

Deligne 3-Cocycles and 2-Transport Transition

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Abstract

3rd Deligne cohomology is the decategorification of the 2-category of transitions of smooth transport 2-functors with values in the 2-group $U(1) \rightarrow 1$.

Let $\Sigma(\Sigma(U(1)))$ be the 2-category with a single object, a single 1-morphism, and one 2-morphism for every element in $U(1)$, with horizontal and vertical composition being the product in $U(1)$.

Let X be some manifold and $\mathcal{U} \rightarrow X$ a good covering by open sets.

We show that a smooth transition tetrahedron

$$\begin{array}{ccc}
 p_2^* \text{tra}_{\mathcal{U}} & \xrightarrow{p_{23}^* g} & p_3^* \text{tra}_{\mathcal{U}} \\
 \uparrow p_{12}^* g & \searrow p_{123}^* f & \nearrow p_{13}^* g \\
 p_1^* \text{tra}_{\mathcal{U}} & \xrightarrow{p_{14}^* g} & p_4^* \text{tra}_{\mathcal{U}} \\
 & & \downarrow p_{134}^* f \\
 & & p_3^* \text{tra}_{\mathcal{U}} \\
 & & \downarrow p_{34}^* g \\
 & & p_4^* \text{tra}_{\mathcal{U}}
 \end{array}
 =
 \begin{array}{ccc}
 p_2^* \text{tra}_{\mathcal{U}} & \xrightarrow{p_{23}^* g} & p_3^* \text{tra}_{\mathcal{U}} \\
 \uparrow p_{12}^* g & \searrow p_{124}^* f & \nearrow p_{24}^* g \\
 p_1^* \text{tra}_{\mathcal{U}} & \xrightarrow{p_{14}^* g} & p_4^* \text{tra}_{\mathcal{U}} \\
 & & \downarrow p_{124}^* f \\
 & & p_2^* \text{tra}_{\mathcal{U}} \\
 & & \downarrow p_{234}^* f \\
 & & p_3^* \text{tra}_{\mathcal{U}} \\
 & & \downarrow p_{34}^* g \\
 & & p_4^* \text{tra}_{\mathcal{U}}
 \end{array}$$

in $\text{Hom}(\mathcal{P}_{\mathcal{U}}, \Sigma(\Sigma(U(1))))$ defines a Deligne 2-cocycle in $\hat{H}^2(X, \mathcal{U})$.

Then we show that 1- and 2-morphisms of local trivializations, correspond to Deligne coboundaries.

This is a particularly simple special case of general nonabelian differential cohomology.

Deligne Cocycles from Transition Tetrahedra. We proceed in four steps, by deriving the data encoded by the transition tetrahedron on objects, 1-morphisms, 2-morphisms and 3-morphisms (where the equality between the two sides of the above equation is regarded as an identity 3-morphism).

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1. On the level of objects, we have an i -trivial 2-functor $\text{tra}_{\mathcal{U}} : \mathcal{P}_{\mathcal{U}} \rightarrow \Sigma(\Sigma(U(1)))$, which associates to each surface element a complex number. We require this to be smooth, so that the number is the integral of a 2-form B over that surface

$$\text{tra}_{\mathcal{U}} : \begin{array}{ccc} & \xrightarrow{\gamma_1} & \\ x & \Downarrow S & y \\ & \xrightarrow{\gamma_2} & \end{array} \mapsto \begin{array}{ccc} & \xrightarrow{\text{Id}} & \\ \bullet & \Downarrow \exp(i \int_S B) & \bullet \\ & \xrightarrow{\text{Id}} & \end{array} .$$

Hence $\text{tra}_{\mathcal{U}}$ defines a 2-form

$$B \in \Omega^2(\mathcal{U}) ,$$

or, equivalently, a collection of 2-forms

$$B_i \in \Omega^2(U_i) ,$$

on each open patch U_i .

2. On the level of 1-morphisms, we have a pseudonatural transformation

$$p_1^* \text{tra}_{\mathcal{U}} \xrightarrow{g} p_2^* \text{tra}_{\mathcal{U}} .$$

This is determined by a functorial assignment

$$g : \begin{array}{ccc} & \xrightarrow{\gamma} & \\ x & & y \end{array} \mapsto \begin{array}{ccc} \bullet & \xrightarrow{\text{Id}} & \bullet \\ \text{Id} \downarrow & \exp(i \int_{\gamma} A) & \downarrow \text{Id} \\ \bullet & \xrightarrow{\text{Id}} & \bullet \end{array} ,$$

given by a 1-form

$$A \in \Omega^1(\mathcal{U}^{[2]}) ,$$

equivalently, by a collection of 1-forms

$$A_{ij} \in \Omega^1(U_{ij})$$

on each double intersection. This assignment has to satisfy the tin can

equation

for all S . In terms of the differential forms this is equivalent to

$$p_1^* B - p_2^* B = dA$$

or

$$B_i - B_j = dA_{ij}$$

on each U_{ij} .

3. On the level of 2-morphisms, we have a modification of pseudonatural transformations

This is determined by an assignment

where f is a $U(1)$ -valued 0-form

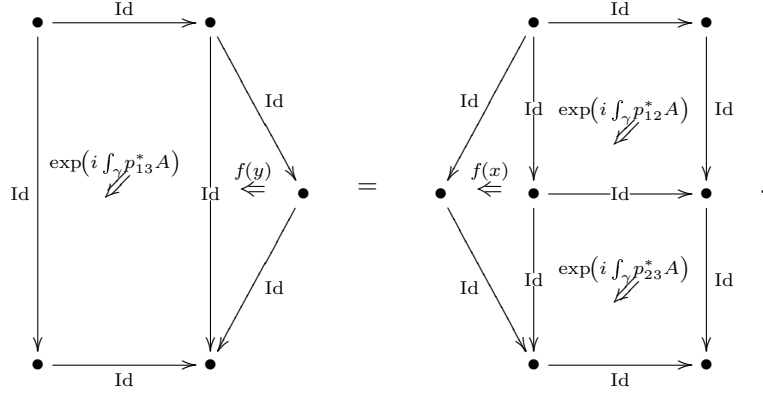
$$f \in C^\infty(\mathcal{U}^{[3]}, U(1)) ,$$

or, equivalently, a collection of such 0-forms

$$f_{ijk} \in C^\infty(U_{ijk}, U(1)) ,$$

one on each triple intersection.

This assignment has to satisfy all modification tin can equations



In terms of the differential forms, this is equivalent to

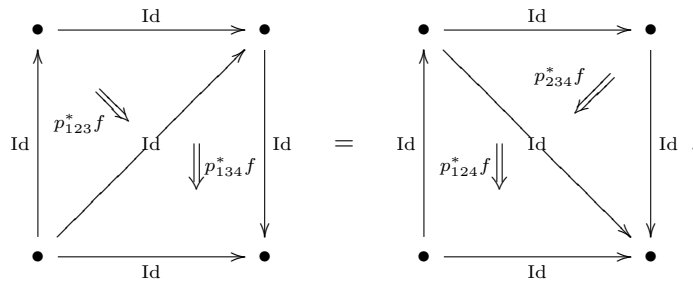
$$p_{12}^* A - p_{13}^* A + p_{23}^* A = d \ln f$$

or

$$A_{ij} - A_{ik} + A_{jk} = d \ln f_{ijk}$$

on each triple intersection.

4. On the level of 3-morphisms, the tetrahedron equation demands that the assignments specifying the modifications of pseudonatural transformations satisfy



This means

$$p_{123}^* \ln f - p_{124}^* \ln f + p_{134}^* \ln f - p_{234}^* \ln f = 0,$$

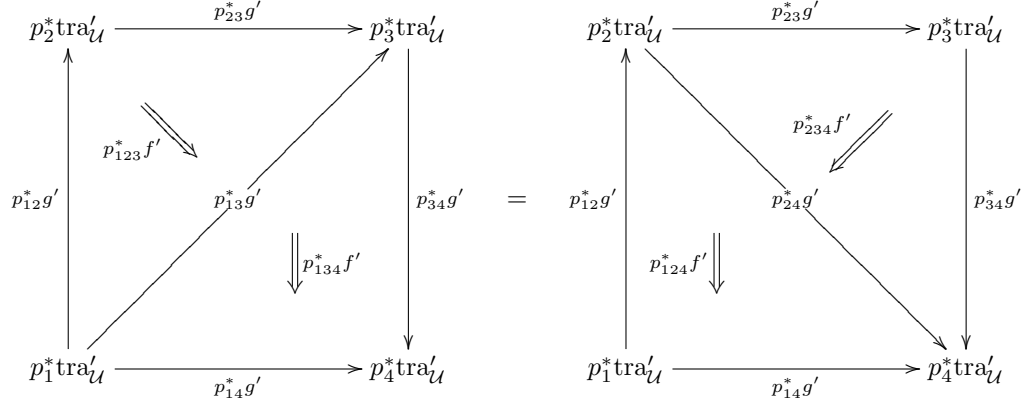
or

$$\ln f_{ijk} - \ln f_{ijl} + \ln f_{ikl} - \ln f_{jkl} = 0$$

on quadruple intersections.

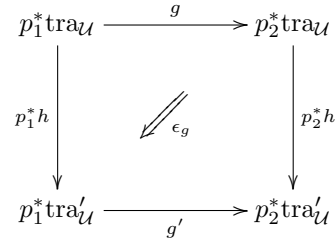
In summary, this shows that the differential forms encoding the transition tetrahedron constitute a Deligne 2-cocycle.

Deligne Coboundaries from Morphisms of Transitions. Let

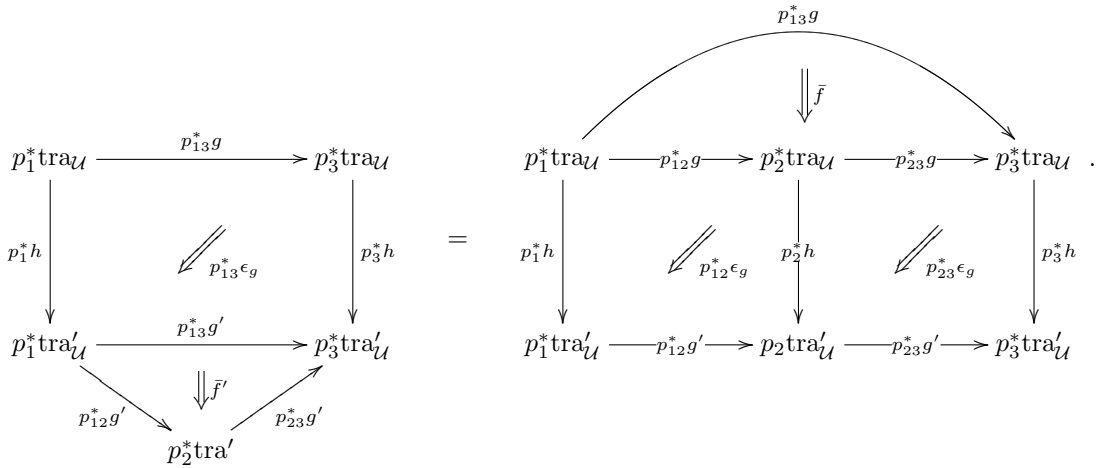


be another transition.

According to section ??, a morphism of transitions is a 2-morphism



satisfying



First of all this involves the 1-morphism

$$\text{tra}_U \xrightarrow{h} \text{tra}'_U .$$

As before, this is determined by a functorial assignment

$$h : x \xrightarrow{\gamma} y \mapsto \begin{array}{ccc} \bullet & \xrightarrow{\text{Id}} & \bullet \\ \text{Id} \downarrow & \exp(i \int_{\gamma} a) & \downarrow \text{Id} \\ \bullet & \xrightarrow{\text{Id}} & \bullet \end{array} ,$$

given by a 1-form

$$a \in \Omega^1(\mathcal{U}^{[1]}),$$

equivalently, by a collection of 1-forms

$$a_i \in \Omega^1(U_i)$$

on each patch. This assignment has to satisfy the tin can equation

$$\begin{array}{ccc} \bullet & \xrightarrow{\text{Id}} & \bullet \\ \text{Id} \downarrow & \exp(i \int_{\gamma_1} a) & \downarrow \text{Id} \\ \bullet & \xrightarrow{\text{Id}} & \bullet \\ & \exp(i \int_S B) & \\ & \text{Id} & \end{array} = \begin{array}{ccc} & \text{Id} & \\ & \exp(i \int_S B') & \\ \bullet & \xrightarrow{\text{Id}} & \bullet \\ \text{Id} \downarrow & \exp(i \int_{\gamma_2} a) & \downarrow \text{Id} \\ \bullet & \xrightarrow{\text{Id}} & \bullet \end{array}$$

for all S . In terms of the differential forms this is equivalent to

$$B - B' = da$$

or

$$B_i - B'_i = da_i$$

on each U_{ij} .

Next, the 2-morphism ϵ_g of 2-functors (a modification of pseudonatural transformations) is given by an assignment

$$x \mapsto \begin{array}{ccc} \bullet & \xrightarrow{\text{Id}} & \bullet \\ \text{Id} \downarrow & \lambda(x) & \downarrow \text{Id} \\ \bullet & \xrightarrow{\text{Id}} & \bullet \end{array} ,$$

where λ is a $U(1)$ -valued 0-form

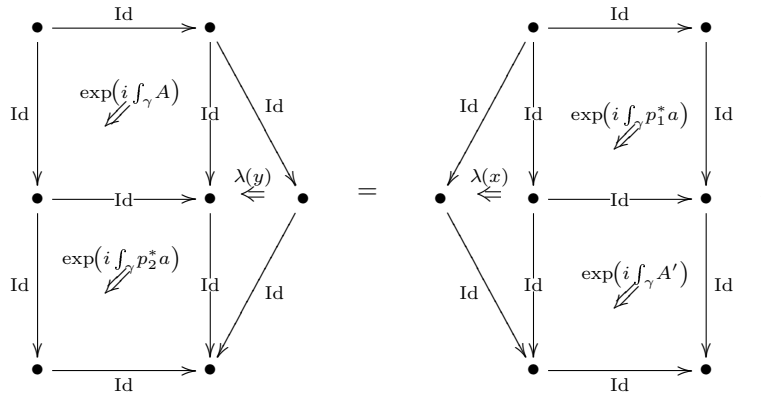
$$\lambda \in C^\infty(\mathcal{U}^{[2]}, U(1)),$$

or, equivalently, a collection of such 0-forms

$$\lambda_{ij} \in C^\infty(U_{ij}, U(1)),$$

one on each double intersection.

This assignment has to satisfy all modification tin can equations



In terms of the differential forms, this is equivalent to

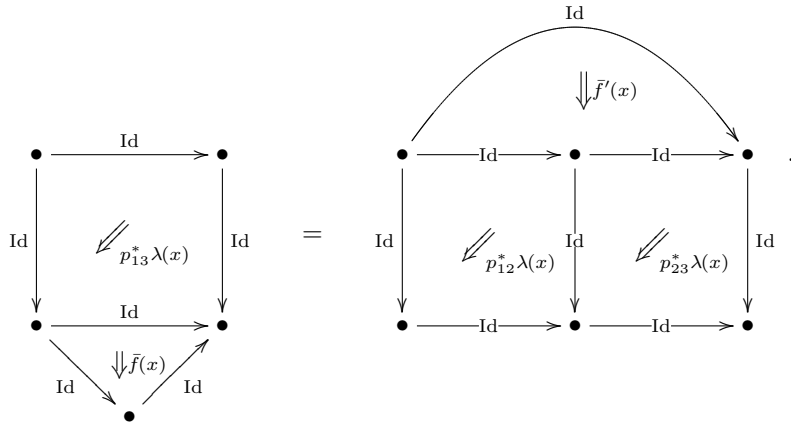
$$A' - A = p_2^* a - p_1^* a + d \ln \lambda$$

or

$$A'_{ij} - A_{ij} = a_j - a_i + d \ln \lambda_{ij}$$

on each double intersection.

Finally, the condition on ϵ_g is equivalent to



for all $x \in \mathcal{U}^{[3]}$. This says that

$$\ln f - \ln f' = \ln p_{12}^* \lambda - \ln p_{13}^* \lambda + \ln p_{23}^* \lambda,$$

or

$$\ln f_{ijk} - \ln f'_{ijk} = \ln \lambda_{ij} - \ln \lambda_{ik} + \ln \lambda_{jk}.$$

In summary, we have found that

$$\begin{aligned} (f_{ijk}, A_{ij}, B_i) - (f'_{ijk}, A'_{ij}, B'_i) &= (\ln \lambda_{ij} - \ln \lambda_{ik} + \ln \lambda_{jk}, a_i - a_j - d \ln \lambda_{ij}, da_i) \\ &= D(\lambda_{ij}, a_i). \end{aligned}$$

A gauge transformation between such gauge transformations is a 2-morphism of transitions, which, according to ??, is a 2-morphism of 2-functors

$$\begin{array}{ccc} & h_1 & \\ & \curvearrowright & \\ \text{tra}_{\mathcal{U}} & \Downarrow E_h & \text{tra}'_{\mathcal{U}} \\ & \curvearrowleft & \\ & h_2 & \end{array}$$

such that

$$\begin{array}{ccc} p_1^* \text{tra}_{\mathcal{U}} & \xrightarrow{g} & p_2^* \text{tra}_{\mathcal{U}} \\ \downarrow p_1^* h_2 \quad \leftarrow p_1^* E_h \quad p_1^* h_1 \quad \swarrow \epsilon_{g_1} \quad \downarrow p_2^* h_1 & & \\ p_1^* \text{tra}'_{\mathcal{U}} & \xrightarrow{g'} & p_2^* \text{tra}'_{\mathcal{U}} \end{array} = \begin{array}{ccc} p_1^* \text{tra}_{\mathcal{U}} & \xrightarrow{g} & p_2^* \text{tra}_{\mathcal{U}} \\ \downarrow p^* h_2 \quad \swarrow \epsilon_{g_2} \quad p_2^* h_2 \quad \leftarrow p_2^* E_h \quad p_2^* h_1 & & \\ p_1^* \text{tra}'_{\mathcal{U}} & \xrightarrow{g'} & p_2^* \text{tra}'_{\mathcal{U}} \end{array}$$

The 2-morphism E_h itself is given by an assignment

$$x \mapsto \begin{array}{ccc} & \text{Id} & \\ & \curvearrowright & \\ \bullet & \Downarrow q(x) & \bullet \\ & \curvearrowleft & \\ & \text{Id} & \end{array}$$

satisfying the tin can equation

$$\begin{array}{ccc} \bullet & \xrightarrow{\text{Id}} & \bullet \\ \downarrow \text{Id} \quad \leftarrow q(x) \quad \downarrow \text{Id} \quad \swarrow \exp(i f_\gamma a_1) \quad \downarrow \text{Id} & & \\ \bullet & \xrightarrow{\text{Id}} & \bullet \end{array} = \begin{array}{ccc} \bullet & \xrightarrow{\text{Id}} & \bullet \\ \downarrow \text{Id} \quad \swarrow \exp(i f_\gamma a_2) \quad \downarrow \text{Id} \quad \leftarrow q(y) \quad \downarrow \text{Id} & & \\ \bullet & \xrightarrow{\text{Id}} & \bullet \end{array}$$

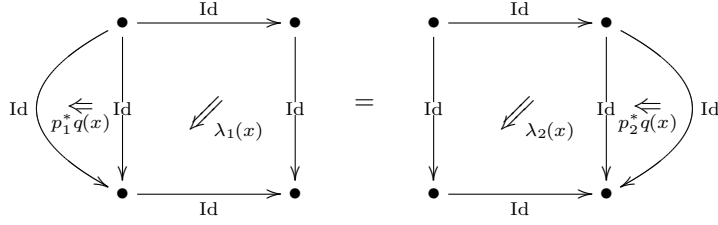
This says that

$$a_1 - a_2 = d \ln q$$

or

$$(a_1)_i - (a_2)_i = d \ln q_i .$$

The condition on E_h says that this satisfies



for all x . Hence

$$\ln \lambda_2 - \ln \lambda_1 = p_1^* q - p_2^* q$$

or

$$\ln(\lambda_2)_{ij} - \ln(\lambda_1)_{ij} = q_i - q_j .$$

In summary, we have

$$\begin{aligned} (\lambda_{ij}, a_i) - (\lambda'_{ij}, a'_i) &= (q_j - q_i, dq_i) \\ &= D(q_i) . \end{aligned}$$