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THE GEOMETRY OF DEGREE-4 CHARACTERISTIC CLASSES AND OF LINE BUNDLES ON LOOP SPACES II

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1. Introduction. In this paper, we continue the study of degree-4 characteristic classes begun in Part I [9]. The underlying theme is to find sheaf-theoretic objects which represent classes $\alpha \in H^4(BG; \mathbb{Z})$ and to explore their geometry. Such an object is a "sheaf of bicategories" and is an example of what L. Breen has called a 2-gerbe [5], [6]. The situation we are dealing with is entirely analogous to that which exists between the first Chern class of a line bundle and the differential geometry of the line bundle itself. We will assume throughout that the reader is familiar with Part I, where this program was carried out in the case of a compact 1-connected Lie group G and its complexification $G_{\mathbb{C}}$.

We begin in Section 2 with the case of the circle S^1 and its complexification $\mathbb{C}^*$. Here we find an explicit 2-gerbe (together with a "notion of connectivity") representing the square of the universal first Chern class c_1^2 (Theorem 2.4 and Remark 2.5). This is done by generalizing Deligne's observation that the construction of a holomorphic line bundle-with-connection from two invertible holomorphic functions can be interpreted geometrically as a cup product [2], [14].

In Section 3, we consider the the natural transgression $\tau: H^4(B\mathbb{C}^*) \to H^3(LB\mathbb{C}^*)$ to the free loop space. This corresponds geometrically to taking the holonomy of the 2-gerbe associated to c_1^2 around a loop. From the Segal-Witten reciprocity law (Theorem 5.9 of Part I) specialized to the case of $\mathbb{C}^*$, we know that τ singles out those extensions of $L\mathbb{C}^*$ by $\mathbb{C}^*$ which have the reciprocity property; these are the extensions that split canonically over loops which extend holomorphically to the interior of any Riemann surface. The main point of Section 3 is to prove that this reciprocity law implies the classical reciprocity theorem of Weil; let f, g be any two meromorphic functions on a Riemann surface with disjoint zeroes and poles, then [14], [26]

$$\prod_p f(p)^{\operatorname{ord} g(p)} = \prod_p g(p)^{\operatorname{ord} f(p)}.$$

While this was certainly known to Witten [42] and Segal [36], we feel that our approach using gerbes is the right framework to understand this phenomenon. Indeed the reciprocity law is exactly what is needed to fill in a 2-arrow between two given 1-arrows in a 2-gerbe.

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For the sake of completeness, we also construct the 2-gerbe associated to a class in $H^4(BG;\mathbb{Z}) \cong H^3(BG;\mathbb{C}^*)$, when G is a finite group (Theorem 4.1). This is done in Section 4 and is just an adaption of the Eilenberg-Mac Lane interpretation of degree-3 group cocycles as equivalence classes of "kernels" [21]. We note that the transgression τ above also makes sense here and leads to a version of the reciprocity law (Theorem 4.3). This is certainly implicit in the paper of Dijkgraaf-Witten [18] as was pointed out by Segal [36]. We use it to explicitly construct the fusion algebra and to derive the Verlinde formula in this context (Theorem 4.5).

Given a holomorphic vector bundle $E \to X$ with trivial determinant, we constructed in Part I a characteristic class $\hat{c}_2$ which lives in the Deligne cohomology group $H^3(X; \underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1)$. This coincides (up to torsion) with Beilinson's second Chern class [2], when X is compact and projective. If $E \to X$ is equipped with a Hermitian structure, then there is a canonical connection ∇ compatible with both the holomorphic structure and the Hermitian structure. We then have the differential character $\tilde{c}_2^{\nabla}$ of Cheeger-Simons [13]. Clearly there must be some compatibility between $\hat{c}_2$ and $\tilde{c}_2^{\nabla}$. This is expressed in Theorem 5.5, where it is shown that both classes combine to form an "enriched" Chern class in what we have called Hermitian holomorphic Deligne cohomology. The proof is by a direct comparison between the explicit Čech cocycles representing $\hat{c}_2$ and $\tilde{c}_2^{\nabla}$ constructed by us in Part I and in [10].

In Section 6, we consider a proper holomorphic fibration $f: X \to Y$ whose fibers are connected Riemann surfaces of genus g. If $E \to X$ is a Hermitian holomorphic vector bundle, then we show how the compatibility between $\hat{c}_2$ and $\tilde{c}_2^{\nabla}$ can be "pushed forward" along the fibers of f to produce a metrized line bundle on Y. This can be done either purely cohomologically as a transgression in Hermitian holomorphic Deligne cohomology (Corollary 6.2) or geometrically using 2-gerbes (Proposition 6.4).

Finally in Section 7, we apply these ideas to construct the Quillen metric on (the rth power of) the determinant line bundle over the moduli space $\mathcal{M}(r, \mathcal{L})$ of stable bundles of rank r and fixed determinant $\mathcal{L}$ on a Riemann surface Σ, in the case where $(r, \deg \mathcal{L}) = 1$. The idea is to interpret the Narasimhan-Seshadri Theorem [32] as defining a Hermitian structure on the universal family over $\mathcal{M}(r, \mathcal{L}) \times \Sigma$ and to push forward along Σ. This bypasses the usual construction based on the analysis of the Laplace operator [34]. The possibility of such an algebro-geometric approach to this metric was first raised by Deligne [16]. From our point of view, the transcendental nature of the metric is already encoded in the Narasimhan-Seshadri Theorem. It should also be emphasized that we obtain more than just a cohomological construction of this metrized line bundle. The geometric approach using 2-gerbes actually leads to a description of the sections themselves which should be relevant to the problem of explicitly constructing noncommutative theta functions.

Elsewhere [11] we have applied these techniques to handle the (considerably more difficult) case of constructing a metric on the noncompact moduli space of stable bundles of degree 0, which extends to the compactification by semistable bundles. This metric has the same curvature as the Quillen metric, but we do not know whether they coincide exactly. The situation in the present paper is quite different as the compact moduli space $\mathcal{M}(r,d)$ is simply-connected so the metric we construct must agree on the nose with Quillen's metric.

This paper owes much to the ideas presented in [15] and [36] and to the paper by Witten [43]. We thank P. Deligne and Shouwu Zhang for helpful conversations.

2. Geometric interpretation of some cup products. Let X be a complex manifold. The Deligne cohomology $H^p(X;\mathbb{Z}(q)_{\mathscr{D}})$ is by definition, the hypercohomology of the truncated complex of sheaves

$$\mathbb{Z}(q)_{\mathscr{D}} := (2\pi i)^q\mathbb{Z} \to \underline{\mathcal{O}}_X \to \underline{\Omega}^1_X \to \cdots \to \underline{\Omega}^{q-1}_X,$$

where $\underline{\mathcal{O}}_X$ denotes the sheaf of holomorphic functions and $\underline{\Omega}^p_X$ denotes the sheaf of holomorphic p-forms. We have a cup product $\mathbb{Z}(p)_{\mathscr{D}} \otimes \mathbb{Z}(q)_{\mathscr{D}} \to \mathbb{Z}(p+q)_{\mathscr{D}}$ given by [2], [22];

$$x \cup y = \begin{cases} x \cdot y & \deg x = 0, \\ x \wedge dy & \deg x \geqslant 0, \quad \deg y = q, \\ 0 & \text{otherwise.} \end{cases}$$

This is a refinement of the usual cup product $\mathbb{Z}(p) \otimes \mathbb{Z}(q) \to \mathbb{Z}(p+q)$, where $\mathbb{Z}(p) = (2\pi i)^p\mathbb{Z}$. The map $\mathcal{O}_X \to \mathcal{O}_X^*$ defined by $x \mapsto \exp(x/(2\pi i)^{q-1})$ induces a quasi isomorphism

$$\mathbb{Z}(q)_{\mathscr{D}} \cong (\underline{\mathcal{O}}_X^* \to \underline{\Omega}^1_X \to \cdots \to \underline{\Omega}^{q-1}_X)[-1],$$

where $K^\bullet[-1]$ denotes the complex $K^\bullet$ shifted by 1 to the right. Therefore, $H^1(X;\mathbb{Z}(1)_{\mathscr{D}}) \cong H^0(X;\underline{\mathcal{O}}_X^*)$—the global, invertible holomorphic functions, and $H^2(X;\mathbb{Z}(2)_{\mathscr{D}}) \cong H^1(X;\underline{\mathcal{O}}_X^* \to \underline{\Omega}^1_X)$—the group of isomorphism classes of holomorphic line bundles with connection [2], [30], [14].

PROPOSITION 2.1 [2], [14], [22]. *Associated to any two invertible holomorphic functions f and g, there is a well-defined holomorphic line bundle with connection, denoted $(f, g]$. This is constructed using the cup product in Deligne cohomology. The isomorphism class of $(f, g]$ in $H^1(X;\underline{\mathcal{O}}_X^* \to \underline{\Omega}^1_X)$ is represented by the Čech cocycle*

$$(g^{m_{\alpha\beta}}, (2\pi i)^{-1} \log_\alpha f \; d \log g),$$

where $\log_\alpha f$ *is a branch of the logarithm of* f *over* U_α *and* $m_{\alpha\beta} := (2\pi i)^{-1}\{\log_\beta f - \log_\alpha f\}$.

A more intrinsic description of this line bundle is given in [14] (see also [8]); every choice of branch $\log f$ of the logarithm of f induces a holomorphic section $\{\log f, g\}$ of $(f, g]$. The effect of choosing a different branch is given by the relation $\{\log f + 2\pi i n, g\} = g^n \cdot \{\log f, g\}$. The connection ∇ is defined by

$$\nabla\{\log f, g\} = -(2\pi i)^{-1} \log f \cdot g^{-1}\, dg \otimes \{\log f, g\}.$$

The curvature of this connection is the 2-form $-(2\pi i)^{-1} d \log f \wedge d \log g$. Over U_α, we have the section $\{\log_\alpha f, g\}$ of $(f, g]$. The corresponding transition cocycles are $\{g^{m_{\alpha\beta}}\}$. There is also a natural isomorphism $(f, g] \otimes (f, g'] \to (f, g \cdot g']$ of holomorphic line bundles-with-connection, which is defined by mapping a local section $\{\log f, g\} \otimes \{\log f, g'\}$ to $\{\log f, g \cdot g'\}$.

Our purpose here is to generalize this discussion and find geometric interpretations for the cup product in Deligne cohomology in other low degrees. First recall the definition of an $\underline{\mathcal{O}}^*$-gerbe $\mathcal{C}$ from Part I. Such an object is a "sheaf of groupoids" with the property that there is a given identification between the automorphisms of any local object over an open set U and $\underline{\mathcal{O}}^*(U)$. This identification must respect restriction to smaller open sets. A connective structure for $\mathcal{C}$ consists of the following data:

(a) the assignment to each object P of $\mathcal{C}$ over an open set U, of an $\underline{\Omega}^1_U$-torsor $\mathrm{Co}(P)$;

(b) the assignment to each isomorphism $\phi: P \to P'$ in $\mathcal{C}_U$, of an isomorphism $\phi_*: \mathrm{Co}(P) \to \mathrm{Co}(P')$ of $\underline{\Omega}^1_U$-torsors.

These assignments must be compatible with restriction to smaller open sets and satisfy the following property; the effect of an automorphism of a local object P induced by the invertible holomorphic function g, is to translate $\mathrm{Co}(P)$ by $-g^{-1}dg$. It was shown in Theorem 3.5 of Part I that the equivalence classes of $\underline{\mathcal{O}}^*$-gerbes with connective structure are classified by $H^3(X; \mathbb{Z}(2)_{\mathcal{D}}) \cong H^2(X; \underline{\mathcal{O}}^*_X \to \underline{\Omega}^1_X)$. As an immediate application of the cup product in Deligne cohomology, we have the following proposition.

PROPOSITION 2.2. *Suppose that* f *is an invertible holomorphic function and that* L *is a holomorphic line bundle on* X. *Then there is a well-defined* $\underline{\mathcal{O}}^*$*-gerbe with connective structure associated to* f *and* L, *denoted* $(f, L]$. *With respect to a trivializing open cover* $\{U_\alpha\}$ *of* L, $(f, L]$ *is represented by the following Čech cocycle with values in* $\underline{\mathcal{O}}^*_X \to \underline{\Omega}^1_X$:

$$(\xi_{\beta\gamma}^{m_{\alpha\beta}}, (2\pi i)^{-1} \log_\alpha f\, d \log \xi_{\alpha\beta}),$$

where the $\{\xi_{\alpha\beta}\}$ *are the transition functions for* L.

We need a more intrinsic description of the gerbe $(f, L]$. For each open set U in X, consider the category $\mathscr{C}_U$ defined as follows. The set of objects of $\mathscr{C}_U$ is the set of nonvanishing sections of L over U. The object associated to such a nonvanishing section s_U will be denoted by $(f, s_U]$. If s'_U is another nonvanishing section of L over U, then $s'_U = g \cdot s_U$, where g is a nonvanishing holomorphic function over U. We define $\mathrm{Hom}_{\mathscr{C}_U}((f, s_U], (f, s'_U])$ to be the set of nowhere vanishing sections of the line bundle $(f, g]$ over U. There is an action of $\underline{\mathcal{O}}^*(U)$ on this set of morphisms since $(f, g]$ is an $\underline{\mathcal{O}}^*(U)$-torsor.

The composition of morphisms

$$\mathrm{Hom}_{\mathscr{C}_U}((f, s'_U], (f, s''_U]) \circ \mathrm{Hom}_{\mathscr{C}_U}((f, s_U], (f, s'_U])$$

is defined to be the tensor product map $(f, s'_U \cdot s_U^{-1}] \otimes (f, s''_U \cdot s'^{-1}_U] \to (f, s''_U \cdot s_U^{-1}]$. The isomorphisms of any object $(f, s]$ over U are then identified with sections of the trivial $\underline{\mathcal{O}}^*$-torsor over U, i.e., invertible holomorphic functions on U. We have an obvious restriction functor on open sets and so we obtain a "presheaf of categories" as defined in Part I, Section 3. However, this is *not* a "sheaf of categories" as the glueing conditions for objects do not hold. Just as in the case of sheaves of abelian groups, one remedies this by *sheafifying* the presheaf $U \to \mathscr{C}_U$ [8], [7] to obtain a "sheaf of categories" which we will denote by $\underline{\mathscr{C}}'$. An object of $\underline{\mathscr{C}}'$ over U is then a collection P_i of objects of $\mathscr{C}_{V_i}$ for some open covering $\{V_i\}$ of U, together with morphisms ϕ_{ij} between the objects P_i and P_j over V_{ij}, satisfying the glueing condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ over V_{ijk}. Morphisms in $\underline{\mathscr{C}}'$ are defined similarly.

Objects of $\underline{\mathscr{C}}'$ certainly exist locally, since objects of the presheaf $U \to \mathscr{C}_U$ are given by local nonvanishing sections of L. Moreover any two objects are locally isomorphic because every $\underline{\mathcal{O}}^*$-torsor is locally trivial. Tensoring with the dual $\underline{\mathcal{O}}^*$-torsor shows that every morphism is invertible.

THEOREM 2.3. (1) *The "sheaf of categories" $\underline{\mathscr{C}}'$ is an $\underline{\mathcal{O}}^*$-gerbe.*

(2) *$\underline{\mathscr{C}}'$ is equipped with a connective structure defined as follows: to each local object of $\underline{\mathscr{C}}$, we associate the trivial Ω^1-torsor. For an isomorphism ϕ: $(f, s] \to (f, s']$ of local objects, we associate the isomorphism defined by the 1-form $\alpha_\phi := -\phi^{-1} \cdot \nabla\phi$, where ∇ is the connection on $(f, s' \cdot s^{-1}]$.*

(3) *The corresponding class of $\underline{\mathscr{C}}'$ in $H^2(X; \underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1)$ is represented by the Čech cocycle of Proposition 2.2.*

Proof. (2) follows easily from the fact that $\alpha_{\phi_1\phi_2} = \alpha_{\phi_1} + \alpha_{\phi_2}$. For the proof of (3), let U_α be an open covering of X, all of whose intersections are either contractible or empty. Over each U_α, choose a branch $\log_\alpha f$ of the logarithm of f and a nonvanishing section s_α of L. Let $\xi_{\alpha\beta} := s_\beta \cdot s_\alpha^{-1}$ denote the transition functions for L. Then $\{\log_\alpha f, \xi_{\alpha\beta}\}$ gives a nonvanishing section of $(f, \xi_{\alpha\beta}]$ over $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and hence an isomorphism $\phi_{\alpha\beta}$: $(f, s_\beta]_{U_{\alpha\beta}} \to (f, s_\alpha]_{U_{\alpha\beta}}$. The corresponding $\underline{\mathcal{O}}^*$-valued degree-2 cocycle $\lambda_{\alpha\beta\gamma}$ defined by $\underline{\mathscr{C}}'$ is then given by the composition $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} \circ \phi_{\gamma\alpha}$. This is the section

$$\{\log_\alpha f, \xi_{\alpha\beta}\} \otimes \{\log_\beta f, \xi_{\beta\gamma}\} \otimes \{\log_\alpha f, \xi_{\alpha\gamma}\}^{-1}$$

of the trivial torsor $(f, 1] \cong \underline{\mathcal{O}}^*$. This product reduces to $\xi_{\beta\gamma}^{m_{\alpha\beta}}$ as required. Finally, the $\underline{\Omega}^1$ component comes from the connection on the torsor $(f, \xi_{\alpha\beta}]$ over $U_\alpha \cap U_\beta$. But this is just $(2\pi i)^{-1} \log_\alpha f \, d \log \xi_{\alpha\beta}$. $\square$

Now consider two holomorphic line bundles L, L' with transition cocycles $\xi_{\alpha\beta}, \xi'_{\alpha\beta}$. Regarding these Čech cohomology classes as elements of $H^2(X; \mathbb{Z}(1)_{\mathcal{D}})$, and taking their cup product in Deligne cohomology, we obtain an element $(L, L']$ in $H^4(X; \mathbb{Z}(2)_{\mathcal{D}}) \cong H^3(X; \underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1)$. This is represented by the Čech cocycle

$$((\xi'_{\gamma\delta})^{n_{\alpha\beta\gamma}}, (2\pi i)^{-1} \log \xi_{\alpha\beta} \, d \log \xi'_{\beta\gamma}),$$

where $n_{\alpha\beta\gamma} := (2\pi i)^{-1}\{\log \xi_{\alpha\beta} - \log \xi_{\alpha\gamma} + \log \xi_{\beta\gamma}\}$.

Recall from Section 7, Part I, that $H^3(X; \underline{\mathcal{O}}_X^*)$ classifies equivalence classes of $\underline{\mathcal{O}}^*$-2-gerbes. Such an $\underline{\mathcal{O}}^*$-2-gerbe is a "sheaf of bicategories" on X, in which the 1-arrows between two local objects form a gerbe, and the 2-arrows between two 1-arrows form an $\underline{\mathcal{O}}^*$-torsor [5], [6]. We shall exhibit an explicit 2-gerbe representing $(L, L']$. For each open set U in X, let $\mathcal{B}_U$ be the bicategory defined as follows; objects of $\mathcal{B}_U$ correspond to nonvanishing sections s_U of L over U, and will be denoted by $(s_U, L']$. For two nonvanishing local sections s_1, s_2 of L over U, with $s_2 = g \cdot s_1$, we define $\mathrm{Hom}_{\mathcal{B}_U}((s_1, L'], (s_2, L'])$ to be the $\underline{\mathcal{O}}^*$-gerbe $(g, L']$ of Proposition 2.2.

To define the composition $\circ$ in $\mathcal{B}_U$, we need the notion of contracted product $\otimes$ of two $\underline{\mathcal{O}}^*$-gerbes $\mathscr{C}_1, \mathscr{C}_2$ [8], [25]. By definition, this is the $\underline{\mathcal{O}}^*$-gerbe obtained by sheafifying the presheaf whose objects are the same as those of the $\underline{\mathcal{O}}^* \times \underline{\mathcal{O}}^*$-gerbe $\mathscr{C}_1 \times \mathscr{C}_2$, but the morphisms are given by

$$\mathrm{Hom}((P_1, P_2), (P'_1, P'_2)) := \mathrm{Hom}_{\mathscr{C}_1}(P_1, P'_1) \otimes \mathrm{Hom}_{\mathscr{C}_2}(P_2, P'_2).$$

The product on the right is just the usual tensor product of $\underline{\mathcal{O}}^*$-torsors, and P_i, P'_i denote local objects of $\mathscr{C}_i$. This defines a group structure on the set of equivalence classes of $\underline{\mathcal{O}}^*$-gerbes over X, where the trivial gerbe of $\underline{\mathcal{O}}^*$-torsors is the identity element.

As a particular case of the contracted product of $\underline{\mathcal{O}}^*$-gerbes, we see that for any two nonvanishing holomorphic functions g_1, g_2, the contracted product $(g_1, L'] \otimes (g_2, L']$ identifies with $(g_1 \cdot g_2, L']$. This allows us to define the composition

$$\circ : \mathrm{Hom}((s_2, L'], (s_3, L']) \times \mathrm{Hom}((s_1, L'], (s_2, L']) \to \mathrm{Hom}((s_1, L'], (s_3, L'])$$

in $\mathscr{B}_U$ to be the equivalence of gerbes

$$(h, L'] \otimes (g, L'] \to (h \cdot g, L'],$$

where $s_2 = g \cdot s_1$ and $s_3 = h \cdot s_2$. Note that this composition is associative.

The upshot of this discussion so far is that the assignment $U \to \mathscr{B}_U$ defines a "presheaf of bicategories" in the sense of Section 7, Part I. However, we do not obtain a "sheaf of bicategories," as the necessary glueing condition for objects does not hold. As before, we remedy this by "sheafifying". A pleasant description of this process is given by Breen in Section 1.10 of [7]. We will denote the resulting sheaf of bicategories by $\underline{\mathscr{B}}'$. It is straightforward to verify that $\underline{\mathscr{B}}'$ is a 2-gerbe. This is similar to the proof of Theorem 7.2 of Part I and relies mainly on the fact that by construction, the 1-arrows form an $\underline{\mathcal{O}}^*$-gerbe. To give a concrete description of $\underline{\mathscr{B}}'$ over an open set U, we must first cover U by open sets V_i. An object of $\underline{\mathscr{B}}'(U)$ is then a collection A_i of objects of $\mathscr{B}_{V_i}$ together with a choice of 1-arrow f_{ij} between A_j and A_i over V_{ij} and a 2-arrow ϕ_{ijk} between the 1-arrows $f_{ij} \circ f_{jk}$ and f_{ik} over V_{ijk}, satisfying the glueing condition

$$\phi_{ijl} * (\phi_{jkl} \circ \mathrm{Id}) = \phi_{ikl} * (\mathrm{Id} \circ \phi_{ijk})$$

(see Section 7 of Part I). The 1-arrows and 2-arrows in $\underline{\mathscr{B}}'(U)$ are described in a similar way.

The 2-gerbe $\underline{\mathscr{B}}'$ carries a natural "concept of connectivity". By definition, this is an assignment of a connective structure to each gerbe of 1-arrows between any two objects of $\underline{\mathscr{B}}'$. This assignment must respect the composition in $\underline{\mathscr{B}}'$ and behave well under restriction to smaller open subsets. The "concept of connectivity" for $\underline{\mathscr{B}}'$ is described as follows. Let $(s_1, L'], (s_2, L']$ be two local objects of $\underline{\mathscr{B}}'$ corresponding to two local sections s_1, s_2 of L. We have $s_2 = g \cdot s_1$ for some invertible holomorphic function g. The gerbe of 1-arrows between these two local objects of $\underline{\mathscr{B}}'$ is by definition the $\underline{\mathcal{O}}^*$-gerbe $(g, L']$. The point is that this gerbe comes equipped with a connective structure by Theorem 2.3. It is then easy to see that this defines a "concept of connectivity" for $\underline{\mathscr{B}}'$. Recall from Part I, Theorem 8.7, that an $\underline{\mathcal{O}}^*$-2-gerbe equipped with a "concept of connectivity" determines a class in $H^3(X; \underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1)$.

Theorem 2.4. *The assignment $U \to \underline{\mathscr{B}}_U$, is a "presheaf of bicategories". The sheafification $\underline{\mathscr{B}}'$ defines an $\underline{\mathcal{O}}^*$-2-gerbe equipped with a "concept of connectivity" which represents the class of $(L, L']$ in $H^3(X; \underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1)$.*

Proof. We must calculate the cohomology class determined by $\underline{\mathscr{B}}'$ and show that it agrees with $(L, L']$. So let $\{U_\alpha\}$ be a covering of X, all of whose intersections are contractible or empty. Over each U_α, we choose some nonvanishing section s_α of L, so that we have an object $(s_\alpha, L']$ of $\underline{\mathscr{B}}'$ over U_α. The equation $s_\beta = \xi_{\alpha\beta} \cdot s_\alpha$ then defines the transition functions $\xi_{\alpha\beta}$ for L. Next, over $U_{\alpha\beta}$ we must choose a 1-arrow from $(s_\alpha, L']$ to $(s_\beta, L']$ in $\underline{\mathscr{B}}'$, which amounts to an object of the gerbe $(\xi_{\alpha\beta}, L']$. We will pick the 1-arrow corresponding to the object $(\xi_{\alpha\beta}, s'_\beta]$. As above, the equation $s'_\beta = \xi'_{\alpha\beta} \cdot s'_\alpha$ defines the transition functions for L'. We must now choose a 2-arrow $\phi_{\alpha\beta\gamma}$ from $(\xi_{\alpha\beta}, s'_\beta] \circ (\xi_{\beta\gamma}, s'_\gamma]$ to $(\xi_{\alpha\gamma}, s'_\gamma]$ over $U_{\alpha\beta\gamma}$, which amounts to a section of the $\underline{\mathcal{O}}^*$-torsor $(\xi_{\alpha\beta}, \xi'_{\beta\gamma}]$; first we choose a branch $\log \xi_{\alpha\beta}$ of the logarithm of $\xi_{\alpha\beta}$. Then we have the section $\phi_{\alpha\beta\gamma} := \{\log \xi_{\alpha\beta}, \xi'_{\beta\gamma}\}$ of $(\xi_{\alpha\beta}, \xi'_{\beta\gamma}]$. Over $U_{\alpha\beta\gamma\delta}$, the tensor product

$$(\xi_{\beta\gamma}, \xi'_{\gamma\delta}] \otimes (\xi_{\alpha\gamma}, \xi'_{\gamma\delta}]^{-1} \otimes (\xi_{\alpha\beta}, \xi'_{\beta\delta}] \otimes (\xi_{\alpha\beta}, \xi'_{\beta\gamma}]^{-1}$$

of $\underline{\mathcal{O}}^*$-torsors is isomorphic to

$$(\xi_{\alpha\beta}, \xi'_{\gamma\delta}]^{-1} \otimes (\xi_{\alpha\beta}, \xi'_{\gamma\delta}] \cong \underline{\mathcal{O}}_X^*$$

and has the nonvanishing section

$$\{\log \xi_{\beta\gamma}, \xi'_{\gamma\delta}\} \otimes \{\log \xi_{\alpha\gamma}, \xi'_{\gamma\delta}\}^{-1} \otimes \{\log \xi_{\alpha\beta}, \xi'_{\beta\delta}\} \otimes \{\log \xi_{\alpha\beta}, \xi'_{\beta\gamma}\}^{-1}.$$

This product reduces to the function $(\xi'_{\gamma\delta})^{n_{\alpha\beta\gamma}}$, which is then the $\underline{\mathcal{O}}^*$ Čech cocycle representing the cohomology class of $\mathscr{B}'$. This agrees with the $\underline{\mathcal{O}}^*$ component of $(L, L']$ computed above.

It remains to compute the $\underline{\Omega}^1$ component of $(L, L']$. Over $U_{\alpha\beta}$, we have the gerbe $(\xi_{\alpha\beta}, L']$ of 1-arrows from $(s_\alpha, L']$ to $(s_\beta, L']$. The connective structure on this gerbe assigns to the object $(\xi_{\alpha\beta}, s'_\beta]$ the trivial $\underline{\mathcal{O}}^*$-torsor. Over $U_{\alpha\beta\gamma}$, the 2-arrow $\phi_{\alpha\beta\gamma}$ above, is then given by the $\underline{\mathcal{O}}^*$-torsor *with connection* $(\xi_{\alpha\beta}, \xi'_{\beta\gamma}]$. By Proposition 2.1, the connection 1-form on this holomorphic line bundle corresponding to the section $\{\log \xi_{\alpha\beta}, \log \xi'_{\beta\gamma}\}$, is given by $(2\pi i)^{-1} \log \xi_{\alpha\beta}\, d \log \xi'_{\beta\gamma}$, and so $\mathscr{B}'$ and $(L, L']$ determine the same 3-cocycle in the complex $\underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1$. $\square$

Remark 2.5. Taking $L' = L$ in Theorem 2.4 gives the 2-gerbe representing $c_1(L)^2$ in $H^4(X; \mathbb{Z}(2)_{\mathscr{D}})$.

3. Weil reciprocity law. In this section, we will derive the classical Weil reciprocity law for Riemann surfaces [14], [26] as a special case of the Segal-Witten reciprocity law, which was proved in Part I, Section 5. Although this is well known [36], [42], our proof is a natural application of the theory of gerbes, and the description of the transgression map in Proposition 3.4 is of independent interest.

Let $X_\bullet$ be a simplicial complex manifold, which is X_p in degree p, with face maps $d_i\colon X_{p+1} \to X_p$. In Part I, Section 5, we defined the notion of a *simplicial* line bundle on $X_\bullet$. This means a line bundle L on X_1, together with a nonvanishing section s of $d_0^* L \otimes d_1^* L^{-1} \otimes d_2^* L$ over X_2, which satisfies the cocycle condition $\prod (-1)^i d_i^* s = 1$ on X_3. Equivalence classes of such objects are classified by the hypercohomology group $H^3(X_{\bullet \geq 1}; \mathbb{Z}(1)_{\mathscr{D}})$ (Theorem 5.7, Part I), where $X_{\bullet\geqslant 1}$ denotes the truncation of $X_\bullet$ in degrees $\geqslant 1$. Similarly, a *simplicial $\underline{\mathcal{O}}^*$-gerbe with connective structure on $X_\bullet$*, consists of an $\underline{\mathcal{O}}^*$-gerbe $\underline{\mathscr{C}}$ with connective structure on X_1, together with an equivalence of gerbes with connective structure $\phi: d_0^*\underline{\mathscr{C}} \otimes d_2^*\underline{\mathscr{C}} \to d_1^*\underline{\mathscr{C}}$ over X_2, and a natural transformation $\psi: d_0^*\phi \otimes d_2^*\phi \to d_1^*\phi \otimes d_3^*\phi$ between the two equivalences of gerbes over X_3, which satisfies $\prod(-1)^i d_i^*\psi = 1$ over X_4. We also require that ϕ should induce the identity isomorphism between the bands of the gerbe $d_0^*\underline{\mathscr{C}} \otimes d_2^*\underline{\mathscr{C}}$ and $d_1^*\underline{\mathscr{C}}$. These simplicial gerbes with connective structure are classified by the simplicial Deligne cohomology group $H^4(X_{\bullet\geqslant 1}; \mathbb{Z}(2)_{\mathscr{D}})$ (Theorem 5.7, Part I).

Now consider the case where $X_\bullet$ is the simplicial manifold $B\mathbb{C}^*_\bullet$, so that $X_p = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (p times) with the usual face maps [37]. Define a simplicial gerbe with connective structure on $B\mathbb{C}^*_\bullet$ as follows: over $\mathbb{C}^*$, we place the trivial gerbe $\underline{\mathscr{C}}$ of $\underline{\mathcal{O}}^*$-torsors with the trivial connective structure, which assigns to an $\underline{\mathcal{O}}^*$-torsor P, the $\underline{\Omega}^1$-torsor of connections on P. Over $\mathbb{C}^* \times \mathbb{C}^*$, we must then specify a trivialization of the trivial gerbe-with-connective-structure, i.e., an $\underline{\mathcal{O}}^*$-torsor with connection. This will be the holomorphic line bundle with connection $(u, v]$ of Proposition 2.1, where (u, v) are the coordinates on $\mathbb{C}^* \times \mathbb{C}^*$. The line bundle

$$(v, w] \otimes (uv, w]^{-1} \otimes (u, vw] \otimes (u, v]^{-1}$$

over $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ has the canonical nonvanishing section $\phi(u, v, w)$ given by

$$w^{-n} \cdot \{\log v, w\} \otimes \{\log uv, w\}^{-1} \otimes \{\log u, vw\} \otimes \{\log u, v\}^{-1}.$$

Here $\log u$, $\log v$, and $\log uv$ are arbitrary branches of logarithms and $\log u - \log uv + \log v := 2\pi i n$; the above section ϕ is well defined independently of the choices of these branches. It is then easy to see that ϕ satisfies the required coherency condition over $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$.

Proposition 3.1. *The cohomology class determined by this simplicial gerbe with connective structure, coincides with $c_1 \cup c_1$ in $H^3(B\mathbb{C}^*_{\geqslant 1}; \underline{\mathcal{O}}^* \to \underline{\Omega}^1) \cong H^4(B\mathbb{C}^*_{\geqslant 1}; \mathbb{Z}(2)_{\mathscr{D}})$, where c_1 is the universal first Chern class in $H^2(B\mathbb{C}^*_{\geqslant 1}; \mathbb{Z}(1)_{\mathscr{D}})$.*

Proof. It is enough to show that for any algebraic line bundle $L \to X$ over a complex manifold, the pullback of the class determined by this simplicial gerbe is $c_1(L) \cup c_1(L)$ in $H^4(X; \mathbb{Z}(2)_{\mathscr{D}})$. The choice of a trivializing open cover of $L \to X$ with transition cocycles $\xi_{\alpha\beta}$ defines a morphism of simplicial manifolds $NX_\bullet \to B\mathbb{C}^*_\bullet$, where $NX_\bullet$ is the nerve of the covering. Then the section ϕ above pulls back to

$$\xi_{\gamma\delta}^{(2\pi i)^{-1}}\{\log \xi_{\alpha\beta} - \log \xi_{\alpha\gamma} + \log \xi_{\beta\gamma}\}.$$

From Section 2, this is exactly the $\underline{\mathcal{O}}^*$ component of the cocycle representing $(L, L]$.

Similarly, the $\underline{\Omega}^1$ component is obtained by pulling back the connection form on the line bundle $(u, v]$. But this gives $(2\pi i)^{-1} \log \xi_{\alpha\beta}\, d \log \xi_{\beta\gamma}$ as required. $\square$

Let G be a compact Lie group with complexification $G_{\mathbb{C}}$ and let $LG_{\mathbb{C}}$ denote the loop group of smooth maps of S^1 to $G_{\mathbb{C}}$ [33]. An extension $\widetilde{LG_{\mathbb{C}}}$ by $\mathbb{C}^*$ is said to have the reciprocity property in the sense of Segal [38], if the following is true for any compact Riemann surface Σ, whose boundary $\partial\Sigma$ is the disjoint union of parametrized circles: the central extension $\widetilde{\mathrm{Map}(\partial\Sigma, G_{\mathbb{C}})}$ obtained by Baer multiplication of the extensions on each boundary component, is canonically split over the subgroup $\mathrm{Hol}(\Sigma, G_{\mathbb{C}})$ of holomorphic maps.

There is a natural transgression map $\tau\colon H^i(X) \to H^{i-1}(LX)$ to the free loop space. This is defined as the composition $\int_{S^1} \circ\, ev^*$, where $ev\colon LX \times S^1 \to X$ is the map which evaluates a loop at an angle and $\int_{S^1}$ is integration over the fiber S^1. As explained in Proposition 6.5.2 of [8], this can be extended to a transgression $\tau\colon H^i(X; \mathbb{Z}(k)_{\mathscr{D}}) \to H^{i-1}(LX; \mathbb{Z}(k-1)_{\mathscr{D}})$ in Deligne cohomology. As noted in Part I, τ can also be defined if X is replaced by any simplicial manifold $X_\bullet$.

In Section 5 of Part I, we interpreted $H^3((BLG_{\mathbb{C}})_{\geqslant 1}; \mathbb{Z}(1)_{\mathscr{D}})$ as the group of central extensions of $LG_{\mathbb{C}}$ by $\mathbb{C}^*$. As a special case of Theorem 5.9 of Part I, we get the following result.

THEOREM 3.2. *Those extensions of $L\mathbb{C}^*$ which lie in the image of the transgression $\tau\colon H^4(B\mathbb{C}^*_{\geqslant 1}; \mathbb{Z}(2)_{\mathscr{D}}) \to H^3(BL\mathbb{C}^*_{\geqslant 1}; \mathbb{Z}(1)_{\mathscr{D}})$, have the reciprocity property.*

Let us describe the group extension of $L\mathbb{C}^*$ obtained by transgressing the simplicial gerbe $\underline{\mathscr{C}}$ of Proposition 3.1. This is the simplicial line bundle on $BL\mathbb{C}^*$ which consists of the trivial line bundle on $L\mathbb{C}^*$ together with the nonvanishing section of the trivial bundle over $L\mathbb{C}^* \times L\mathbb{C}^*$ given by the function h, which to each pair of loops (γ, μ) in $\mathbb{C}^*$, assigns the holonomy of the line bundle with connection $(u, v]$ around the loop (γ, μ) in $\mathbb{C}^* \times \mathbb{C}^*$. Explicitly, this is given by

$$h(\gamma, \mu) := \exp\left((2\pi i)^{-1}\left\{-\int_{S^1} \log \gamma \, d \log \mu + \log \mu(0) \int_{S^1} d \log \gamma\right\}\right).$$

Then for a Riemann surface Σ whose boundary is a disjoint union of circles, we obtain a simplicial line bundle over $\mathrm{Map}(\partial\Sigma, \mathbb{C}^*)$. This is the data of the trivial line bundle over $\mathrm{Map}(\partial\Sigma, \mathbb{C}^*)$ together with the section over $\mathrm{Map}(\partial\Sigma, \mathbb{C}^*) \times \mathrm{Map}(\partial\Sigma, \mathbb{C}^*)$ induced by the product of each of the sections on the boundary components.

Now pull this construction back by the obvious map $\mathrm{Hol}(\Sigma, \mathbb{C}^*) \to \mathrm{Map}(\partial\Sigma, \mathbb{C}^*)$ to obtain a simplicial line bundle on $B\,\mathrm{Hol}(\Sigma, \mathbb{C}^*)_\bullet$. It follows from Stokes's Theorem that the part of this data over $\mathrm{Hol}(\Sigma, \mathbb{C}^*) \times \mathrm{Hol}(\Sigma, \mathbb{C}^*)$ is the invertible function

$$H(\phi, \psi) := \exp\left((2\pi i)^{-1} \int_\Sigma d \log \phi \wedge d \log \psi\right) = 1.$$

By Theorem 3.2, the simplicial line bundle has a canonical section s, i.e., a section s of the trivial line bundle over $\mathrm{Hol}(\Sigma, \mathbb{C}^*)$ satisfying

$$(d_0^* s \otimes d_1^* s^{-1} \otimes d_2^* s)_{(\phi,\psi)} = 1$$

over $\mathrm{Hol}(\Sigma, \mathbb{C}^*) \times \mathrm{Hol}(\Sigma, \mathbb{C}^*)$. This is equivalent to saying that s is a group homomorphism.

Next let S be a compact Riemann surface and ϕ, ψ two meromorphic functions with disjoint zeroes and poles. For any point p in S and γ a small loop encircling p, it is well known (e.g., [14]) that the holonomy of the line bundle with connection $(\phi, \psi]$ is equal to the Tate symbol

$$(\phi, \psi)_p := (-1)^{\mathrm{ord}(\phi)\cdot\mathrm{ord}(\psi)} \left\{ \frac{\phi^{\mathrm{ord}(\psi)}}{\psi^{\mathrm{ord}(\phi)}} \right\}_p .$$

This is 1, unless p is a zero or pole of ϕ or ψ. Since these points are distinct, we may choose disjoint small loops γ_i encircling them. Let Σ be the Riemann surface with boundary which is obtained from S by cutting out the open discs bounded by the γ_i. Then $\partial\Sigma$ is a disjoint union of circles and ϕ, ψ are in $\mathrm{Hol}(\Sigma, \mathbb{C}^*)$. Now consider the commutator $[s(\phi), s(\psi)]$, where s is the canonical trivializing section above. Since $L\mathbb{C}^*$ is abelian, this must equal 1. On the other hand, the commutator is given by $H(\phi, \psi) \cdot H(\psi, \phi)^{-1} = H(\phi, \psi)^2$. But

$$H(\phi, \psi) = \exp\left((2\pi i)^{-1} \left\{ -\int_{\partial\Sigma} \log \phi \, d \log \psi + \sum_i \log \psi_i(0) \int_{S^1} d \log \phi_i \right\} \right),$$

where ϕ_i, ψ_i denote the restrictions to the ith boundary circle. This is the product of the holonomies of $(\phi, \psi]$ around each γ_i and therefore equals the product of the Tate symbols taken over each of the zeroes and poles of f and g. We have shown the following.

Corollary 3.3 [14], [26], [42]. *Let f, g be two meromorphic functions on a compact Riemann surface S having disjoint zeroes and poles. Then*

$$\left(\prod_p (f, g)_p \right)^2 = 1.$$

This is the square of the classical Weil reciprocity law. The reason that we do not obtain the reciprocity law on the nose comes from the following proposition. To state it, we note that BS^1 is an H-space, so that LBS^1 is homeomorphic to the product $BS^1 \times \Omega BS^1 \cong BS^1 \times S^1$, where ΩBS^1 denotes the based loop space.

Therefore the cohomology groups $H^4(BS^1;\mathbb{Z})$ and $H^3(LBS^1;\mathbb{Z})$ are both isomorphic to $\mathbb{Z}$.

PROPOSITION 3.4 [36]. *The natural map*

$$\tau\colon H^4(BS^1;\mathbb{Z}) \to H^3(LBS^1;\mathbb{Z})$$

is multiplication by 2.

Proof. First note that LS^1 is homotopic to $S^1 \times \mathbb{Z}$, where the S^1 factor corresponds to the constant loops and the $\mathbb{Z}$ factor to the maps $\theta \mapsto n \cdot \theta$. Since $LBS^1 = BLS^1$ [31], the map ev restricts to a homotopically equivalent map $BS^1 \times B\mathbb{Z} \times S^1 \to BS^1$, which can be described as follows: let $\mu\colon B\mathbb{Z} \times S^1 \to BS^1$ be the natural map which on the level of simplicial sets is given by sending $(a_1, \dots, a_n, \theta)$ in $\mathbb{Z}^n \times S^1$ to the element $(a_1 \cdot \theta, \dots, a_n \cdot \theta)$ in $(S^1)^n$. Then $ev(x, y, \theta) = x + \mu(y, \theta)$, where $(x, y, \theta) \in BS^1 \times B\mathbb{Z} \times S^1$, and $+$ denotes the H-space composition in BS^1. If α, β, γ are the generators of $H^2(BS^1)$, $H^1(B\mathbb{Z})$, and $H^1(S^1)$ respectively, then $ev^*\alpha = \alpha + \mu^*\alpha = \alpha + \beta \wedge \gamma$. Therefore $ev^*\alpha^2 = \alpha^2 + 2 \cdot \alpha \wedge \beta \wedge \gamma$, and so $\tau(\alpha^2)$ is twice $\alpha \wedge \beta$. □

This proposition implies that the central extension $\widetilde{L\mathbb{C}^*}$ constructed by transgressing the simplicial gerbe corresponding to c_1^2, must have a square root. However, that square root is not unique, as the set of possible square roots correspond to the two spin structures on S^1 [38]. One way to remove the square in Corollary 3.3, is then to specify a spin structure on the surface Σ obtained from S by cutting out small discs around the supports of f and g. This leads to the notion of a topological spin theory [38], [18].

There is however a more direct way to obtain the Weil reciprocity law from Theorem 5.9 of Part I. One may consider the case of $\mathbb{C}^* \times \mathbb{C}^*$ rather than just $\mathbb{C}^*$ and those extensions of $L\mathbb{C}^* \times L\mathbb{C}^*$ which lie in the image of $\tau\colon H^4(B(\mathbb{C}^* \times \mathbb{C}^*)_{\geqslant 1}; \mathbb{Z}(2)_{\mathcal{D}}) \to H^3(B(L\mathbb{C}^* \times L\mathbb{C}^*)_{\geqslant 1}; \mathbb{Z}(1)_{\mathcal{D}})$ have the reciprocity property. Now consider the simplicial gerbe $\underline{\mathscr{C}}'$ on $B(\mathbb{C}^* \times \mathbb{C}^*)$ defined as follows. Over $\mathbb{C}^* \times \mathbb{C}^*$, one just has the trivial gerbe with trivial connective structure. The part of the data over each point (u_1, v_1, u_2, v_2) in $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ is the line bundle with connection $(u_1, v_2]$. It is easy to check that this is indeed a simplicial gerbe. The simplicial line bundle $\tau(\underline{\mathscr{C}}')$ over $B(L\mathbb{C}^* \times L\mathbb{C}^*)$ is then just the trivial line bundle on $L\mathbb{C}^* \times L\mathbb{C}^*$ together with the functional $(\gamma_1, \mu_1, \gamma_2, \mu_2) \to h(\gamma_1, \mu_2)$ over $L\mathbb{C}^* \times L\mathbb{C}^* \times L\mathbb{C}^* \times L\mathbb{C}^*$, where h is the holonomy functional defined above.

Now follows the proof of Corollary 3.3. The induced extension of $\mathrm{Hol}(\Sigma, \mathbb{C}^* \times \mathbb{C}^*)$ has a canonical section s' and one computes that the commutator $[s'(\phi, 1), s'(1, \psi)]$ is now the functional $H(\phi, \psi)$ above (rather than its square). We then deduce the following result.

COROLLARY 3.5 (Weil Reciprocity). *Let f, g be two meromorphic functions on* *a compact Riemann surface S having disjoint zeroes and poles. Then*

$$\prod_p (f, g)_p = 1.$$

Remark 3.6. In fact it was pointed out by Deligne [14] that Corollary 3.5 follows immediately once one knows that the Tate symbol can be interpreted as the holonomy of a line bundle-with-connection. We have given this roundabout derivation simply because its real meaning and subsequent generalization (Theorem 5.9, Part I) is best understood in the language of gerbes.

4. The case of a finite group. Throughout this section, $[\alpha]$ will represent a "characteristic class" in $H^4(BG; \mathbb{Z})$, for G a finite group. Since the cohomology of BG is all torsion and G is discrete,

$$H^4(BG; \mathbb{Z}) \cong H^3(BG; \mathbb{C}^*) \cong H^3_{\text{group}}(G; \mathbb{C}^*),$$

where the latter is group cohomology. We will construct a 2-gerbe representing $[\alpha]$. This was essentially done by Eilenberg and Mac Lane [21] in their original paper on group cohomology, but they used the language of kernels, which we now recall.

To each extension

$$1 \to K \to E \to G \to 1$$

of the group G by a nonabelian group K with center $\mathscr{Z}$, there is associated a group of operators $\theta \in \text{Hom}(G; \text{Out}(K))$. Here $\text{Out}(K)$ denotes the group of outer automorphisms. Conversely, given a group K with center $\mathscr{Z}$ and a homomorphism $\theta \colon G \to \text{Out}(K)$, one can ask whether there is a group extension which realizes this group of operators. The answer in general is no; for each element $g \in G$, choose an automorphism $\tilde{\theta}(g)$ of K mapping to $\theta(g)$ in $\text{Out}(K)$. Then the composition $m(g, h) := \tilde{\theta}(gh)^{-1} \circ \tilde{\theta}(g) \circ \tilde{\theta}(h)$ is an inner automorphism of K. For each g, h, we choose an element $\phi(g, h)$ of K inducing the inner automorphism $m(g, h)$. By associativity, the two ways of computing the composition $\tilde{\theta}(g) \circ \tilde{\theta}(h) \circ \tilde{\theta}(l)$ must be equal. This implies that the inner automorphisms determined by the two elements $\phi(g, h \cdot l) \cdot \phi(h, l)$ and $\phi(g \cdot h, l) \cdot \tilde{\theta}(l)[\phi(g, h)]$ must in fact be equal. Therefore these two elements of K must differ by some element $\alpha(g, h, l)$ of the center $\mathscr{Z}$. If there exists an extension of G by K with a group of operators θ, the $m(g, h)$ would then be a factor system, and the associativity of the group law would force α to be the identity. Therefore α is the obstruction to constructing such an extension. It is easy to check that α is a group cocycle.

The pair (θ, K) is called a "kernel" and we have associated an element of $H^3(G; \mathscr{Z})$ to each such kernel. Eilenberg and Mac Lane showed that every element of $H^3(G; \mathscr{Z})$ arises in this way.

Fix a kernel (θ, K) representing α. For each $g \in G$, choose an automorphism $\tilde{\theta}(g)$ as above. To each element g in G, we may associate a category $\mathscr{C}_g$ defined as follows. The objects of $\mathscr{C}_g$ are sets S with two commuting actions of K, each of which is free and transitive, and such that for some $s \in S$ we have $k \cdot s = s \cdot \tilde{\theta}(g)(k)$. Morphisms in $\mathscr{C}_g$ are (K, K)-equivariant maps. It is clear that the automorphism group of any object identifies with $\mathscr{Z}$, and that any two objects are isomorphic. Therefore, the assignment $g \mapsto \mathscr{C}_g$ defines a $\mathscr{Z}$-gerbe $\mathscr{C}$ on G regarded as a discrete set of points. There is a composition functor $\mathscr{C}_g \circ \mathscr{C}_h \to \mathscr{C}_{gh}$, defined by the contracted product of (K, K) sets. Note that to define $\mathscr{C}_g$, it is not necessary to assume that θ is a homomorphism. This assumption is only needed to define the composition functor $\mathscr{C}_g \otimes \mathscr{C}_h \to \mathscr{C}_{gh}$, since the construction requires the inner automorphism $m(g, h)$.

For the remainder of this section, assume that $\mathscr{Z} = \mathbb{C}^*$.

THEOREM 4.1. *The assignment to each group element g of the category $\mathscr{C}_g$ defines a simplicial gerbe on BG. The corresponding class in $H^3(BG; \mathbb{C}^*) \cong H^3_{\mathrm{group}}(G; \mathbb{C}^*)$ is the cohomology class of α.*

Proof. Over G, we have the $\mathscr{Z}$-gerbe $\mathscr{C}$. The choice of $m(g, h)$ as above determines an equivalence of gerbes ϕ: $d_0^*\mathscr{C} \circ d_2^*\mathscr{C} \to d_1^*\mathscr{C}$ over $G \times G$. Then the elements $\alpha(g, h, l) \in \mathbb{C}^*$ determine the natural transformation (call it α) between the two equivalences $d_0^*\phi \otimes d_2^*\phi \to d_1^*\phi \otimes d_3^*\phi$ over $G \times G \times G$. The requirement that $\prod d_i^*\alpha = 1$ on $G \times G \times G \times G$ is automatically satisfied, since it is exactly the cocycle condition for α. This shows that $\mathscr{C}$ is a simplicial gerbe and that it has the required cohomology class.

We could also have regarded $\mathscr{C}$ as a groupoid with tensor product and used the method of Sinh [40] to determine the group cocycle. □

Remark 4.2. Since G is discrete, we can pull back this simplicial gerbe on BG to obtain a description of the 2-gerbe $\mathscr{B}$ corresponding to the characteristic class $[\alpha](P)$ for any principal G-bundle $P \to M$. To each open set U in M, we associate a bicategory $\mathscr{B}_U$ as follows. Objects of $\mathscr{B}_U$ are in 1-1 correspondence with sections s of P over U, and are denoted A_s. For another section s' with $s' = s \cdot g$, we set $\mathrm{Hom}(A_s, A_{s'})$ equal to the gerbe $\mathscr{C}_{g^{-1}}$ above. Composition is then given by the functor $\mathscr{C}_g \circ \mathscr{C}_h \to \mathscr{C}_{gh}$. If $\{U_i\}$ is a covering all of whose intersections are contractible or empty, the cocycle α on $G \times G \times G$ pulls back to a degree 3 Čech cocycle for this covering with coefficients in the constant sheaf $\mathbb{C}^*$. The value of this cocycle on U_{ijkl} is just α evaluated at the transition functions g_{ij}, g_{ik}, g_{il}. It is easy to check that $U \to \mathscr{B}_U$ is a $\mathbb{C}^*$ 2-gerbe representing the characteristic class $[\alpha](P)$.

In [12], we showed how one can construct all levels of the Dijkgraaf-Witten topological quantum field theory, directly from this 2-gerbe. This involved a reciprocity law for surfaces with boundary, but the presentation there was rather abstract. We will now make this more concrete and the relationship with [18], [17], and [23] will become more apparent.

For any space X, let $\mathscr{C}_X$ denote the following category. The objects of $\mathscr{C}_X$ are principal G-bundles on X. The morphisms are isomorphisms of G-bundles, i.e., gauge transformations. It is easy to see that the classifying space $B\mathscr{C}_X$ of this category is homotopy equivalent to $\mathrm{Map}(X, BG)$. The case $X = S^1$ is of special interest. Notice that if G is connected, then $\mathscr{C}_{S^1}$ has only one isomorphism class of objects, namely the trivial G-bundle over S^1. The morphisms correspond to elements of the loop group LG so that $B\mathscr{C}_{S^1} = BLG \cong LBG$. Therefore, if G is discrete, for instance finite, the category $\mathscr{C}_{S^1}$ is the correct analogue of the loop group. In this case, $\mathscr{C}_{S^1}$ can be described noncanonically as follows; choose representatives g_i of each conjugacy class of G. The isomorphism classes of objects of $\mathscr{C}_{S^1}$ correspond to conjugacy classes and the automorphism group of the object associated to g_i identifies (noncanonically) with the centralizer Z_{g_i} of g_i. Then

$$B\mathscr{C}_{S^1} \cong \coprod_i BZ_{g_i}$$

and so

$$H^2(LBG; \mathbb{C}^*) \cong \oplus_i H^2(BZ_{g_i}; \mathbb{C}^*) \cong \oplus_i H^2_{\mathrm{group}}(Z_{g_i}; \mathbb{C}^*).$$

For any space X, let us define a *central extension* of $\mathscr{C}_X$ by $\mathbb{C}^*$ to be a category $\tilde{\mathscr{C}}_X$ mapping to $\mathscr{C}_X$, such that the automorphism groups in $\tilde{\mathscr{C}}_X$ are central extensions by $\mathbb{C}^*$ of automorphism groups in $\mathscr{C}_X$. More precisely, for each object A in $\tilde{\mathscr{C}}_X$, there should be a group homomorphism $\mathbb{C}^* \to \mathrm{Aut}_{\tilde{\mathscr{C}}_X}(A)$ which satisfies the following; for any two objects A, B, the left and right actions of $\mathbb{C}^*$ on $\mathrm{Hom}_{\tilde{\mathscr{C}}_X}(A, B)$ should coincide. Furthermore, we require that $\mathrm{Hom}_{\tilde{\mathscr{C}}_X}(A, B)/\mathbb{C}^*$ be isomorphic to $\mathrm{Hom}_{\mathscr{C}_X}(F(A), F(B))$, where $F: \tilde{\mathscr{C}}_X \to \mathscr{C}_X$ is the canonical functor. Then for each object A, the group $\mathrm{Aut}_{\tilde{\mathscr{C}}_X}(A)$ is a central extension of $\mathrm{Aut}_{\mathscr{C}_X}(F(A))$ by $\mathbb{C}^*$.

Concretely, a central extension $\tilde{\mathscr{C}}_{S^1}$ of the category $\mathscr{C}_{S^1}$ can be described (noncanonically) by specifying an extension of each centralizer Z_{g_i} by $\mathbb{C}^*$. It follows from the above that these central extensions are classified by elements of $H^2(LBG; \mathbb{C}^*) \cong H^3(LBG; \mathbb{Z})$.

There is an obvious notion of product on this set of extensions of $\mathscr{C}_{S^1}$. A given extension is said to have the reciprocity property if the following is true for every compact oriented surface Σ, whose boundary $\partial\Sigma$ is a disjoint union of circles; the extension $\widetilde{\mathscr{C}_{\partial\Sigma}}$ of $\mathscr{C}_{\partial\Sigma}$ obtained by Baer multiplication of the extensions on each boundary component, splits when pulled back to the category $\mathscr{C}_\Sigma$. The following theorem was originally observed by G. Segal.

THEOREM 4.3 (Reciprocity). *Let G be a finite group. Those extensions of $\mathscr{C}_{S^1}$ by $\mathbb{C}^*$ which lie in the image of the natural map $\tau: H^4(BG; \mathbb{Z}) \to H^3(LBG; \mathbb{Z})$ have the reciprocity property.*

Recall from Section 3 that the map τ is the pullback by the evaluation $ev\colon LBG \times S^1 \to BG$ composed with integration over S^1, which we denote by $\int_{S^1} \circ\, ev^*$. Once representatives $g_1, \dots, g_n$ have been chosen for each conjugacy class, τ can be described as the map which sends a degree-3 cocycle α to the n-tuple $((\tau\alpha)_{g_1}, \dots, (\tau\alpha)_{g_n})$, where $(\tau\alpha)_{g_i}$ is the degree-2 cocycle on Z_{g_i} defined by

$$(\tau\alpha)_{g_i}(h,k) := \alpha(g_i,h,k)\alpha(h,g_i,k)^{-1}\alpha(h,k,g_i).$$

We will write $(\tau\alpha)_i$ for $(\tau\alpha)_{g_i}$. This map plays a key role in [18]. The following is easy to prove by repeatedly using the cocycle condition for α.

PROPOSITION 4.4. *Fix g_i and g_j. Define*

$$r(h) := \alpha(h,g_i,g_j), s(h) := \alpha(g_i,h,g_j), t(h) := \alpha(g_i,g_j,h).$$

If δ denotes the coboundary in group cohomology, then for h,k in $Z_{g_i} \cap Z_{g_j}$, we have

$$(\tau\alpha)_{g_i \cdot g_j}(h,k) = (\tau\alpha)_i(h,k) \cdot (\tau\alpha)_j(h,k) \cdot (\delta r)(h,k) \cdot (\delta s)^{-1}(h,k) \cdot (\delta t)(h,k),$$

i.e., $(\tau\alpha)_{g_i \cdot g_j}$ is cohomologous to $(\tau\alpha)_i \cdot (\tau\alpha)_j$ on $Z_{g_i} \cap Z_{g_j}$.

Proof of 4.3. Without loss of generality, we may assume that α is a *normalized* group cocycle, i.e., that $\alpha(g,h,k) = 1$, if one of the arguments is the identity. Note that this means that $(\tau\alpha)_i$ is also normalized. Fix a surface Σ whose boundary is a disjoint union of n circles. The extension of $\mathscr{C}_{\partial\Sigma}$ defined by τ can be described (noncanonically) by specifying for each n-tuple $g_{i_1}, \dots, g_{i_n}$ (determining an isomorphism class of G-bundle on $\partial\Sigma$), a group extension of $Z_{g_{i_1}} \times \cdots \times Z_{g_{i_n}}$. This will be the extension given by the cocycle $(\tau\alpha)_{i_1}(h_{i_1}, k_{i_1}) \cdot \cdots \cdot (\tau\alpha)_{i_n}(h_{i_n}, k_{i_n})$, where $h_{i_l}, k_{i_l} \in Z_{g_l}$ for each l. The category $\mathscr{C}_\Sigma$ can be described (noncanonically) as follows. The objects correspond to representatives of conjugacy classes of homomorphisms $\pi_1(\Sigma) \to G$, i.e., to $(2m+n)$-tuples $a_1, b_1, \dots, a_m, b_m, g_{i_1}, \dots, g_{i_n}$ satisfying

$$[a_1,b_1] \cdot \cdots \cdot [a_m,b_m] \cdot g_{i_1} \cdot \cdots \cdot g_{i_n} = 1.$$

Here m is the genus of Σ. The automorphism group is then the simultaneous centralizer of these $2m+n$ elements, and will be denoted by $Z_{a_1,b_1,\dots,a_m,b_m,g_{i_1},\dots,g_{i_n}}$. To describe the induced extension $\mathscr{C}_\Sigma$, it is enough to specify the extension of each such centralizer. This is given by the formula $\prod_l (\tau\alpha)_{i_l}(h,k)$, where h,k lie in $Z_{a_1,b_1,\dots,g_{i_n}}$. Now apply Proposition 4.4 and use the relation imposed by the fundamental group of Σ to see that this cocycle is cohomologous to $(\tau\alpha)_{\mathrm{Id}}(h,k)$. But this last cocycle is identically 1, since α is normalized. $\square$

We will now derive the Verlinde algebra associated to α. First recall from [38] how in the case of a compact group G, one can construct the fusion algebra starting from an extension $\widetilde{LG_{\mathbb{C}^*}}$ which has the reciprocity property. To every parametrized circle, associate the category of irreducible representations R_i of $\widetilde{LG_{\mathbb{C}}}$. Fix a pair of pants Σ. If one assigns representations R_i, R_j to the "incoming circles" and R_k to the "outgoing circle" of Σ, then $R_i \otimes R_j \otimes R_k^*$ is a projective representation of $\mathrm{Map}(\partial\Sigma, G_{\mathbb{C}})$. By restriction, one obtains a projective representation of $\mathrm{Hol}(\Sigma, G_{\mathbb{C}})$. But if the extension $\widetilde{LG_{\mathbb{C}}}$ has the reciprocity property, $R_i \otimes R_j \otimes R_k^*$ is an actual (not projective) representation of $\mathrm{Hol}(\Sigma, G_{\mathbb{C}})$. Let V_{ijk} denote the invariant part. It is proved in [38] that V_{ijk} is finite-dimensional. The fusion algebra is then defined by setting

$$R_i \cdot R_j = \sum_k n_{ijk} R_k,$$

where $n_{ijk} := \dim V_{ijk}$.

Now carry out the analogous procedure in case G is finite. Start with an extension $\widetilde{\mathscr{C}_{S^1}}$ which has the reciprocity property. To each parametrized circle, associate the set of (finite-dimensional) irreducible representations of $\widetilde{\mathscr{C}_{S^1}}$, i.e., all "irreducible" functors from $\widetilde{\mathscr{C}_{S^1}}$ to the category **Vec** of finite-dimensional vector spaces and linear transformations. If one chooses representatives $g_1, \ldots, g_n$ for the conjugacy classes of G, then such a functor is given (noncanonically) by specifying a projective representation of each centralizer Z_{g_i}. As before, we fix a pair of pants Σ. If one assigns the irreducible functors F_a, F_b to the "incoming circles" and F_c to the "outgoing circle" of Σ, the functor $F_a \otimes F_b \otimes F_c^*$ defines a representation of the category $\widetilde{\mathscr{C}_{\partial\Sigma}}$. The notion of tensor product and $*$ are the obvious ones. By composition, we obtain a functor from $\widetilde{\mathscr{C}_\Sigma}$ to **Vec**. But $\widetilde{\mathscr{C}_{S^1}}$ has the reciprocity property, so that we have in fact a genuine representation of $\mathscr{C}_\Sigma$, not just a projective one. Taking the invariant part as above, gives a finite-dimensional vector space V_{abc} of dimension n_{abc}. The fusion rule is then defined by the formula

$$F_a \cdot F_b := \sum_c n_{abc} F_c.$$

Our goal is to find a formula for $\dim V_{abc} = n_{abc}$ in the case where the extension $\widetilde{\mathscr{C}_{S^1}}$ satisfying the reciprocity property, is constructed from a 3-cocycle α by the transgression in Theorem 4.3.

The objects of the category $\mathscr{C}_\Sigma$ correspond to the set S of orbits of G acting by simultaneous conjugation on the set of triples g, h, k satisfying $ghk = 1$. The automorphism group of the object of $\mathscr{C}_\Sigma$ corresponding to a given orbit $[g, h, k]$ is (noncanonically) the centralizer $Z_g \cap Z_h$. The vector space V_{abc} then decomposes as a direct sum $\oplus_{s\in S} V_{abc,s}$, and we need only compute $\dim V_{abc,s} := n_{abc,s}$.

Fix an orbit s in S and suppose it is represented by the triple (g, h, k). The functor F_a associates to the circle labelled by g an irreducible representation R_g^a of the extension $\widetilde{Z_g}$ defined by the transgression of α. Denote its character by ρ_g^a.

The tensor product $R_g^a \otimes R_h^b \otimes R_k^{c^*}$ is then a projective representation of $Z_g \times Z_h \times Z_{k^{-1}}$. Restricting, we obtain an (a priori) projective representation of $Z_g \cap Z_h$. But in view of Theorem 4.3, it can be "rescaled" to give a genuine representation and $V_{abc,s}$ is then the invariant part. From Proposition 4.2, the effect of rescaling the action is to multiply the character $\rho_g^a(l) \cdot \rho_h^b(l) \cdot \overline{\rho_k^c(l)}$ by a factor

$$(\tau\alpha)_l(g,h) := \alpha(l,g,h) \cdot \alpha^{-1}(g,l,h) \cdot \alpha(g,h,l).$$

Schur orthogonality immediately gives the following result.

THEOREM 4.5 [18], [17]. *In the Verlinde algebra constructed from a group cocycle α on a finite group G, the fusion coefficients are (with the above notation) $n_{abc} = \sum_{s\in S} n_{abc,s}$, where*

$$n_{abc,s} = \frac{1}{|Z_g \cap Z_h|} \sum_{l \in Z_g \cap Z_h} \rho_g^a(l) \cdot \rho_h^b(l) \cdot \overline{\rho_{gh}^c(l)} \cdot (\tau\alpha)_l(g,h).$$

5. Characteristic classes of Hermitian holomorphic bundles. Given a holomorphic bundle with Hermitian structure over a complex manifold X, there are two kinds of classes one can define; the first is the Chern class $\hat{c}_2$ studied in the last two sections of Part I. This lives in $H^4(X; \mathbb{Z}(2)_D)$ and depends only on the holomorphic structure. The other class is the differential character of Chern-Cheeger-Simons $\tilde{c}_2^\nabla$ associated to the unique connection ∇ compatible with both the holomorphic and Hermitian structure [13]. Following [10] and [44], one can regard $\tilde{c}_2^\nabla$ as living in $H^3(X; \underline{\mathbb{T}}_X \to i\underline{A}_X^1 \to i\underline{A}_X^2 \to i\underline{A}_X^3)$, where $\underline{A}_X^p$ denotes the sheaf of smooth, real-valued p-forms. Explicit cocycles representing $\tilde{c}_2^\nabla$ and indeed all the differential characters were given in [10]. In this section, we will study the compatibility between $\hat{c}_2$ and $\tilde{c}_2^\nabla$.

To guide our discussion, we will begin with the first Chern class. So let $\mathscr{L} \to X$ be a holomorphic line bundle which admits a Hermitian structure. By definition, this is a smooth reduction of the structure group from $\mathbb{C}^*$ to $\mathbb{T}$—the complex numbers of norm 1. This is equivalent to specifying a Hermitian form h on $\mathscr{L}$. Clearly the set of isomorphism classes of holomorphic bundles with Hermitian structure form a group under $\otimes$. In keeping with [11], we will denote this group by $\mathrm{Pic}_{\mathrm{hh}}(X)$. Our first task is to describe $\mathrm{Pic}_{\mathrm{hh}}(X)$ cohomologically.

Let $\{U_i\}$ be a trivializing open covering of $\mathscr{L} \to X$. Let s_i be an invertible holomorphic section over U_i, and let t_i be a smooth section of $\mathscr{L}$ over U_i with $h(t_i) = 1$. We then have $t_i = \rho_i \cdot s_i$, for some smooth $\mathbb{C}^*$-valued function ρ_i on U_i. The equations $s_j = s_i \cdot g_{ij}, t_j = t_i \cdot u_{ij}$ define transition cocycles $\{g_{ij}\}$ and $\{u_{ij}\}$ for $\mathscr{L}$ and satisfy $\rho_j \cdot \rho_i^{-1} = g_{ij} u_{ij}^{-1}$. Therefore the triple $(g_{ij}^{-1}, u_{ij}; \rho_i)$ defines a Čech 1-cocycle with coefficients in the complex of sheaves $\underline{\mathcal{O}}_X^* \oplus \underline{\mathbb{T}}_X \xrightarrow{(\mathrm{incl},\mathrm{incl})} \underline{\mathbb{C}}_X^*$, where $(\mathrm{incl}, \mathrm{incl})$ denotes the natural inclusions. It is easy to see that the cohomology class of $(g_{ij}^{-1}, u_{ij}; \rho_i)$ is independent of all choices involved. Also it is clear that every 1-cocycle for this complex determines a unique holomorphic line bundle with Hermitian structure.

PROPOSITION 5.1. *The group* $\mathrm{Pic}_{\mathrm{hh}}(X)$ *is isomorphic to* $H^1(X; \underline{\mathcal{O}}_X^* \oplus \underline{\mathbb{T}}_X \to \underline{\mathbb{C}}_X^*)$.

This should be compared to the classification of holomorphic line bundles with connection by the group $H^2(X; \mathbb{Z}(2)_D)$ [2], [30]. For this reason, the complex $\underline{\mathcal{O}}_X^* \oplus \underline{\mathbb{T}}_X \to \underline{\mathbb{C}}_X^*$, with $\underline{\mathcal{O}}_X^* \oplus \underline{\mathbb{T}}_X$ placed in degree *one*, was called the hermitian holomorphic Deligne complex of order 1 in [11]. It will be denoted by $\mathbb{Z}(1)_{\mathrm{D.h.h}}$. We then have $H^1(X; \underline{\mathcal{O}}_X^* \oplus \underline{\mathbb{T}}_X \to \underline{\mathbb{C}}_X^*) \cong H^2(X; \mathbb{Z}(1)_{\mathrm{D.h.h}})$.

Next, we compute the curvature of a Hermitian holomorphic line bundle. This is well known and was done classically by Weil [41]. Nevertheless, it will be useful to recall the computation by purely cohomological methods given in [11].

PROPOSITION 5.2. *The curvature of a Hermitian holomorphic line bundle is given by* $K = \bar{\partial}\partial \log h(s_i)$, *where* h *is the Hermitian form and* s_i *is an invertible holomorphic section over* U_i.

Proof. Consider the double complex of sheaves

$$\begin{array}{ccc} \underline{A}_X^{1,0} \oplus i\underline{A}^1_X & \to & \underline{A}^1_{X;\mathbb{C}} \\ \uparrow & & \uparrow \\ \underline{\mathcal{O}}_X^* \oplus \underline{\mathbb{T}}_X & \to & \underline{\mathbb{C}}^*. \end{array}$$

Here the vertical arrows are $d\log$, $\underline{A}^p_{X;\mathbb{C}}$ is the sheaf of smooth complex p-forms, $\underline{A}_X^{1,0}$ is the sheaf of 1-forms of type $(1,0)$, and $i\underline{A}^1_X$ denotes the sheaf of purely imaginary 1-forms. The top row of this complex is acyclic, so the hypercohomology of the double complex is isomorphic to $H^*(X; \underline{\mathcal{O}}_X^* \oplus \underline{\mathbb{T}}_X \to \underline{\mathbb{C}}_X^*)$. Therefore a hermitian holomorphic line bundle determines a cohomology class in this double complex. To construct a cocycle representing this class, choose a trivializing open cover $\{U_i\}$ and let g_{ij}, u_{ij}, ρ_i be as above. The components of the representative cocycle in the bottom row will again be $(g_{ij}^{-1}, u_{ij}; \rho_i)$. The only contribution from the top row will be a degree-zero cocycle with values in $\underline{A}^{1,0}_{X;\mathbb{C}} \oplus i\underline{A}^1_X$ and the only possibility for this is $(-2\partial \log \rho_i, (\partial \log \rho_i - \bar{\partial} \log \rho_i))$.

There is a natural morphism from our double complex to the complex $\underline{\mathcal{O}}_X^* \to \underline{A}_X^{1,0}$, obtained by projecting to the first column. This induces a homomorphism $\mathrm{Pic}_{\mathrm{hh}} \to H^1(X; \underline{\mathcal{O}}_X^* \to \underline{A}_X^{1,0})$. This latter group is just the equivalence classes of holomorphic line bundles with a connection compatible with the holomorphic structure. This interpretation comes from the fact that any such connection is of type $(1,0)$.

On the level of cocycles, the above homomorphism maps $(g_{ij}^{-1}, u_{ij}; \rho_i, -2\partial \log \rho_i, \bar{\partial} \log \rho_i - \partial \log \rho_i)$ to $(g_{ij}^{-1}, -2\partial \log \rho_i)$. The latter corresponds to a holomorphic line bundle with transition functions g_{ij} which is equipped with a connection represented by the 1-form $-2\partial \log \rho_i$. Therefore the curvature is the exterior derivative of this 1-form, which is $-2\bar{\partial}\partial \log \rho_i$. Finally, note that one can choose $t_i = h(s_i)^{-1/2} \cdot s_i$, so that $\rho_i = h(s_i)^{-1/2}$. This finishes the proof. □

Let $\Lambda^2(X)$ denote the group of closed smooth forms ω on X of type $(1,1)$, such that $(2\pi i)^{-1}\omega$ has integral periods. The following is the analogue for Hermitian holomorphic line bundles of a well-known result of Weil and Kostant [41], [29].

PROPOSITION 5.3. *The following sequence is exact:*

$$0 \to H^1(X;\mathbb{T}) \to \mathrm{Pic}_{\mathrm{hh}}(X) \to \Lambda^2(X) \to 0,$$

where the map to Λ^2 is given by taking the curvature of the Hermitian holomorphic line bundle.

A straightforward proof is given in [11].

Remark 5.4. Since $H^1(X;\mathbb{T})$ classifies isomorphism classes of flat unitary line bundles, the map $H^1(X;\mathbb{T}) \to \mathrm{Pic}_{\mathrm{hh}}(X)$ of Proposition 5.3 just gives the Hermitian holomorphic bundle defined by a flat unitary bundle.

So far, we have been studying the first Chern class of a Hermitian holomorphic line bundle in two ways; from the holomorphic viewpoint, we obtain a class in $H^1(X;\underline{\mathcal{O}}_X^*)$, and from the smooth perspective, we obtain a class in $H^1(X;\underline{\mathbb{T}}_X)$. Not only are these classes compatible, but we have a "Hermitian holomorphic first Chern class" in $H^1(X;\underline{\mathcal{O}}_X^* \oplus \underline{\mathbb{T}}_X \to \underline{\mathbb{C}}_X^*)$ which induces both of them. We now carry out the analogous discussion for the second Chern class of a holomorphic bundle with Hermitian structure.

Let $p\colon P \to X$ be a holomorphic principal $SL_n(\mathbb{C})$-bundle which admits a Hermitian structure, i.e., a smooth reduction of the structure group to a principal $SU(n)$-bundle $q\colon Q \to X$. Choose a trivializing open cover $\{U_i\}$ and let s_i be a holomorphic section of $P \to X$ over U_i. The equations $s_j = s_i \cdot g_{ij}$ define the transition cocycles g_{ij}, which are holomorphic $SL_n(\mathbb{C})$-valued functions. In Section 9 of Part I, we constructed a class $\hat{c}_2 \in H^3(X;\underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1)$ for any holomorphic $G_{\mathbb{C}}$-bundle. This class refines the usual topological second Chern class and agrees (up to torsion) with the Deligne-Beilinson class in the case where X is compact and projective. Let us recall the construction of the explicit cocycle representing $\hat{c}_2$.

For each $x \in U_{ijkl}$, let $\tilde{\gamma}_{ij}(x)$ be a path from $s_i(x)$ to $s_j(x)$ in $p^{-1}(x) \cong SL_n(\mathbb{C})$ depending holomorphically on x. The composition $\tilde{\gamma}_{ij}(x) * \tilde{\gamma}_{jk}(x) * \tilde{\gamma}_{ik}^{-1}(x)$ is a loop and so bounds some 2-simplex $\tilde{\sigma}_{ijk}(x)$, which we may assume to depend holomorphically on x. The formal linear combination of 2-simplices $\tilde{\sigma}_{jkl}(x) - \tilde{\sigma}_{ikl}(x) + \tilde{\sigma}_{ijl}(x) - \tilde{\sigma}_{ijk}(x)$ is then a cycle. Therefore it bounds some 3-simplex $\tilde{T}_{ijkl}(x)$, which again can be chosen to depend holomorphically on x. Now let $\nu = k \cdot \mathrm{Tr}(g^{-1}dgg^{-1}dgg^{-1}dg)$ be the canonical bi-invariant 3-form on $SL_n(\mathbb{C})$. The value of the constant k is $1/(24\pi^2)$. Note that in Section 5 of Part I, k was incorrectly given as $1/(8\pi^2)$. Let β be the 2-form on $SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$ given by $3k \cdot \mathrm{Tr}(g_1^{-1}dg_1 dg_2 g_2^{-1})$. Take $v_i = s_i^* v$ and β_{ij} the 2-form defined in Lemma 8.1 of Part I; $\beta_{ij} := r_j^* F_{ij}^* \beta$, where $F_{ij}\colon U_{ij} \times SL_n(\mathbb{C}) \to SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$ is the map $F_{ij}(x,h) := (g_{ij}(x), h)$ and $r_i\colon p^{-1}(U_i) \to U_i \times SL_n(\mathbb{C})$ is the isomorphism $r_i(y) := (x, s_i(x)^{-1} \cdot y)$, for $x = p(y)$.

Define

$$h_{ijkl}(x) = \exp\left(2\pi i \int_{\tilde{T}_{ijkl}(x)} v_i\right) \cdot \exp\left(2\pi i \int_{\tilde{\sigma}_{jkl}(x)} \beta_{ij}\right)$$

$$\omega_{ijk}^1 = 2\pi i \int_{\tilde{\sigma}_{ijk} \to U_{ijk}} v_i$$

$$\omega_{ijk}^2 = -2\pi i \int_{\tilde{\gamma}_{jk} \to U_{jk}} \beta_{ij}.$$

Then $(h_{ijkl}, \omega_{ijk} := \omega_{ijk}^1 + \omega_{ijk}^2)$ is a Čech cocycle with coefficients in the complex $\underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1$ representing $\hat{c}_2$.

Now suppose that ∇ is a connection on the principal $SU(n)$-bundle $Q \to X$. Let us recall the explicit cocycle for the Cheeger-Chern-Simons class $\tilde{c}_2^{\nabla}$, which was constructed in [10]. Choose sections t_i of Q over each open set U_i. Then $t_j(x) = t_i(x) \cdot u_{ij}(x)$, for smooth $SU(n)$-valued functions u_{ij}. Let $\tilde{\Gamma}_{ij}(x), \tilde{\Sigma}_{ijk}(x)$, $\tilde{\mathcal{T}}_{ijkl}(x)$ be simplices in $q^{-1}(x)$ depending smoothly on x, which are constructed in a similar fashion to the simplices $\tilde{\gamma}_{ij}(x), \tilde{\sigma}_{ijk}(x), \tilde{T}_{ijkl}(x)$ above. Let $CS(\nabla)$ denote the Chern-Simons 3-form associated to ∇. This is the 3-form on Q given by $(1/(8\pi^2))\,\mathrm{Tr}(A \wedge dA + (2/3)A \wedge A \wedge A)$, where A is the connection 1-form.

Define

$$f_{ijkl}(x) = \exp\left(2\pi i \int_{\tilde{\mathcal{T}}_{ijkl}(x)} CS(\nabla)\right)$$

$$\eta_{ijk}^1 = 2\pi i \int_{\tilde{\Sigma}_{ijk} \to U_{ijk}} CS(\nabla)$$

$$\eta_{ij}^2 = -2\pi i \int_{\tilde{\Gamma}_{ij} \to U_{ij}} CS(\nabla)$$

$$\eta_i^3 = (2\pi i) \cdot s_i^* CS(\nabla).$$

Then $(f_{ijkl}, \eta_{ijk}^1, \eta_{ij}^2, \eta_i^3)$ is a degree-3 Čech cocycle for the complex $\underline{\mathbb{T}}_X \to i\underline{A}_X^1 \to i\underline{A}_X^2 \to i\underline{A}_X^3$ representing the differential character $\tilde{c}_2^{\nabla}$.

Suppose that ∇ is the unique connection $P \to X$ which is compatible with both the holomorphic and the Hermitian structure. It can be uniquely characterized as the connection on the subbundle $Q \to X$ whose connection form extends to a connection form of type $(1,0)$ on $P \to X$ [28]. We want to relate the classes $\hat{c}_2$ and $\tilde{c}_2^{\nabla}$. The basic idea, as outlined in [11], is to introduce the complex of sheaves $\underline{\mathbb{C}}_X^* \to \underline{A}^1_{X,\mathbb{C}} \to \underline{A}^2_{X,\mathbb{C}}/F^2_{\text{Hodge}} \to \underline{A}^3_{X,\mathbb{C}}/F^2_{\text{Hodge}}$, where F^2_{Hodge} denotes the Hodge filtration

$$F^2_{\text{Hodge}}\, \underline{A}^m_{X,\mathbb{C}} := \oplus_{p\geqslant 2}\, \underline{A}_X^{p,m-p}.$$

Forgetting the holomorphic structure gives a morphism of complexes ϕ:

$$\begin{array}{ccccccc}
\underline{\mathcal{O}}_X^* & \to & \underline{\Omega}_X^1 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\underline{\mathbb{C}}_X^* & \to & \underline{A}^1_{X,\mathbb{C}} & \to & \underline{A}^2_{X,\mathbb{C}}/F^2_{\text{Hodge}} & \to & \underline{A}^3_{X,\mathbb{C}}/F^2.
\end{array}$$

On the other hand, the purely imaginary forms sit inside the complex forms, giving a morphism ψ:

$$\begin{array}{ccccccc}
\underline{\mathbb{T}}_X & \to & i\underline{A}^1_X & \to & i\underline{A}^2_X & \to & i\underline{A}^3_X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\underline{\mathbb{C}}^* & \to & \underline{A}^1_{X,\mathbb{C}} & \to & \underline{A}^2_{X,\mathbb{C}}/F^2_{\text{Hodge}} & \to & \underline{A}^3_{X,\mathbb{C}}/F^2_{\text{Hodge}}.
\end{array}$$

Therefore, both $\tilde{c}_2^{\nabla}$ and $\hat{c}_2$ map to cohomology classes in the complex $\underline{\mathbb{C}}_X^* \to \underline{A}^1_{X,\mathbb{C}} \to \underline{A}^2_{X,\mathbb{C}}/F^2_{\text{Hodge}} \to \underline{A}^3_{X,\mathbb{C}}/F^2_{\text{Hodge}}$. We will show that they have the same image in the cohomology of the truncated complex $\underline{\mathbb{C}}_X^* \to \underline{A}^1_{X,\mathbb{C}} \to \underline{A}^2_{X,\mathbb{C}}/F^2_{\text{Hodge}}$.

THEOREM 5.5. *There exists a 2-cochain $\alpha = (\alpha^0_{ijk}, \alpha^1_{ij}, \alpha^2_i)$ with coefficients in the complex of sheaves $\underline{\mathbb{C}}^* \to \underline{A}^1_{X,\mathbb{C}} \to \underline{A}^2_{X,\mathbb{C}}/F^2_{\text{Hodge}}$ such that $\psi(f_{ijkl}, \eta^1_{ijk}, \eta^2_{ij}, \eta^3_i) - \phi(h_{ijkl}, \omega_{ijk})$ is the coboundary of α.*

The proof will depend on the following lemma.

LEMMA 5.6. *There exists a 2-form μ_i on $p^{-1}(U_i)$ with the following properties:*
(1) $v_i - CS(\nabla) = d\mu_i +$ basic.
(2) $\mu_j - \mu_i = \beta_{ij} +$ basic.

Proof. Let A be the connection 1-form for ∇. This is a globally defined 1-form on the total space of the bundle $p\colon P \to X$. There is also a canonical flat connection on $p^{-1}(U_i)$ given by the section s_i and we will denote its connection 1-form by A_i'. Then the 3-form v_i is the Chern-Simons 3-form $(1/(8\pi^2))\,\mathrm{Tr}(A_i' \wedge dA_i' + (2/3)A_i' \wedge A_i' \wedge A_i')$ associated to A_i'.

Let $A_i^t := tA + (1-t)A_i'$ be the segment of connections from $A_{|U_i}$ to A_i'. Then from [24], we have

$$\mathrm{Tr}(A_i' \wedge dA_i' + \tfrac{2}{3}A_i'^3) - \mathrm{Tr}(A \wedge dA + \tfrac{2}{3}A^3) = 2\, ch_2^1(A, A_i') + d\,\mathrm{Tr}(A \wedge A_i'),$$

where

$$ch_2^1 := \mathrm{Tr}\int_0^1 K_t^2\, dt$$

and K_t is the curvature of A_i^t. Notice that the integrand here is basic and therefore so is ch_2^1. We may then choose $\mu_i := (1/(8\pi^2))\,\mathrm{Tr}(A \wedge A_i')$. This proves (1).

It is easy to see that

$$A_j' - A_i' = g_i^{-1}(p^*[g_{ij}dg_{ij}^{-1}])g_i,$$

where $g_i(y) := s_i(p(y))^{-1} \cdot y$ is the unique element of $SL_n(\mathbf{C})$ satisfying $s_i(p(y)) \cdot g_i(y) = y$. Notice that $A_i' = g_i^{-1}dg_i$. Now

$$\begin{aligned} 8\pi^2(\mu_j - \mu_i) &= \mathrm{Tr}(A \wedge (A_j' - A_i')) \\ &= \mathrm{Tr}((A - A_i') \wedge (A_j' - A_i')) + \mathrm{Tr}(A_i' \wedge (A_j' - A_i')). \end{aligned}$$

The first term in the last line is a basic 2-form, since A, A_i', A_j' all restrict to the Maurer-Cartan form in each fiber. Using the transformation law for A_i', we then have (modulo basic 2-forms)

$$\begin{aligned} 8\pi^2(\mu_j - \mu_i) &\equiv -\mathrm{Tr}(g_i^{-1}g_{ij}dg_{ij}^{-1}g_i \wedge g_i^{-1}dg_i) \\ &\equiv \mathrm{Tr}(g_{ji}^{-1}dg_{ji} \wedge dg_i g_i^{-1}). \end{aligned}$$

The last term here is exactly the 2-form β_{ji}. But β is a Čech cocycle modulo basic forms, so that $\beta ji = -\beta_{ij} +$ basic. This finishes the proof of (2). □

Proof of Theorem 5.5. We regard $Q \to X$ as a subbundle of $P \to X$, so we can think of the smooth sections t_i as taking values in P. Let $(\hat{f}_{ijkl}, \hat{\eta}^1_{ijk}, \hat{\eta}^2_{ij}, \hat{\eta}^3_i)$ denote the cocycle constructed in the same way as $(f_{ijkl}, \eta^1_{ijk}, \eta^2_{ij}, \eta^3_i)$, but using the sections s_i instead of the sections t_i. It was shown in [10] that the cohomology class of $(f, \eta^1, \eta^2, \eta^3)$ in the complex of sheaves $\mathbf{C}_X^* \to \underline{A}^1_{X,\mathbf{C}} \to \underline{A}^2_{X,\mathbf{C}} \to \underline{A}^3_{X,\mathbf{C}}$ depends only on the bundle $P \to X$ and the connection ∇ on it. Therefore the difference

$$(f, \eta^1, \eta^2, \eta^3) - (\hat{f}, \hat{\eta}^1, \hat{\eta}^2, \hat{\eta}^3)$$

is a coboundary δa in this complex. In fact, an explicit formula was given for a in [10]. Let $\bar{a}$ denote the image of a in the truncated quotient complex $\mathbb{C}_X^* \to \underline{A}^1_{X,\mathbb{C}} \to \underline{A}^2_{X,\mathbb{C}}/F^2_{\text{Hodge}}$.

To finish the proof, we will show that $(\hat{f}, \hat{\eta}^1, \hat{\eta}^2) - \phi(h, \omega)$ is the coboundary of a Čech cochain $b := (b^0, b^1, b^2)$ in the complex $\mathbb{C}_X^* \to \underline{A}^1_{X,\mathbb{C}} \to \underline{A}^2_{X,\mathbb{C}}/F^2_{\text{Hodge}}$. Define

$$b^0_{ijk} = \exp\left(-2\pi i \int_{\tilde{\sigma}_{ijk}} \mu_i\right)$$

$$b^1_{ij} = 2\pi i \int_{\tilde{\gamma}_{ij} \to U_{ij}} \mu_i$$

$$b^2_i = 2\pi i \cdot s_i^* \mu_i.$$

From 5.5 (2) and the fact that the tangent space to $\tilde{\sigma}$ is purely vertical, we see that

$$(\delta b^0)_{ijkl} = \exp\left\{-2\pi i \int_{\tilde{\sigma}_{jkl}} (\mu_j - \mu_i) - 2\pi i \int_{\partial \tilde{T}_{ijkl}} \mu_i\right\}$$

$$= \exp\left\{-2\pi i \int_{\tilde{\sigma}_{jkl}} \beta_{ij} - 2\pi i \int_{\tilde{T}_{ijkl}} d\mu_i\right\}.$$

But from 5.5 (1), this is exactly the quotient $\underset{\sim}{\psi}(f_{ijkl}) \cdot \phi(h_{ijkl})^{-1}$.

Similarly, for ξ a vector field on U_{ijk} and $\tilde{\xi}$ any lift of ξ, we find that

$$\eta^1_{ijk}(\xi) - \omega_{ijk}(\xi) - d \log b^0_{ijk}(\xi) = 2\pi i \int_{\tilde{\gamma}_{jk}} i(\tilde{\xi}) \cdot \beta_{ij} + 2\pi i \int_{\partial \tilde{\sigma}_{ijk}} i(\tilde{\xi}) \cdot \mu_i,$$

again using Theorem 5.5, and this is the coboundary of $b^1_{ij}(\xi)$.

Finally, we note that the 2-form $(\xi_1, \xi_2) \mapsto -2\pi i \int_{\tilde{\gamma}_{ij}} i(\tilde{\xi}_1) \cdot i(\tilde{\xi}_2) \cdot CS(\nabla)$ is equal to the 2-form $(\xi_1, \xi_2) \to 2\pi i \int_{\tilde{\gamma}_{ij}} i(\tilde{\xi}_1) \cdot i(\tilde{\xi}_2) \cdot d\mu_i$, modulo F^2_{Hodge}. We may assume that $[\xi_1, \xi_2] = [\tilde{\xi}_1, \tilde{\xi}_2] = 0$, so that $[L_{\tilde{\xi}_1}, i(\tilde{\xi}_2)] = [L_{\tilde{\xi}_2}, i(\tilde{\xi}_1)] = 0$, where L denotes the Lie derivative. Using the identity $L_V = d \circ i(V) + i(V) \circ d$ repeatedly, we find that

$$\eta^2_{ij}(\xi_1, \xi_2) + db^1_{ij}(\xi_1, \xi_2) = 2\pi i \int_{\tilde{\gamma}_{ij}} d \circ i(\tilde{\xi}_1) \cdot i(\tilde{\xi}_2) \cdot \mu_i$$

$$= 2\pi i s_j^* \mu_i(\tilde{\xi}_1, \tilde{\xi}_2) - 2\pi i s_i^* \mu_i(\tilde{\xi}_1, \tilde{\xi}_2).$$

But this is equal to the coboundary of b^2, because $s_j^*\mu_j - s_j^*\mu_i = s_j^*(\mu_j - \mu_i) = s_j^*\beta_{ij} +$ basic and $s_j^*\beta_{ij}$ is zero.

We may then define the cochain α in the statement of the theorem by $\alpha := \bar{a} + b$. This finishes the proof of Theorem 5.5. $\square$

It follows from this theorem that the data

$$(\psi(f_{ijkl}, \eta^1_{ij}, \eta^2_i), \phi(h_{ijkl}, \omega^1_{ijk}), \alpha)$$

combine to give a cocycle κ with coefficients in the double complex of sheaves

$$\begin{array}{ccccc}
\underline{\mathbb{C}}^*_X & \to & \underline{A}^1_{X,\mathbb{C}} & \to & \underline{A}^2_{X,\mathbb{C}}/F^2_{\mathrm{Hodge}} \\
\uparrow & & \uparrow & & \uparrow \\
\underline{\mathcal{O}}^*_X \oplus \underline{\mathbb{T}}_X & \to & \underline{\Omega}^1_X \oplus i\underline{A}^1_X & \to & i\underline{A}^2_X.
\end{array}$$

This complex with the bottom row placed in degree 1 will be denoted by $\mathbb{Z}(2)_{\mathrm{D.h.h}}$. With this convention, κ determines a cohomology class in $H^4(X; \mathbb{Z}(2)_{\mathrm{D.h.h}})$.

The cocycle κ maps to (h_{ijkl}, ω_{ijk}) and $(f_{ijkl}, \eta^1_{ijk}, \eta^2_{ij})$ under the obvious projections from $\mathbb{Z}(2)_{\mathrm{D.h.h}}$. We can recover the full differential character $\tilde{c}^\nabla_2$ and not just its truncation, if we use the double complex of sheaves

$$\begin{array}{ccccccc}
\underline{\mathbb{C}}^*_X & \to & \underline{A}^1_{X,\mathbb{C}} & \to & \underline{A}^2_{X,\mathbb{C}}/F^2 & \to & \underline{A}^3_{X,\mathbb{C}}/F^2 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\underline{\mathcal{O}}^*_X \oplus \underline{\mathbb{T}}_X & \to & \underline{\Omega}^1_X \oplus i\underline{A}^1_X & \to & i\underline{A}^2_X & \to & i\underline{A}^3_X.
\end{array}$$

Since the last column is acyclic, this complex is quasi-isomorphic to $\mathbb{Z}(2)_{\mathrm{D.h.h}}$. Therefore, κ gives rise to a cocycle with coefficients in this completed complex and maps to $\tilde{c}^\nabla_2$ by projection to $\underline{\mathbb{T}}_X \to i\underline{A}^1_X \to i\underline{A}^2_X \to i\underline{A}^3_X$.

COROLLARY 5.7. *Let $A^p(X)_{\mathbb{C}}$ denote the global complex-valued p-forms. The image of κ under the natural map*

$$H^4(X; \mathbb{Z}(2)_{\mathrm{D.h.h}}) \to A^4(X)_{\mathbb{C}}$$

induced by the exterior derivative, is $-2\pi i$ times the Chern-Weil representative for the second Chern class.

Proof. The map $H^4(X; \mathbb{Z}(2)_{\mathrm{D.h.h}}) \to A^4(X)_{\mathbb{C}}$ is defined by applying the exterior derivative to the $i\underline{A}^3_X$ component in the completed complex above. For the cocycle κ, this component is $-2\pi i$ times the Chern-Simons 3-form. $\square$

6. Integration over the fiber in Deligne cohomology. Let $f: X \to Y$ be a proper holomorphic fibration, where the fibers are connected Riemann surfaces of genus g. We will show the following theorem.

THEOREM 6.1. *There exists a natural map*

$$\int_f : H^4(X; \mathbb{Z}(2)_{\mathrm{D.h.h}}) \to H^2(Y; \mathbb{Z}(1)_{\mathrm{D.h.h}})$$

and a commutative diagram

$$\begin{array}{ccc} H^4(X; \mathbb{Z}(2)_{\mathrm{D.h.h}}) & \to & H^2(Y; \mathbb{Z}(1)_{\mathrm{D.h.h}}) \\ \downarrow & & \downarrow \\ A^4(X)_{\mathbb{C}} & \xrightarrow{\int_f} & A^2(Y)_{\mathbb{C}}. \end{array}$$

The bottom horizontal arrow is just the usual integration of differential forms over the fiber.

Proof. We will construct a natural trace map

$$\mathrm{tr}\colon Rf_* \mathbb{Z}(2)_{\mathrm{D.h.h}}[2] \to \mathbb{Z}(1)_{\mathrm{D.h.h}}$$

in the derived category of bounded complexes of sheaves on Y. Set

$$\mathscr{A}_X^\bullet := (\underline{\mathbb{T}}_X \to i\underline{A}_X^1 \to i\underline{A}_X^2)[-1]$$

$$\mathscr{B}_X^\bullet := (\underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1)[-1]$$

$$C_X^\bullet := (\underline{\mathbb{C}}_X^* \to \underline{A}_{X,\mathbb{C}}^1 \to \underline{A}_{X,\mathbb{C}}^2/F^2)[-1].$$

Then $\mathbb{Z}(2)_{\mathrm{D.h.h}}$ is the cone

$$\mathrm{Cone}\{\mathscr{A}_X^\bullet \oplus \mathscr{B}_X^\bullet \to C_X^\bullet\}[-1]$$

of the mapping of complexes $\psi - \phi$, where ϕ, ψ are the natural inclusions considered in Section 5. We will first construct trace maps on each of the complexes $\mathscr{A}_X^\bullet, \mathscr{B}_X^\bullet, C_X^\bullet$ separately.

Choose an open covering $\mathscr{U} = \{U_i\}$ of X such that

(1) for each $i_0, \ldots, i_p$ with $U_{i_0 \ldots i_p}$ non-empty, the restriction of f to $U_{i_0 \ldots i_p}$ is a surjective fibration with contractible fibers;

(2) the intersection of any four or more distinct open sets of $\mathscr{U}$ is empty.

Note that an orientation of f amounts to an orientation of the nerve of the covering. Fix such an orientation.

The complex $\mathscr{A}_X^\bullet$ is quasi-isomorphic to $\mathrm{Cone}\{(2\pi i)\mathbb{Z} \oplus i\underline{A}_X^{\bullet \geqslant 3} \to i\underline{A}^\bullet_X\}$, and it is easy to describe the direct image of this complex. Indeed, $Rf_*(2\pi i)\mathbb{Z}$ is realized by the complex of sheaves

$$\mathscr{C}^p(\mathscr{U};(2\pi i)\mathbb{Z}) := \oplus_{i_0\dots i_p}(f_{i_0\dots i_p})_*(2\pi i)\mathbb{Z},$$

where $f_{i_0\dots i_p}$ denotes the restriction of f to $U_{i_0\dots i_p}$. Note that the direct image sheaf $(f_{i_0\dots i_p})_*(2\pi i)\mathbb{Z}$ is either zero or the constant sheaf $(2\pi i)\mathbb{Z}$. By (2) above, the complex of sheaves $\mathscr{C}^\bullet(\mathscr{U},(2\pi i)\mathbb{Z})$ is concentrated in degrees $\leqslant 4$. We define tr: $\mathscr{C}^p(\mathscr{U},(2\pi i)\mathbb{Z}) \to (2\pi i)\mathbb{Z}$ on each nonzero factor $(f_{i_0\dots i_{p*}})(2\pi i)\mathbb{Z} \simeq (2\pi i)\mathbb{Z}$ as plus or minus the identity, where the sign is given by the orientation of the p-simplex $(i_0\dots i_p)$ in the nerve of $\mathscr{U}$.

The direct image complex $Rf_* i\underline{A}^q_X$ is realized by the double complex of sheaves

$$\mathscr{C}^p(\mathscr{U};i\underline{A}^q_X) := \oplus_{i_0\dots i_p}(f_{i_0\dots i_p})_* i\underline{A}^q_X.$$

To describe a concrete trace morphism

$$\mathrm{tr}\colon \mathscr{C}^\bullet(\mathscr{U};i\underline{A}^\bullet_X) \to i\underline{A}^{\bullet-2}_Y,$$

we need to choose a partition of unity $\{\tau_i\}$ subordinate to $\mathscr{U}$, where the support of each τ_i maps properly to Y. Then for an open set V of Y and for

$$\omega \in iA^q(U_{i_0\dots i_p} \cap f^{-1}(V)),$$

we define

$$\mathrm{tr}(\omega) := \int_f \tau_{i_0}\, d\tau_{i_1} \wedge \cdots \wedge d\tau_{i_p} \wedge \omega.$$

This is indeed a $(p+q-2)$-form on V. Observe that this trace map is the composition of the quasi isomorphism $\mathscr{C}^\bullet(\mathscr{U};\, \underline{A}^\bullet_X) \to f_*\,\underline{A}^\bullet_X$ given by the partition of unity [4], with the integration map for differential forms $\int_f\colon \underline{A}^\bullet_X \to \underline{A}^{\bullet-2}_Y$.

Since the map tr in singular cohomology is compatible with fiber integration of differential forms, we have a commutative diagram

$$\begin{array}{ccc}
\mathscr{C}^\bullet(\mathscr{U},(2\pi i)\mathbb{Z}) & \xrightarrow{\mathrm{tr}} & (2\pi i)\mathbb{Z}_Y[-2] \\
\downarrow & & \downarrow \\
\mathscr{C}^\bullet(\mathscr{U},i\underline{A}^\bullet_X) & \xrightarrow{\mathrm{tr}} & i\underline{A}^\bullet_Y[-2].
\end{array}$$

Note that the morphism $\mathrm{tr}\colon \mathscr{C}^\bullet(\mathscr{U},i\underline{A}^\bullet_X) \to i\underline{A}^{\bullet-2}_Y$ maps $\mathscr{C}^\bullet(\mathscr{U},i\underline{A}^{\bullet\geqslant m}_X)$ to

$i\underline{A}_Y^{\bullet \geqslant m-2}$. Hence we obtain a morphism of mapping cones from

$$\mathrm{Cone}\{\mathscr{C}^p(\mathscr{U},(2\pi i)\mathbb{Z}) \oplus \mathscr{C}^\bullet(\mathscr{U}, i\underline{A}_X^{\bullet\geqslant m}) \to \mathscr{C}^\bullet(\mathscr{U}, i\underline{A}_X^\bullet)\}$$

to

$$\mathrm{Cone}\{(2\pi i)\mathbb{Z} \oplus i\underline{A}_Y^{\bullet\geqslant m-2} \to i\underline{A}_Y^\bullet\}[-2].$$

In particular, after a quasi isomorphism, this gives a trace map tr: $Rf_*\mathscr{A}^\bullet \to \mathbb{I}_Y$.

Next, the direct image $Rf_*C_X^\bullet$ is quasi-isomorphic to

$$\mathrm{Cone}\{Rf_*(2\pi i)\mathbb{Z} \oplus Rf_*\underline{A}_{X,\mathbb{C}}^{\bullet\geqslant 3}/F^2 \to Rf_*\,\underline{A}_{X,\mathbb{C}}^\bullet\}$$

and this maps to $\mathrm{Cone}\{(2\pi i)\mathbb{Z} \to \underline{A}_{X,\mathbb{C}}^0\}$ in exactly the same manner as above.

For the third trace, note that $\mathscr{B}^\bullet$ is quasi-isomorphic to $\mathrm{Cone}\{(2\pi i)\mathbb{Z} \oplus \underline{\Omega}_X^{\bullet\geqslant 2} \to \underline{\Omega}_X^\bullet\}$. This in turn, is quasi-isomorphic to

$$\mathrm{Cone}\{(2\pi i)\mathbb{Z} \oplus F^2_{\mathrm{Hodge}}\,\underline{A}_{X,\mathbb{C}}^\bullet \to \underline{A}_{X,\mathbb{C}}^\bullet\}.$$

Indeed, $F^m_{\mathrm{Hodge}}\,\underline{A}_{X,\mathbb{C}}^\bullet$ is a resolution of the truncated holomorphic de Rham complex of sheaves. Using exactly the same maps as above, we obtain a natural trace from the direct image of this last complex to $2\pi i\mathbb{Z} \to \underline{\mathscr{O}}_Y$ as desired.

Observe that the above trace maps on $\mathscr{A}^\bullet, \mathscr{B}^\bullet, C^\bullet$ are all compatible; indeed, they were all defined using the same formulas on the complexes $\mathscr{C}^\bullet(\mathscr{U},(2\pi i)\mathbb{Z})$ and $\mathscr{C}^\bullet(\mathscr{U},\underline{A}_{X,\mathbb{C}}^\bullet)$. Therefore, all the individual trace maps combine to give a natural trace $Rf_*\mathbb{Z}(2)_{\mathrm{D.h.h}} \to \mathbb{Z}(1)_{\mathrm{D.h.h}}$ in the derived catgory and hence a map on cohomology groups

$$\int_f : H^4(X;\mathbb{Z}(2)_{\mathrm{D.h.h}}) \to H^2(X;\mathbb{Z}(1)_{\mathrm{D.h.h}}).$$

Finally, the commutative diagram in the statement of Theorem 6.1 follows directly from Stokes's Theorem along the fibers of a fibration. □

It would be interesting to have an explicit description of $\int_f$ on the level of Čech cocycles. Applying Theorem 6.1 to the "enriched Chern class" $\kappa \in H^4(X;\mathbb{Z}(2)_{\mathrm{D.h.h}})$ constructed in Theorem 5.5, we obtain the following result, which was exploited in [11].

COROLLARY 6.2. *Let $f\colon X \to Y$ be a proper holomorphic map whose fibers are connected Riemann surfaces. Suppose that $P \to X$ is a holomorphic $SL_n(\mathbb{C})$-bundle over X which is equipped with a smooth reduction of the structure group to an $SU(n)$-bundle $Q \to X$. Then there exists a natural Hermitian holomorphic line bundle on Y whose curvature is the 2-form $\int_f \Omega$, where Ω is the Chern-Weil representative for the second Chern class of P with respect to the canonical connection determined by the Hermitian structure Q.*

We will now give a more geometric construction of a Hermitian line bundle on Y, starting from the Hermitian holomorphic $SL_n(\mathbb{C})$-bundle $p: P \to X$. This will use the canonical 2-gerbe $\mathscr{B}$ constructed in Section 8 of Part I. The idea is to "push forward" $\mathscr{B}$ along the fibers of f, in a purely sheaf-theoretic manner. This results in a holomorphic $\mathbb{C}^*$-bundle on Y, which we will describe explicitly in Theorem 6.3.

To state the theorem, we must review some definitions from Part I. First recallthat an object of the canonical 2-gerbe $\mathscr{B}$ over an open set U of X is itself agerbe $\mathscr{C}_U$ on $p^{-1}(U)$ equipped with a fiberwise holomorphic connective structure. For each open subset V of $p^{-1}(U)$, this is an assignment to each object A of the category $\mathscr{C}_U(V)$ of a relative $\underline{\Omega}^1$-torsor $\mathrm{Co}(A)$. We say that a connective structure $A \mapsto \widetilde{\mathrm{Co}}(A)$ is *admissible* if we have a given identification of $\mathrm{Co}(A)$ with the relative $\underline{\Omega}^1$-torsor $\widetilde{\mathrm{Co}}(A) \times^{\underline{\Omega}^1_P} \underline{\Omega}^1_{P|X}$, which is compatible with restrictions to smaller open sets. A *twisted curving* of an admissible connective structure $\widetilde{\mathrm{Co}}$ is an assignment to each $\tilde{\nabla} \in \widetilde{\mathrm{Co}}(A)$ of a family of 2-forms $\tilde{K}_i = \tilde{K}_i(A, \tilde{\nabla})$ on $V \cap p^{-1}(U_i)$. These 2-forms are only defined modulo $F^2(\underline{\Omega}^2)$ and must satisfy $\tilde{K}_j - \tilde{K}_i = 2\pi i \cdot \beta_{ij} \pmod{F^2(\underline{\Omega}^2)}$ on $V \cap p^{-1}(U_{ij})$. Here F^2 refers to the Cartan filtration on differential forms, so that $F^2(\underline{\Omega}^2)$ are the horizontal forms on P. The fiberwise connective structure Co is also equipped with a curving $\nabla \mapsto K(\nabla)$ which is defined mod F^1 and we say that the twisted curving $\tilde{K}$ of $\widetilde{\mathrm{Co}}$ is *admissible* if $\tilde{K}(\tilde{\nabla}) = K(\nabla) \bmod F^1$.

THEOREM 6.3. *Let $p: P \to X$ be a holomorphic principal $SL_n(\mathbb{C})$-bundle and let $f: X \to Y$ be as above. With the notation of Section 8, Part I, let S be the set of quintuples $(y, \underline{\mathscr{C}}_y, \widetilde{\mathrm{Co}}, \tilde{K}, z)$, where*

(1) *$y \in Y$ and $\underline{\mathscr{C}}_y$ is an object of the restriction of the 2-gerbe $\mathscr{B}$ to $f^{-1}(y)$;*
(2) *$\widetilde{\mathrm{Co}}$ is an admissible connective structure on $\underline{\mathscr{C}}_y$, and $\tilde{K}$ is an admissible twisted curving;*
(3) *$z \in \mathbb{C}^*$.*

Let $\simeq$ be the equivalence relation on S generated by the following two relations:

(a) *$(y, \underline{\mathscr{C}}_{1y}, \widetilde{\mathrm{Co}_1}, \tilde{K}_1, z)$ is equivalent to $(y, \underline{\mathscr{C}}_{2y}, \widetilde{\mathrm{Co}_2}, \tilde{K}_2, z)$ if there exists an equivalence of gerbes with connective structure and twisted curving on $p^{-1}f^{-1}(y)$, between $(\underline{\mathscr{C}}_{1y}, \widetilde{\mathrm{Co}_1}, \tilde{K}_1)$ and $(\underline{\mathscr{C}}_{2y}, \widetilde{\mathrm{Co}_2}, \tilde{K}_2)$.*
(b) *$(y, \underline{\mathscr{C}}_y, \widetilde{\mathrm{Co}}, \tilde{K}, z)$ is equivalent to $(y, \underline{\mathscr{C}}_y, \widetilde{\mathrm{Co}} \otimes N, \tilde{K}, z \cdot \exp(\tau_N))$, where N is an $\underline{\Omega}^1_{f^{-1}(y)}$-torsor over $f^{-1}(y)$, $\widetilde{\mathrm{Co}} \otimes N$ is the corresponding twisted connective structure, with the admissible twisted curving $\tilde{K}$, and $\tau_N := \int_{f^{-1}(y)}[N]$ gives the canonical isomorphism $H^1(f^{-1}(y); \underline{\Omega}^1) \to \mathbb{C}$.*

Then the quotient $\mathscr{L}$ of S by $\simeq$ is a complex manifold on which $\mathbb{C}^*$ acts by

$$w \cdot (y, \underline{\mathscr{C}}_y, \widetilde{\mathrm{Co}}, \tilde{K}, z) = (y, \underline{\mathscr{C}}_y, \widetilde{\mathrm{Co}}, \tilde{K}, w \cdot z),$$

so that $\mathscr{L} \to Y$ is a holomorphic principal $\mathbb{C}^*$-bundle.

We can also use the differential geometry of the 2-gerbe to find a Hermitian metric on $\mathscr{L} \to Y$.

PROPOSITION 6.4. *Suppose that the principal bundle $P \to X$ of Theorem 6.3 admits a smooth reduction of the structure group to an $SU(n)$-bundle $Q \to X$. Then the $\mathbb{C}^*$-bundle $\mathscr{L} \to Y$ admits a corresponding smooth reduction of its structure group to $\mathbb{T}$, i.e., $\mathscr{L} \to Y$ has a Hermitian metric.*

Proof. There is a natural version of the constructions of Section 8, Part I for $SU(n)$-bundles rather than $SL_n(\mathbb{C})$-bundles. One considers unitary connective structures and curvings and uses smooth local sections of the $SU(n)$-bundle $Q \to X$ instead of holomorphic sections of $P \to X$. Here Q is to be regarded as a subbundle of P. The "push forward" of Q along the fibers of f is then a smooth $\mathbb{T}$-bundle $L \to Y$ and admits a description similar to that of $\mathscr{L}$ in Theorem 6.3. It follows from Lemma 8.6 of Part I, that the line bundle obtained by "push forward" does not depend on the choice of local smooth sections of the bundle $P \to X$. Therefore L must identify with the smooth line bundle associated to $\mathscr{L}$. □

The next step is to identify the isomorphism class of the line bundle $\mathscr{L}$. Recall that the exterior derivative induces a map $d\colon H^{2p-1}(X; \underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1 \to \cdots \to \underline{\Omega}_X^{p-1}) \to H^p(X; \underline{\Omega}_X^p)$. We have the following result.

PROPOSITION 6.5. *The image of $\mathscr{L}$ in $H^1(Y; \underline{\Omega}_Y^1)$ under the exterior derivative is obtained from the image of $\mathscr{B}$ in $H^2(X; \underline{\Omega}_X^2)$ by applying the trace map $\int_f\colon H^2(X; \underline{\Omega}_X^2) \to H^1(Y; \underline{\Omega}_Y^1)$.*

Proof. This follows from Proposition 8.4 of Part I. The key point is that any two connective structures on a local object of $\mathscr{B}$ differ by a well-defined $\underline{\Omega}_X^1$-torsor N. Therefore, any two local sections of $\mathscr{L}$ must differ on overlaps by multiplication by $\exp(\tau_N)$, for some such N. Applying d log we obtain $d \int [N]$, which equals $\int d[N]$ by Stokes's Theorem along the fibers of a fibration. In Section 8 of Part I, we showed how to construct an explicit cocycle representing the class of $\mathscr{B}$ in $H^3(X; \underline{\mathcal{O}}_X^* \to \underline{\Omega}_X^1)$. The $\underline{\Omega}^1$ component α of this cocycle is exactly a section of the torsor N, and so the class of $\mathscr{B}$ in $H^2(X; \underline{\Omega}_X^2)$ is given by $[d\alpha]$. □

Remark 6.6. We do not know the exact relationship between $\mathscr{L}$ and the holomorphic line bundle constructed in 6.2. However, in the case where X is projective, the cohomology class determined by $\mathscr{B}$ agrees (mod torsion) with $\hat{c}_2$. It then follows from 6.5 that in this case, both line bundles must differ by tensoring with a flat line bundle.

7. Algebro-geometric construction of metrized line bundles on moduli spaces of vector bundles. Throughout this section, Σ will be a fixed compact Riemann surface of genus $g \geqslant 2$. A holomorphic vector bundle $E \to \Sigma$ is said to be *stable*

[32], if for every proper holomorphic subbundle $F \to \Sigma$ we have

$$\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E},$$

where $\deg E := c_1(E)[\Sigma]$. One defines the notion of a semistable bundle in a similar fashion, by allowing $\leqslant$ in the inequality.

If r and d are coprime, the set of isomorphism classes of stable bundles of rank r and degree d form a smooth, compact projective variety $\mathscr{M}(r,d)$. For general r and d, the space $\mathscr{M}(r,d)$ has a natural compactification $\bar{\mathscr{M}}(r,d)$ by allowing semistable bundles [39].

We will be especially interested in the subset $\mathscr{M}(r,\mathscr{L})$ of $\mathscr{M}(r,d)$ consisting of those stable bundles $E \to \Sigma$, for which $\mathrm{Det}(E)$ is isomorphic to a fixed line bundle $\mathscr{L}$ of degree d. If $(r,d) = 1$, this subvariety $\mathscr{M}(r,\mathscr{L})$ is simply-connected, and its homology is torsion-free [1].

A Poincaré family for Σ is by definition a holomorphic vector bundle $P \to \mathscr{M}(r,d) \times \Sigma$ with the property that for each point $\{E\}$ in $\mathscr{M}(r,d)$, the restriction of the bundle P to $\{E\} \times \Sigma$ is in the isomorphism class of $\{E\}$. A Poincaré family is not unique; any two choices differ by tensoring with a holomorphic bundle $L \to \mathscr{M}(r,d) \times \Sigma$. The universal property implies that in the Σ-direction, L must be flat and satisfy $L^r = 1$ [1]. Although a Poincaré family $P \to \mathscr{M}(r,d) \times \Sigma$ exists if and only if $(r,d) = 1$ [35], nevertheless the tensor product $P \otimes P^*$ always exists and is unique (see [11]); this is the holomorphic vector bundle on $\mathscr{M}(r,d) \times \Sigma$, whose restriction to $\{E\} \times \Sigma$ is in the isomorphism class of the bundle $E \otimes E^* \to \Sigma$.

The tangent space to $\mathscr{M}(r,d)$ at the point $\{E\}$ is the cohomology group $H^1(\Sigma; \mathrm{ad}\, E)$. Its complex structure is given by the Hodge theory for this cohomology group. Concretely, as explained in Section 4 of [1], we can view any tangent vector as a Lie algebra valued 1-form $A \in \Omega^1(\Sigma; \mathrm{ad}\, E)$.

Much of the differential geometry of $\mathscr{M}(r,d)$ is captured by the fundamental Theorem of Narasimhan-Seshadri [32] (see also Donaldson [19]).

THEOREM 7.1. *Fix a Hermitian metric on Σ, normalized to have unit volume. Then an indecomposable holomorphic vector bundle $E \to \Sigma$ of rank r and degree d is stable if and only if there is a unitary connection on E having constant central curvature $*F = -2\pi i d/r$. Such a connection is unique up to isomorphism.*

Our observation is that the Narasimhan-Seshadri Theorem defines a Hermitian structure on any Poincaré family and we can compute its curvature directly.

THEOREM 7.2. *Suppose that $(r,d) = 1$. Fix a line bundle $\mathscr{L} \to \Sigma$ of degree d and a Poincaré family $P \to \mathscr{M}(r,\mathscr{L}) \times \Sigma$.*

(1) *The Narasimhan-Seshadri Theorem gives a smooth reduction of the structure group of $P \otimes P^* \to \mathscr{M}(r,\mathscr{L}) \times \Sigma$ from $SL_{r^2}(\mathbb{C})$ to $SU(r^2)$.*

(2) *Let K be the curvature of the unique connection ∇ on $P \otimes P^*$ defined by this unitary structure. Then K is flat in the Σ-direction and the Chern-Weil representa-*

tive $\Omega := (1/(8\pi^2))\,\mathrm{Tr}(K \wedge K)$ *satisfies*

$$\Omega(A, A', v, v') = \frac{2r}{8\pi^2}\,\mathrm{Tr}(\langle A, v\rangle \cdot \langle A', v'\rangle + \langle A, v'\rangle \cdot \langle A', v\rangle),$$

where $\langle\ ,\ \rangle$ *denotes the natural pairing between the tangent vectors* A, A' *to* $\mathscr{M}(r, \mathscr{L})$ *viewed as 1-forms on* Σ, *and the vectors* v, v' *on* Σ.

Proof. (1) is clear. For the proof of (2), it is obvious that $P \otimes P^*$ is flat in the Σ-direction, so we need only compute Ω. Choose a Galois covering $\pi\colon \Sigma_1 \to \Sigma$ of degree r. Let $\mathscr{L}_1 \to \Sigma_1$ be any holomorphic line bundle with the property $\pi^*\mathscr{L} \simeq \mathscr{L}_1^r$. Define a holomorphic map $\phi\colon \mathscr{M}_\Sigma(r, \mathscr{L}) \to \mathscr{M}_{\Sigma_1}(r, 0)$ by the formula

$$\phi(E) := \pi^*E \otimes \mathscr{L}_1^{-1}.$$

Here we use subscripts in our notation to emphasize that ϕ is a mapping of moduli spaces of bundles over *different* surfaces. Although a Poincaré family $\mathscr{F}$ over $\mathscr{M}_{\Sigma_1}(r, 0) \times \Sigma_1$ does not exist, nevertheless the bundle $\mathscr{F} \otimes \mathscr{F}^*$ exists and is unique. It is invariant under the action of the Galois group of the covering π and therefore descends to a holomorphic bundle $V \to \mathscr{M}_\Sigma(r, \mathscr{L}) \times \Sigma$. The universal property of the family $\mathscr{F} \otimes \mathscr{F}^*$ then implies that the $SL_{r^2}(\mathbf{C})$-bundle V must identify with $P \otimes P^*$. Similarly, the Hermitian structure on $\mathscr{F} \otimes \mathscr{F}^*$ coming from the Narasimhan-Seshadri Theorem, descends to the Hermitian structure on $P \otimes P^*$. This reduces the problem to computing the curvature K_1 of the connection on the bundle $\mathscr{F} \otimes \mathscr{F}^* \to \mathscr{M}_{\Sigma_1}(r, 0) \times \Sigma_1$ defined by the Narasimhan-Seshadri Theorem.

This last computation was carried out by us in Section 6 of [11]. There we found that the Chern-Weil representative Ω_1 for K_1 satisfies

$$\Omega_1(A_1, A_1', v_1, v_1') = \frac{r}{4\pi^2}\,\mathrm{Tr}(\langle A_1, v_1\rangle \langle A_1', v_1'\rangle + \langle A_1, v_1'\rangle \langle A_1', v_1\rangle),$$

where A_1, A_1' are tangent vectors to $\mathscr{M}_{\Sigma_1}(r, 0)$ and v_1, v_2 are tangent vectors to Σ_1. This then descends to the 2-form over $\mathscr{M}_\Sigma(r, d) \times \Sigma$ given in (2). $\square$

As an immediate application of Corollary 6.2, we obtain the following result.

COROLLARY 7.3. *Fix a holomorphic line bundle* $\mathscr{L}$ *of degree* d *over* Σ *and suppose that* $(r, d) = 1$. *Let* $\mathscr{M}_\Sigma(r, \mathscr{L})$ *denote the moduli space of stable bundles with fixed determinant* $\mathscr{L}$. *Choose a Poincaré family* $P \to \mathscr{M}_\Sigma(r, \mathscr{L}) \times \Sigma$. *Then*

(1) *The universal family* $P \otimes P^* \to \mathscr{M}_\Sigma(r, \mathscr{L}) \times \Sigma$ *transgresses to a holomorphic line bundle* $L \to \mathscr{M}_\Sigma(r, \mathscr{L})$.

(2) *The Hermitian metric on* $P \otimes P^*$ *defined by the Narasimhan-Seshadri Theorem induces a Hermitian metric* h *on* L. *The curvature of the canonical connection coming from* h *is the 2-form* $(r/2\pi i)\int_\Sigma \mathrm{Tr}(A \wedge B)$, *where* A, B *are tangent vectors to* $\mathscr{M}_\Sigma(r, \mathscr{L})$.

On the other hand, it is known that the Picard group of $\mathscr{M}_\Sigma(r,\mathscr{L})$ is infinite cyclic and its generator is the determinant line bundle $\mathrm{Det} \to \mathscr{M}_\Sigma(r,\mathscr{L})$ [1], [20], [27]. By definition, Det is the holomorphic line bundle whose fiber at any point $\{E \to \Sigma\}$ is the complex line $\wedge^{\max} H^1(\Sigma, \mathrm{ad}\, E)^{\otimes^{-1}}$ (there are no global sections of ad E since E is stable). The line bundle Det carries a natural Hermitian structure given by its Quillen metric [34]. The curvature of the associated canonical connection is the symplectic form

$$\omega(A,B) := \frac{1}{2\pi i}\int_\Sigma \mathrm{Tr}(A \wedge B),$$

considered by many authors (cf. [1] and the references therein).

Remark 7.4. In view of Proposition 5.3 and the fact that $\mathscr{M}_\Sigma(r,\mathscr{L})$ is simply-connected, the metrized line bundle (L,h), constructed in 7.3, identifies with $(\mathrm{Det},\omega)^r$—the rth power of the determinant line bundle equipped with its Quillen metric. To obtain the determinant line bundle by our transgression procedure rather than its rth power, one would have to extend Corollary 6.2 to $GL_r(\mathbb{C})$-bundles and then apply it to a Poincaré family $P \to \mathscr{M}_\Sigma(r,\mathscr{L}) \times \Sigma$ instead of $P \otimes P^*$. We shall not carry this out here.

It is important to note that 7.3 only gives the cohomolgy class of $(\mathrm{Det},h)^r$ in $H^2(\mathscr{M}_\Sigma(r,\mathscr{L}); \mathbb{Z}(1)_{\mathrm{D.h.h}})$. However it is possible to geometrically recover the actual line bundle itself using Theorem 6.3.

THEOREM 7.5. *Let $\mathscr{L}$ be a fixed holomorphic line bundle on Σ and suppose that $(r, \deg \mathscr{L}) = 1$. Choose any Poincaré family $P \to \mathscr{M}_\Sigma(r,\mathscr{L}) \times \Sigma$ and form the unique universal bundle $P \otimes P^* \to \mathscr{M}_\Sigma(r,\mathscr{L}) \times \Sigma$. Suppose that the bundle $P \otimes P^*$ is endowed with the Hermitian metric coming from the Narasimhan-Seshadri Theorem. Let $\underline{\mathscr{B}}$ be the 2-gerbe representing the second Chern class of $P \otimes P^*$. Then the "push forward" of $\underline{\mathscr{B}}$ by the projection $\mathscr{M}_\Sigma(r,\mathscr{L}) \times \Sigma \to \mathscr{M}_\Sigma(r,\mathscr{L})$ defined in Theorem 6.3 identifies with the metrized line bundle* $(\mathrm{Det},\omega)^r$ *equipped with its Quillen metric.*

Proof. This uses the fact that $\mathscr{M}_\Sigma(r,\mathscr{L})$ is simply connected and torsion free, so by Remark 6.6 the "push forward" of the 2-gerbe $\underline{\mathscr{B}}$ must identify with the metrized line bundle (L,h) constructed in Corollary 7.3 by transgression in Hermitian holomorphic Deligne cohomology. $\square$

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