Deligne 3-Cocycles and 2-Transport Transition

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Abstract

3rd Deligne cohomology is the decategorification of the 2-category of transitions of smooth transport 2-functors with values in the 2-group $U(1) \to 1$.

Let $\Sigma(\Sigma(U(1)))$ be the 2-category with a single object, a single 1-morphism, and one 2-morphism for every element in $U(1)$, with horizontal and vertical composition being the product in $U(1)$.

Let X be some manifold and $\mathcal{U} \to X$ a good covering by open sets.

We show that a smooth transition tetrahedron

$$
\begin{tikzcd}
p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_{23}^* g"] \arrow[dr, phantom, near start, "\Rightarrow p_{123}^* f"] \arrow[dr, phantom, near end, "\Downarrow p_{134}^* f"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_{34}^* g"] \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[u, "p_{12}^* g"] \arrow[ur, "p_{13}^* g"] \arrow[r, "p_{14}^* g"'] & p_4^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
=
\begin{tikzcd}
p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "p_{23}^* g"] \arrow[dr, "p_{24}^* g"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_{34}^* g"] \\
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[u, "p_{12}^* g"] \arrow[r, "p_{14}^* g"'] \arrow[ur, phantom, near start, "\Downarrow p_{124}^* f"] \arrow[ur, phantom, near end, "\Leftarrow p_{234}^* f"] & p_4^*\mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
$$

in $\mathrm{Hom}(\mathcal{P}_{\mathcal{U}}, \Sigma(\Sigma(U(1))))$ defines a Deligne 2-cocycle in $\hat{H}^2(X, \mathcal{U})$.

Then we show that 1- and 2-morphisms of local trivializations, correspond to Deligne coboundaries.

This is a particularly simple special case of general nonabelian differential cohomology.

Deligne Cocycles from Transition Tetrahedra. We proceed in four steps, by deriving the data encoded by the transition tetrahedron on objects, 1-morphisms, 2-morphisms and 3-morphisms (where the equality between the two sides of the above equation is regarded as an identity 3-morphism).

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1. On the level of objects, we have an i-trivial 2-functor $\mathrm{tra}_{\mathcal{U}} : \mathcal{P}_{\mathcal{U}} \to \Sigma(\Sigma(U(1)))$, which associates to each surface element a complex number. We require this to be smooth, so that the number is the integral of a 2-form B over that surface

$$\begin{tikzcd} \mathrm{tra}_{\mathcal{U}} : x \arrow[rr, bend left, "\gamma_1"] \arrow[rr, bend right, "\gamma_2"'] \arrow[rr, phantom, "\Downarrow S"] & & y \end{tikzcd} \mapsto \begin{tikzcd} \bullet \arrow[rr, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"'] \arrow[rr, phantom, "\Downarrow \exp\left(i\int_S B\right)"] & & \bullet \end{tikzcd} .$$

   Hence $\mathrm{tra}_{\mathcal{U}}$ defines a 2-form

$$B \in \Omega^2(\mathcal{U}) ,$$

   or, equivalently, a collection of 2-forms

$$B_i \in \Omega^2(U_i) ,$$

   on each open patch U_i.

2. On the level of 1-morphisms, we have a pseudonatural transformation

$$p_1^*\mathrm{tra}_{\mathcal{U}} \xrightarrow{g} p_2^*\mathrm{tra}_{\mathcal{U}} .$$

   This is determined by a functorial assignment

$$g : \begin{tikzcd} x \arrow[r, bend left, "\gamma"] & y \end{tikzcd} \mapsto \begin{tikzcd} \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\swarrow \exp\left(i\int_\gamma A\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\ \bullet \arrow[r, "\mathrm{Id}"'] & \bullet \end{tikzcd} ,$$

   given by a 1-form

$$A \in \Omega^1\left(\mathcal{U}^{[2]}\right) ,$$

   equivalently, by a collection of 1-forms

$$A_{ij} \in \Omega^1(U_{ij})$$

   on each double intersection. This assignment has to satisfy the tin can equation

$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\Downarrow \exp\left(i\int_{\gamma_1} A\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\
\bullet \arrow[r, "\mathrm{Id}"] \arrow[r, bend right=60, "\mathrm{Id}"'] \arrow[r, phantom, bend right=30, "\Downarrow \exp\left(i\int_S p_1^* B\right)"] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[r, bend left=60, "\mathrm{Id}"] \arrow[r, phantom, bend left=30, "\Downarrow \exp\left(i\int_S p_2^* B\right)"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\Downarrow \exp\left(i\int_{\gamma_2} A\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\
\bullet \arrow[r, "\mathrm{Id}"] & \bullet
\end{tikzcd}
$$

for all S. In terms of the differential forms this is equivalent to

$$p_1^* B - p_2^* B = dA$$

or

$$B_i - B_j = dA_{ij}$$

on each U_{ij}.

3. On the level of 2-morphisms, we have a modification of pseudonatural transformations

$$
\begin{tikzcd}
 & p_2^* \mathrm{tra}_{\mathcal{U}} \arrow[dr, "p_{23}^* g"] \arrow[d, phantom, "\Downarrow f"] & \\
p_1^* \mathrm{tra}_{\mathcal{U}} \arrow[ur, "p_{12}^* g"] \arrow[rr, "p_{13}^* g"'] & {} & p_3^* \mathrm{tra}_{\mathcal{U}}
\end{tikzcd}
.
$$

This is determined by an assignment

$$
x \mapsto
\begin{tikzcd}
 & \bullet \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow f(x)"] & \\
\bullet \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}"'] & {} & \bullet
\end{tikzcd}
,
$$

where f is a $U(1)$-valued 0-form

$$f \in C^\infty\left(\mathcal{U}^{[3]}, U(1)\right) ,$$

or, equivalently, a collection of such 0-forms

$$f_{ijk} \in C^\infty\left(U_{ijk}, U(1)\right) ,$$

one on each triple intersection.

This assignment has to satisfy all modification tin can equations

$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[dd, "\mathrm{Id}"'] & \bullet \arrow[dd, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"] & \\
{} \arrow[r, phantom, "\swarrow\, \exp\left(i \int_\gamma p_{13}^* A\right)"] & {} & \bullet \arrow[dl, "\mathrm{Id}"] \arrow[l, phantom, "\overset{f(y)}{\Leftarrow}"] \\
\bullet \arrow[r, "\mathrm{Id}"'] & \bullet &
\end{tikzcd}
=
\begin{tikzcd}
& \bullet \arrow[r, "\mathrm{Id}"] \arrow[dl, "\mathrm{Id}"'] \arrow[d, "\mathrm{Id}"] \arrow[dr, phantom, "\swarrow\, \exp\left(i \int_\gamma p_{12}^* A\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\
\bullet \arrow[dr, "\mathrm{Id}"'] \arrow[r, phantom, "\overset{f(x)}{\Leftarrow}"] & \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"] \arrow[dr, phantom, "\swarrow\, \exp\left(i \int_\gamma p_{23}^* A\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\
& \bullet \arrow[r, "\mathrm{Id}"'] & \bullet
\end{tikzcd}
.
$$

In terms of the differential forms, this is equivalent to

$$p_{12}^* A - p_{13}^* A + p_{23}^* A = d \ln f$$

or

$$A_{ij} - A_{ik} + A_{jk} = d \ln f_{ijk}$$

on each triple intersection.

4. On the level of 3-morphisms, the tetrahedron equation demands that the assignments specifying the modifications of pseudonatural transformations satisfy

$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[dr, phantom, "p_{123}^* f \searrow"] & \bullet \arrow[d, "\mathrm{Id}"] \\
\bullet \arrow[u, "\mathrm{Id}"] \arrow[ur, "\mathrm{Id}"] \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Downarrow p_{134}^* f"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"] \arrow[dr, phantom, "\swarrow p_{234}^* f"] & \bullet \arrow[d, "\mathrm{Id}"] \\
\bullet \arrow[u, "\mathrm{Id}"] \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "p_{124}^* f \Downarrow"'] & \bullet
\end{tikzcd}
.
$$

This means

$$p_{123}^* \ln f - p_{124}^* \ln f + p_{134}^* \ln f - p_{234}^* \ln f = 0\,,$$

or

$$\ln f_{ijk} - \ln f_{ijl} + \ln f_{ikl} - \ln f_{jkl} = 0$$

on quadruple intersections.

In summary, this shows that the differential forms encoding the transition tetrahedron constitute a Deligne 2-cocycle.

Deligne Coboundaries from Morphisms of Transitions. Let

$$
\begin{tikzcd}
p_2^*\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "p_{23}^*g'"] \arrow[dr, phantom, "\Searrow p_{123}^*f'"] & & p_3^*\mathrm{tra}'_{\mathcal{U}} \arrow[dd, "p_{34}^*g'"] \\
& & \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[uu, "p_{12}^*g'"] \arrow[uurr, "p_{13}^*g'" description] \arrow[rr, "p_{14}^*g'"'] & & p_4^*\mathrm{tra}'_{\mathcal{U}} \arrow[ul, phantom, "\Downarrow p_{134}^*f'"]
\end{tikzcd}
=
\begin{tikzcd}
p_2^*\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "p_{23}^*g'"] \arrow[ddrr, "p_{24}^*g'" description] & & p_3^*\mathrm{tra}'_{\mathcal{U}} \arrow[dd, "p_{34}^*g'"] \arrow[dl, phantom, "\Swarrow p_{234}^*f'"] \\
& & \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[uu, "p_{12}^*g'"] \arrow[rr, "p_{14}^*g'"'] \arrow[ur, phantom, "p_{124}^*f' \Downarrow"] & & p_4^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

be another transition.

According to section **??**, a morphism of transitions is a 2-morphism

$$
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p_1^*h"'] \arrow[dr, phantom, "\Swarrow \epsilon_g"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "g'"'] & p_2^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
$$

satisfying

$$
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, "p_{13}^*g"] \arrow[d, "p_1^*h"'] \arrow[drr, phantom, "\Swarrow p_{13}^*\epsilon_g"] & & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_3^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[rr, "p_{13}^*g'"] \arrow[dr, "p_{12}^*g'"'] & \Downarrow \bar{f}' & p_3^*\mathrm{tra}'_{\mathcal{U}} \\
& p_2^*\mathrm{tra}' \arrow[ur, "p_{23}^*g'"'] &
\end{tikzcd}
=
\begin{tikzcd}
p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[rr, bend left, "p_{13}^*g"] \arrow[r, "p_{12}^*g" description] \arrow[d, "p_1^*h"'] \arrow[dr, phantom, "\Swarrow p_{12}^*\epsilon_g"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[u, phantom, "\Downarrow \bar{f}"] \arrow[r, "p_{23}^*g" description] \arrow[d, "p_2^*h" description] \arrow[dr, phantom, "\Swarrow p_{23}^*\epsilon_g"] & p_3^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_3^*h"] \\
p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "p_{12}^*g'" description] & p_2\mathrm{tra}'_{\mathcal{U}} \arrow[r, "p_{23}^*g'" description] & p_3^*\mathrm{tra}'_{\mathcal{U}}
\end{tikzcd}
.
$$

First of all this involves the 1-morphism

$$
\mathrm{tra}_{\mathcal{U}} \xrightarrow{h} \mathrm{tra}'_{\mathcal{U}} .
$$

As before, this is determined by a functorial assignment

$$h \;:\; \begin{tikzcd} x \arrow[rr, bend left, "\gamma"] & & y \end{tikzcd} \;\mapsto\; \begin{tikzcd} \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow \exp\left(i \int_\gamma a\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\ \bullet \arrow[r, "\mathrm{Id}"'] & \bullet \end{tikzcd} ,$$

given by a 1-form

$$a \in \Omega^1\left(\mathcal{U}^{[1]}\right) ,$$

equivalently, by a collection of 1-forms

$$a_i \in \Omega^1\left(U_i\right)$$

on each patch. This assignment has to satisfy the tin can equation

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow \exp\left(i \int_{\gamma_1} a\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\ \bullet \arrow[r, "\mathrm{Id}"] \arrow[r, bend right=70, "\mathrm{Id}"'] \arrow[r, phantom, shift right=3ex, "\Downarrow \exp\left(i \int_S B\right)"] & \bullet \end{tikzcd} \;=\; \begin{tikzcd} \bullet \arrow[r, bend left=70, "\mathrm{Id}"] \arrow[r, phantom, shift left=3ex, "\Downarrow \exp\left(i \int_S B'\right)"] \arrow[r, "\mathrm{Id}"'] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow \exp\left(i \int_{\gamma_2} a\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\ \bullet \arrow[r, "\mathrm{Id}"] & \bullet \end{tikzcd}$$

for all S. In terms of the differential forms this is equivalent to

$$B - B' = da$$

or

$$B_i - B'_i = da_i$$

on each U_{ij}.

Next, the 2-morphism ϵ_g of 2-functors (a modification of pseudonatural transformations) is given by an assignment

$$x \;\mapsto\; \begin{tikzcd} \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow \lambda(x)"] & \bullet \arrow[d, "\mathrm{Id}"] \\ \bullet \arrow[r, "\mathrm{Id}"'] & \bullet \end{tikzcd} ,$$

where λ is a $U(1)$-valued 0-form

$$\lambda \in C^{\infty}\left(\mathcal{U}^{[2]}, U(1)\right) ,$$

or, equivalently, a collection of such 0-forms

$$\lambda_{ij} \in C^{\infty}\left(U_{ij}, U(1)\right) ,$$

one on each double intersection.

This assignment has to satisfy all modification tin can equations

$$\begin{tikzcd}
\bullet \arrow[r,"\mathrm{Id}"] \arrow[d,"\mathrm{Id}"'] \arrow[dr, phantom, "\exp\left(i\int_\gamma A\right)"] & \bullet \arrow[d,"\mathrm{Id}"] \arrow[dr,"\mathrm{Id}"] & \\
\bullet \arrow[r,"\mathrm{Id}"] \arrow[d,"\mathrm{Id}"'] \arrow[dr, phantom, "\exp\left(i\int_\gamma p_2^* a\right)"] & \bullet \arrow[d,"\mathrm{Id}"] & \bullet \arrow[l, phantom, "\overset{\lambda(y)}{\Leftarrow}"] \arrow[dl,"\mathrm{Id}"] \\
\bullet \arrow[r,"\mathrm{Id}"'] & \bullet &
\end{tikzcd}
=
\begin{tikzcd}
& \bullet \arrow[r,"\mathrm{Id}"] \arrow[d,"\mathrm{Id}"] \arrow[dl,"\mathrm{Id}"'] \arrow[dr, phantom, "\exp\left(i\int_\gamma p_1^* a\right)"] & \bullet \arrow[d,"\mathrm{Id}"] \\
\bullet \arrow[r, phantom, "\overset{\lambda(x)}{\Leftarrow}"] \arrow[dr,"\mathrm{Id}"'] & \bullet \arrow[r,"\mathrm{Id}"] \arrow[d,"\mathrm{Id}"] \arrow[dr, phantom, "\exp\left(i\int_\gamma A'\right)"] & \bullet \arrow[d,"\mathrm{Id}"] \\
& \bullet \arrow[r,"\mathrm{Id}"'] & \bullet
\end{tikzcd} .$$

In terms of the differential forms, this is equivalent to

$$A' - A = p_2^* a - p_1^* a + d \ln \lambda$$

or

$$A'_{ij} - A_{ij} = a_j - a_i + d \ln \lambda_{ij}$$

on each double intersection.

Finally, the condition on ϵ_g is equivalent to

$$\begin{tikzcd}
\bullet \arrow[rr,"\mathrm{Id}"] \arrow[d,"\mathrm{Id}"'] \arrow[drr, phantom, "p_{13}^*\lambda(x)"] & & \bullet \arrow[d,"\mathrm{Id}"] \\
\bullet \arrow[rr,"\mathrm{Id"] \arrow[dr,"\mathrm{Id}"'] & \Downarrow \bar{f}(x) & \bullet \\
& \bullet \arrow[ur,"\mathrm{Id}"'] &
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r,"\mathrm{Id}"] \arrow[d,"\mathrm{Id}"'] \arrow[rr, bend left, "\mathrm{Id}"] \arrow[dr, phantom, "p_{12}^*\lambda(x)"] & \bullet \arrow[r,"\mathrm{Id}"] \arrow[d,"\mathrm{Id}"] \arrow[u, phantom, "\Downarrow \bar{f}'(x)"] \arrow[dr, phantom, "p_{23}^*\lambda(x)"] & \bullet \arrow[d,"\mathrm{Id}"] \\
\bullet \arrow[r,"\mathrm{Id}"'] & \bullet \arrow[r,"\mathrm{Id}"'] & \bullet
\end{tikzcd} .$$

for all $x \in \mathcal{U}^{[3]}$. This says that

$$\ln f - \ln f' = \ln p_{12}^*\lambda - \ln p_{13}^*\lambda + \ln p_{23}^*\lambda\,,$$

or

$$\ln f_{ijk} - \ln f'_{ijk} = \ln \lambda_{ij} - \ln \lambda_{ik} + \ln \lambda_{jk}\,.$$

In summary, we have found that

$$\begin{aligned}(f_{ijk}, A_{ij}, B_i) - (f'_{ijk}, A'_{ij}, B'_i) &= (\ln \lambda_{ij} - \ln \lambda_{ik} + \ln \lambda_{jk}\,, a_i - a_j - d\ln \lambda_{ij}\,, da_i) \\ &= D(\lambda_{ij}, a_i)\,.\end{aligned}$$

A gauge transformation between such gauge transformations is a 2-morphisms of transitions, which, according to **??**, is a 2-morphism of 2-functors

$$\begin{tikzcd} \mathrm{tra}_{\mathcal{U}} \arrow[rr, bend left, "h_1"] \arrow[rr, bend right, "h_2"'] \arrow[rr, phantom, "\Downarrow E_h"] & & \mathrm{tra}'_{\mathcal{U}} \end{tikzcd}$$

such that

$$\begin{tikzcd} p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p_1^*h_1"] \arrow[d, bend right=60, "p_1^*h_2"'] \arrow[dr, phantom, "\swarrow \epsilon_{g_1}"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h_1"] \\ p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "g'"'] & p_2^*\mathrm{tra}'_{\mathcal{U}} \end{tikzcd} \quad \Leftarrow p_1^*E_h \quad = \quad \begin{tikzcd} p_1^*\mathrm{tra}_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "p^*h_2"] \arrow[dr, phantom, "\swarrow \epsilon_{g_2}"] & p_2^*\mathrm{tra}_{\mathcal{U}} \arrow[d, "p_2^*h_2"] \arrow[d, bend left=60, "p_2^*h_1"] \\ p_1^*\mathrm{tra}'_{\mathcal{U}} \arrow[r, "g'"'] & p_2^*\mathrm{tra}'_{\mathcal{U}} \end{tikzcd} \quad \Leftarrow p_2^*E_h\,.$$

The 2-morphism E_h itself is given by an assignment

$$x \mapsto \begin{tikzcd} \bullet \arrow[rr, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"'] \arrow[rr, phantom, "\Downarrow q(x)"] & & \bullet \end{tikzcd}$$

satisfying the tin can equation

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"] \arrow[d, bend right=60, "\mathrm{Id}"'] \arrow[dr, phantom, "\swarrow \exp\left(i\int_\gamma a_1\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \\ \bullet \arrow[r, "\mathrm{Id}"'] & \bullet \end{tikzcd} \quad \Leftarrow q(x) \quad = \quad \begin{tikzcd} \bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"] \arrow[dr, phantom, "\swarrow \exp\left(i\int_\gamma a_2\right)"] & \bullet \arrow[d, "\mathrm{Id}"] \arrow[d, bend left=60, "\mathrm{Id}"] \\ \bullet \arrow[r, "\mathrm{Id}"'] & \bullet \end{tikzcd} \quad \Leftarrow q(y)\,.$$

This says that

$$a_1 - a_2 = d \ln q$$

or

$$(a_1)_i - (a_2)_i = d \ln q_i \,.$$

The condition on E_h says that this satisfies

$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"] \arrow[d, bend right=60, "\mathrm{Id}"'] \arrow[d, phantom, bend right=30, "\Leftarrow p_1^* q(x)"] \arrow[dr, phantom, "\swarrow \lambda_1(x)"] & \bullet \arrow[d, "\mathrm{Id}"] \\
\bullet \arrow[r, "\mathrm{Id}"'] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"] \arrow[dr, phantom, "\swarrow \lambda_2(x)"] & \bullet \arrow[d, "\mathrm{Id}"] \arrow[d, bend left=60, "\mathrm{Id}"] \arrow[d, phantom, bend left=30, "\Leftarrow p_2^* q(x)"] \\
\bullet \arrow[r, "\mathrm{Id}"'] & \bullet
\end{tikzcd}
$$

for all x. Hence

$$\ln \lambda_2 - \ln \lambda_1 = p_1^* q - p_2^* q$$

or

$$\ln(\lambda_2)_{ij} - \ln(\lambda_1)_{ij} = q_i - q_j \,.$$

In summary, we have

$$
\begin{aligned}
(\lambda_{ij}, a_i) - (\lambda'_{ij}, a'_i) \quad &= \quad (q_j - q_i, dq_i) \\
&= \quad D(q_i) \,.
\end{aligned}
$$