Paths in Categories

Urs Schreiber*

February 2, 2006

Abstract

1 Introduction

As argued in particular in [1], orbifolds are best thought of as decategorified groupoids. A point in the orbifold hence appears as an isomorphism class of objects in some category.

Motivated by parallel transport along paths in orbifolds as well as by the study of strings propagating on orbifolds, one would like to similarly understand *paths* and *loops* in orbifolds in terms of the representing groupoids.

In the context of what is being called *orbifold string topology* [3] Lupercio and Uribe had introduced [2] a certain notion of a loop space of a groupoid G, called the *loop groupoid* of G.

Their approach rests on the strategy to regard the circle S^1 as a groupoid itself in a suitable sense and define the loop space of G as the category of (smooth) functors from S^1 to G.

Heuristically, a loop in G defined this way is a an alternating concatenation of smooth paths in the object space of G formally composed with morphisms in G.

The purpose of the following notes is to indicate that this concept admits also a 2-functorial perspective, which provides a nice way to describe higher order equivariant structures on orbifolds (like (nonabelian) gerbes with connection) in terms of transport 2-functors.

In general, given any *smooth* category S (groupoid or not), there are generally two different ways to "move" from a to b inside of S, where a and b are objects of S.

First, there might be a morphism $a \longrightarrow b$ in $\mathrm{Mor}(S)$. But second, since S is smooth, there might be a smooth path running through the space of objects of S, from a to b.

*E-mail: urs.schreiber at math.uni-hamburg.de

We formalize this and introduce the general concept of a **category of paths inside a smooth category** S, whose objects are those of S and whose morphisms are formal composites of smooth paths in the object space of S with morphisms of S, subject to certain compatibility relations.

In fact, the main point is that this concept easily categorfies. Given any smooth *2-category* S (2-groupoid or not) we can consider the **2-category of 2-paths inside** S. This has 2-morphisms being formal compositions of smooth surface elements in the object space of S with 2-morphsism of S.

We demonstrate that for S representing an orbifold, this concept refines the loop groupoid given by Lupercio and Uribe in that it suspends a 1-category of loops and cobordisms to a 2-category of points, paths and cobordisms. (See the introduction of [5] for why this is desirable.)

Moreover, 2-paths in 2-categories as defined here generalize the definition by Lupercio and Uribe in that it admits cobordisms between paths that are true surfaces, not just "jumps" between orbifold sectors.

We claim that 2-functors from a 2-category of paths inside a groupoid represent equivariant gerbes with connection and parallel surface transport on orbifolds. This applies to abelian bundle gerbes [6] just as well as to nonabelian bundle gerbes [7]. The main concepts are described below. Details of this construction however will be discussed elsewhere.

A special case for this has already been discussed at length. Choosing a good covering of any space gives rise to the *Čech-groupoid* associated to that space. Regarding this groupoid as an orbifold (it is in fact the *embedding groupoid* of the trivial orbifold, as defined in section 3.5 of [1]), the cocycle conditions for a locally trivialized 1- or 2-bundle over this space are nothing but the equivariance conditions with respect to this "orbifold" [4, 8].

For this special case the 1- and 2-path categories of paths inside the Čech groupoid have already been studied in section 12.1 of [8]. The following definitions are a straightforward generalization of this concept to arbitary smooth categories. The reader interested in more technical details should hence consult section 12 of [8].

2 Bundles with Connection over Orbifolds

Definition 1 *A* **smooth category** *or* **smooth 2-space** *is a category internal to the category of smooth spaces.*

Hence a smooth category has a smooth space of morphisms, a smooth space of objects and smooth source, target, identity and composition maps.

Definition 2 *For the purposes of the present discussion, a* **path** *in a smooth space M is always a path with* **sitting instant** *at both endpoints. This means that a path in M is a smooth map*

$$\gamma : I \to M$$

from an interval $I = [a, b] \subset \mathbb{R}$ to M, such that all derivatives of γ vanish in some neighborhood $U_a \subset \mathbb{R}$ of a and in some neighborhood $U_b \subset \mathbb{R}$ of b.

Definition 3 *Let M be any smooth space.*

1. *Denote by* $\mathrm{Moore}_1(M)$ *the smooth category whose*
   (a) *space of objects is M*
   (b) *space of morphisms is that of Moore paths in M*

   with the obvious composition of morphisms by concatenation of paths.

2. *Denote by* $\mathcal{P}_1(M)$ *the smooth groupoid whose*
   (a) *space of objects is M*
   (b) *space of morphisms is that of* thin *homotopy classes of paths in M with*

   with the obvious composition of morphisms by concatenation of paths.

Remark. Every smooth space may trivially be regarded as a smooth 2-space with only identity morphisms. From this point of view the above defines paths inside a smooth category. This is one way to motivate the following constructions.

Definition 4 *Let S be a smooth category. We say that S is* **orbifold-like** *if the following is true. Given any morphism $x \xrightarrow{g} y$ of S and any path $\gamma : I \to \mathrm{Obj}(S)$ with $\gamma(0) = x$, there is a* unique *path*

$$g_\gamma : I \to \mathrm{Mor}(S)$$

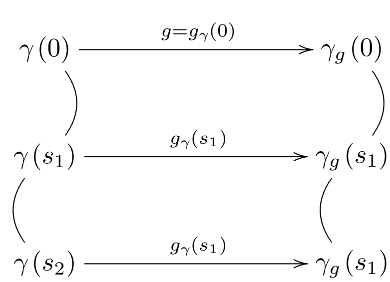
such that $g_\gamma(0) = g$ and

$$\gamma = s \circ g_\gamma .$$

We write

$$\gamma_g \equiv t \circ g_\gamma$$

for the path traces out by the target object of this unique path of morphisms.



Definition 5 *Let S be an orbifold-like category.*

1. *Denote by* $\mathrm{Moore}_1(S)$ *the smooth category whose*

   (a) *space of objects is that of* S

   (b) *space of morphisms is that* generated *by formal compositions of morphisms in* $\mathrm{Moore}(\mathrm{Obj}(S))$ *with morphisms in* S, *subject to the* relations

$$\begin{array}{ccc} \gamma(0) & \xrightarrow{g_\gamma(0)} & \gamma_g(0) \\ \downarrow{\scriptstyle \gamma} & /\!/ & \downarrow{\scriptstyle \gamma_g} \\ \gamma(s) & \xrightarrow[g_\gamma(s)]{} & \gamma_g(s) \end{array} \qquad (1)$$

   for all $\gamma(0) \xrightarrow{\gamma} \gamma(s) \in \mathrm{Mor}(\mathrm{Moore}_1(\mathrm{Obj}(S)))$ *and all* $\gamma(0) \xrightarrow{g} \gamma_g(0) \in \mathrm{Mor}(S)$, *with* g_γ *the unique smooth continuation of* g, *according to def. 4.*

2. *Denote by* $\mathcal{P}_1(S)$ *the smooth groupoid obtained similarly from generators and relations, but with Moore paths in* $\mathrm{Obj}(S)$ *replaced by thin homotopy classes of paths.*

Proposition 1 *Let* X *be an orbifold represented by the Lie groupoid* G. *Vector bundles* $E \longrightarrow X$ *with connection on* X *are smooth functors*

$$\mathrm{tra} : \mathcal{P}_1(G) \to \mathbf{Vect}\,.$$

Proof. Applying such a functor to a morphism of the groupoid yields isomorphisms

$$\mathrm{tra}\left(x \xrightarrow{g} gx \right) = E_x \xrightarrow{\mathrm{tra}(g)} E_y$$

of fibers which glue to bundle isomorphisms. Commutativity of triangles

$$\mathrm{tra}\left(\begin{array}{ccccc} & & g_1 x & & \\ & {\scriptstyle g_1}\nearrow & & \searrow{\scriptstyle g_2} & \\ x & & \xrightarrow[g_2 g_1]{} & & g_2 g_1 x \end{array} \right) = \begin{array}{ccccc} & & E_{g_1 x} & & \\ & {\scriptstyle \mathrm{tra}(g_1)}\nearrow & & \searrow{\scriptstyle \mathrm{tra}(g_2)} & \\ E_x & & \xrightarrow[\mathrm{tra}(g_2 g_1)]{} & & E_{\mathrm{tra}(g_2 g_1 x)} \end{array}$$

encodes the equivariance of the bundle on X.

Applying such a functor to paths in $\mathrm{Obj}(G)$ encodes the parallel transport of the connection. Applying the functor to the mixed commuting diagrams (1) yields naturality squares

$$\mathrm{tra}\left(\begin{array}{ccc} \gamma(0) & \xrightarrow{\gamma} & \gamma(s) \\ {\scriptstyle g_\gamma(0)}\downarrow & & \downarrow{\scriptstyle g_\gamma(s)} \\ \gamma_g(0) & \xrightarrow[\gamma_g]{} & \gamma_g(s) \end{array}\right) = \begin{array}{ccc} E_{\gamma(0)} & \xrightarrow{\mathrm{tra}(\gamma)} & E_{\mathrm{tra}(\gamma(s))} \\ {\scriptstyle \mathrm{tra}(g_\gamma(0))}\downarrow & & \downarrow{\scriptstyle \mathrm{tra} g_\gamma(s)} \\ E_{\gamma_g(0)} & \xrightarrow[\mathrm{tra}(\gamma_g)]{} & E_{\mathrm{tra}(\gamma_g(s))} \end{array}$$

encoding natural transformations between transport along paths related by morphisms in G. This yields the equivariance condition on the connection of the bundle. For more details see section 2.1 of [8]. $\square$

Example 1

For any locally trivializable bundle we can choose a good cover of the base space, pull the bundle back to that good cover and trivialize it there. The good cover can be regarded as a groupoid, the **Čech groupoid** and the cocycle conditions on the bundle are equivalent to the trivialized bundle being equivariant with respect to this groupoid.

This has been discussed in full detail in section 12.1 of [8]. There the path category of the Čech groupoid had been called the **Čech-extended path groupoid**.

3 2-Bundles with 2-Connection over Orbifolds

Definition 6 *For the purposes of the present discussion, a* **2-path** *in a smooth space M is always a 2-path with* **sitting instant** *along its boundary. This means that a 2-path in M is a smooth map*

$$\gamma : I_1 \times I_2 \to M$$

from $I_1 \times I_2 = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}$ to M, such that all derivatives of γ vanish in some neighborhood of $\partial(I_1 \times I_2)$. In addition, all our 2-paths have to be constant along their 'vertical' boundary.

Definition 7 *Let M be any smooth space.*

1. *Denote by $\mathrm{Moore}_2(M)$ the smooth 2-category whose*
   (a) *space of objects is M*
   (b) *space of 1-morphisms is that of Moore paths in M*

(c) *space of 2-morphisms is that of Moore 2-paths in* M

with the obvious composition of 2-morphisms by horizontal and vertical concatenation of 2-paths.

2. *Denote by* $\mathcal{P}_1(M)$ *the smooth 2-groupoid whose*

   (a) *space of objects is* M

   (b) *space of 1-morphisms is that of* thin *homotopy classes of paths in* M

   (c) *space of 2-morphisms is that of* thin *homotopy classes of 2-paths in* M

   with the obvious composition of 2-morphisms by horizontal and vertical concatenation of 2-paths.

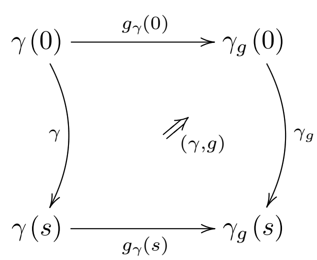
Definition 8 *Let* S *be an orbifold-like category.*

1. *Denote by* $\mathrm{Moore}_2(S)$ *the smooth 2-category whose*

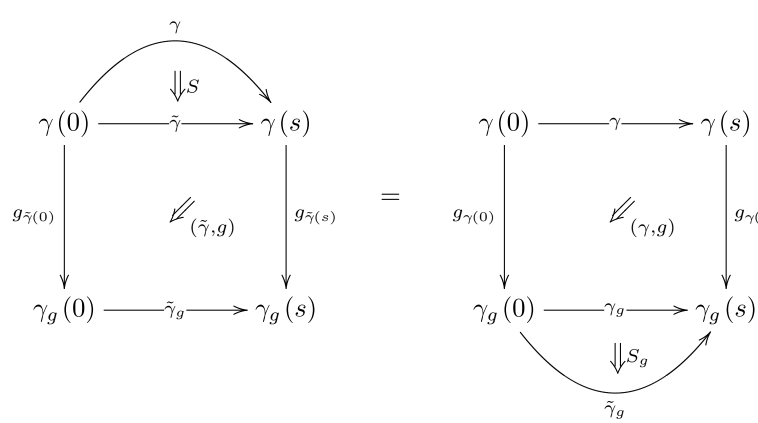
   (a) *space of objects is that of* S

   (b) *space of morphisms is that* generated *by formal compositions of morphisms in* $\mathrm{Moore}(\mathrm{Obj}(S))$ *with morphisms in* S,

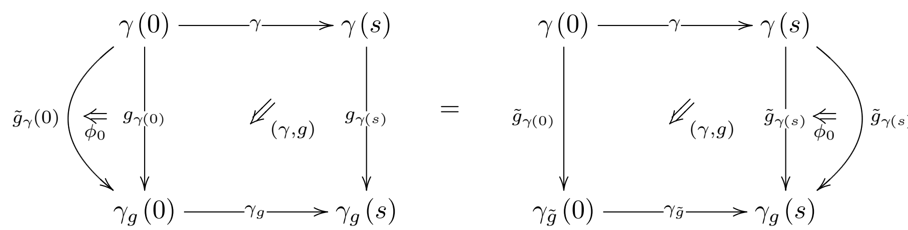
   (c) *space of 2-morphisms is that* generated *by formal compositions of 2-morphisms in* $\mathrm{Moore}(\mathrm{Obj}(S))$ *with 2-morphisms in* S *as well as unique 2-isomorphisms filling all squares*



subject to the relations



and



whenever the 2-morphisms in these equations exist.

2. *Denote by* $\mathcal{P}_1(S)$ *the smooth groupoid obtained similarly from generators and relations, but with Moore paths in* $\mathrm{Obj}(S)$ *replaced by thin homotopy classes of paths.*

Proposition 2 *If set up correctly, a 2-bundle with 2-connection on an orbifold* X *represented by a 2-groupoid* G *is a 2-functor*

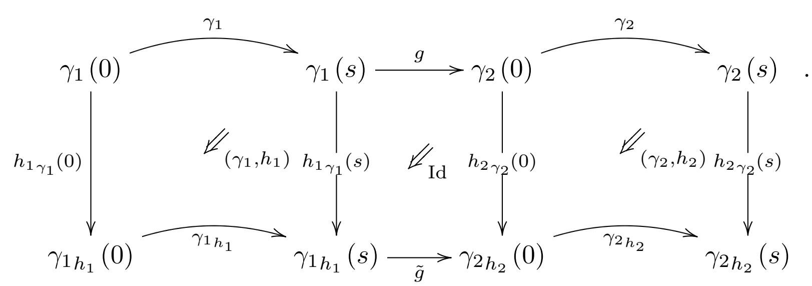
$$\mathrm{tra} : \mathcal{P}_2(G) \to C\,.$$

Proof. The idea is a generalization of that of prop. 1. Applying the transport functor to the relations stated in def 8 yields the tin can equations for the pseudonatural transformations and their modification of the 2-connection tranport 2-functor on paths in G. (For a review of pseudonatural transformations see the appendix of [6].) For more details see section 12.2 of [8]. □

Remark. A typical path in $\mathcal{P}_2(G)$ is an alternating concatenation of paths γ_i in $\mathcal{P}_1(\mathrm{Obj}(G))$ and morphisms in $g \in \mathrm{Mor}_1(G)$:

$$\gamma_1(0) \xrightarrow{\gamma_1} \gamma_1(s) \xrightarrow{g} \gamma_2(0) \xrightarrow{\gamma_2} \gamma_2(s)\;.$$

A typical 2-morphism between such paths is a piecewise orbifold shift



For paths being closed loops, this reproduces the morphisms of the **loop groupoid** discussed in section 3 of [2].

Example 2

Again, a simple example is that of a locally trivialized 2-bundle over a good cover of some base space, regarded as a **Čech-2-groupoid**. This is worked out in detail in section 12.2 of [8].

Proposition 3 *Applying this formalism to transport 2-functors with values in the suspension of* **Vect** *yields equivariant bundle gerbes classified by discrete torsion. Computing surface holonomy of these objects over closed surfaces in* $\mathcal{P}_2(G)$ *yields the known twisted sector phase shifts.*

Proof. Top secret. :-) □

References

[1] I. Moerdijk, Orbifolds as Groupoids: an Introduction, available as `math.DG/0203100`

[2] E. Lupercio, B. Uribe, Loop Groupoids, Gerbes, and Twisted Sectors on Orbifolds, available as `math.AT/0110207`

[3] E. Lupercio, B. Uribe, A. Xicoténcatl, Orbifold String Topology, available as `math.AT/0512658`

[4] J. Baez, U. Schreiber, Higher Gauge Theory, contribution to Streetfest proceedings (2005), available as `math.DG/0511710`.

[5] U. Schreiber, FRS Formalism from 2-Transport, private notes, available at `http://www.math.uni-hamburg.de/home/schreiber/FRSfrom2Transport.pdf`

[6] U. Schreiber, Line-2-Bundles and Bundle Gerbes, draft available at `http://www.math.uni-hamburg.de/home/schreiber/Line2Bundle.pdf`

[7] U. Schreiber, Nonabelian Bundle Gerbes from 2-Transport, Part I: Synthetic Bibundles, draft available at `http://www.math.uni-hamburg.de/home/schreiber/NABGfrom2Trans.pdf`

[8] U. Schreiber, From Loop Space Mechanics to Nonabelian Strings, PhD thesis, available as `hep-th/0509163`