Smooth 2-Functors and Differential p-Forms

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Abstract

Smooth 2-functors from 2-paths to a strict 2-group are characterized by certain differential p-form data in a way that generalizes the familiar relation of parallel transport to path ordered exponentials, $\mathrm{Pexp}(\int_\gamma A)$, of a Lie-algebra valued 1-form A.

Contents

1 Smooth functors from n-paths to n-groupoids

A smooth functor on paths is a functor between smooth categories that is smooth on objects and on morphisms.

By its smoothness, it is completely specified by its differential. By functoriality, it is completly specified already by its differential evaluated at identity morphisms of the domain category.

These differentials at all identity morphisms combine into a smooth 1-form on the space of objects of the domain.

Similarly, smooth 2-functors on 2-paths are determined by a collection of 1-forms and 2-forms on the space of objects of their domain.

1.1 Smooth structure on path spaces

In order to make sense of the concept of a smooth functor on an n-groupoid of paths, we need to define what we mean by a smooth structure on a space of paths.

Definition 1 *A* **smooth structure** *on a set X is*

- *a collection of maps (of sets) – called* **plots** *– of the form*

$$c_U : U \to X\,,$$

  with U any manifold diffeomorphic to $\mathbb{R}^n$, for some n;

- *such that this collection is closed under pullback by smooth maps of domains, i.e such that for*

$$c_U : U \to X$$

  any plot and

$$f : V \to U$$

  any smooth map, also

$$f^* c_U : \; V \xrightarrow{f} U \xrightarrow{c_U} X$$

  is a plot.

- *such that every map from the point $\mathbb{R}^0$ to X is a plot;*

- *and such that*

$$c_U : U \to X$$

  is a plot when there exists

$$c_{\mathbf{U}} : \mathbf{U} \to U$$

  a surjective submersion with

$$c_U \circ c_{\mathbf{U}}$$

  a plot.

The idea is that we specify the smoothness of X by saying which images of $\mathbb{R}^n$s in X we consider smooth.

Notice that in particular any ordinary smooth manifold canonically has a smooth structure in the above sense, obtained by taking the plots to be the charts of a maximal atlas.

But the point of the above definition is that it allows us to consider smooth structures also on spaces that cannot be equipped with the structure of a manifold.

Definition 2 (smooth structure on spaces of maps) *For X and Y smooth manifolds, the canonical smooth structure on the topological space of maps*

$$\mathrm{Hom}_{\mathrm{Top}}(X, Y)$$

is that where a map

$$c : U \to \mathrm{Hom}_{\mathrm{Top}}(X, Y)$$

is a plot if and only if the composite map

$$U \times X \xrightarrow{c \times X} \mathrm{Hom}_{\mathrm{Top}}(X, Y) \times X \xrightarrow{\mathrm{ev}} Y$$

is an ordinary smooth map of smooth manifolds.

We are frequently interested in spaces of maps from the unit cube $[0,1]^n$ into some manifold X modulo some equivalence relation, like for instance reparameterization of the n-cube. We shall equip these quotients with the canonical smooth structure on quotient spaces:

Definition 3 (push-forward of smooth structure) *For* X *any diffeological space and $p : X \to Y$ any map of sets, we take the smooth structure on Y induced by p to be that where*

$$c : U \to Y$$

is a plot if and only if c factors through p by a plot c_X of X:

$$c = p \circ c_X \,,$$

for any c_X.

A smooth map between smooth spaces is one that preserves the notion of plots:

Definition 4 (smooth maps) *For X and Y spaces with smooth structure as above, we say that a map (of sets) $f : X \to Y$ is smooth, if for every plot c_U of X the composite*

$$f_* c_U : \; U \xrightarrow{c_U} X \xrightarrow{f} Y$$

is a plot of Y.

We shall regard the groupoid

$$\mathcal{P}_1(X)$$

of thin-homotopy classes of paths in a manifold X as a category internal to the category of spaces with smooth structure by

- equipping the space of objects with the smooth structure coming from the manifold structure of X;
- equipping the space of morphisms with the smooth structure obtained from the canonical smooth structure on spaces of maps.

1.2 Smooth 1-functors from paths to groupoids

The groupoid $\mathcal{P}_1(\mathbb{R})$ of paths on the real line is particularly simple. We first see how smooth functors on that come from 1-forms on $\mathbb{R}$.

Then we use the groupoid $\mathcal{P}_1(\mathbb{R})$ as a standard probe for groupoids $\mathcal{P}_1(X)$ of paths in general manifolds X. Pulling back functors on $\mathcal{P}_1(X)$ along all "plots"

$$\mathcal{P}_1(\mathbb{R}) \xrightarrow{\gamma} \mathcal{P}_1(X)$$

allows to express them in terms of a collection $\{A_\gamma \mid \gamma : \mathbb{R} \to X\}$ of 1-forms on the real line. Using the smooth structure on $\mathcal{P}_1(X)$ then allows to show that this collection comes from a single 1-form A on X.

1.2.1 Paths on the real line

Concerning paths in one dimension, the following categories are all canonically isomorphic:

- the groupoid of thin-homotopy classes of paths in $\mathbb{R}$
- the fundamental groupoid of $\mathbb{R}$
- the pair groupoid of $\mathbb{R}$

We write $\mathcal{P}_1(\mathbb{R})$ for this path groupoid and note that it is a category internal to manifolds. We have

- $\mathrm{Obj}(\mathcal{P}_1(X)) = \mathbb{R}$
- $\mathrm{Mor}(\mathcal{P}_1(X)) = \mathbb{R} \times \mathbb{R}$

Source and target are the obvious projection maps $s, t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and composition

$$\circ : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$$

is the obvious projection that forgets the second argument.

Now let Gr be any Lie groupoid.

Definition 5 *A* **Lie groupoid** *is a groupoid internal to the category of smooth manifolds, such that source and target map are submersions that admit local sections.*

We want to characterize smooth functors

$$\mathrm{tra} : \mathcal{P}_1(\mathbb{R}) \to \mathrm{Gr}\,,$$

i.e. functors whose maps on objects and on morphisms are smooth maps between smooth manifolds.

Remark: exponentials in the universal enveloping algebra of $\mathrm{Lie}(G)$**.** Whenever we encounter Lie groups in the following, it will be convenient to use exponentials in the universal enveloping algebra of G, which provides a useful model of G in a neighbourhood of the identity where the exponential map has an inverse.

Writing $\{t^i\}_{i=1}^n$ for a choice of basis of $\mathrm{Lie}(G)$, and writing

$$v \cdot t := \sum_{i=1}^n v_i t^i \in \mathrm{Lie}(G)$$

for any $v \in \mathbb{R}^n$, the assignment

$$\mathbb{R}^n \ni v \mapsto \exp(v \cdot t) = 1 + v \cdot t + \frac{1}{2}(v \cdot t)^2 + \cdots \in U(\mathrm{Lie}(G))$$

for all v with $|v| < r$ for sufficiently small r provides a chart of a neighbourhood of the neutral element of G.

Proposition 1 *Let* Gr *be a Lie group G regarded as a Lie groupoid with a single object. Then smooth functors*

$$\mathcal{P}_1(X) \to \mathrm{Gr}$$

are in bijection with $\mathrm{Lie}(G)$*-valued 1-forms*

$$A \in \Omega^1(\mathbb{R}, \mathrm{Lie}(G))\,.$$

Proof. Write

$$A(x)(\frac{\partial}{\partial y}) := \frac{\partial}{\partial y}\mathrm{tra}_1(x,y)|_{y=x}$$

for the differential of tra at all identity morphisms.

We see that tra is uniquely specified by A as follows.

Let $x_0 \to x_1$ be any morphism in $\mathcal{P}_1(\mathbb{R})$ such that for all $x_0 \leq x \leq y \leq x_1$ we have $\mathrm{tra}(x,y) \in U_e$, for $U \subset G$ a neighbourhood of the identity on which the exponential map is invertible.

By subdividing the interval $[x_0, x_1]$ into N pieces we get

$$\mathrm{tra}(x_0,x_1) = \mathrm{tra}(x_0,x_0{+}\delta(N))\mathrm{tra}(x_0{+}\delta,x_0{+}2\delta(N))\cdots\mathrm{tra}(x_0{+}(N{-}1)\delta(N),x_1)\,,$$

where

$$\delta(N) = \frac{x_1 - x_0}{N}\,,$$

and using the functoriality of tra.

By assumption on our chart, there is a unique element $t(x_0, N) \in \mathrm{Lie}(G)$ such that

$$\mathrm{tra}(x_0,x_0{+}\delta(x_0,N)) = 1{+}(\mathrm{tra}(x_0,x_0{+}\delta(x_0,N)){-}1) = 1{+}\delta(N)t(x_0,N){+}\frac{1}{2}\delta(x_0,N)^2t(x_0,N)^2{+}\cdots\,.$$

Dividing this equation by $\frac{x_1 - x_0}{N}$ and taking the limit $N \to \infty$ implies that this t tends to $A(x_0)$ as N grows:

$$\lim_{N\to\infty} t(x_0, N) = A(x_0)(\frac{\partial}{\partial x})\,.$$

Moreover, inserting the exponential expansion of $\mathrm{tra}(x, x+\delta(N))$ into the above decomposition of $\mathrm{tra}(x_0, x_1)$ we get

$$\mathrm{tra}(x_0,x_1) = (1 + \delta(N)t(x_0,N))\,(1 + \delta(N)t(x_0+\delta(N),N))\cdots + \mathcal{O}(\delta(N))\,.$$

This involves carefully tracking how many terms of which power of $\delta(N)$ appear.

We may rewrite this expression as an iterated sum

$$\begin{aligned}
\mathrm{tra}(x_0,x_1) = 1 \quad &+ \quad \delta(N) \sum_{x_0+n\delta(N)=x_0}^{x_1} t(x_0+n\delta(N),N) \\
&+ \quad \delta(N)^2 \sum_{x_0+m\delta(N)=x_0}^{x_1} \left(\sum_{x_0+n\delta(N)=x_0}^{x_0+m\delta(N)} t(x_0+n\delta(N),N) \right) \left(\sum_{x_0+n\delta(N)=x_0+m\delta(N)}^{x_1} t(x_0+n\delta(N),N) \right) \\
&+ \quad \cdots \\
&+ \quad \mathcal{O}(\delta(N))\,.
\end{aligned}$$

Taking now $N \to \infty$, the sums over t become Riemann integrals

$$\lim_{N\to\infty} \delta(N) \sum_{a+n\delta(N)=x_0}^{b} t(x_0+n\delta(N),N) = \int_{(a,b)\subset\mathbb{R}} A\,.$$

This way we arrive at the **iterated intral** representation

$$\mathrm{tra}(x_0,x_1) = 1 + \int_{x_0 \text{---} x_1} A + \int_{\substack{x_0,\,x_0 \\ x_0,\,x_1 \text{---} x_1,\,x_1}} A \wedge A + \cdots$$

The right hand side is often denoted by the symbols

$$\mathrm{tra}(x_0,x_1) = \mathrm{P}\exp\left(\int_{x_0}^{x_1} A\,dx\right)$$

and then called the **path ordered exponential** of A. $\Box$

We can generalize this to functors with values in Lie groupoids with more than one object.

Proposition 2 *Smooth functors* $\mathrm{tra} : \mathcal{P}_1(\mathbb{R}) \to \mathrm{Gr}$ *are in bijection with pairs* (f, A)*, where*

- $f : \mathrm{Obj}(\mathcal{P}_1(\mathbb{R})) \to \mathrm{Obj}(\mathrm{Gr})$ *is a smooth map*

- $A \in \Omega^1(\mathbb{R}, T\mathrm{Mor}(\mathrm{Gr}))$ *is a smooth 1-form with values in tangent vectors to the space of morphisms of* Gr*, such that A sends tangent vectors at x to tangent vectors at the identity morphism in the s-fiber of $f(x)$:*

$$A(x) : T\mathbb{R} \to T_{\mathrm{Id}_{f(x)}} s^{-1}(f(x)) \,.$$

Proof. By definition of Lie groupoids, their source and target maps admits local sections. So choose a cover $U \to \mathbb{R}$ of the real line by open intervals such that tra restricted to each such interval admits a section on the interval's image in Obj(Gr). Using that section, we can identify all points in that image. This reduces the problem to transport with values in a group. $\square$

Proposition 3 *For* $\mathrm{Gr} = \Sigma(G)$ *a suspended Lie group, and*

$$\mathrm{tra}_{A,\tilde{A}} : \mathcal{P}_1(\mathbb{R}) \to \Sigma(G)$$

two smooth functors coming from 1-forms A and $\tilde{A}$ as above, a smooth natural transformation

$$g : \mathrm{tra}_A \to \mathrm{tra}_{\tilde{A}}$$

is a smooth map

$$g : \mathbb{R} \to G$$

such that

$$A = g\tilde{A}g^{-1} + gdg^{-1} \,.$$

Proof. Differentiate the naturality square

$$\begin{tikzcd} \bullet \arrow[r, "g(x)"] \arrow[d, "\mathrm{tra}_A(x,y)"'] & \bullet \arrow[d, "\mathrm{tra}_{\tilde{A}}(x,y)"] \\ \bullet \arrow[r, "g(y)"'] & \bullet \end{tikzcd}$$

at $y = x$ with respect to y:

$$\frac{\partial}{\partial y}\mathrm{tra}_A(x,y)|_{y=x} = \frac{\partial}{\partial y} g(x)\mathrm{tra}_{\tilde{A}}(x,y)g^{-1}(x,y)|_{y=x} \,,$$

which yields

$$A(x)(\frac{\partial}{\partial y}) = g(x)\tilde{A}(x)(\frac{\partial}{\partial y})g(x)^{-1} + g(x)dg^{-1}(x)(\frac{\partial}{\partial y}) \,.$$

$\square$

In summary, then, we have found

Theorem 1 *The category*

$$[\mathcal{P}_1(\mathbb{R}), \Sigma(G)]$$

of smooth functors and smooth natural transformations is isomorphic to the category $\bar{H}^1(\mathbb{R}, G)$*, whose objects are 1-forms* $A \in \Omega^1(\mathbb{R}, \mathrm{Lie}(G))$ *and whose morphisms*

$$g : A \to \tilde{A}$$

are smooth functions $g : \mathbb{R} \to G$ *such that* $A = g\tilde{A}g^{-1} + gdg^{-1}$*.*

1.2.2 Paths on general manifolds

We can now use $\mathcal{P}_1(\mathbb{R})$ to probe paths in higher dimensional spaces.

Fix some manifold X and let $\mathcal{P}_1(X)$ be the groupoid of thin-homotopy classes of paths in X. This is a groupoid internal to smooth spaces.

We want to characterize smooth functors

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Gr}\,,$$

i.e. functors whose maps on objects and on morphisms are smooth maps.

Proposition 4 *Smooth functors* $\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Gr}$ *are in bijection with pairs* (f, A)*, where*

- $f : \mathrm{Obj}(\mathcal{P}_1(X)) \to \mathrm{Obj}(\mathrm{Gr})$ *is a smooth map*
- $A \in \Omega^1(X, T\mathrm{Mor}(\mathrm{Gr}))$ *is a smooth 1-form with values in tangent vectors to the space of morphisms of* Gr*, such that* A *sends tangent vectors at* x *to tangent vectors at the identity morphism in the s-fiber of* $f(x)$*:*

$$A(x) : TX \to T_{\mathrm{Id}_{f(x)}} s^{-1}(f(x))\,.$$

Proof. The idea is to probe $\mathcal{P}_1(X)$ by mapping $\mathcal{P}_1(\mathbb{R})$ into it. Pulling back tra along any such map gives a 1-form on its image, by the above theorem. It then remains to be shown that all these 1-forms combine to a 1-form on X.

So let

$$\bigsqcup_{[\mathcal{P}_1(\mathbb{R}), \mathcal{P}_1(X)]} \mathcal{P}_1(\mathbb{R})$$

be the disjoint union of path groupoids of the real line, one for each smooth functor

$$l : \mathcal{P}_1(\mathbb{R}) \to \mathcal{P}_1(X)\,.$$

This is equipped with the obvious smooth structure of a disjoint union of smooth spaces.

We have an obvious smooth functor

$$p : \bigsqcup_{[\mathcal{P}_1(\mathbb{R}), \mathcal{P}_1(X)]} \mathcal{P}_1(\mathbb{R}) \twoheadrightarrow \mathcal{P}_1(X)\ ,$$

which is surjective on morphisms.

By the above theorem, we find that $p^*\mathrm{tra}$ is characterized by a set of 1-forms

$$\{A_l | l \in [\mathcal{P}_1(\mathbb{R}), \mathcal{P}_1(X)]\}$$

on $\mathbb{R}$, one for each map of the real line into X.

To see that these 1-forms all arise as the pull-back along the given l of a single 1-form A on X we show that A_l and $A_{l'}$ coincide at every point where l and l' are tangent. This will follow from the fact that smoothness of tra means that it is smooth on every smooth family of paths.

So let $l, l' : \mathbb{R} \to X$ be smooth maps that both go through some point $x \in X$ where they are tangent. We may assume without loss of generality that x is the image of $0 \in \mathbb{R}$ for both l and l' and that both l and l' are injective in a neightbourhood of this point.

Now pick any neighbourhood $U_x \subset X$ with the topology of an n-ball and pick a neighbourhood $(a, b) \subset \mathbb{R}$ of 0, such that its image under both l and l' is inside U_x and such that both l and l' are injective on (a, b).

If there is such interval such that the images of l and l' coincide on all of it, then the identity of the corresponding 1-forms is trivial.

So assume that l and l' coincide in an isolated point.

Then pick a smooth map

$$\Sigma : (a, b) \times [0, 1] \to U_x$$

which is

- injective away from $\{0\} \times [0, 1]$,
- interpolates between l and l' in that

  $$l = \Sigma(\cdot, 0)$$

  and

  $$l' = \Sigma(\cdot, 1)\,,$$

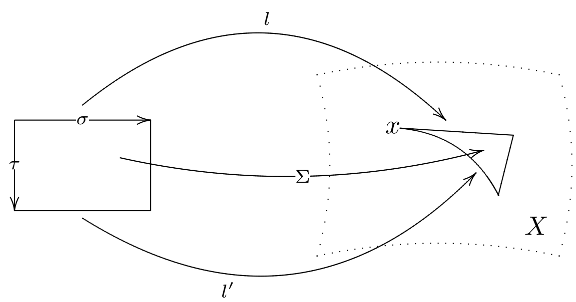
- such that

  $$\frac{\partial}{\partial \tau}\frac{\partial}{\partial \sigma}\Sigma(0, \tau) = 0\,.$$

For instance, with any choice of coordinates on U_x, we may set

$$\Sigma : (\sigma, \tau) \mapsto (1 - \tau)l(\sigma) + \tau l'(\sigma)\,,$$

where the linear combination is with respect to the chosen coordinate chart. By the assumption that l and l' coincide only in an isolated point, this is injective away from $\{0\} \times [0,1]$ for sufficiently small U_x.



We may also think of Σ as a map from $(a,b) \times [0,1]$ into paths in X by sending

$$\tilde{\Sigma} : (\sigma, \tau) \mapsto (\gamma_{\sigma,\tau} : [0,1] \ni s \mapsto \Sigma(k(s)\sigma, \tau)) \,,$$

where the smoothing function $k : [0,1] \to [0,1]$ is any smooth bijective function with $k(0) = 0$ and $k(1) = 1$ and with all derivatives vanishing in a neighbourhood of $\{0,1\}$.

Since $\tilde{\Sigma}$ is hence a plot for the space of paths in X, it follows that

$$\mathrm{tra}_1 \circ \tilde{\Sigma} : (a,b) \times [0,1] \to G$$

is a smooth function. Since Σ is injective away from $\{0\} \times [0,1]$, there is a unique smooth function $F : \mathrm{Im}(\Sigma) \to G$ such that

$$F \circ \Sigma = \mathrm{tra}_1 \circ \tilde{\Sigma} \,.$$

By the chain rule we then find

$$\frac{\partial}{\partial \sigma}(\mathrm{tra}_1 \circ \tilde{\Sigma})(\sigma,\tau) = \frac{\partial}{\partial \sigma}(F \circ \Sigma)(\sigma,\tau) = dF(\frac{\partial}{\partial \sigma}\Sigma)(\sigma,\tau) \,.$$

But $\frac{\partial}{\partial \sigma}\Sigma(0,\cdot)$ is constant, hence $\frac{\partial}{\partial \sigma}(\mathrm{tra}_1 \circ \tilde{\Sigma})(0,\tau)$ is constant. Using

$$\frac{\partial}{\partial \sigma}(\mathrm{tra}_1 \circ \tilde{\Sigma})(0,0) = A_l(0)(\frac{\partial}{\partial \sigma})$$

and

$$\frac{\partial}{\partial \sigma}(\mathrm{tra}_1 \circ \tilde{\Sigma})(0,1) = A_{l'}(0)(\frac{\partial}{\partial \sigma})$$

it follows that

$$A_l(0) = A_{l'}(0) \,.$$

$\square$

It is now straightforward to characterize natural transformations between our smooth functors.

Proposition 5 *For* $\mathrm{Gr} = \Sigma(G)$ *a suspended Lie group, and*

$$\mathrm{tra}_{A,\tilde{A}} : \mathcal{P}_1(X) \to \Sigma(G)$$

two smooth functors coming from 1-forms $A, \tilde{A} \in \Omega^1(X, \mathrm{Lie}(G))$ *as above, a smooth natural transformation*

$$g : \mathrm{tra}_A \to \mathrm{tra}_{\tilde{A}}$$

is a smooth map

$$g : X \to G$$

such that

$$A = g\tilde{A}g^{-1} + gdg^{-1} .$$

Proof. Pull back tra_A and $\mathrm{tra}_{\tilde{A}}$ to paths on the real line. Then use prop. 3. $\Box$

In summary, then, we have found

Theorem 2 *The category*

$$[\mathcal{P}_1(X), \Sigma(G)]$$

of smooth functors and smooth natural transformations is isomorphic to the category $\bar{H}^1(X, G)$*, whose objects are 1-forms* $A \in \Omega^1(X, \mathrm{Lie}(G))$ *and whose morphisms*

$$g : A \to \tilde{A}$$

are smooth functions $g : X \to G$ *such that* $A = g\tilde{A}g^{-1} + gdg^{-1}$*.*

1.3 Smooth 2-functors from 2-paths to 2-groupoids

1.3.1 Double paths on the real plane

Before looking at 2-categories of paths, it is helpful to first study the *double* category

$$\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$$

of rectangular 2-paths in the plane.

The category of horizontal morphisms in $\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$ is the disjoint union of copies of $\mathcal{P}_1(\mathbb{R})$, one for each line parallel to the first canonical coordinare axis in $\mathbb{R}^2$.

The category of vertical morphisms in $\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$ is the disjoint union of copies of $\mathcal{P}_1(\mathbb{R})$, one for each line parallel to the second canonical coordinare axis in $\mathbb{R}^2$.

2-morphisms in $\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$ are rectangles in $\mathbb{R}^2$ whose sides are parallel to the canonical coordinate axes. Composition of 2-morphism is the obvious concatenation of rectangles.

The double category $\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$ is a double category internal to manifolds. The manifold of objects is $\mathbb{R}^2$, that of horizontal morphism is $\mathbb{R}^2 \times \mathbb{R}$, as is that of vertical morphisms. The manifold of 2-morphisms is $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$.

$$
\begin{tikzcd}
x_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & x_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & x_2 \arrow[d] \\
y_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & y_2 \arrow[d] \\
z_0 \arrow[r] & z_1 \arrow[r] & z_2
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x_0 \arrow[rr] \arrow[dd] \arrow[ddrr, phantom, "\swarrow"] & & x_2 \arrow[dd] \\
& & \\
z_0 \arrow[rr] & & z_2
\end{tikzcd}
$$

Figure 1: **Composition of 2-morphisms** in $\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$.

For every strict 2-category C we canonically obtain a double category

$$\mathrm{Squares}(C)$$

whose 2-morphisms are squares in C.

Definition 6 *To every smooth 1-functor*

$$\mathrm{tra}_A : \mathcal{P}_1(\mathbb{R}^2) \to \Sigma(G)$$

from paths in the plane to a Lie group, we associate its **curvature double-functor**

$$\mathrm{curv}_A \quad : \quad \mathcal{P}_2^{\mathrm{doub}}\left(\mathbb{R}^2\right) \quad \to \quad \mathrm{Squares}(\Sigma\left(G \xrightarrow{\mathrm{Id}} G\right))$$

$$
\begin{tikzcd}
x_s \arrow[r, "\gamma_1"] \arrow[d, "\gamma_3"'] \arrow[dr, phantom, "\swarrow_S"] & x_1 \arrow[d, "\gamma_2"] \\
x_2 \arrow[r, "\gamma_4"'] & x_t
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}_A(\gamma_1)"] \arrow[d, "\mathrm{tra}_A(\gamma_3)"'] \arrow[dr, phantom, "\swarrow\,\mathrm{tra}_A(\partial S)"] & \bullet \arrow[d, "\mathrm{tra}_A(\gamma_2)"] \\
\bullet \arrow[r, "\mathrm{tra}_A(\gamma_4)"'] & \bullet
\end{tikzcd}
\,.
$$

Here

$$\partial S \equiv \bar{\gamma}_1 \circ \bar{\gamma}_2 \circ \gamma_4 \circ \gamma_3$$

denotes the **boundary** of S (a morphism in $\mathcal{P}_1(\mathbb{R}^2)$).

Lemma 1 *To first nonvanishing order in the length $|\gamma_i|$ of the sides γ_i of a rectangle in $\mathbb{R}^2$, the value of curv_A is*

$$
\mathrm{curv}_A \; : \;
\begin{tikzcd}
x_s \arrow[r, "\gamma_1"] \arrow[d, "\gamma_3"'] \arrow[dr, phantom, "\swarrow_S"] & x_1 \arrow[d, "\gamma_2"] \\
x_2 \arrow[r, "\gamma_4"'] & x_t
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\bullet \arrow[r, "1+A(\gamma_1)+\cdots"] \arrow[d, "1+A(\gamma_3)+\cdots"'] \arrow[dr, phantom, "\swarrow\,1+F_A(\gamma_3,\gamma_1)+\cdots"] & \bullet \arrow[d, "1+A(\gamma_2)+\cdots"] \\
\bullet \arrow[r, "1+A(\gamma_4)+\cdots"'] & \bullet
\end{tikzcd}
\,.
$$

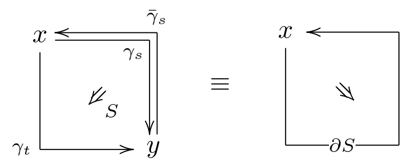


Figure 2: The **boundary of a 2-path**.

Here $F_A \in \Omega^2(U, \mathrm{Lie}(G))$ is the curvature 2-form of A.

Proof. Let γ_1 be the path of length l_i along the i-th axis and γ_3 the path of length l_j along the j-th axis. Then, as an equation in the universal enveloping algebra of $\mathrm{Lie}(G)$, we find

$$\begin{aligned}
\mathrm{tra}_A(\partial S) &= \exp(A_j(x)\, l_j) \exp((A_i(x) + l_j \partial_j A_i(x)) l_i) \cdot \\
&\quad \cdot \exp(-(A_j(x) + l_i \partial_i A_j(x)) l_j) \exp(-A_i(x)) \\
&\quad + \mathcal{O}(l_1^2, l_2^2) \\
&= 1 + l_i l_j (\partial_j A_i - \partial_i A_j + A_j A_i - A_i A_j)(x) + \mathcal{O}(l_1^2, l_2^2) \\
&= 1 + F_A(\gamma_3, \gamma_1) + \mathcal{O}(l_1^2, l_2^2)\,.
\end{aligned}$$

□

Proposition 6 *For $\Sigma(G_2)$ the suspension of a strict 2-group that comes from the crossed module $(t : H \to G)$, smooth double functors*

$$\mathrm{tra} : \mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2) \to \mathrm{Squares}(\Sigma(G_2))$$

are in bijection with pairs (A, B) where

- $A \in \Omega^1(\mathbb{R}^2, \mathrm{Lie}(G))$
- $B \in \Omega^2(\mathbb{R}^2, \mathrm{Lie}(H))$
- *such that* $F_A + t_*(B) = 0$.

Proof.

We want to reduce this problem to the 1-functorial case that we already understand. In order to do so, we restrict tra to the various 1-categories inside the double category of paths, where it becomes an ordinary smooth 1-functor.

More precisely, for every interval

$$I = [a, b] \subset \mathbb{R}$$

on the y-axis of $\mathbb{R}^2$, we get an isomorphism of $\mathcal{P}_1(\mathbb{R})$ with a 1-category of squares in $\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$ and hence a smooth 1-functor

$$\mathrm{tra}_I : \mathcal{P}_1(\mathbb{R}) \to \mathrm{Squares}(\Sigma(G_2))\,.$$

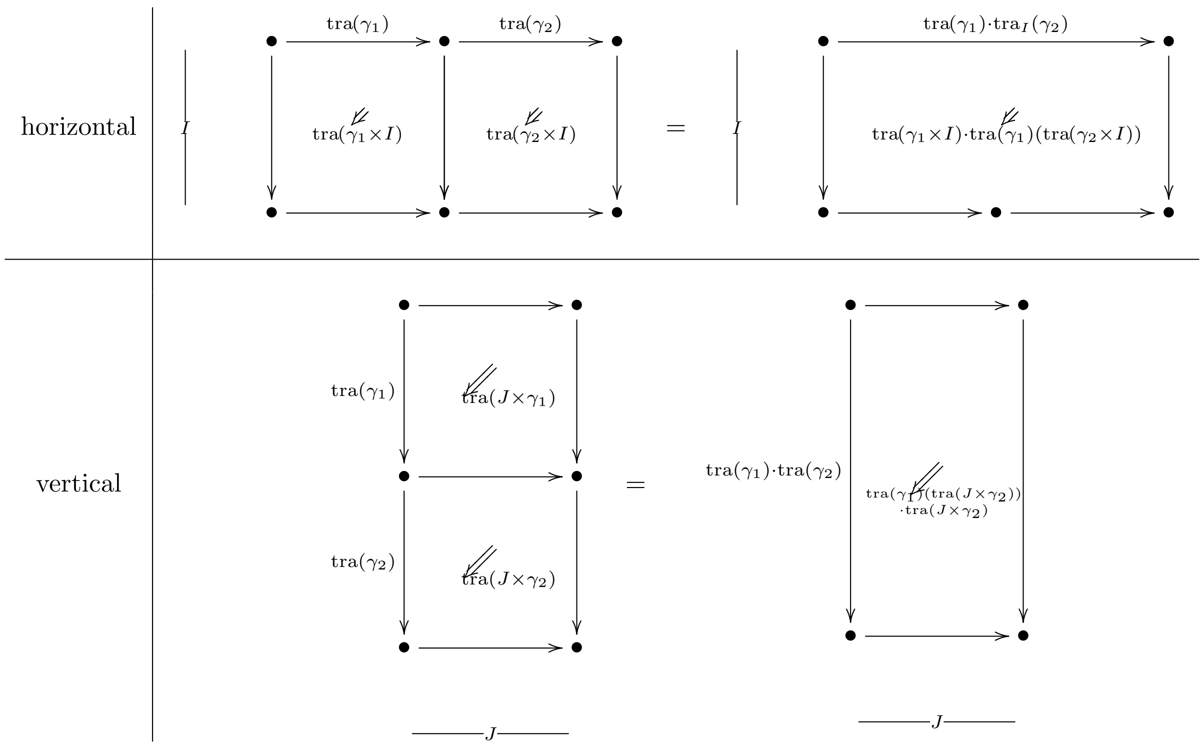


Figure 3: **Composition of squares in** $\Sigma(G_2)$ induces, in two different ways, a semidirect product group structure $H \ltimes G$ on pairs consisting of one of the four 1-morphisms and the 2-morphism filling the square.

$$
\mathrm{tra}_I : \left(x \longrightarrow y \right) \;\mapsto\;
\begin{tikzcd}
(x,a) \arrow[r, "\gamma_1"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & (y,a) \arrow[d] \\
(x,b) \arrow[r] & (y,b)
\end{tikzcd}
\;\mapsto\;
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}(\gamma_1)"] \arrow[d] \arrow[dr, phantom, "\swarrow\, \mathrm{tra}(\gamma_1 \times I)"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\,.
$$

By restricting attention to the value of the top horizontal morphism of the square on the right, as indicated, and by using the composition law in the 2-group as detailed in figure 3, we can regard this as a 1-functor with values in the semidirect product group $H \ltimes G$

$$
\mathrm{tra}_I : \mathcal{P}_1(\mathbb{R}) \to \Sigma(H \ltimes G)\,.
$$

By prop. 1 this defines a Lie$(H \ltimes G)$-valued 1-form

$$
B_I \in \Omega^1(\mathbb{R}, \mathrm{Lie}(H \ltimes G))
$$

on $\mathbb{R}$:

$$
\mathrm{tra}\;:\;
\begin{tikzcd}
{} \arrow[d, dash, "I"] \\
{}
\end{tikzcd}
\quad
\begin{tikzcd}
(x,a) \arrow[r, "\gamma_1"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & (y,a) \arrow[d] \\
(x,b) \arrow[r] & (y,b)
\end{tikzcd}
\;\mapsto\;
\begin{tikzcd}
\bullet \arrow[r, "1+A(\gamma_1)+\cdots"] \arrow[d] \arrow[dr, phantom, "\swarrow\, 1+B_I(\gamma_1)+\cdots"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\,.
$$

Consider now two composable intervals

$$I_1 = [a,b]$$

$$I_2 = [b,c]\,.$$

By using vertical composition as shown in figure 3 the functor

$$\mathrm{tra}_{I_1 \cup I_2}$$

is found to come from a 1-form given by

$$B_{I_1 \cup I_2}(x) = B_{I_1}(x) + \mathrm{tra}((x,a) \to (x,b))_* B_{I_2}(x)$$

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\mathrm{tra}(\gamma_1)"'] \arrow[dr, phantom, "\swarrow\, 1+B_{I_1}(J)+\cdots"] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d, "\mathrm{tra}(\gamma_2)"'] \arrow[dr, phantom, "\swarrow\, 1+B_{I_2}(J)+\cdots"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \\
{} \arrow[r, dash, "J"] & {}
\end{tikzcd}
\;=\;
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\mathrm{tra}(\gamma_1)\cdot\mathrm{tra}(\gamma_2)"'] \arrow[dr, phantom, "\swarrow\, {1+B_{I_1}(J)+ \atop \mathrm{tra}(\gamma_1)_* B_{I_2}(J)+\cdots}"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \\
{} \arrow[r, dash, "J"] & {}
\end{tikzcd}
\,.
$$

The composition here is again a semidirect group product. The group in question,

$$\Omega^1(\mathbb{R}, \mathrm{Lie}(H)) \ltimes \Omega^0(\mathbb{R}, G)\,,$$

comes from the group of G-valued functions on $\mathbb{R}$ and the additive group of $\mathrm{Lie}(H)$-valued 1-forms on $\mathbb{R}$.

Under vertical composition, tra hence induces a smooth 1-functor with values in this group.

Since the Lie algebra of a semidirect product Lie group is a semidirect sum of Lie algebras, we obtain a 1-form-valued 1-form

$$B \in \Omega^1(\mathbb{R}, \Omega^1(\mathbb{R}, \mathrm{Lie}(H)))$$

$$\mathrm{tra}\;:\; \begin{tikzcd} {} \arrow[d, dash, "I"'] & (x,a) \arrow[r, "\gamma_1"] \arrow[d, "\gamma_3"'] \arrow[dr, phantom, "\swarrow"] & (y,a) \arrow[d, "\gamma_2"] \\ {} & (x,b) \arrow[r, "\gamma_4"'] & (y,b) \end{tikzcd} \;\mapsto\; \begin{tikzcd} \bullet \arrow[r, "1+A(\gamma_1)+\cdots"] \arrow[d, "1+A(\gamma_3)+\cdots"'] \arrow[dr, phantom, "\swarrow 1+B_I(\gamma_3)+\cdots"] & \bullet \arrow[d, "1+A(\gamma_2)+\cdots"] \\ \bullet \arrow[r, "1+A(\gamma_4)+\cdots"'] & \bullet \end{tikzcd}$$

$$= \begin{tikzcd} \bullet \arrow[r, "1+A(\gamma_1)+\cdots"] \arrow[d, "1+A(\gamma_3)+\cdots"'] \arrow[dr, phantom, "\swarrow 1+B(\gamma_1,\gamma_3)+\cdots"] & \bullet \arrow[d, "1+A(\gamma_2)+\cdots"] \\ \bullet \arrow[r, "1+A(\gamma_4)+\cdots"'] & \bullet \end{tikzcd} .$$

We may equally well regard B as a 2-form on $\mathbb{R}^2$:

$$B \in \Omega^2(\mathbb{R}^2, \mathrm{Lie}(H)) \,.$$

This way, we have extracted from our smooth double functor the advertized pair (A, B).

Not every such pair can arise. The double functoriality of tra induces a constraint on the relation between the various 1-functors that we extracted from it-

Assuming that 2-morphisms close to an identity 2-morphism are indeed given by a 1-form and a 2-form

$$\mathrm{tra}\;:\; \begin{tikzcd} 0 \arrow[r, "\gamma_1"] \arrow[d, "\gamma_3"'] \arrow[dr, phantom, "\swarrow"] & x \arrow[d, "\gamma_2"] \\ y \arrow[r, "\gamma_4"'] & x+y \end{tikzcd} \;\mapsto\; \begin{tikzcd} \bullet \arrow[r, "1+A(\gamma_1)+\cdots"] \arrow[d, "1+A(\gamma_3)+\cdots"'] \arrow[dr, phantom, "\swarrow 1+B(\gamma_1,\gamma_3)+\cdots"] & \bullet \arrow[d, "1+A(\gamma_2)+\cdots"] \\ \bullet \arrow[r, "1+A(\gamma_4)+\cdots"'] & \bullet \end{tikzcd} ,$$

then the constraint $F_A + \delta(B) = 0$ follows from using lemma 1 in the condition

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \in \mathrm{Mor}_2\Big(\Sigma\Big(H \xrightarrow{\delta} G\Big)\Big) \quad \Leftrightarrow \quad \delta(h) = g' g^{-1} \,,$$

which yields

$$\delta\left(1+B\left(\gamma_1,\gamma_3\right)+\cdots\right)=1+F_A\left(\gamma_3,\gamma_1\right)+\cdots.$$

Any such pair (A,B) defines a smooth 2-functor $\mathcal{P}_2^{\mathrm{doub}}(U)\to\Sigma\left(G_2\right)$ by taking the limit of the assignment

$$\begin{tikzcd}
x_0 \arrow[rr] \arrow[dd] & & x_2 \arrow[dd] & & x_0 \arrow[r, "1+A_{x_0,x_1}"] \arrow[d] \arrow[dr, phantom, "\Swarrow 1+B(x_0)"] & x_1 \arrow[r, "1+A_{x_0,x_1}"] \arrow[d] \arrow[dr, phantom, "\Swarrow 1+B(x_1)"] & x_2 \arrow[d] \\
& \Swarrow_{\Sigma} & & \mapsto & y_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow 1+B(y_0)"] & y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow 1+B(y_1)"] & y_2 \arrow[d] \\
z_0 \arrow[rr] & & z_2 & & z_0 \arrow[r] & z_1 \arrow[r] & z_2
\end{tikzcd}$$

over successive refinements of the decompositions of Σ. $\square$

1.3.2 Cubical 2-paths on the real plane

With smooth double functors on the double category $\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$ of rectangular 2-paths in the plane thus understood, we can now pass without much further effort to a 2-category of 2-paths in the plane:

Definition 7 *Write $\mathcal{P}_2^{\mathrm{cub}}(\mathbb{R}^2)$ for the strict 2-category of* **cubical 2-paths in the plane***, that is generated from the double category $\mathcal{P}_2^{\mathrm{doub}}(\mathbb{R}^2)$.*

$$\begin{tikzcd}
x_0 \arrow[r] & x_1 \arrow[dd] & & \\
& & x_4 \arrow[r] & x_5 \arrow[dd] \\
& x_2 \arrow[r] & x_3 \arrow[u] & \\
& & & x_6
\end{tikzcd}$$

Figure 4: **Typical 1-morphism** in $\mathcal{P}_n^{\mathrm{cub}}\left(\mathbb{R}^2\right)$.

This is a smooth 2-category whose spaces of 1- and 2-morphisms are disjoint unions of finite dimensional manifolds (albeit the dimension of the manifolds in this union is not bounded).

Since it is generated by double paths, we record that prop. 6 now reads

$$
\begin{tikzcd}
x_0 \arrow[r] \arrow[ddd] & x_1 \arrow[dd] & & \\
& & x_4 \arrow[r] & x_5 \arrow[dd] \\
& x_2 \arrow[r] \arrow[d, phantom, "\Swarrow"] & x_3 \arrow[u] & \\
y_1 \arrow[rrr] & & & x_6
\end{tikzcd}
$$

Figure 5: **Typical 2-morphism** in $\mathcal{P}_2^{\mathrm{cub}}\left(\mathbb{R}^2\right)$.

$$
\begin{tikzcd}
x_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & x_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & x_2 \arrow[d] \\
y_0 \arrow[r] & y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & y_2 \arrow[d] \\
& z_1 \arrow[r] & z_2
\end{tikzcd}
=
\begin{tikzcd}
x_0 \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Swarrow"] & & x_2 \arrow[dd] \\
y_0 \arrow[r] & y_1 \arrow[d] & \\
& z_1 \arrow[r] & z_2
\end{tikzcd}
$$

Figure 6: **Composition of 2-morphisms** in $\mathcal{P}_2^{\mathrm{cub}}\left(\mathbb{R}^2\right)$.

Proposition 7 *For $\Sigma(G_2)$ the suspension of a strict 2-group that comes from the crossed module $(t : H \to G)$, smooth 2-functors*

$$\mathrm{tra} : \mathcal{P}_2^{\mathrm{cub}}(\mathbb{R}^2) \to \Sigma(G_2)$$

are in bijection with pairs (A, B) where

- $A \in \Omega^1(\mathbb{R}^2, \mathrm{Lie}(G))$
- $B \in \Omega^2(\mathbb{R}^2, \mathrm{Lie}(H))$
- *such that $F_A + t_*(B) = 0$.*

We shall write $\mathrm{tra}_{(A,B)}$ for the smooth 2-functor coming from a pair (A, B) this way.

Proposition 8 *Smooth isomorphisms*

$$g : \mathrm{tra}_{(A,B)} \to \mathrm{tra}_{(A',B')}$$

are in bijection with pairs $g \in \Omega^0\left(\mathbb{R}^2, G\right)$ and $a \in \Omega^1\left(\mathbb{R}^2, \mathrm{Lie}(H)\right)$ satisfying

$$A + \delta_* a = gA'g^{-1} + gdg^{-1}$$

$$
\begin{tikzcd}
x \arrow[r, dash, "\gamma_1"] \arrow[d, dash, "\gamma_3"'] \arrow[dr, phantom, "\Swarrow"] & {} \arrow[d, "\gamma_2"] \\
{} \arrow[r, "\gamma_4"'] & y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, dash, "g"] \arrow[d, dash, "g'"'] \arrow[dr, phantom, "\Swarrow h"] & {} \arrow[d] \\
{} \arrow[r] & \bullet
\end{tikzcd}
$$

$$
\Leftrightarrow \quad \mathrm{tra}\left(
\begin{tikzcd}
x \arrow[r, dash, "\gamma_1"] \arrow[d, dash, "\gamma_3"'] \arrow[dr, phantom, "\Nearrow"] & {} \arrow[d, "\gamma_2"] \\
{} \arrow[r, "\gamma_4"'] & y
\end{tikzcd}
\right)
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, dash, "g"] \arrow[d, dash, "g'"'] \arrow[dr, phantom, "\Nearrow h^{-1}"] & {} \arrow[d] \\
{} \arrow[r] & \bullet
\end{tikzcd}
$$

Figure 7: **Antisymmetry** $B(\gamma_1, \gamma_3) = -B(\gamma_3, \gamma_1)$ can be regarded as coming from the 2-groupoid nature of $\mathcal{P}_2^{\mathrm{doub}}(U)$.

and

$$B = g(B') + F_a \,,$$

where

$$F_a = da + a \wedge a + A(a) \;.$$

Proof. The isomorphism g is defined by a smooth functorial assignment

$$
g \;\;:\;\; (\, x \xrightarrow{\gamma} y \,) \quad \mapsto \quad
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}_A(\gamma)"] \arrow[d, "g(x)"'] \arrow[dr, phantom, "\Swarrow g(\gamma)"] & \bullet \arrow[d, "g(y)"] \\
\bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma)"'] & \bullet
\end{tikzcd}
\;.
$$

With g and A' being given, this can be regarded as an assignment

$$g \;\;:\;\; (\, x \xrightarrow{\gamma} y \,) \quad \mapsto \quad (\mathrm{tra}_A(\gamma), g(\gamma)) \in H \ltimes G \,,$$

i.e. as a smooth functor $\mathcal{P}_1(\mathbb{R}) \to \Sigma(H \ltimes G)$. By prop. 1 this means that

$$g(\gamma) = 1 + a(\gamma) + \cdots$$

for some

$$a \in \Omega^1(\mathbb{R}^2, \mathrm{Lie}(H)) \,.$$

Then the mere existence of the 2-cell on the right above is equivalent to the first proposed equation.

By further regarding g as a functor with values in $\Sigma(H \ltimes G)$ we obtain, by lemma 1, a $\mathrm{Lie}(H \ltimes G)$-valued curvature 2-form associated to this functor, obtained by differentiating

$$\mathrm{curv}_{(A,a)}(S) \;=\; \begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}_A(\gamma_1)"] \arrow[d, "g(x_s)"'] \arrow[dr, phantom, "\swarrow g(\gamma_1)"] & \bullet \arrow[r, "\mathrm{tra}_A(\gamma_2)"] \arrow[d, "g(x_1)"'] \arrow[dr, phantom, "\swarrow g(\gamma_2)"] & \bullet \arrow[r, "\mathrm{tra}_A(\bar\gamma_4)"] \arrow[d, "g(x_t)"'] \arrow[dr, phantom, "\swarrow g(\bar\gamma_4)"] & \bullet \arrow[r, "\mathrm{tra}_A(\bar\gamma_3)"] \arrow[d, "g(x_2)"'] \arrow[dr, phantom, "\swarrow g(\bar\gamma_3)"] & \bullet \arrow[d, "g(x_s)"'] \\
\bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma_1)"'] & \bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma_2)"'] & \bullet \arrow[r, "\mathrm{tra}_{A'}(\bar\gamma_4)"'] & \bullet \arrow[r, "\mathrm{tra}_{A'}(\bar\gamma_3)"'] & \bullet
\end{tikzcd},$$

with the right hand regarded as an element in $H \ltimes G$. Notice that the $\mathrm{Lie}(G)$-part of this connection is equal to A. By definition of the semidirect product, we have

$$\begin{aligned}
F_{(A,a)} &= d(A,a) + (A,a)\wedge(A,a) \\
&= (dA, da) + A^a \wedge A^b \frac{1}{2}[t_a, t_b] + a^i \wedge a^j \frac{1}{2}[b_i, b_j] + A^a \wedge a^i \alpha(t_a)(b_i) \\
&= (dA + A\wedge A,\ da + a \wedge a + A(a)) \\
&\equiv (F_A, F_a)\,.
\end{aligned}$$

In order for g to qualify as a pseudonatural transformation, we must require

$$\begin{tikzcd}[column sep=small]
& \bullet \arrow[rr, "\mathrm{tra}_A(\gamma_1)"] \arrow[dl, "g(x_s)"'] \arrow[dr, phantom, "\swarrow g(\gamma_1)"] & & \bullet \arrow[dl, "g(x_1)"'] \arrow[dd, "\mathrm{tra}_A(\gamma_2)"] \arrow[ddl, phantom, "\swarrow g(\gamma_2)"] \\
\bullet \arrow[rr, "\mathrm{tra}_{A'}(\gamma_1)"] \arrow[dd, "\mathrm{tra}_{A'}(\gamma_3)"'] \arrow[ddrr, phantom, "\swarrow \mathrm{tra}_{(A',B')}(S)"] & & \bullet \arrow[dd, "\mathrm{tra}_{A'}(\gamma_2)"'] & \\
& & & \bullet \arrow[dl, "g(x_t)"] \\
\bullet \arrow[rr, "\mathrm{tra}_{A'}(\gamma_4)"'] & & \bullet &
\end{tikzcd}
\;=\;
\begin{tikzcd}[column sep=small]
& \bullet \arrow[rr, "\mathrm{tra}_A(\gamma_1)"] \arrow[dl, "g(x_s)"'] \arrow[dd, "\mathrm{tra}_A(\gamma_3)"] \arrow[ddrr, phantom, "\swarrow \mathrm{tra}_{(A,B)}(S)"] & & \bullet \arrow[dd, "\mathrm{tra}_A(\gamma_2)"] \\
\bullet \arrow[dd, "\mathrm{tra}_{A'}(\gamma_3)"'] \arrow[r, phantom, "\swarrow g(\gamma_3)"] & & & \\
& \bullet \arrow[rr, "\mathrm{tra}_A(\gamma_4)"] \arrow[dl, "g(x_2)"'] \arrow[dr, phantom, "\swarrow g(\gamma_4)"] & & \bullet \arrow[dl, "g(x_t)"] \\
\bullet \arrow[rr, "\mathrm{tra}_{A'}(\gamma_4)"'] & & \bullet &
\end{tikzcd}\,.$$

It is convenient to rewhisker on the right to obtain the equivalent equation

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}_A(\gamma_1)"] \arrow[d, "g(x_s)"'] \arrow[dr, phantom, "\swarrow g(\gamma_1)"] & \bullet \arrow[r, "\mathrm{tra}_A(\gamma_2)"] \arrow[d, "g(x_1)"'] \arrow[dr, phantom, "\swarrow g(\gamma_2)"] & \bullet \arrow[r, "\mathrm{tra}_A(\bar\gamma_4)"] \arrow[d, "g(x_t)"'] \arrow[dr, phantom, "\swarrow g(\bar\gamma_4)"] & \bullet \arrow[r, "\mathrm{tra}_A(\bar\gamma_3)"] \arrow[d, "g(x_2)"'] \arrow[dr, phantom, "\swarrow g(\bar\gamma_3)"] & \bullet \arrow[d, "g(x_s)"'] \\
\bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma_1)"'] \arrow[d, "\mathrm{Id}"'] & \bullet \arrow[r, "\mathrm{tra}_{A'}(\gamma_2)"'] & \bullet \arrow[r, "\mathrm{tra}_{A'}(\bar\gamma_4)"'] \arrow[d, phantom, "\Downarrow \mathrm{tra}_{(A',B')}"] & \bullet \arrow[r, "\mathrm{tra}_{A'}(\bar\gamma_3)"'] & \bullet \arrow[d, "\mathrm{Id}"] \\
\bullet \arrow[rrrr, "\mathrm{Id}"'] & & {} & & \bullet
\end{tikzcd}
\;=\;
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}_A(\gamma_1)"] \arrow[d, "\mathrm{Id}"'] & \bullet \arrow[r, "\mathrm{tra}_A(\gamma_2)"] & \bullet \arrow[r, "\mathrm{tra}_A(\bar\gamma_4)"] \arrow[d, phantom, "\Downarrow \mathrm{tra}_{(A,B)}"] & \bullet \arrow[r, "\mathrm{tra}_A(\bar\gamma_3)"] & \bullet \arrow[d, "\mathrm{Id}"] \\
\bullet \arrow[rrrr, "\mathrm{Id}"'] & & {} & & \bullet
\end{tikzcd}\,.$$

Using lemma 1 together with the above considerations, this implies the second advertised equation. $\square$

Proposition 9 *Smooth 2-isomorphisms*

$$
\begin{tikzcd}
 & \mathrm{tra}_{(A_2,B_2)} \arrow[dr, "g_{23}"] \arrow[d, phantom, "\Downarrow f"] & \\
\mathrm{tra}_{(A_1,B_1)} \arrow[ur, "g_{12}"] \arrow[rr, "g_{13}"'] & {} & \mathrm{tra}_{(A_3,B_3)}
\end{tikzcd}
$$

are in bijection with $f \in \Omega^0\left(\mathbb{R}^2, H\right)$ *satisfying*

$$\delta(f)\, g_{12} g_{23} = g_{13}$$

and

$$a_{12} + g_{12}(a_{23}) = f a_{13} f^{-1} + f\, df^{-1} + f^{-1} A_1(f)\ .$$

Proof. Differentiate the relevant tin can equations

$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}_{A_1}(\gamma)"] \arrow[d, "g_{12}(x)"] \arrow[dd, bend right=60, "g_{13}(x)"'] \arrow[dr, phantom, "\swarrow g_{12}(\gamma)"] & \bullet \arrow[d, "g_{12}(y)"] \\
\bullet \arrow[r, "\mathrm{tra}_{A_2}(\gamma)"] \arrow[d, "g_{23}(x)"] \arrow[dr, phantom, "\swarrow g_{23}(\gamma)"] & \bullet \arrow[d, "g_{23}(y)"] \\
\bullet \arrow[r, "\mathrm{tra}_{A_3}(\gamma)"'] & \bullet
\end{tikzcd}
\overset{f(x)}{\Leftarrow}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}_{A_1}(\gamma)"] \arrow[dd, "g_{13}(x)"'] \arrow[ddr, phantom, "\swarrow g_{13}(\gamma)"] & \bullet \arrow[dd, "g_{13}(y)"] \arrow[dr, "g_{12}(y)"] & \\
 & \overset{f(y)}{\Leftarrow} & \bullet \arrow[dl, "g_{23}(y)"] \\
\bullet \arrow[r, "\mathrm{tra}_{A_3}(\gamma)"'] & \bullet &
\end{tikzcd}
$$

at all identity paths to obtain

$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{tra}_{A_1}(\gamma)"] \arrow[d, "g_{12}(x)"'] \arrow[dr, phantom, "\swarrow 1+a_{12}(\gamma)+\cdots"] & \bullet \arrow[d, "g_{12}(y)"] \\
\bullet \arrow[r, "\mathrm{tra}_{A_2}(\gamma)"] \arrow[d, "g_{23}(x)"'] \arrow[dr, phantom, "\swarrow 1+a_{23}(\gamma)+\cdots"] & \bullet \arrow[d, "g_{23}(y)"] \\
\bullet \arrow[r, "\mathrm{tra}_{A_3}(\gamma)"'] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \bullet \arrow[r, "\mathrm{tra}_{A_1}(\gamma)"] \arrow[dl, "g_{12}(x)"'] \arrow[dd, "a_{13}(x)"] \arrow[ddr, phantom, "\swarrow 1+a_{13}(\gamma)+\cdots"] & \bullet \arrow[dd, "g_{13}(y)"'] \arrow[dr, "g_{12}(y)"] & \\
\bullet \arrow[dr, "g_{23}(x)"'] & \overset{\bar{f}(x)}{\Leftarrow} & \overset{f(y)}{\Leftarrow} & \bullet \arrow[dl, "g_{23}(y)"] \\
 & \bullet \arrow[r, "\mathrm{tra}_{A_3}(\gamma)"'] & \bullet &
\end{tikzcd}
\ .
$$

$\square$

Proposition 10 *These 2-isomorphism satisfy the tetrahedral equation*

$$
\begin{tikzcd}
\mathrm{tra}_{(A_2,B_2)} \arrow[r, "g_{23}"] \arrow[dr, phantom, pos=0.25, "f_{123}\,\Downarrow"] \arrow[dr, phantom, pos=0.75, "\Downarrow f_{134}"] & \mathrm{tra}_{(A_3,B_3)} \arrow[d, "g_{34}"] \\
\mathrm{tra}_{(A_1,B_1)} \arrow[u, "g_{12}"] \arrow[ur, "g_{13}"] \arrow[r, "g_{14}"'] & \mathrm{tra}_{(A_4,B_4)}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{tra}_{(A_2,B_2)} \arrow[r, "g_{23}"] \arrow[dr, "g_{24}"] & \mathrm{tra}_{(A_3,B_3)} \arrow[d, "g_{34}"] \\
\mathrm{tra}_{(A_1,B_1)} \arrow[u, "g_{12}"] \arrow[r, "g_{14}"'] \arrow[ur, phantom, pos=0.25, "f_{124}\,\Downarrow"] \arrow[ur, phantom, pos=0.75, "f_{234}\,\Downarrow"] & \mathrm{tra}_{(A_4,B_4)}
\end{tikzcd}
$$

precisely if

$$f_{134}\, f_{123} = f_{124}\, g_{12}\,(f_{234}) \,.$$

We collect all this data in

Definition 8 *For G_2 a strict 2-category coming from the crossed module $\delta : H \to G$, the 2-category*

$$\bar{H}^2(\mathbb{R}^2, G_2)$$

of ("fake-flat") **differential G_2-cocycles** *on $\mathbb{R}^2$ is*

- *objects are pairs*
$$(A, B)$$
with $A \in \Omega^1(\mathbb{R}^2, \mathrm{Lie}(G))$, $B \in \Omega^2(\mathbb{R}^2, \mathrm{Lie}(H))$ and such that $F_a + \delta_* B = 0$,

- *morphisms*
$$(A, B) \xrightarrow{(g,a)} (A', B')$$
are pairs $g \in \Omega^0(\mathbb{R}^2, G)$, $a \in \Omega^1(\mathbb{R}^2, \mathrm{Lie}(H))$ and such that $A + \delta_* a = gA'g^{-1} + gdg^{-1}$ and $B = g\,(B') + F_a$

- *2-morphisms*
$$
\begin{tikzcd}
(A,B) \arrow[r, bend left=50, "{(g,a)}"] \arrow[r, bend right=50, "{(g',a')}"'] \arrow[r, phantom, "\Downarrow f"] & (A',B')
\end{tikzcd}
$$
are $f \in \Omega^0(\mathbb{R}^2, H)$ such that $\delta\,(f)\,g = g'$ and $a = f a' f^{-1} + f\,df^{-1} + f^{-1}A(f)$.

Composition is that induced from the way this p-form data is obtained from smooth 2-functors according to the above propositions.

This way we have, in summary:

Theorem 3 *There is a canonical isomorphism*

$$[\mathcal{P}_2^{\mathrm{cub}}(\mathbb{R}^2), \Sigma(G_2)] \simeq \bar{H}^2(\mathbb{R}^2, G_2)\,.$$

1.3.3 2-paths on general manifolds

For a general manifold X and a strict 2-group G_2, write $\bar{H}^2_{\mathrm{glob}}(X, G_2)$ for the analog of the 2-category from def. 8.

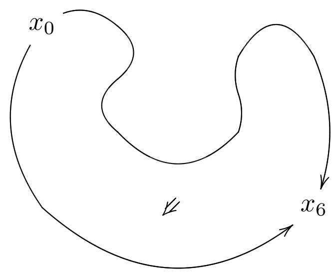


Figure 8: **Typical 2-morphism** in $\mathcal{P}_n\left(\mathbb{R}^2\right)$.

We want to generalize theorem 3

Theorem 4 *For X a smooth manifold and G_2 a strict Lie-2-group, there is a canonical isomorphism*

$$[\mathcal{P}_2(X), \Sigma(G_2)] \simeq \bar{H}^2_{\mathrm{glob}}(X, G_2)\,.$$

Idea of proof.

Probe 2-paths in $\mathcal{P}_2(X)$ in all possible ways with cubical 2-paths in the plane by considering smooth 2-functors

$$l : \mathcal{P}_2^{\mathrm{cub}}(\mathbb{R}^2) \to \mathcal{P}_2(X)\,.$$

Use the analogous reasoning as in the proof of prop. 4 to find that the p-form data obtained by pulling back along all such l glues to the respective p-form data on X: show that the p-form data coincides at all points at which l and l' are tangent. $\square$

$$\begin{tikzcd} x \arrow[r, dash, "\gamma_1"] \arrow[d, dash, "\gamma_3"'] & {} \arrow[d, "\gamma_2"] \\ {} \arrow[r, "\gamma_4"'] \arrow[ur, phantom, "\Swarrow"] & y \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, dash] \arrow[d, dash] & {} \arrow[d, "g"] \\ {} \arrow[r, "g'"'] \arrow[ur, phantom, "\Swarrow h"] & \bullet \end{tikzcd}$$

$$\Leftrightarrow \quad \mathrm{tra}\left(\begin{tikzcd} x \arrow[r, dash, "\gamma_1"] \arrow[d, dash, "\gamma_3"'] & {} \arrow[d, "\gamma_2"] \\ {} \arrow[r, "\gamma_4"'] \arrow[ur, phantom, "\Nearrow"] & y \end{tikzcd} \right) \quad = \quad \begin{tikzcd} \bullet \arrow[r, dash] \arrow[d, dash] & {} \arrow[d, "g"] \\ {} \arrow[r, "g'"'] \arrow[ur, phantom, "\Nearrow h^{-1}"] & \bullet \end{tikzcd}$$

Figure 9: **Antisymmetry** $B(\gamma_1, \gamma_3) = -B(\gamma_3, \gamma_1)$ can be regarded as coming from the 2-groupoid nature of $\mathcal{P}_2^{\mathrm{doub}}(U)$.