adjunctions from §2.4.1 to the constructions used in [18], section 2.4.

What relates the two constructions is the isomorphism between a special symmetric Frobenius algebra A and its dual from def. 7.

We need this simple

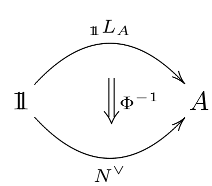
Proposition 11 *The morphism*



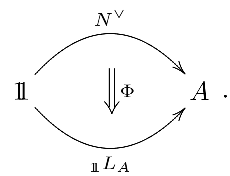
is in particular also a morphism of right A-modules.

Proof. Attach an A-line to the incoming A-line, use associativity to pass it past the product, observe that the result is the right A-action on $A^\vee$. □

Let ${}_{\mathbb{1}}L_A$ be, as before, A regarded as a right A-module over itself, and let $N^\vee = A^\vee$ be $A^\vee$ regarded as a right A-module. This means we have isomorphisms of $\mathbb{1} - A$ bimodules



and



By pasting these 2-cells wherever appropriate, we can transform the diagrams corresponding to the adjunction on ${}_{\mathbb{1}}L_A$ and ${}_AR_{\mathbb{1}}$ to those of the adjunction in N and $N^\vee$, where $N = A$ as a right A-module over itself. And vice versa.

2.5 Opposite Algebras

2.5.1 Definitions

We recall some facts and definitions on opposite algebras in ribbon categories from section 3.5 of [17] and section 2.1 of [18].

In the following, let $\mathcal{C}$ be a *ribbon category*. Denote its braiding morphisms by

$$U \otimes V \xrightarrow[\simeq]{c_{U,V}} V \otimes U$$

and its twist morphisms by

$$U \xrightarrow[\simeq]{\theta_U} U \ .$$

Definition 18 *Let A be an algebra with product*

$$A \otimes A \xrightarrow{m} A$$

internal to some ribbon category $\mathcal{C}$.

The **opposite algebra** A_{op} *is the internal algebra based on the same object, $A_{\mathrm{op}} = A$, but with product m_{op} given by*

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m_{\mathrm{op}}} & A \\ & {\scriptstyle c_{A,A}} \searrow \quad \nearrow {\scriptstyle m} & \\ & A \otimes A & \end{array} \ .$$

Remark. In general $(A_{\mathrm{op}})_{\mathrm{op}}$ is not isomorphic to A. We write

$$A = A^{(0)}$$

and

$$A^{(n+1)} \equiv (A^{(n)})_{\mathrm{op}} \, .$$

Definition 19 *A morphism*

$$A \xrightarrow{\sigma} A$$

is called an **algebra antihomomorphism** *if regarded as a morphism*

$$A \xrightarrow{\sigma} A^{\mathrm{op}}$$

it is an ordinary algebra homomorphism.

Hence σ is an algebra antihomomorphism iff

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\sigma\otimes\sigma} & A\otimes A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m^{\mathrm{op}}} \\ A & \xrightarrow[\sigma]{} & A \end{array}$$

and

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{i} & A \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \sigma} & \\ & A & \end{array}$$

commute.

Definition 20 *An algebra antihomomorphism is called a* **reversion** *if it squares to the twist, i.e. if*

$$\begin{array}{ccc} A & \xrightarrow{\theta} & A \\ & {\scriptstyle \sigma}\searrow \quad \nearrow{\scriptstyle \sigma} & \\ & A & \end{array}$$

Definition 21 *If A is also a coalgebra we let the coproduct Δ^{op} on A^{op} be given by*

$$\begin{array}{ccc} A & \xrightarrow{\Delta^{\mathrm{op}}} & A\otimes A \\ & {\scriptstyle \Delta}\searrow \quad \nearrow{\scriptstyle \bar{c}_{AA}} & \\ & A\otimes A & \end{array} .$$

An antihomomorphism of coalgebras then has to satisfy

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & A \\ {\scriptstyle \Delta^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \Delta} \\ A\otimes A & \xrightarrow[\sigma\otimes\sigma]{} & A\otimes A \end{array} .$$

Proposition 12 *An algebra A with reversion is Morita equivalent to its opposite algebra A^{op}. If A is special Frobenius then so is A^{op}, and we have*

$$\begin{aligned} \beta_A &= \beta_{A^{\mathrm{op}}} \\ (\beta_{\mathbb{1}})_A &= (\beta_{\mathbb{1}})_{A^{\mathrm{op}}} . \end{aligned}$$

Proof. A reversion is an algebra homomorphism and hence induces invertible left-induced bimodules

$$ {}_A N_{\sigma\, A^{\mathrm{op}}} = \; A \overset{m}{- - \succ} A \overset{m\circ\sigma}{\prec - -} A^{\mathrm{op}} $$

and

$$ {}_{A^{\mathrm{op}}} N_{\sigma\, A} = \; A^{\mathrm{op}} \overset{m^{\mathrm{op}}}{- - \succ} A^{\mathrm{op}} \overset{m\circ\bar{\sigma}}{\prec - -} A $$

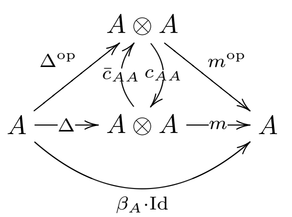
whose product is, according to prop. 5 (p. 24),

$$ {}_A N_{\sigma\, A^{\mathrm{op}}} \;\otimes_{A^{\mathrm{op}}}\; {}_{A^{\mathrm{op}}} N_{\sigma\, A} = {}_A A_A $$

and

$$ {}_{A^{\mathrm{op}}} N_{\sigma\, A} \;\otimes_A\; {}_A N_{\sigma\, A^{\mathrm{op}}} = {}_{A^{\mathrm{op}}} A^{\mathrm{op}}{}_{A^{\mathrm{op}}} \, . $$

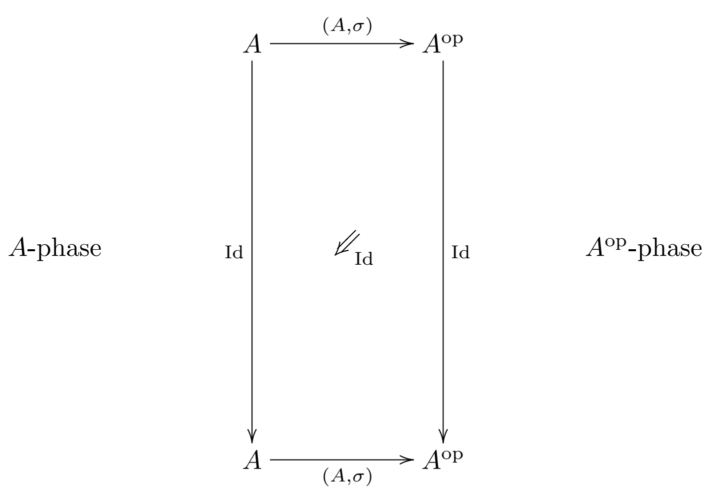
The fact that A^{op} is special with the same constants as A follows from the commutativity of the diagram



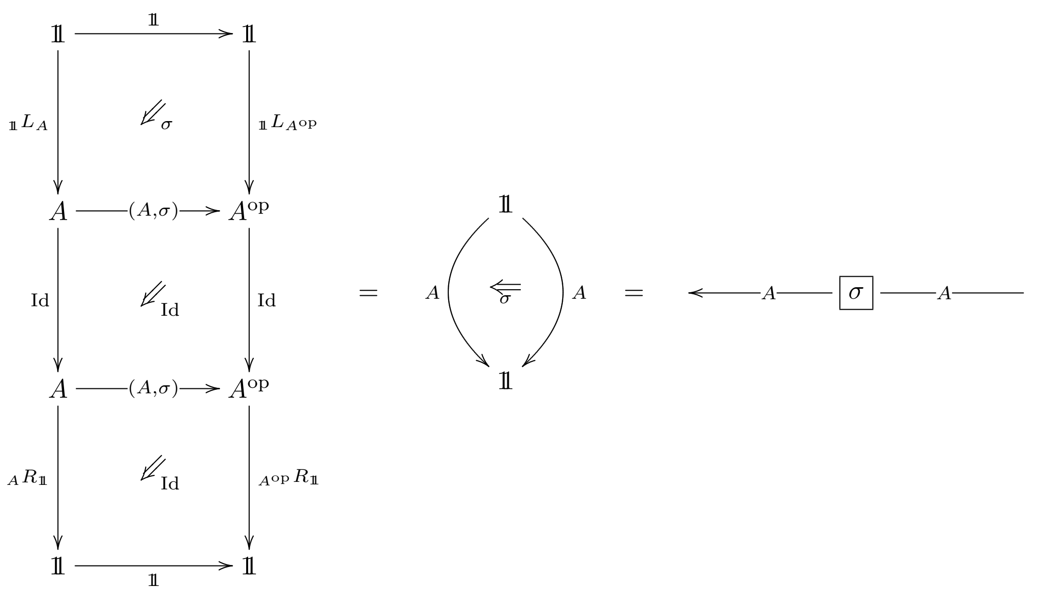
That the $\beta_{\mathbb{1}}$ coincide is trivial, since unit and counit of A and A^{op} coincide. □

Example 4

Consider a bimodule which relates an A-phase with an A^{op}-phase



According to §14, local trivialization turns this defect locally into



On the left we have here the 2-morphism in $\Sigma(\mathcal{C})$ which is obtained from the above 2-morphism in $\mathbf{BiMod}(\mathcal{C})$ by the trivialization procedure described in §14. In the middle the same 2-morphism is depicted, now with the products of bimodules explicitly evaluated. On the right the Poincaé-dual string diagram of this 2-morphism is given. This is simply an A-line with a reversion. Compare section 3 of [18].

2.5.2 The Half-Twist in terms of Adjunctions

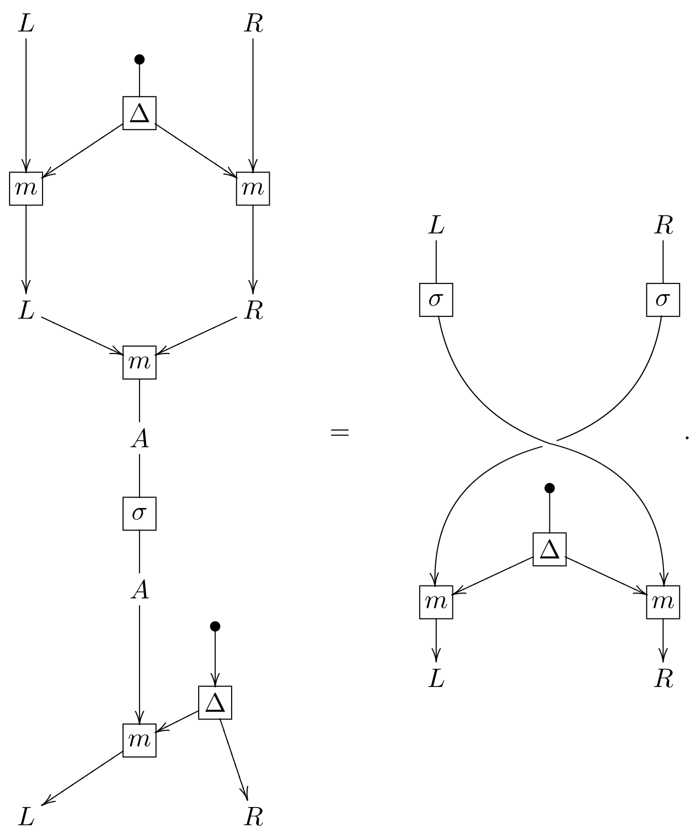
We would like to understand how the reversion

$$A \xrightarrow{\sigma} A$$

looks like in terms of the ambidextrous adjunction that A is made of. Using the formalism from [18], section 2 (def. 17) we can resolve A as the image of the map $L \otimes R \to L \otimes_A R$ and push σ through to the right A-module ${}_{\mathbb{1}}L_A = (A, m)$ and the left module ${}_A R_{\mathbb{1}} = (A, m)$.

Doing so, one finds

Proposition 13



Proof. Use the defining properties of σ as an (invertible) algebra antihomomorphism. □

Notice how both L and R are, as objects, nothing but A itself.

If we think of two lines, one labelled by L, one by R, as the boundary of a ribbon which is decorated by $L \otimes_A A \simeq A$, then the above proposition says that a reversion acts on these ribbons by acting as σ on L and R and by performing a half-twist of that ribbon.

2.5.3 $\mathbf{Vect}_1$

$$\mathbb{C} \xrightarrow{\sigma} \mathbb{C}^{\mathrm{op}}$$

$$V \otimes \mathbb{C}_\sigma \simeq V_\sigma$$

$$vc \simeq (v,1)c = (v,\bar{c}) = (v\bar{c},1) = \bar{c}(v,1) \simeq \bar{c}v$$

$$\mathbb{C}_\sigma \otimes V \simeq {}_\sigma V$$

$$cv \simeq c(1, v) = (c, v) = (1\bar{c}, v) = (1, \bar{c}v) = (1, v)\bar{c} \simeq v\bar{c}$$

2.5.4 Involutions

For any real vector space V with complex structure, let $\bar{V}$ be the same real vector space, but with opposite complex structure.

Denote by

$$\begin{array}{ccccc} \sigma & : & \mathbb{C} & \to & \mathbb{C} \\ & & c & \mapsto & \bar{c} \end{array}$$

the conjugation involution on $\mathbb{C}$ and by $\mathbb{C}_\sigma$ the $\mathbb{C}$-$\mathbb{C}$ bimodule which, as an object, is $\mathbb{C}$ itself, with the left $\mathbb{C}$-action being multiplication in $\mathbb{C}$ and the right $\mathbb{C}$ action given by first acting with σ and then multiplying in $\mathbb{C}$:

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}_\sigma & \xrightarrow{l} & \mathbb{C}_\sigma \\ (c, d) & \mapsto & cd \end{array}$$

$$\begin{array}{ccc} \mathbb{C}_\sigma \times \mathbb{C} & \xrightarrow{r} & \mathbb{C}_\sigma \\ (d, c) & \mapsto & \bar{c}d \end{array}.$$

Similarly, for any complex vector space V, let

$$V_\sigma \simeq V \otimes \mathbb{C}_\sigma$$

and

$${}_\sigma V \simeq \mathbb{C}_\sigma \otimes V$$

be the $\mathbb{C}$-$\mathbb{C}$-bimodule V, as an object, but with the left or right $\mathbb{C}$ action twisted, as indicated.

Notice that we have the canonical isomorphism

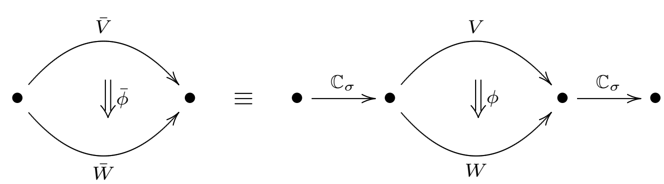
$${}_\sigma V_\sigma \simeq \bar{V}$$

and in particular the canonical identitfication

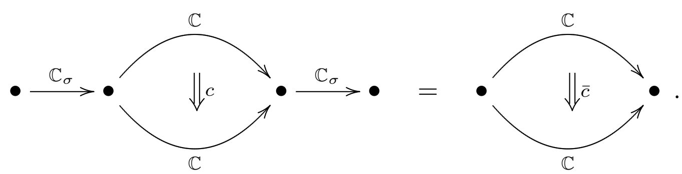
$$\mathbb{C}_\sigma \otimes \mathbb{C}_\sigma \simeq \bar{\mathbb{C}} \simeq \mathbb{C}\,.$$

Denote by $\mathbf{BiMod}_{\mathbb{C}}$ the 2-category of $\mathbb{C}$-$\mathbb{C}$-bimodules, with single object $\mathbb{C}$, bimodules up to canonical isomorphism as 1-morphisms and bimodule intertwiners as 2-morphisms.

We write

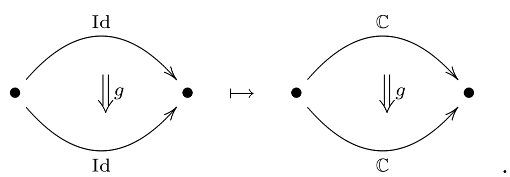


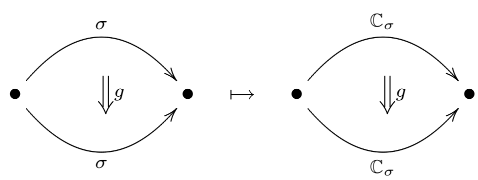
and find in particular



It follows that we obtain a representation of the automorphism 2-group $\mathrm{Aut}(U(1))$ of $U(1)$, on $\mathbf{BiMod}_{\mathbb{C}}$ by setting

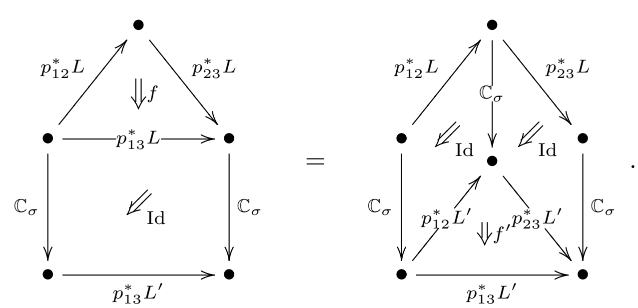
$$\rho \quad : \quad \Sigma(\mathrm{Aut}(U(1))) \quad \to \quad \mathbf{BiMod}_{\mathbb{C}}$$





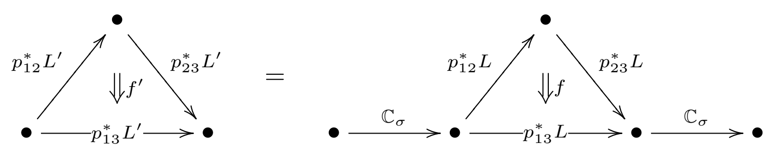
Here we have denoted the nontrivial element in the automorphism group $\mathbb{Z}_2$ of $U(1)$ also by σ.

We are interested in transition morphisms in $\mathbf{Trans}(\mathcal{P}, \mathbf{BiMod}_{\mathbb{C}})$. Consider the case where such a morphism involves $\mathbb{C}_\sigma$ in its defining tin can equation as follows



The existence of the identity-2-morphisms here says that the transition line bundles are related by $L' = \bar{L}$.

When we equivalently rewrite this equation as



,

which says that

$$f' = \bar{f}\,.$$

3 Transport 2-Functors

Fix once and for all some monoidal category $\mathcal{C}$. We are interested in 2-functors

$$\mathrm{tra} : \mathcal{P}_2 \to \mathbf{BiMod}(\mathcal{C})$$

from some geometric 2-category $\mathcal{P}_2$ to the 2-category of algebra bimodules (of special symmetric Frobenius algebras) internal to $\mathcal{C}$ (def. 11). In the context of the present discussion these 2-functors shall be called **transport 2-functors**.

There is a general theory of transport 2-functors and in particular of local trivialization of 2-transport. All of the following constructions are just special instances of that general theory. For more details see [9].

3.1 Trivial 2-Transport

Definition 22 *We say a transport 2-functor* $\mathrm{tra} : \mathcal{P}_2 \to \mathbf{BiMod}(\mathcal{C})$ *is* **trivial** *precisely if it takes values only in* $\mathcal{C} \simeq \mathrm{Hom}_{\mathbf{BiMod}(\mathcal{C})}(\mathbb{1}, \mathbb{1}) \subset \mathbf{BiMod}(\mathcal{C})$.

We write

$$\mathrm{tra}_{\mathbb{1}} : \mathcal{P}_2 \to \mathcal{C} \subset \mathbf{BiMod}(\mathcal{C})$$

for a trivial transport 2-functor $\mathrm{tra}_{\mathbb{1}}$.

Remark. The terminology "trivial" here is motivated from a similar condition on 2-transport in 2-bundles. It is not supposed to suggest that a trivial transport 2-functor encodes no interesting information. Rather, one should think of a general transport 2-functor as defining an algebra-bundle over the space of objects of $\mathcal{P}_2$. For a trivial transport 2-functor this bundle is trivial in that all its fibers are identified with the (trivial) algebra $\mathbb{1}$.

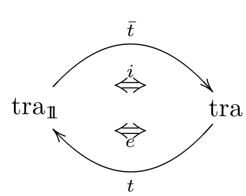
Definition 23 *A* **trivialization** *of a transport 2-functor*

$$\mathrm{tra} : \mathcal{P}_2(M) \to \mathbf{BiMod}(\mathcal{C})$$

is a choice *of a trivial transport 2-functor*

$$\mathrm{tra}_{\mathbb{1}} : \mathcal{P}_2 \to \mathcal{C} \subset \mathbf{BiMod}(\mathcal{C})$$

together with a choice *of special ambidextrous adjunction (defs. 1, 2, 4)*



A transport 2-functor is called **trivializable** if it admits a trivialization.

The most general condition under which a transport 2-functor is trivializable is not investigated here. We shall be content with showing that all transport 2-functors of the following form are trivializable.

Theorem 1 *Transport 2-functors to left-induced bimodules*

$$\mathrm{tra} : \mathcal{P}_2 \to \mathbf{LFBiMod}(\mathcal{C}) \subset \mathbf{BiMod}(\mathcal{C})$$

are trivializable if *every 2-morphism*

$$\mathrm{tra}\left(x \underset{\gamma_2}{\overset{\gamma_1}{\Downarrow S}} y \right) = A_x \underset{(A_x \otimes V_{\gamma_2}, \phi_{\gamma_2})}{\overset{(A_x \otimes V_{\gamma_1}, \phi_{\gamma_1})}{\Downarrow \mathrm{tra}(S)}} A_y \qquad \in \mathrm{Mor}_2(\mathbf{LFBiMod}(\mathcal{C}))$$

for all $S \in \mathrm{Mor}_2(\mathcal{P}_2)$ *is of the form*

$$\begin{array}{c} A_x \otimes V_{\gamma_1} \\ \downarrow \mathrm{tra}(S) \\ A_x \otimes V_{\gamma_2} \end{array} = \begin{array}{c} A_x \otimes V_{\gamma_1} \\ \downarrow A_x \otimes \lambda_S \\ A_x \otimes V_{\gamma_2} \end{array}$$

for some $\lambda_S \in \mathrm{Mor}(\mathcal{C})$.

For the proof of prop. 12 below it is crucial to note by prop. 3 (p. 21) $\mathrm{tra}(S)$ being a morphism of bimodules implies that λ_S is such that the diagrams

$$\begin{array}{ccc} V_{\gamma_1} \otimes A_y & \xrightarrow{\phi_{\gamma_1}} & A_x \otimes V_{\gamma_1} \\ \Big\downarrow{\scriptstyle \lambda_s \otimes A_y} & & \Big\downarrow{\scriptstyle A_x \otimes \lambda_S} \\ V_{\gamma_2} \otimes A_y & \xrightarrow{\phi_{\gamma_2}} & A_x \otimes V_{\gamma_2} \end{array} \qquad (1)$$

commute.

Remark. In the application to the FRS formalism $\mathrm{tra}(S)$ plays the role of the morphism which connects the field insertions on the *connecting 3-manifold*. Hence the curious restriction on the nature of $\mathrm{tra}(S)$, which is crucial for the above theorem to be true, is precisely the property that $\mathrm{tra}(S)$ is assumed to have in FRS formalism.

The proof of proposition 1 amounts to constructing a trivialization and checking its properties. This is the content of the following subsection.

3.2 Trivialization of trivializable 2-Transport

In order to prove theorem 1 we need to construct a trivial transport 2-functor as well as all ingredients of a special ambidextrous adjunction such that all the required conditions are satisfied.

1. the trivial transport functor

   Define

$$\mathrm{tra}_{1\!\!1} : \mathcal{P}_2 \to \mathcal{C}$$

   by

$$\mathrm{tra}\left(x \underset{\gamma_2}{\overset{\gamma_1}{\underset{\Downarrow S}{\rightrightarrows}}} y \right) = 1\!\!1 \underset{V_{\gamma_2}}{\overset{V_{\gamma_1}}{\underset{\Downarrow \lambda_S}{\rightrightarrows}}} 1\!\!1 \qquad \in \mathrm{Mor}_2\left(\mathcal{C}\right)$$

   for all $S \in \mathrm{Mor}_2\left(\mathcal{P}_2\right)$.

2. the trivialization morphism

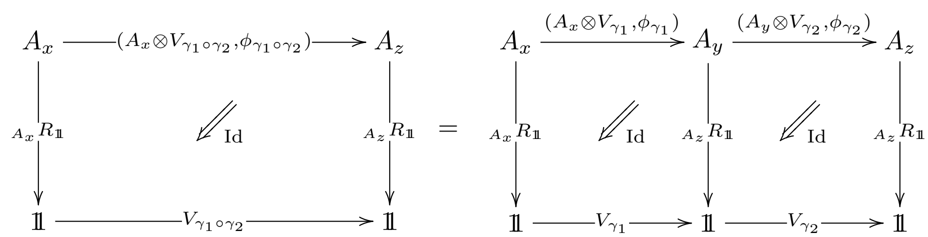
   Define the pseudonatural transformation $\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{1\!\!1}$ to be that given by the map

$$x \xrightarrow{\gamma} y \quad \mapsto \quad \begin{array}{ccc} \mathrm{tra}\left(x\right) & \xrightarrow{\mathrm{tra}(\gamma)} & \mathrm{tra}\left(y\right) \\ {\scriptstyle t(x)}\downarrow & \swarrow\!\!\!\swarrow\ {\scriptstyle t(\gamma)} & \downarrow{\scriptstyle t(y)} \\ \mathrm{tra}\left(x\right) & \xrightarrow{\mathrm{tra}(\gamma)} & \mathrm{tra}_i\left(y\right) \end{array} \quad \equiv \quad \begin{array}{ccc} A_x & \xrightarrow{(A_x \otimes V_\gamma, \phi_\gamma)} & A_y \\ {\scriptstyle {}_{A_x}R_{1\!\!1}}\downarrow & \swarrow\!\!\!\swarrow\ {\scriptstyle \mathrm{Id}} & \downarrow{\scriptstyle {}_{A_y}R_{1\!\!1}} \\ 1\!\!1 & \xrightarrow{V_\gamma} & 1\!\!1 \end{array} .$$

   That we really have an identity 2-morphism on the right hand side follows from def 15. This is readily seen to satisfy the required tin can equation

$$\begin{array}{ccc} A_x & \underset{(A_x\otimes V_{\gamma_2},\phi_{\gamma_2})}{\overset{(A_x\otimes V_{\gamma_1},\phi_{\gamma_1})}{\underset{\Downarrow \mathrm{tra}(S)}{\rightrightarrows}}} & A_y \\ {\scriptstyle {}_{A_x}R_{1\!\!1}}\downarrow & \swarrow\!\!\!\swarrow\ {\scriptstyle \mathrm{Id}} & \downarrow{\scriptstyle {}_{A_y}R_{1\!\!1}} \\ 1\!\!1 & \xrightarrow{V_\gamma} & 1\!\!1 \end{array} \quad = \quad \begin{array}{ccc} A_x & \xrightarrow{(A_x\otimes V_{\gamma_1},\phi_{\gamma_1})} & A_y \\ {\scriptstyle {}_{A_x}R_{1\!\!1}}\downarrow & \swarrow\!\!\!\swarrow\ {\scriptstyle \mathrm{Id}} & \downarrow{\scriptstyle {}_{A_y}R_{1\!\!1}} \\ 1\!\!1 & \underset{V_{\gamma_2}}{\overset{V_{\gamma_1}}{\underset{\Downarrow \lambda_S}{\rightrightarrows}}} & 1\!\!1 \end{array} .$$

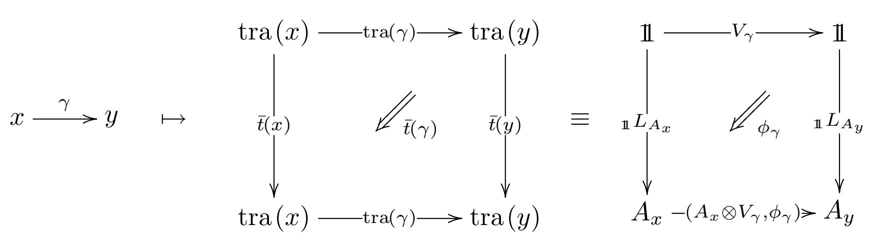
The functoriality condition



follows from the functoriality of tra.

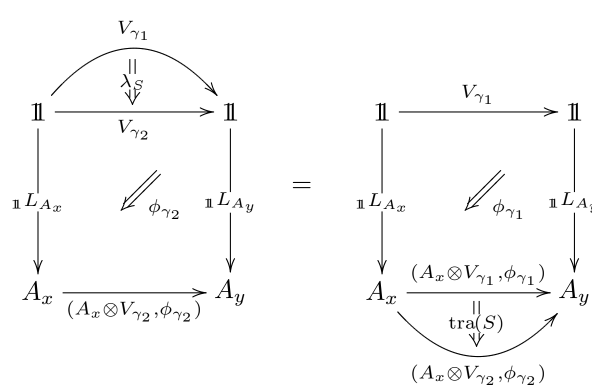
3. <u>the adjoint of the trivialization morphism</u>

   Define the pseudonatural transformation $\mathrm{tra}_{\mathbb{1}} \xrightarrow{\bar{t}} \mathrm{tra}$ to be that given by



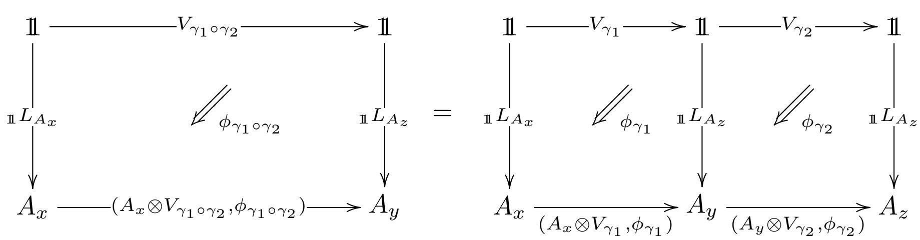
   This is well defined (i.e. this 2-morphism really gives a morphism of bimodules $V_\gamma \otimes_{\mathbb{1}} {}_{\mathbb{1}}L_{A_y} \xrightarrow{\phi_\gamma} {}_{\mathbb{1}}L_{A_x} \otimes_{A_x} (A_x \otimes V_\gamma, \phi_\gamma)$) due to the fact that ϕ_γ is compatible with the product (see def. 12).

   The required tin can equation



   holds by assumption on $\mathrm{tra}(S)$ (namely using the commutativity of the diagram (1), on p. 44).

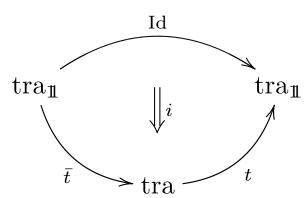
The functoriality condition



again follows from the functoriality of tra.

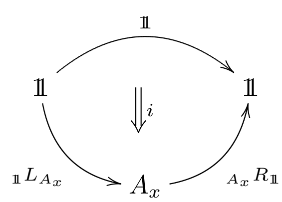
4. <u>the left unit</u>

Define the modification



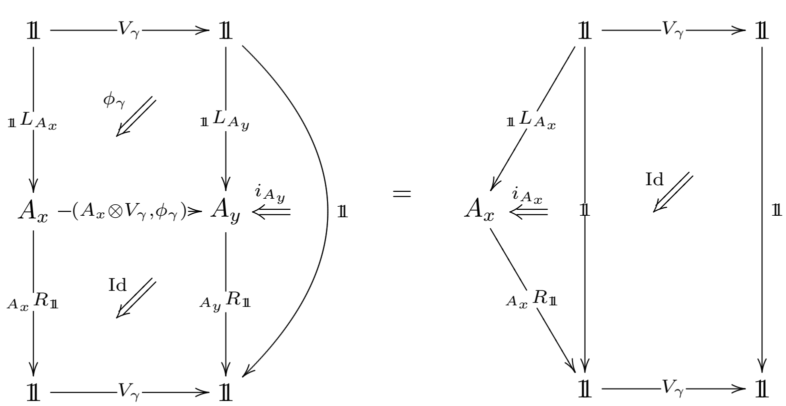
to be that given by the map

$\mathrm{Obj}(\mathcal{P}_2) \ni x \mapsto$



$\in \mathrm{Mor}_2(\mathbf{BiMod}(\mathcal{C}))$

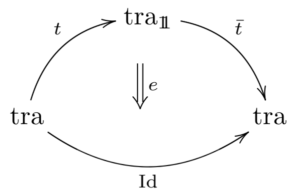
The required tin can equation



follows from the compatibility of ϕ with the unit (see def. 12).

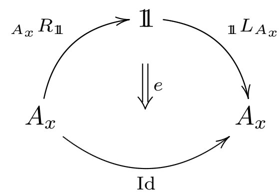
5. <u>the left counit</u>

Define the modification



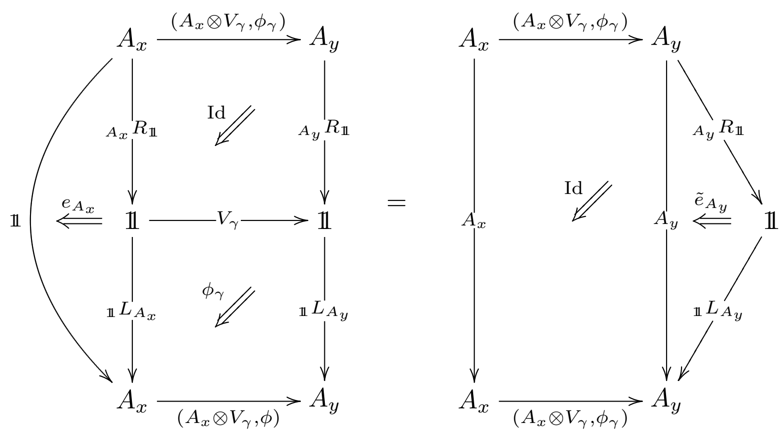
to be that given by the map

$\mathrm{Obj}(\mathcal{P}_2) \ni x \ \mapsto$



$\in \mathrm{Mor}_2(\mathbf{BiMod}(\mathcal{C}))$.

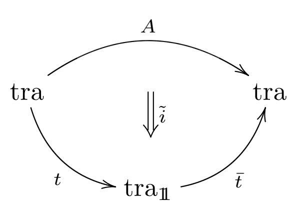
The required tin can equation



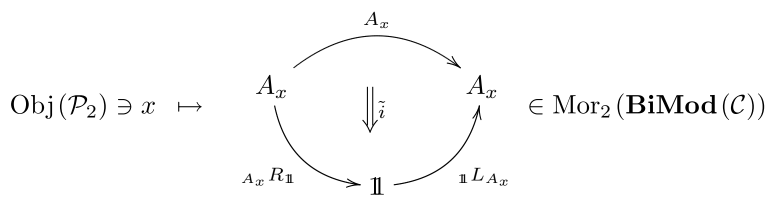
holds due to the compatibility of ϕ_γ with the counit.

6. <u>the right unit</u>

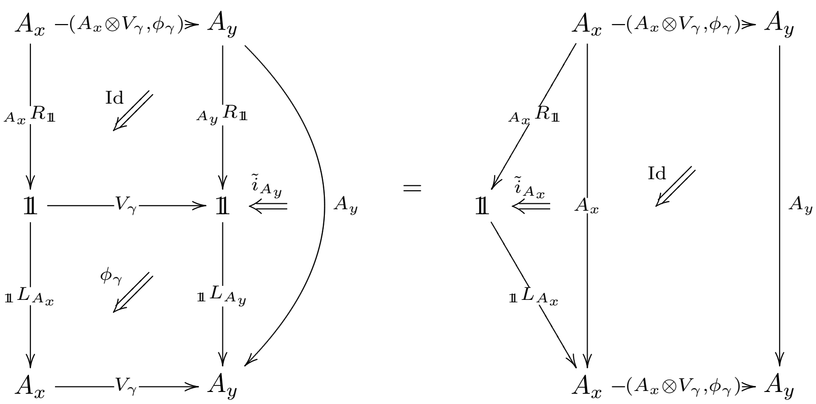
Define the modification



to be that given by the map

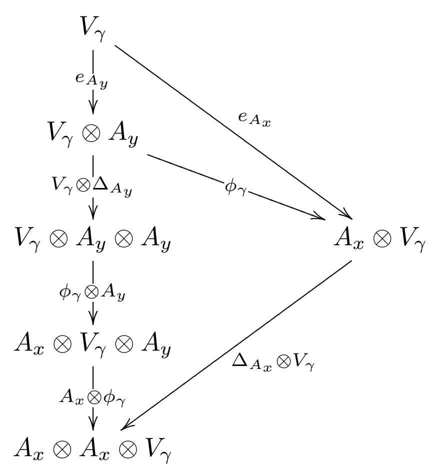


The required tin can equation



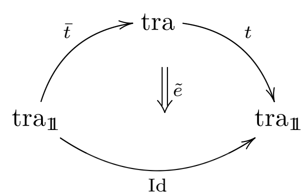
holds due to the compatibility of ϕ_γ with the coproduct.

Notice at this point that all these statements are straightforward to check, but require a careful application of all the definitions governing composition of morphisms of left free bimodules. The truth of the above statement for instance involves the commutativity of the diagram



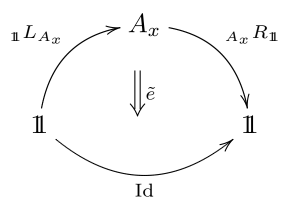
which holds due to compatibility with counit (upper part) and coproduct (lower part).

7. <u>the right counit</u> Define the modification



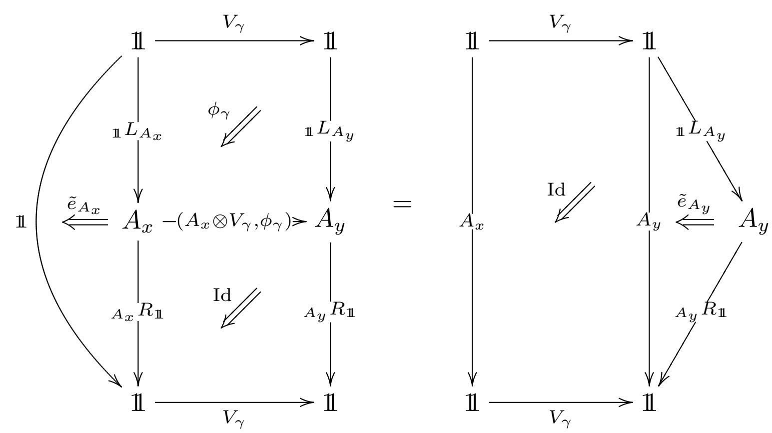
to be that given by the map

$\mathrm{Obj}(\mathcal{P}_2) \ni x \mapsto$



.

The required tin can equation



holds due to the compatibility of ϕ_γ with the the counit.

Finally, we need to check that the zig-zag identities are satisfied. But this is automatic, since the composition of modifications of pseudonatural transformations corresponds to the compositon of the respective 2-morphisms in the target 2-category. But these 2-morphisms, as defined above, are precisely those of the underlying ambijunction itself.

This then completes the proof. □

3.3 Expressing 2-Transport in Terms of Trivial 2-Transport

The crucial aspect of a trivialization of a 2-transport 2-functor is that it allows to express tra completely in terms of $\mathrm{tra}_{1\!\!1}$, t and $\bar{t}$.

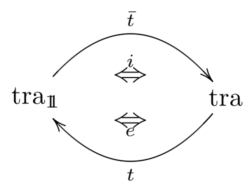
This involves two proposition, which are stated and proven in this section

1. Trivializable 2-transport is expressible completely in terms of trivial 2-transport (prop. 14).

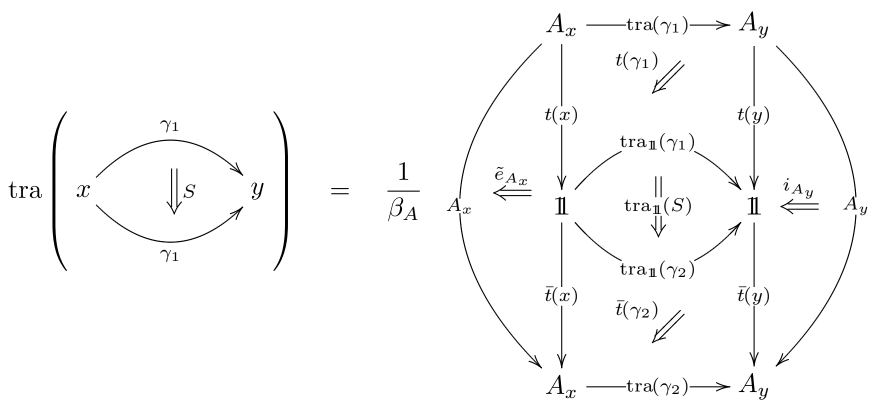
2. Possibly non-trivializable 2-transport gives rise to transitions between trivializable 2-transport (prop. 15).

3.3.1 Trivializable 2-Transport

Proposition 14 *The image of a trivializable 2-transport* tra *trvialized by*



can be expressed in terms of the trivialization as follows:



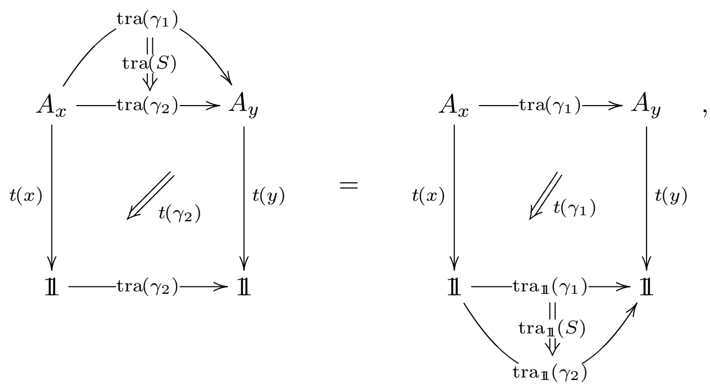
(2)

for all $S \in \mathrm{Mor}_2(\mathcal{P}_2)$.

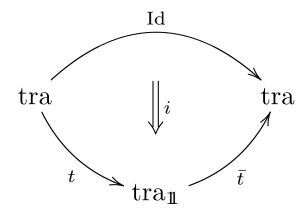
Proof. Use the tin can equation for the pseudonatural transformation

$$\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathbb{1}}$$

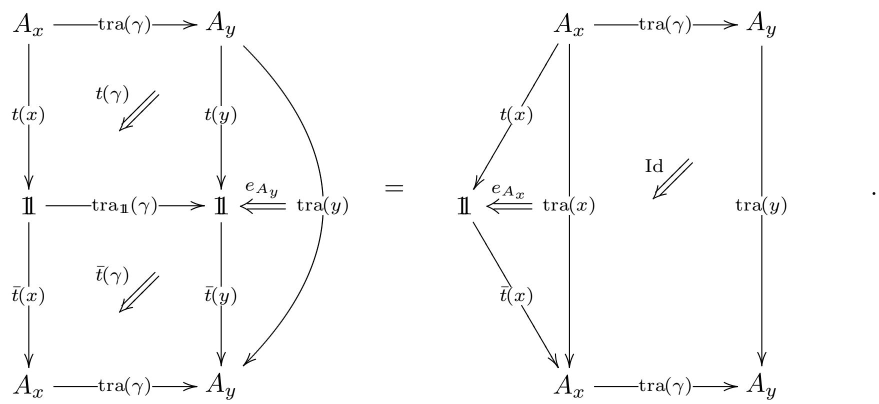
which reads



as well as the tin can equation for the modification



which reads



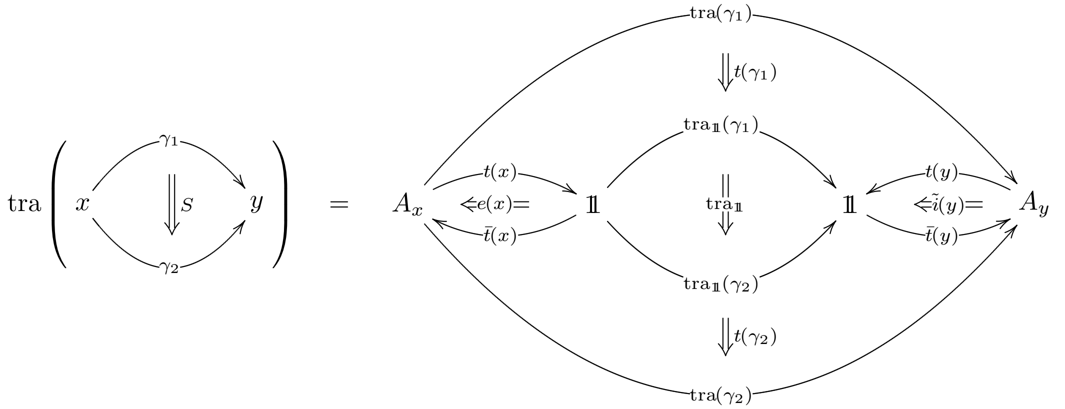
Finally, use the condition that the adjunction is *special*. □

Remark. It is precisely the above construction which makes us want to consider *special* ambidextrous adjunctios. In the following we will assume that we have arranged that

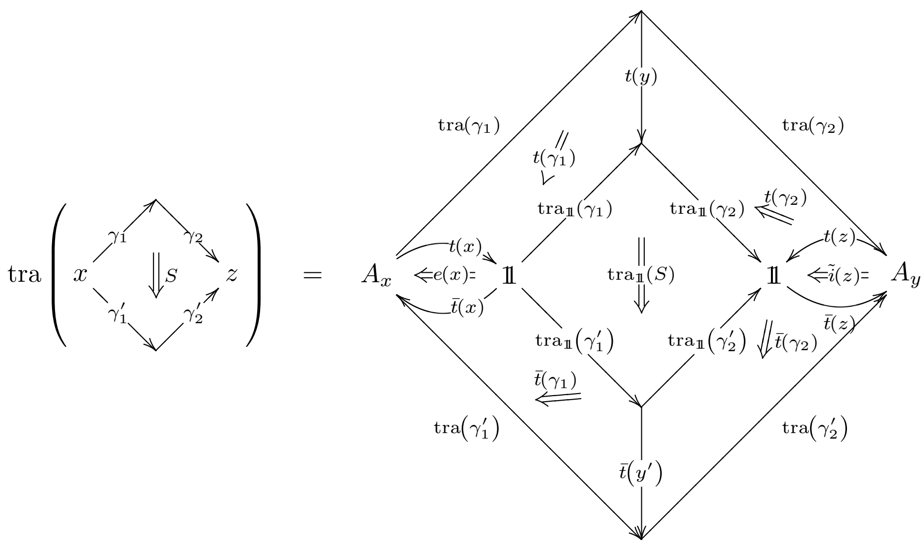
$$\beta_{A_x} = \beta_{A_y} = 1\,,$$

which can always be done.

By contracting the identity morphisms in (2) to a point, we can redraw this diagram more suggestively as

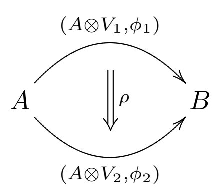


Often it is helpful to use transport over bigons which have a square-like appearance. Using the functoriality of pseudonatural transformations we can write

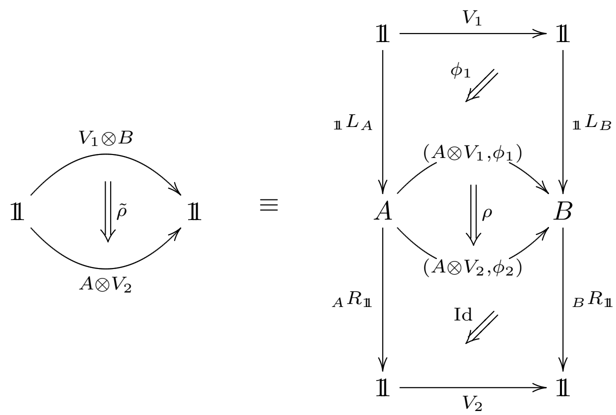


3.3.2 Not-necessarily trivializable 2-transport

Definition 24 *We can use the trivialization morphism and its adjoint as defined in def. 3.2 to assign to every 2-morphism*



in **LFBiMod**$(\mathcal{C})$ *the 2-morphism*

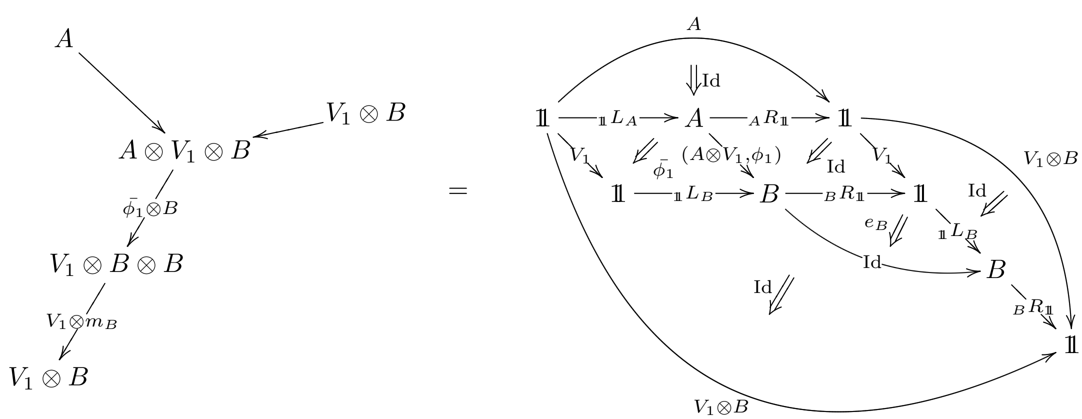


in $\Sigma(\mathcal{C})$.

Definition 25 *We make*

$$\mathbb{1} \xrightarrow{V_1 \otimes B} \mathbb{1} \;=\; \mathbb{1} \xrightarrow{V_1} \mathbb{1} \xrightarrow{{}_{\mathbb{1}}L_B} B \xrightarrow{{}_BR_{\mathbb{1}}} \mathbb{1}$$

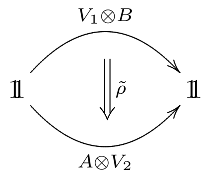
into an internal A-B-bimodule by using the obvious right action by B and left action by A. More precisely, the left A-action is that given by



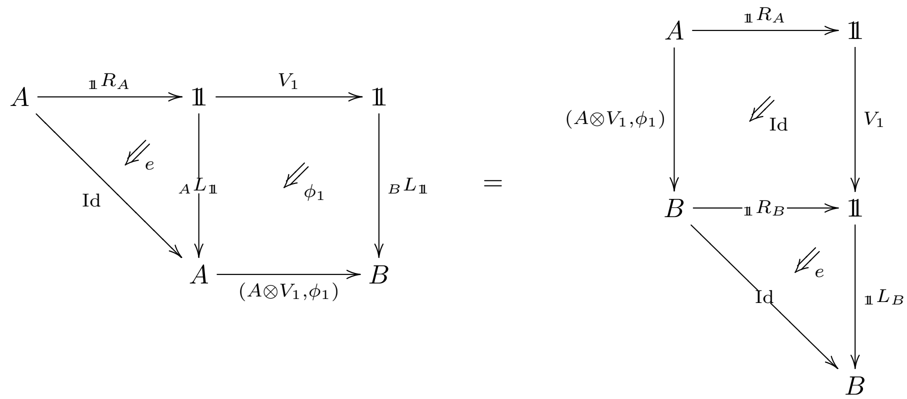
We make

$$\mathbb{1} \xrightarrow{A \otimes V_2} \mathbb{1} \;=\; \mathbb{1} \xrightarrow{{}_{\mathbb{1}}L_A} A \xrightarrow{{}_AR_{\mathbb{1}}} \mathbb{1} \xrightarrow{V_1} \mathbb{1}$$

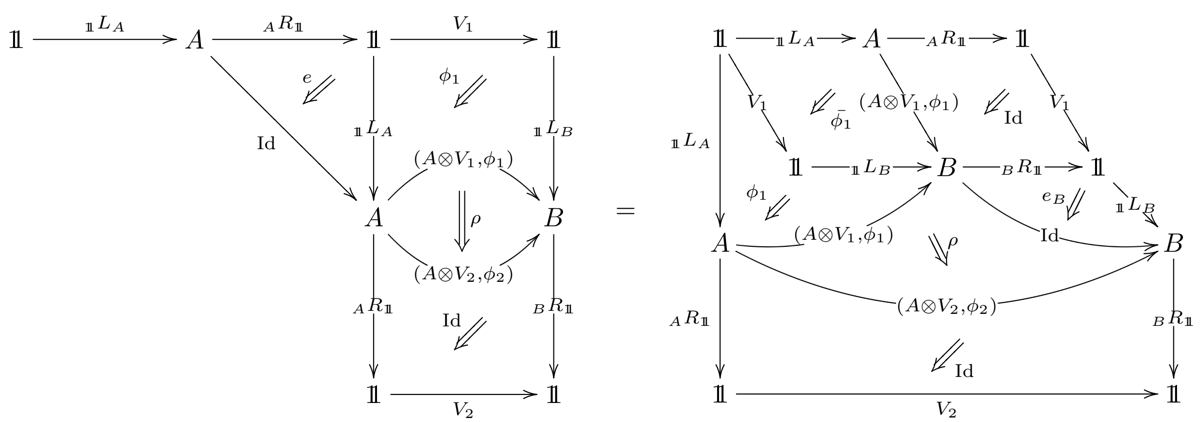
into an internal A-B-bimodule by using the obvious right action by B and left action by A. More precisely, the right B-action is that given by [. . .].

Proposition 15 *With the A-B-bimodule structure on $V_1 \otimes B$ and $A \otimes V_2$ as defined above,*  *(def. 24) is indeed a bimodule homomorphism.*

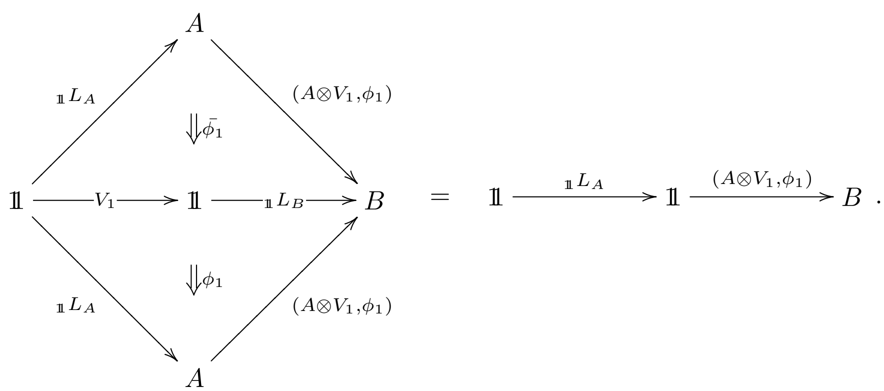
Proof. We need the equality



which is readily seen to be equivalent to the tin can equation in item 5 on p. 48. Using this equation, we get



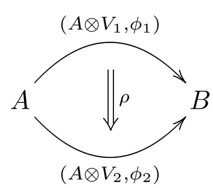
where in the top left corner we have inserted the identity 2-morphism



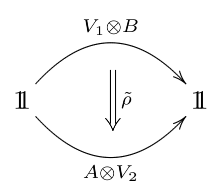
But according to the definition def. 25 of the left A-action on $V_1 \otimes B$ this says nothing but that $\tilde\rho$ respects the left A-action.

A precisely analogous argument applies to the right B-action. □

Remark. By the very definition of $\mathbf{BiMod}(\mathcal{C})$, the 2-morphism



is an internal homomorphism of bimodules internal to $\mathcal{C}$. Above we have constructed (def. 24) a 2-morphism



in $\Sigma(\mathcal{C})$ by composing ρ with the trivialization data obtained in §3.2. Remarkably, $\tilde\rho$ is *not* quite the same internal bimodule homomorphism as ρ, it does not even relate the same internal bimodules. But the difference between the two is small. The source bimodule of $\tilde\rho$ is obtained from the source bimodule of ρ by acting with the isomorphism $\bar\phi_1$. This is a direct consequence of the fact that our trivialization data had to contain this isomorphism in order to constitute a trivialization of the transport 2-functors in theorem refproposition on trivializations of 2-transport.

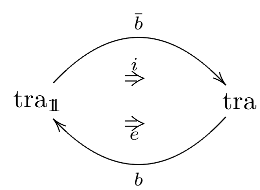
This has the following interesting consequence. Recall that we used only left-induced bimodules in **BiMod**$(\mathcal{C})$, because **LFBiMod**$(\mathcal{C})$ is precisely large enough to accomodate an ambidextrous adjunction realizing every Frobenius algebra in $\mathcal{C}$. But by sending these left-induced bimodules and their homomorphisms to $\mathcal{C}$ by means of our trivialization, some *right*-free bimodules appear automatically. In particular, all homomorphisms of left-induced bimodules become, as shown above, homomorphism between one right-free and one left-induced bimdoule.

This is important, because precisely these latter types of bimodule homomorphisms do appear in FRS formalism (e.g. p.5 of [17]), where they encode the insertion of bulk fields. We will see in example §4.3 that this is precisely reproduced by locally trivialized 2-transport.

3.4 Boundary Trivialization of 2-Transport

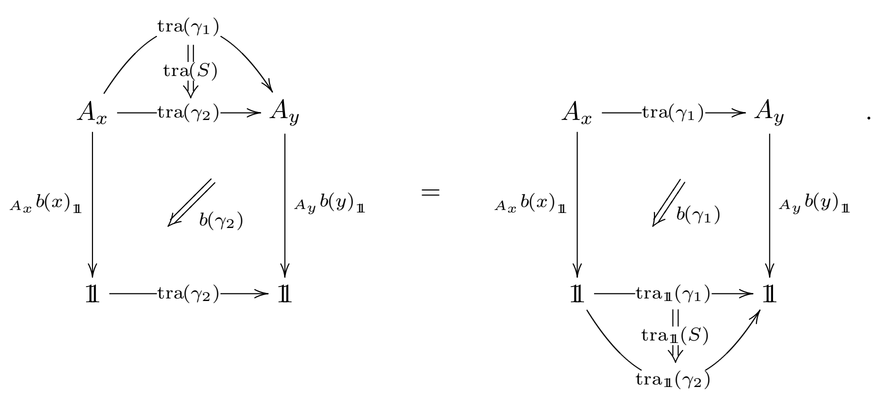
Precisely at the boundary of the surface whose 2-transport we want to compute there is another possibility to express it in terms of trivial 2-transport.

Suppose we are given an adjunction



not necessarily a special ambedextrous one.

The pseudonatural transformation b gives rise to a tin can equation



An analogous statement holds for non-invertible morphism

$$\mathrm{tra}_{\mathbb{1}} \xrightarrow{\bar{b}} \mathrm{tra} \ .$$

Therefore, assigning 1-sided A-modules to the boundaries of a surface allows to completely express the surface transport, which originally takes values in **BiMod**$(\mathcal{C})$, in terms of 2-morphisms in $\mathcal{C}$.

4 Examples

This section lists some examples demonstrating how locally trivializing transport 2-functors yields dual triangulations decorated according to the rules of FRS formalism.

4.1 Field Insertions passing Triangulation Lines

When locally trivializing transport 2-functors one frequently encounters 2-morphisms of the form

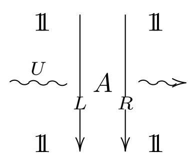
$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{U} & \mathbb{1} \\ {\scriptstyle L}\downarrow & \swarrow_{\phi} & \downarrow{\scriptstyle L} \\ A & \xrightarrow{(A\otimes U,\phi)} & A \\ {\scriptstyle R}\downarrow & \swarrow_{\mathrm{Id}} & \downarrow{\scriptstyle R} \\ \mathbb{1} & \xrightarrow[U]{} & \mathbb{1} \end{array} .$$

These come from the tin can faces of the pseudonatural transformations $\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathbb{1}}$ and $\mathrm{tra}_{\mathbb{1}} \xrightarrow{\bar{t}} \mathrm{tra}$ which have been introduced in items 2 and 3 of §3.2.

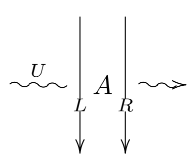
Assume that ϕ is the braiding

$$\phi = c_{U,A}$$

with U passing beneath A. Then the Poincaré dual to this diagram looks as follows (when rotated by $\pi/2$)



Here the bimodules $L \equiv {}_{\mathbb{1}}L_A$ and $R \equiv {}_A R_{\mathbb{1}}$ (introduced in def. 15) combine to yield the algebra object A regarded as a $\mathbb{1} - \mathbb{1}$-bimodule, as described in §2.4. We shall often suppress the symbol "$\mathbb{1}$" from string diagrams, such that all unlabelled regions are implicitly to be thought of as labelled by $\mathbb{1}$:



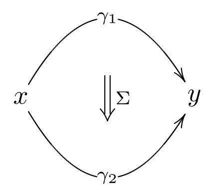
Here the U-line is supposed to pass beneath the A-line, depicting the braiding morphism

$$\begin{array}{c} U \otimes A \\ \downarrow{\scriptstyle c_{U,A}} \\ A \otimes U \end{array} .$$

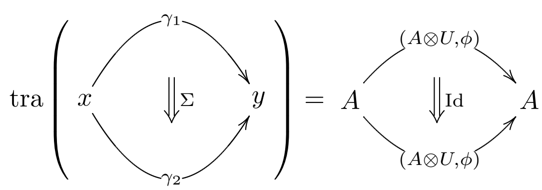
That this does indeed represent the above globular diagram follows by applying the rules for horizontal and vertical compositon of morphism of left-induced bimodules.

4.2 Disk without Insertions

Let the worldsheet Σ be a disk



and let the transport 2-functor tra be such that

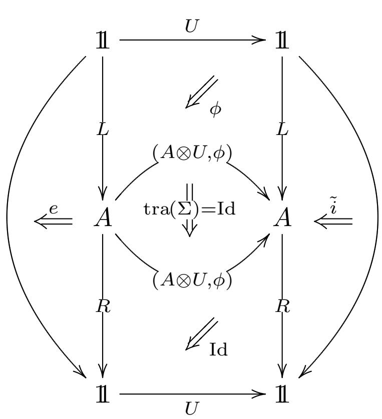


for A some algebra and $(A \otimes U, \phi)$ a left-induced A-module induced by some arbitrary object U (this was introduced in §2.3.1), with ϕ taken to be the braiding.

Let there be a physical boundary (as discussed in §3.4) given by A itself, i.e.

$$\mathrm{tra} \xrightarrow{b} \mathrm{tra}_{\mathbb{1}} \equiv \mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathbb{1}} \ .$$

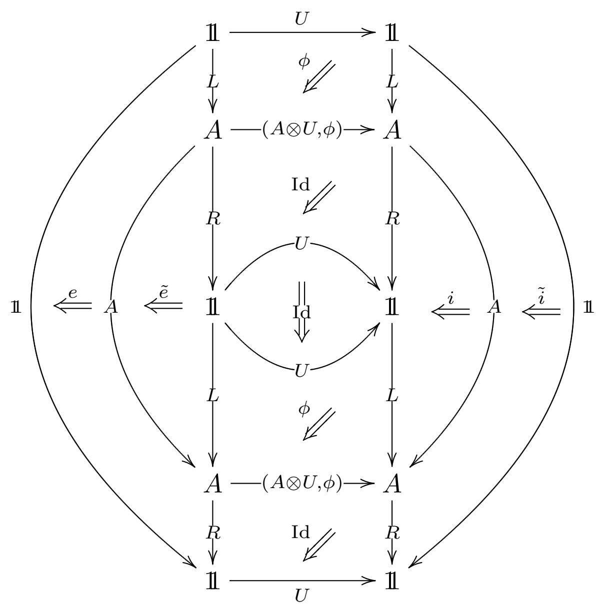
Attaching this boundary condition to $\mathrm{tra}(\Sigma)$ yields the 2-morphism.



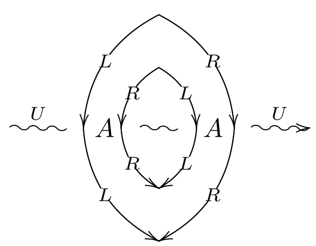
$$\in \mathrm{Mor}_2(\mathcal{C}) \subset \mathrm{Mor}_2(\mathbf{BiMod}(\mathcal{C})) \ .$$

This now entirely lives in $\mathcal{C}$.

Now trivializing tra(Σ), by substituting the right hand side of the equation in proposition 14, yields



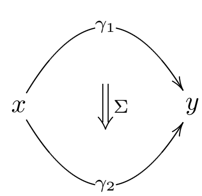
This is the locally trivialized form of the surface transport describing the disk with no insertions and trivial boundary conditions. In order to compare this to the FRS prescription we pass to the string diagram which is Poincaré-dual to the above globular diagram. It is easily seen that this looks as follows:



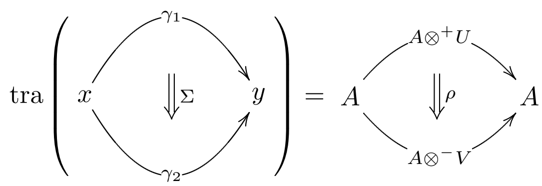
(This is rotated by $\pi/2$ with respect to the above globular diagram.)

4.3 Disk with One Bulk Insertion

Let the worldsheet Σ be a disk

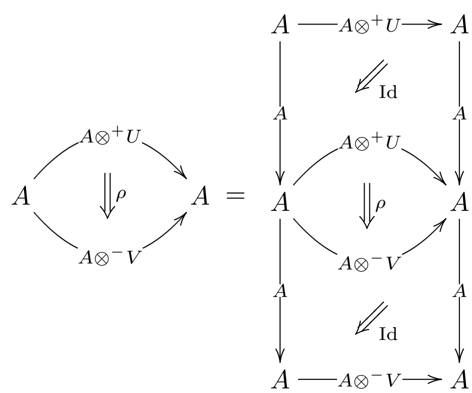


and let the transport 2-functor tra be such that

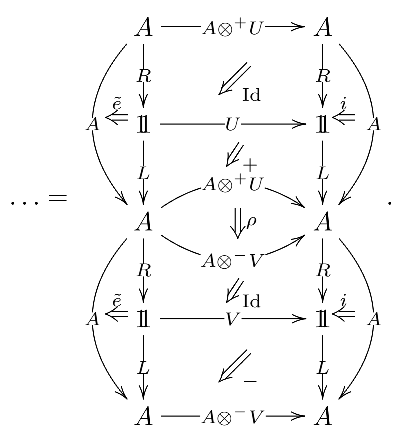


for A some algebra, $A \otimes^+ U$, $A \otimes^- V$ left-induced A-bimodules induced by some objects U and V with right action induced by left braiding ($\otimes^+$) and right braiding ($\otimes^-$), respectively.

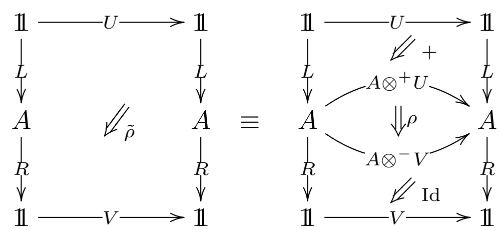
In general, this tra cannot be trivialized itself. Instead, we can write



and trivialize the two identity 2-morphisms (using prop. 14). This yields

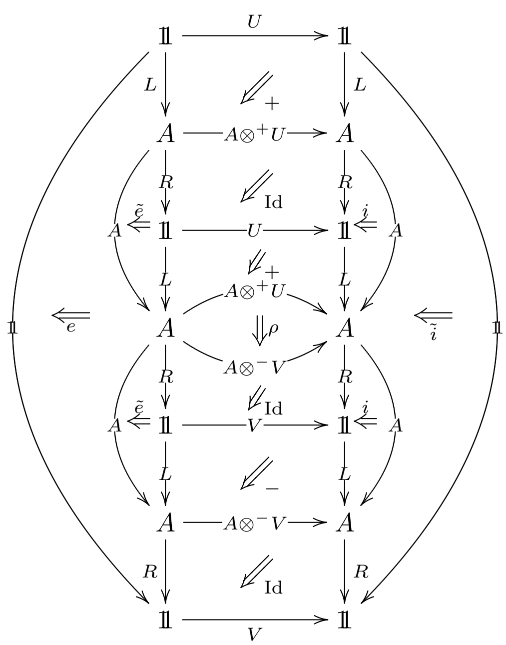


The 2-morphism in the center of this diagram



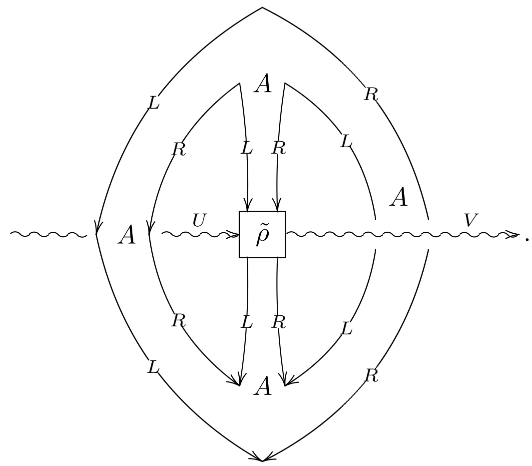
plays the role of the trivialization of ρ. This is precisely the situation studied in §3.3.2.

Let again the boundary conditions be the trivial ones, as in the previous example §4.2. Attaching these to the transport along the disk gives the 2-morphism



$\in \mathrm{Mor}_2(\mathcal{C}) \subset \mathrm{Mor}_2(\mathbf{BiMod}(\mathcal{C}))$

living in $\mathcal{C}$. The Poincaré-dual string diagram of this globular diagram is

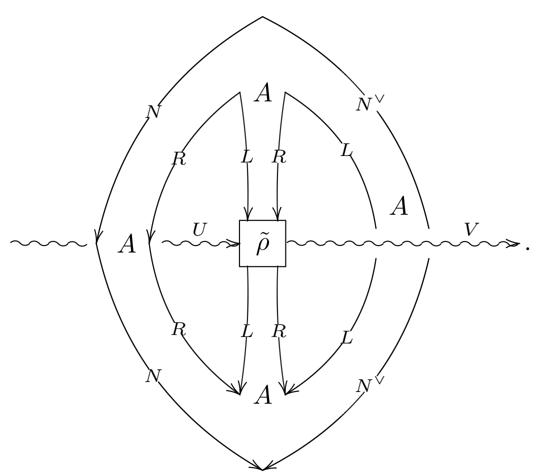


This diagram is again rotated by $\pi/2$ with respect to the above globular diagram. $\tilde{\rho}$ is the abbreviation for the center 2-morphisms introduced above.

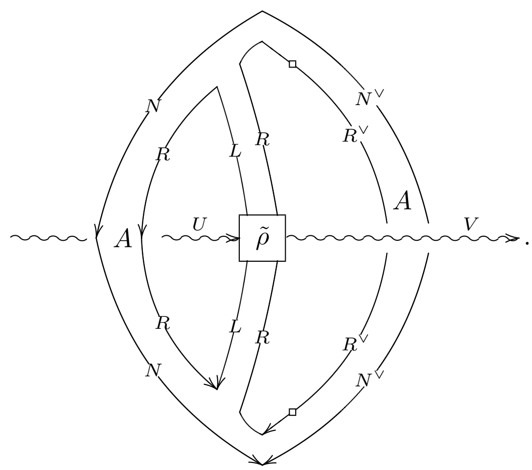
This is indeed the graph and its decoration in $\mathcal{C}$ as it appears in FRS formalism in section 4.3 of [19]. This is seen in detail by performing some standard manipulations. Applying the move of example 1 (p. 15) and then rewriting right module actions in terms of dual left module actions turns our graph above into the graph (4.19) of [19] (without the boundary insertion, which we have here not considered yet). Notice that, by prop. 15, $\tilde{\rho}$ is indeed an internal bimodule homomorphism in $\mathcal{C}$.

In [19] the corresponding correlator would be obtained by connecting the incoming U and the outgoing V by means of some morphism. This is a step not yet considered here. It will presumeably involve taking some sort of trace of 2-transport.

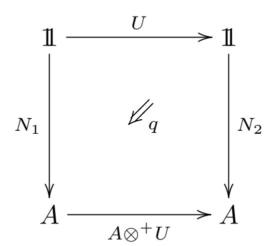
More generally, we would take the boundary to be an arbitrary A-module N. Then we get



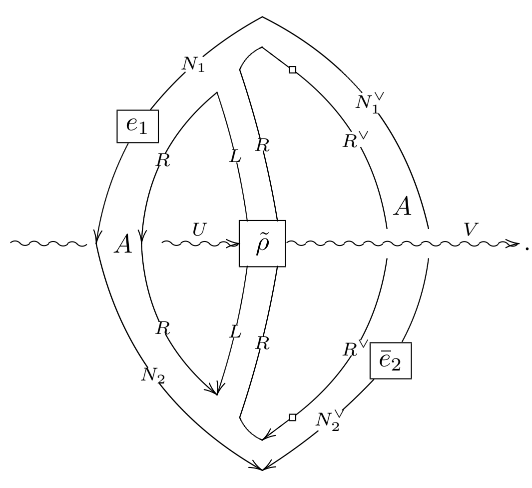
Slightly deforming this suggestively and inserting the isomorphism $L \simeq R^\vee$ we get



In addition, we can consider inserting a nontrivial morphisms

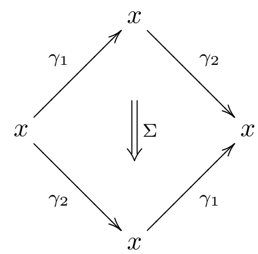


at the boundary and similarly at the opposite boundary:

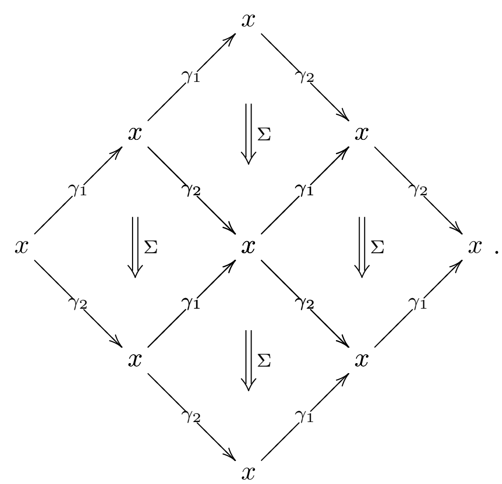


4.4 Torus without Insertions

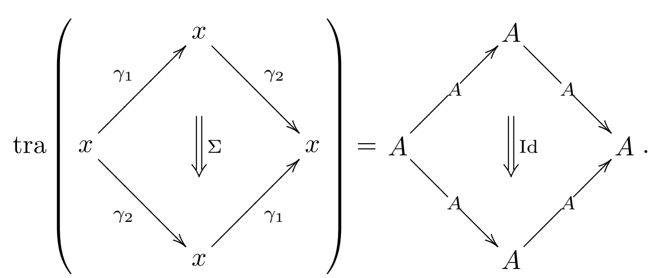
Let



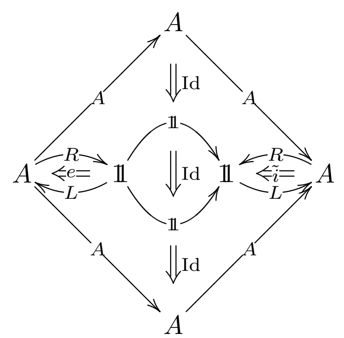
be a torus. Consider the periodic continuation



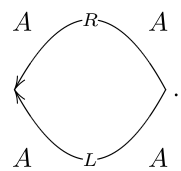
Let the 2-transport be such that



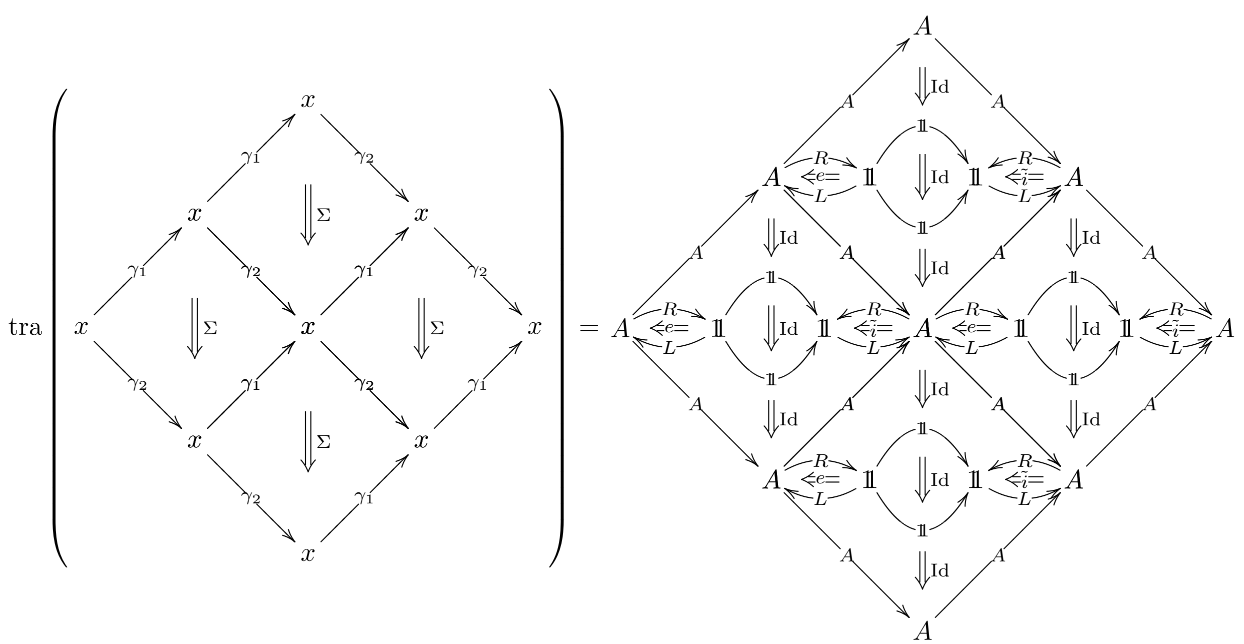
Locally trivializing this yields

$\ldots =$ 

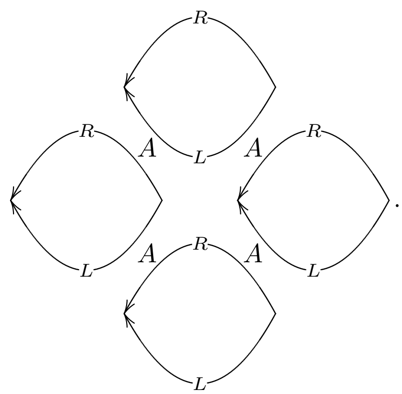
The Poincaré-dual string diagram corresponding to this globular diagram is simply



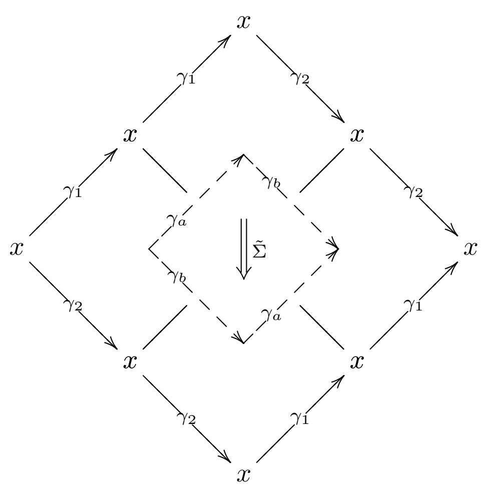
In globular diagrams we have



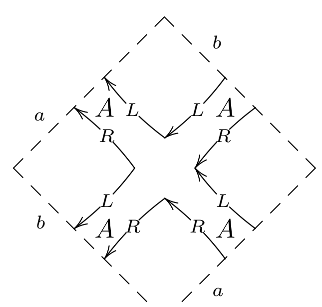
which in string diagrams looks like



Concentrate on the fundamental domain indicated by $\tilde{\Sigma}$ in the following diagram:



In terms of the above trivialization, tra($\tilde{\Sigma}$) in string diagram form looks like



This is the torus graph as it is used in section 5.3 of [17].

4.5 Klein Bottle without Insertions

As described in [9], we are really studying 2-transport

$$\tilde{\mathrm{tra}} : \mathcal{P}_2 \to {}_{\mathcal{C}}\mathbf{Mod}$$

pulled back along the chain of injections

$$\Sigma(\mathcal{C}) \xrightarrow{i_1} \mathbf{BiMod}(\mathcal{C}) \xrightarrow{i_2} {}_{\mathcal{C}}\mathbf{Mod} \ .$$

As for equivariant gerbes with connection, i_2-trivialization over unoriented surfaces gives rise to $\mathbb{Z}_2$-transitions ("defect lines") in $\mathbf{BiMod}(\mathcal{C})$.

The reasoning is completely analogous to that in [10]. To every point x on the unoriented surface Σ, $\tilde{\mathrm{tra}}$ assigns a module category

$$\tilde{\mathrm{tra}}(x) =\in \mathrm{Obj}({}_{\mathcal{C}}\mathbf{Mod}) \ .$$

x has two lifts, x_1 and x_2 to the the double $\hat{\Sigma}$ of Σ. We construct on $\mathcal{P}_2(\hat{\Sigma})$ a 2-transport tra with values in $\mathbf{BiMod}(\mathcal{C})$ by requiring its local pullback along local sections $\Sigma \to \hat{\Sigma}$ to be equivalent to $\tilde{\mathrm{tra}}$ (actually, to be related by an ambijunction).

This in particular implies two algebras A, A' internal to $\mathcal{C}$ such that $\mathrm{tra}(x_1) = \mathbf{Mod}_A$ and $\mathrm{tra}(x_2) = \mathbf{Mod}_{A'}$, with

$$\begin{array}{c} \mathbf{Mod}_A \xrightarrow{\simeq} \tilde{\mathrm{tra}}(x) \xrightarrow{\simeq} \mathbf{Mod}_{A'} \ , \\ \mathbf{Mod}_A \xrightarrow[\simeq]{g} \mathbf{Mod}_{A'} \end{array}$$

where g comes from a $\mathbb{Z}_2$-equivariant structure on tra.

At least one way to realize this nontrivially (I don't know if there are others) is to let

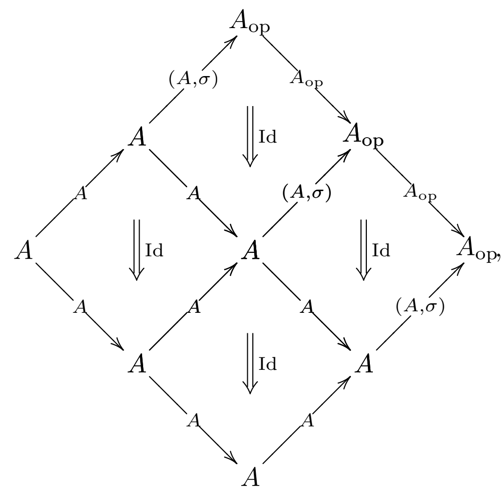
$$A' = A_{\mathrm{op}}$$

and

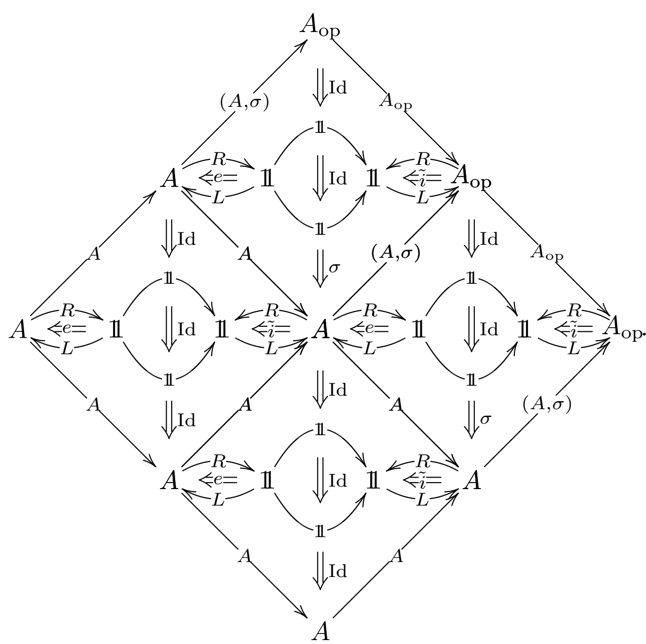
$$\mathbf{Mod}_A \xrightarrow{g} \mathbf{Mod}_{A_{\mathrm{op}}} \ \equiv \ \mathbf{Mod}_A \xrightarrow{{}_A(A,\sigma)_{A_{\mathrm{op}}}} \mathbf{Mod}_{A_{\mathrm{op}}} \ ,$$

with the bimodule ${}_A(A,\sigma)_{A_{\mathrm{op}}}$ as defined in §2.5.

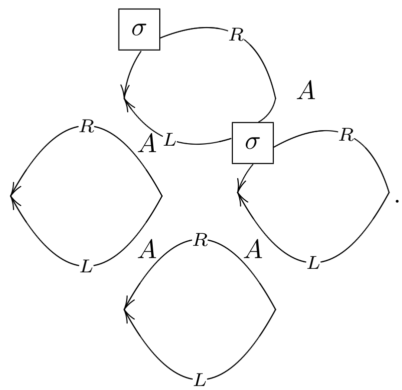
Therefore, on the Klein bottle without any insertions, tra produces an image like



In terms of trivial 2-transport this reads, using prop. 14 (p. 51),



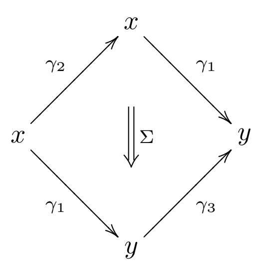
By the reasoning of example 4 (p. 37) the corresponding string diagram looks like



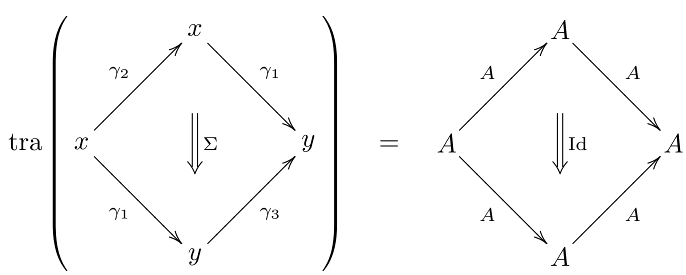
Analogous to the above discussion for the torus, this reproduces the FRS diagram for the Klein bottle (equation (3.52b) in [18]).

4.6 Annulus without Insertions

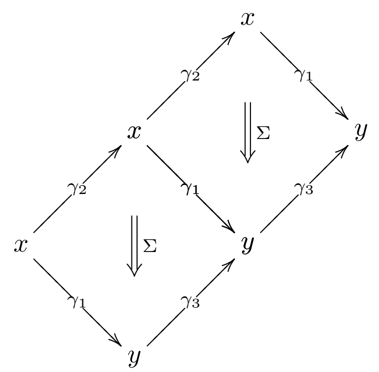
Let



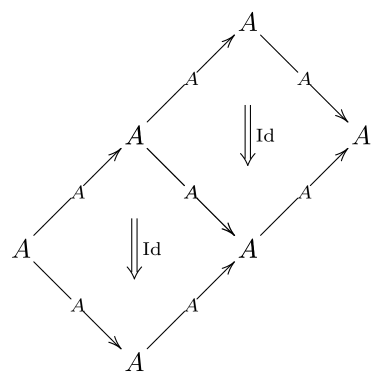
be an annulus. Assume the 2-transport is simply



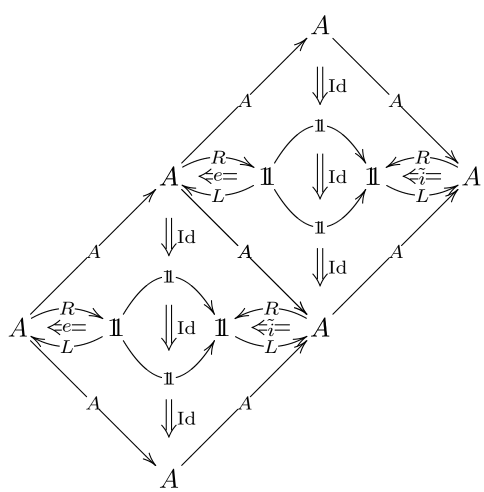
Consider the periodic continuation



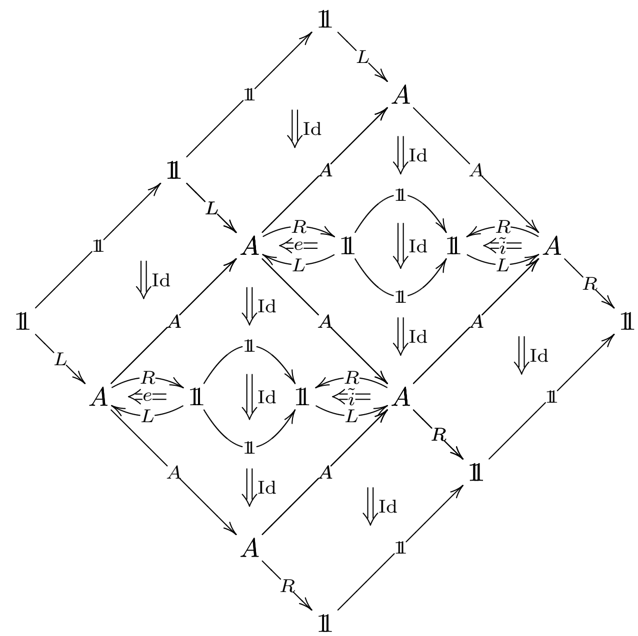
Applying tra to that yields



In terms of local trivialization this equals

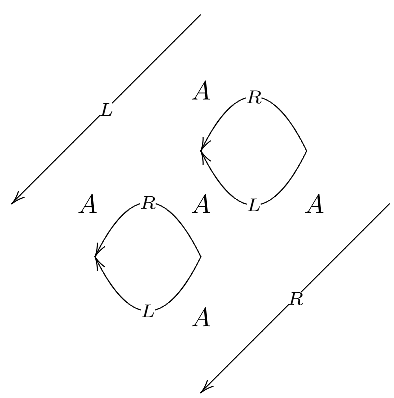


Attaching A-boundary conditions at γ_2 and γ_3 yields



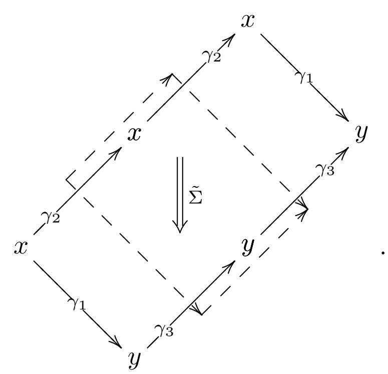
.

The string diagram dual to this globular diagram is

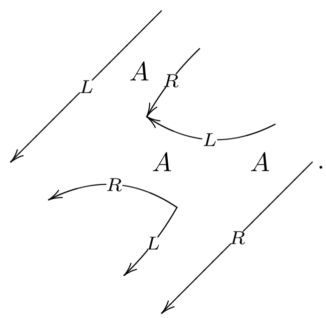


.

Now concentrate on the fundamental domain denoted $\tilde{\Sigma}$ in the following diagram:



The dual string diagram of the 2-transport over $\tilde{\Sigma}$ is



This is the annulus diagram as used in section 5.8 of [17] (in the form of eq. (5.117)).

Acknowledgements. I benefited immensely from discussions with Ingo Runkel and Christoph Schweigert on what I here call the "FRS formalism".

Many thanks go to Aaron Lauda for helpful conversation about his work on adjunctions, and for pointing me to the latest version [11] of his recent paper.

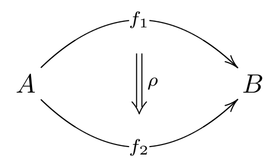
All diagrams were created using the LaTeXpackage XY-pic written by Kristoffer Rose and Ross Moore.

A 2-Categories

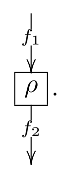
We are dealing mostly with strict 2-categories, which are categories enriched in Cat. This is the same notion as that of a stricly associative bicategory and the same as a double category with all vertical morphisms being identities.

A.1 Globular Diagrams and String Diagrams

A 2-morphism (2-cell) of such a 2-category can be depicted in terms of a **globular diagram**

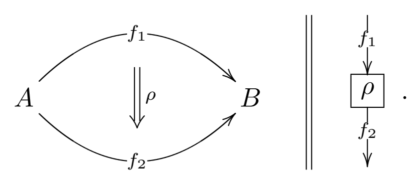


or, equivalently, in terms of a **string diagram**



These are related by Poincaré duality.

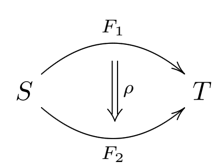
For our purposes it is convenient to pass back and forth between these two manifestations of 2-morphism. At several occasions both notions will be given simultaneously, seperated by a vertical double line:



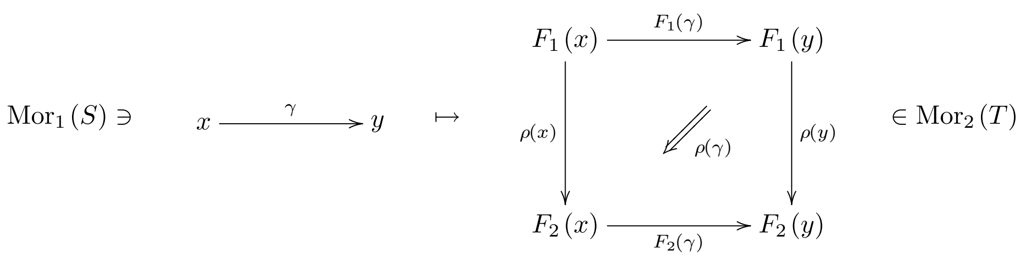
A.2 1- and 2-Morphisms of 2-Functors

1-morphisms and 2-morphisms between 2-functors are called pseudonatural transformations and modifications, respectively. These are defined as follows (*cf.* [20]).

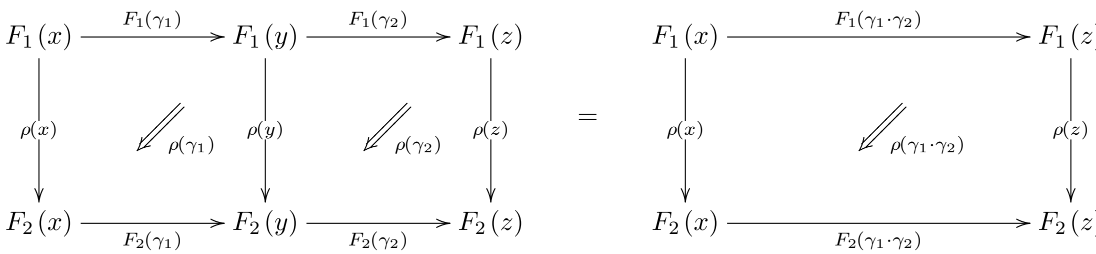
Definition 26 *Let* $S \xrightarrow{F_1} T$ *and* $S \xrightarrow{F_2} T$ *be two 2-functors. A* **pseudonatural transformation**



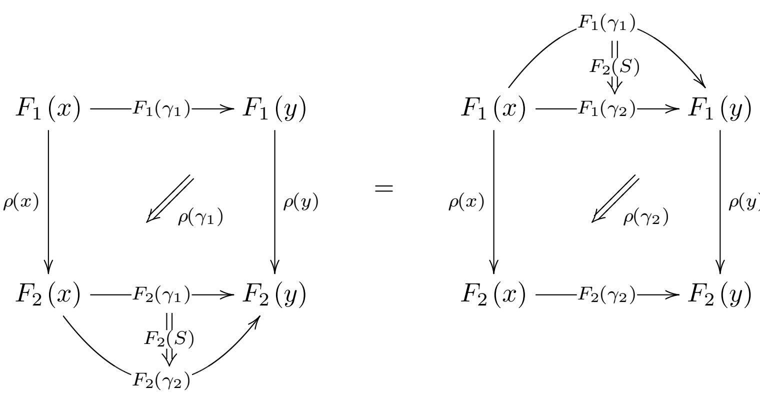
is a map

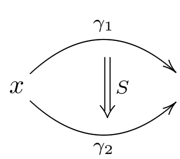


which is functorial in the sense that



and which makes the pseudonaturality tin can 2-commute



for all  $\in \mathrm{Mor}_2\,(S)$.

Definition 27 *The vertical* **composition of pseudonatural transforma-**

tions

$$S \underset{F_3}{\overset{F_1}{\Downarrow \rho}} T \;\equiv\; S \underset{F_3}{\overset{F_1}{\xrightarrow[\Downarrow \rho_2]{\Downarrow \rho_1 \;\; F_2}}} T$$

is given by

$$\begin{array}{ccc} F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\ {\scriptstyle \rho(x)}\downarrow & \swarrow\!\!\swarrow \rho(\gamma) & \downarrow{\scriptstyle \rho(y)} \\ F_3(x) & \xrightarrow{F_3(\gamma)} & F_3(y) \end{array} \quad \equiv \quad \begin{array}{ccc} F_1(x) & \xrightarrow{F_1(\gamma)} & F_1(y) \\ {\scriptstyle \rho_1(x)}\downarrow & \swarrow\!\!\swarrow \rho_1(\gamma) & \downarrow{\scriptstyle \rho_1(y)} \\ F_2(x) & \xrightarrow{F_2(\gamma)} & F_2(y) \\ {\scriptstyle \rho_2(x)}\downarrow & \swarrow\!\!\swarrow \rho_2(\gamma) & \downarrow{\scriptstyle \rho_2(y)} \\ F_3(x) & \xrightarrow{F_3(\gamma)} & F_3(y) \end{array}$$

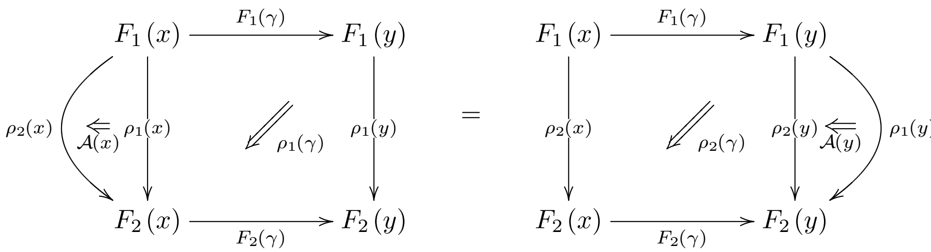
Definition 28 *Let* $F_1 \xrightarrow{\rho_1} F_2 \quad F_1 \xrightarrow{\rho_2} F_2$ *be two pseudonatural transformations. A* **modification** *(of pseudonatural transformations)*

$$F_1 \underset{\rho_2}{\overset{\rho_1}{\Downarrow \mathcal{A}}} F_2$$

is a map

$$\mathrm{Obj}(S) \ni x \mapsto \; F_1(x) \underset{\rho_2(x)}{\overset{\rho_1(x)}{\Downarrow \mathcal{A}(x)}} F_2(x) \;\in \mathrm{Mor}_2(T)$$

such that

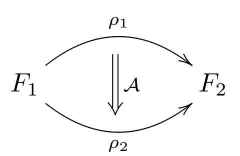


for all $x \xrightarrow{\gamma} y \in \mathrm{Mor}_1(S)$.

Definition 29 *The horizontal and vertical composite of modifications is, respectively, given by the horizontal and vertical composites of the maps to 2-morphisms in* $\mathrm{Mor}_2(T)$.

Definition 30 *Let* S *and* T *be two 2-categories. The* **2-functor 2-category** T^S *is the 2-category*

1. *whose objects are functors* $F : S \to T$
2. *whose 1-morphisms are pseudonatural transformations* $F_1 \xrightarrow{\rho} F_2$
3. *whose 2-morphisms are modifications*



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